THE THEORY OF MATHEMATICAL MACHINES

FRANCIS J. MURRAY

ASSOCIATE PROFESSOR OF MATHEMATICS, COLUMBIA UNIVERSITY

REVISED EDITION

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FOREWORD

1. Mathematical machines are widely utilized because they can perform certain calculations more swiftly and more accurately than any human calculator could.

We define a mathematical machine as follows: A mathematical machine is a mechanism which provides information concerning the relationships between a specified set of mathematical concepts. Normally, this requires that the machine contain a realization of the set of mathematical concepts. Thus, a counter will contain a method of realizing certain of the natural numbers, an adding machine will have a realization of these and a way of realizing the operation of addition, while a differential analyzer will contain a realization of the notion of function and integral. However, the essential part of the definition of a mathematical machine is that it provides information concerning the mathematical concepts involved. Thus, an electrical network may constitute a perfect realization of a system of differential equations but we shall not regard it as a mathematical machine unless it provides information concerning the system; for instance, concerning the solutions of the system of differential equations.

2. It is interesting to note that this process of utilizing a mathematical machine is the converse of the usual applications of mathematics. Ordinarily, we have a natural system, which we want to analyze mathematically. We endeavor to find a set of mathematical concepts, in terms of which we can describe the system. In a mathematical machine, however, it is the mathematical concepts which are given and we set up a natural system which realizes the concepts. In a number of cases, it will be obvious that both processes occur. For instance, if we have a natural system, we may be able to describe it in mathematical terms. However, the associated calculations may be too laborious or time consuming. Hence, we set up a mathematical machine to perform these. Notice then that the mathematical machine must be abstractly equivalent to the original system.

Thus the theory of mathematical machines is quite interesting in connection with the foundations of mathematics. For, if we define a system of mathematical concepts as one which is determined by "postulates," i.e., certain statements which can be used as the basis of a logical discussion, in general, such a system is obtained by a process of "abstraction" and the theory of mathematical machines may be of considerable interest in the study of this latter operation.

3. However, besides its importance in calculation and from the theoretical point of view, our subject is fascinating for its own sake. We shall find many ingenious devices, we shall come upon many interesting mathematical problems and incidentally we shall make a survey of the elementary applications of mathematical analysis.

We must, however, have an exclusion principle and it will be this. We shall be interested in a device as far as it realizes and gives information concerning mathematical concepts. It may have a very complex use, although it may be relatively simple from the viewpoint of actual realization of mathematical concepts. We will not, in general, concern ourselves with such a complicated use of a simple device.

For a bibliography, the student is referred to the very interesting lecture, "Instrumental Analysis," by Vannevar Bush, published in the *Bulletin of the American Mathematical Society*, Vol. 42 (1936), pp. 649-69. There have been, however, a number of recent papers of considerable interest to which reference will be made.

4. The use of mathematical machines is of increasing importance for scientific and technical purposes. Various theoretical developments require elaborate mathematical procedures which can be carried out only by machine methods.

Many aspects of the subject are of interest to mathematicians. It is connected in numerous ways with basic logical questions and the relationship between mathematics and the other sciences. There are innumerable contacts with the extraordinary technological advances which are characteristic of our civilization. In mathematical machines, one finds examples of practically every essential modern technique, either of a mechanical or electrical nature. These techniques have a very respectable mathematical basis with which the analyst should be familiar.

In addition the subject is rapidly approaching the intellectual maturity corresponding to the research level. The earlier theoretical developments were geometrical in character, although the paper of Shannon on the differential analyzer was based on algebraic reasoning and is definitely of a more abstract nature. Recently, the problems of computation on sequential digital machines have indicated the importance of stability and this is deep. The stability problem in continuous machines is precisely analogous to the design of computations in the sequential digital case.

The recent papers of Goldstine and von Neumann show that computational questions may require advanced mathematical techniques. There are, of course, many scientific and technical problems for which no mathematical solution is known. But even in the case where mathematically satisfactory solutions have been obtained considerable research may be necessary before satisfactory computational procedures can be set up.

FOREWORD TO THE REVISED EDITION

Books and mathematical devices have one regrettable aspect in common. Initially, they have "bugs." It is hoped that in this second edition, we have eradicated most of the errors. Planographing should at least make the fumigation monotonic.

Part I has been expanded somewhat and a summary added. The discussion of trigger circuits in Part II has been altered and augmented. In Part III, new chapters on "Electronic Digital Computers" and "Noise, Accuracy and Stability" have been added. The pages have been renumbered and an index added.

The author wishes to thank the many persons who have shown an interest in the book, indicated errors and pointed out other ways in which the book could be improved. The assistance and comments of H. H. Goldstine were particularly valuable as well as the author's contact with Robert Walker. The kind cooperation of Miss Russell, Miss Pelliciari and Mr. Aichroth is gratefully acknowledged.

TABLE OF CONTENTS

PART ONE: DIGITAL MACHINES

I. Counters

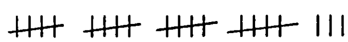
1. Let us consider the natural numbers 1, 2, 3, ... The natural numbers can be regarded either as ordinal numbers or cardinal numbers. From the elementary standpoint both these notions can be associated with the operation of counting. The ordinals are used in the operation itself, the cardinal denote the result. For instance, if we count seven objects, we might say one, two, three, four, five, six, seven, and this is clearly an ordering. When we say that there are seven objects, we give the cardinal meaning to seven.

Thus both aspects of the natural numbers are immediately associated with the operation of counting and it is in connection with aids to counting, that we will find the natural numbers realized.

Any sequence of distinguishable objects which can be precisely labeled can be used as a realization of a finite part of the set of ordinals. When we "count out loud," there is a sequence of sounds memorized by the counter, which constitutes a realization. It is difficult to say whether this or "counting on one's fingers" is the most elementary. The word digit, of course, refers to finger. It is customary to assume that the Roman numbers I, II, III, IIII, represent fingers, and it has been proposed that V stands for the hand with four fingers closed and thumb extended. X stands for two hands. (Cf. Clodd, E. *History of the Alphabet*. New York: Appleton-Century Co., 1938, p. 92.)

Of course there are very many elementary methods of "keeping count." For instance, we may put pebbles in a vase, cut notches in a stick like Robinson Crusoe, or use the "quipus" or knotted cords of Peruvian Indians. (Cf. *loc. cit.*, p. 36.)

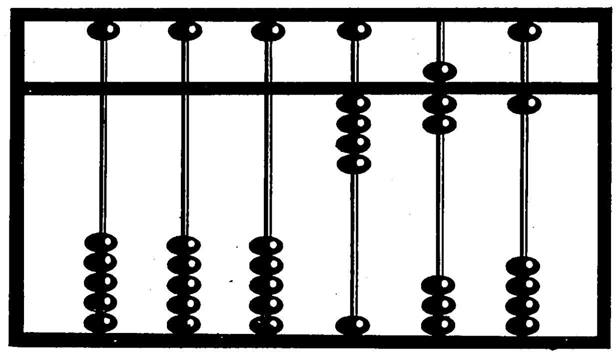
However, keeping score by means of the chalk marks, as for instance in the method illustrated



in which the chalk marks are grouped in certain ways has certain advantages. Note that the score can be immediately read off as 23 from the four full sets. The essential point here is apparent from our use of the decimal itself. We do not use 100 symbols for the numbers 0 to 99; instead we use ten symbols and position. We use the fact that the numbers from 0 to 99 are abstractly equivalent to the pairs of numbers $\alpha\beta$, where α and β run between 0 and nine. This correspondence is order preserving and hence inferentially it preserves the usual operations.

In a sense, then the decimal system of notation satisfies certain aspects of our definition of a mathematical machine. It is a realization of the concept of natural number, not as we would imagine from the definition by a simple sequence of elements but by pairs or n'tuples which have the same ordering.

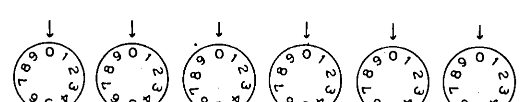
All practical modern counters use the decimal system, and it is important to realize that we have avoided the actual realization of the set of ordinals and are using something abstractly equivalent to it, i.e., the set of ordered n'tuples of digits.



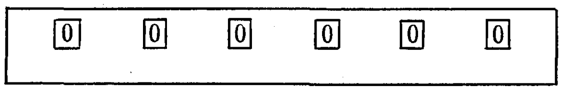
One of the simplest and most convenient methods of realizing an n-tuple of digits is the abacus (ab'a-kus). This consists of a number of beads strung on a sequence of wires in a framework. The beads are divided into two sets by a dividing line. In the upper portion, "heaven," we have one bead on each wire. This bead counts five. In the lower portion, "earth," there are five beads, which are of value one on each wire. The zero position is the position in which all the beads are against the exterior boundary and a digit is represented by pushing beads of the proper value against the middle dividing line. Thus the abacus in the diagram is set up to represent the number 471.

The diagram represents the Japanese abacus. In a picture of a Chinese abacus, I have seen there are two beads in "heaven." For a description of the use of the Japanese abacus, see Yoshino, Y. *The Japanese Abacus Explained*. Tokyo, Japan.

The abacus then realizes an n'tuple of digits. A counter is a device which realizes an n'tuple and also performs the operation of carrying. For instance, we may have a series of dials (or cylinders) covered by a plate with windows, arranged in such a way that only the digit at which the arrow is pointing will show. This then, of course, will represent

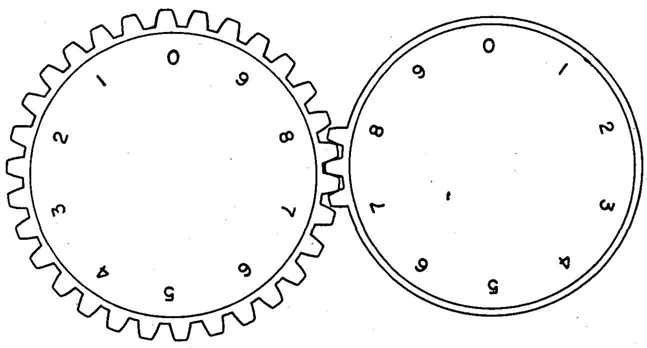


an n'tuple of digits. If we have a method of "carrying," from one place to another such a machine



can be used as a counter since it will register ten times the number of turns of the wheel furtherest to the right.

Now there are essentially two ways in which "carrying" from one place to the next can be achieved. One way is to use a pair of equal gears, with one stripped of all but 1/10' of its teeth. When the

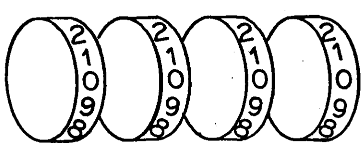


unit dial passes from 9 to 0, the teeth on it engage the teeth on the tens dial and move it a tenth of a revolution, thus carrying one. Thus, if the units dial is turned once completely, the ten's dial will move 1/10. Ten revolutions of the unit dial will yield a complete revolution of the tens dial and one-tenth of the hundred wheel. Notice that if the wheels are engaged directly as in our illustration, then adjacent wheels must turn in opposite directions. It would be preferable to insert an idler gear between the two gears in which case all dials turn in the same direction.

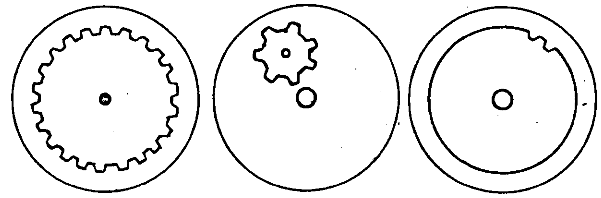
In this arrangement then, each dial except the first and the last has two gears on it, one to receive from below, the other to transmit to the next highest.

The dial form of the counter is very common in the older instruments. For instance one of these is pictured in Dyck, W., *Katalog mathematische unsw. Instrumente*. Munich: Wolf & Sohn, 1892, p. 148.

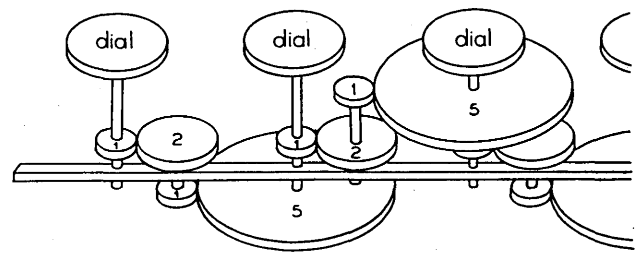
In modern calculating machines, a cylinder is preferred to a dial, because of the greater compactness and greater ease in the reading of the answer.



Each cylinder is wheel-like with a broad rim. The interior of the rim on the side next to the lower decimal place is fully toothed; that on the other side is partly toothed. There is an idler gear on a fixed partition between these. When a given cylinder passes from 9 to 0, the partly toothed rim engages the idler which is in mesh with the fully toothed part of the cylinder for the next highest place.

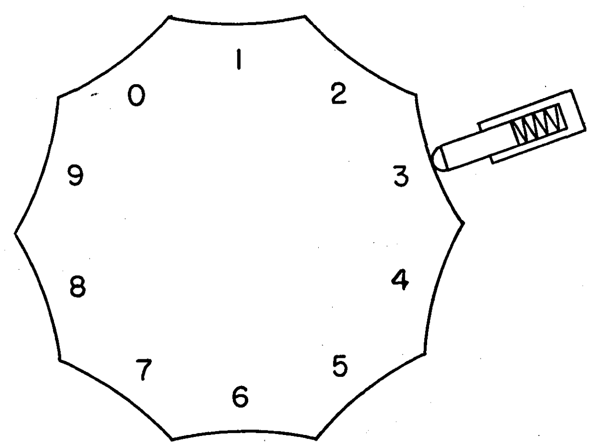


The alternate method for tens transmission in a counter is to gear each dial or cylinder directly to the next by a gear ratio of ten to one. This method has the greatest accuracy, there is no chance for a failure to transmit the one to be carried, and this is, of course, the reason why it is preferred by the public utility corporations in gas and electric meters. Such a gear ratio is indicated in the following diagram. Here we look down on such an arrangement of gears and dials.



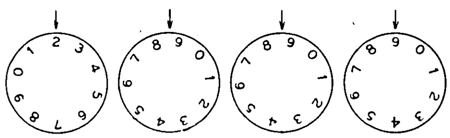
Each relative unit shaft is geared to an idler shaft in a 1 to 2 ratio and the idler shaft is geared to the relative ten shaft in a 1 to 5 ratio.

In the case of the stripped gear carry, a wheel or dial should come to rest in one of ten possible positions. This can be done by means of a positioning wheel whose circumference is dented. A simple spring device, consisting of a plunger sliding in a sleeve and pressed by a spring can be



used to insure that the dial comes to rest in one of the ten possible positions. This is very desirable to insure accuracy and ease of reading. It also insures that if the stripped gear has more than one-twentieth of the original teeth, the carrying will take place.

In the case of the complete gear carry, accuracy in reading is assured if we remember that each dial represents approximately the reading of all later dials combined. Thus, in the dials illustrated, it is clear that the reading should be



1,889, rather than 2,889, since the first dial reads almost 2, not almost 3. A certain amount of skill is required to read such dials accurately, but there is no ambiguity.

Another method of indicating the digit which appears in a decimal place in a counter is by means of a multiposition switch. Attached to the cylinder for the place is a brush which makes different contacts in accordance with the position of the cylinder. In electrically operated apparatus this means that the digit represented as in this place corresponds to a specific closed circuit.

Besides the radix 10 system for representing natural numbers, there is also the radix 2 system. In this system there are only two digits, instead of ten and it is frequently preferred by designers of electrical computing devices. An example will illustrate the representation. Thus the number $1{,}443 = 1.10^3 + 4.10^2 + 4.10 + 3 = 1.2^{10} + 0.2^9 + 1.2^8 + 1.2^7 + 0.2^6 + 1.2^5 + 0.2^4 + 0.2^3 + 0.2^2 + 1.2^1 + 1.2^0 = 1{,}01101{,}00011$. The conversion is readily accomplished if we have a table of powers of two. Thus the highest power of two is less than 1,443 is $2^{10}. = 1024$. The expansion begins therefore with 1.2^{10}. Subtracting 1,024 from 1,443, we obtain 419. Since this is smaller than $512 = 2^9$, the next term in the expansion is 0.2^9. We can, however, subtract $2^8 = 256$ and obtain the term 1.2^8. The process should be clear now.

The radix two system has certain advantages due to the fact that two is the smallest number that can be used as the base of a radix system. Thus a monetary system based on a radix two system will permit one to make up any sum of money in a given range with the least number of bills.

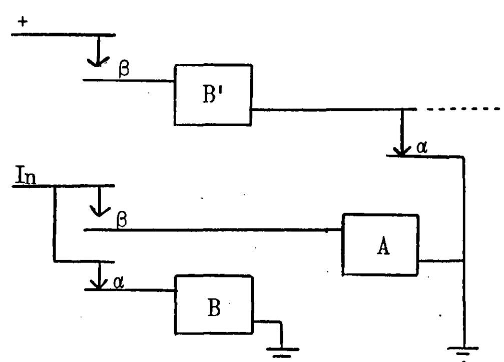
2. In general, electro mechanical counters are based on the usual counter wheel which has a clutch connection to a continuously turning shaft. The clutch is magnetically controlled. When the clutch is energized by a signal it connects the shaft to the counter wheel, causing the latter to turn and register. (Cf. *A Manual of Operation for the Automatic Sequence Controlled Calculator*. Cambridge: Harvard University Press.)

On the other hand, relays may be used directly to form counters. We describe a dyadic counter based on these.

A relay is an electrically controlled switch. It consists of an electromagnet which activates an armature upon which switch contacts are mounted. Some of these are closed when the electromagnet is not operating and these are called "normally closed." We will indicate these by α. The normally open contacts we will denote by β.

Now if an electromagnet can also be energized by a circuit through a β contact, the relay will hold itself when an impulse is received. If the holding circuit goes through an α contact of the second relay, the second relay can be used to release the first. Suppose, furthermore, that the signal impulses are channelled so that when the first relay is not activated, the signal will pass through an α contact to its (the B) coil but when it is activated a signal impulse will pass through a β contact to the second (A) relay. Thus the first impulse will set the first relay and a second impulse will return the system to its original state. It is clear that a number of relays can be used in this fashion to obtain a cyclic system of period n.

In the accompanying diagram, B and B' refer to coils on the first relay. The first impulse received passes through the α contact and causes this first relay to operate. This opens the α contact and causes the two β contacts to close. (If the current is d.c., a condenser across the α contact helps this action.) When a second impulse is received it passes through the lower β contact to the A coil on the second relay. The α contact on this relay is opened which also opens the holding circuit for the first relay. Theoretically a number of these holding relay combinations could be connected in series to constitute a dyadic counter The dotted line indicates the connection from the output of the present stage to the input of the next. Actually it would be necessary to insert a pulsing circuit between each stage.



For completeness, at this point, we mention the electronic counter. These will be discussed later in more detail when we consider the vacuum tube. There are a variety of these designed for use in radiation counters and the most modern large computers are based on them because of their high speed.

The basic element is what is called a "trigger circuit." This circuit has two states just like the above pair of relays and an impulse can change it from one state to the other. Consequently, the binary system is favored in these counters. While a good relay pair can act in about one thousandth of a second, trigger circuits are perhaps five hundred times as fast.

There are a variety of ways in which radix 10 is reintroduced into such a system. One way is to take a four-place binary system and have it arranged to reset itself when $2^3 + 2$ appears, i.e., permit it to lose a number of its positions. It should be clear how this can be accomplished with relays and there is an electronic equivalent. There is another way involving ten pair of trigger circuits in which the impulse is sent to the next pair each time and each pair resets the previous one. This latter is very similar to the cyclic arrangement of relays mentioned above.

II. Digital Adders

1. The fundamental step in counting is the passing from a number to its successor. Addition can be defined in terms of counting as a process in which one first counts up to a and then proceeding; one, also, counts the further steps until one has counted b. This yields a + b.

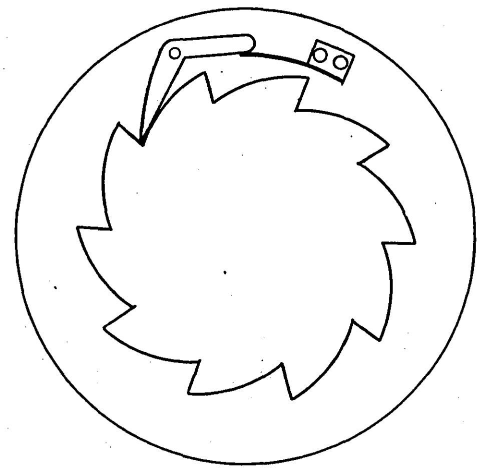
Thus two counters can be used as an adder. We begin by setting both counters at zero. We then count a on the first, then we count simultaneously on both until the second reads b. The first then reads a + b. To add another number c to the result, we set the second counter back to zero and then proceed to count on both simultaneously until the second reads c.

This illustrates two essential parts of an adder based on counters. The first counter is the basic counter, the second gives the measure of the addend.

However, the above does not utilize the advantages present in the decimal system. We can, of course, modify the basic counter so that one can count not only by units, but by tens, hundreds, and so forth. Addition can then be accomplished with a much shorter feeding process. For example, suppose we have a four-digit counter and let us impress, say, a number $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ on it. To add $\beta_1, \beta_2, \beta_3, \beta_4$, we successively add β_4 to the unit wheel, β_3 to the tens, and so forth.

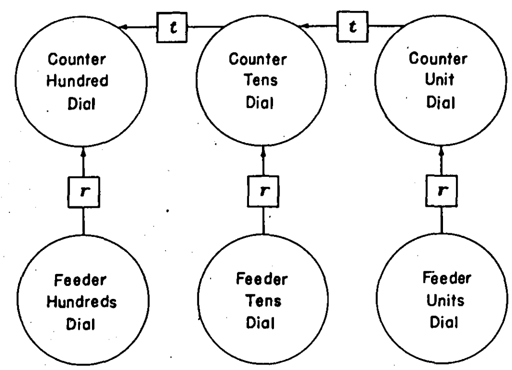
Naturally this raises the question of how this can be done conveniently. An essential device for this is the rachet connection. It was introduced quite early, I believe, by Pascal.

A rachet arrangement connects two shafts in such a way that when one shaft moves in a certain direction, the second shaft is constrained to move with it, but if the first shaft moves in the opposite direction, the second may remain fixed. The connection involves a cog-wheel with teeth flat on one side and rounded on the other, and a wedge-shaped piece of metal called the ratchet. The cog-wheel is on one shaft, the ratchet is mounted on a wheel on the other shaft in such a way that its point is pressed against the cog-wheel by a spring. Thus in the accompanying diagram, if the cog-wheel is turned clockwise, the rachet catches against the flat part of the tooth and the rear wheel rotates with the cog-wheel. If the cog-wheel moves in the



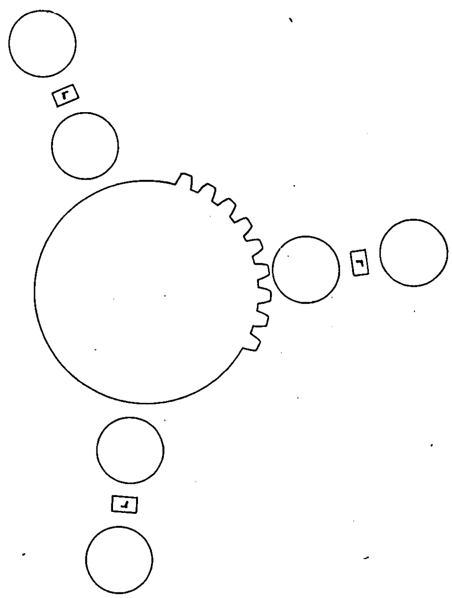
opposite direction, i.e., counter-clockwise, the ratchet is supposed to slip over the smooth part of the tooth and the rear wheel is supposed to remain stationary under the expected load conditions.

2. The simplest way to introduce a convenient feed for a counter for the purpose of making it an adder is by means of a dial and rachet arrangement on each position of the counter. We indicate this, schematically, in the following diagram:

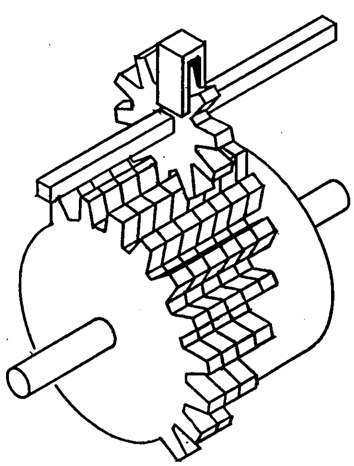


Here the letter t indicates the tens transmission and r denotes a rachet connection between the feeder dial and the counter dial. It is clear that the top line of dials with their tens transmission constitute the basic counter. To add $\beta_1, \beta_2, \beta_3$ to the number present on the basic counter, we first turn the unit feeder dial to the number β_3. This adds β_3 to the counter unit dial. We next turn the tens feeder dial to β_2, thus adding β_2 0 to the basic counter. (In this operation the units counter dial should be disconnected either by the nature of the tens transmission arrangement [for instance, it may contain a rachet connection], or by a special clutch for this purpose, which we have not indicated.) We then turn the hundreds feeder dial to β_1, thus adding β_1 00 to the basic counter. (The tens counter dial should be disconnected during this process. Cf. above.) We now set each dial back to zero. Owing to the rachet connection, this last operation does not disturb the counter.

A variation of this is to have the rachet work in the opposite direction so that setting the dial does not affect the basic counter. However, the feed dials are arranged in a semicircle and their circumference is, say, half toothed. Within the circle is a wheel with teeth on part of its circumference. After all the dials are set, the inner wheel is revolved once. The teeth on the inner wheel engage in turning the feeder dials, setting them back to zero and feeding in the proper value through the rachet connection. This method involves fewer operations than the previous arrangement and has certain more desirable operational characteristics. It is also more adaptable for multiplication as we shall see.

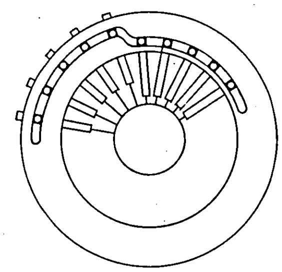


3. Another way in which a feed can be arranged is by means of the Leibnitz wheel. The latter is a cylinder along whose lateral surface teeth of varying length are placed. Associated with this wheel is a smaller gear which meshes with a varying number of teeth, depending upon its position.

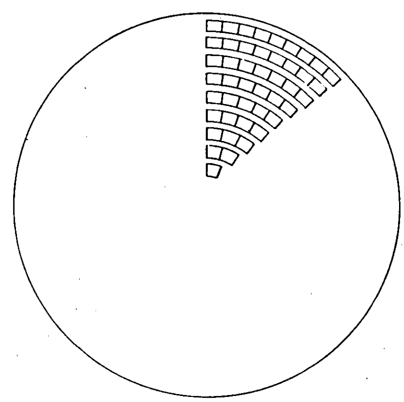


The cylinder and smaller wheel have parallel axis and there is a small pusher, which permits one to displace the smaller wheel along its axle. The axle of the smaller wheel has a square cross-section.

Each tooth starts at the same plane perpendicular to the axis but extend different amounts along the lateral surface of the cylinder. Thus, if we assign a length of nine units to the longest tooth the next longest will be eight and each will be one shorter than the next longest. Consequently, in one revolution of the cylinder, the small wheel will engage 0, 1, 2, etc., up to nine teeth depending upon its position as determined by the pusher. (Cf. Galle, A. *Mathematische Instrumente*. Leipzig: B. G. Teubner, 1912, pp. 25-29, in particular figures 8 and 9.) The earliest commercial machines, those of Thomas and Burkhardt used the Leibnitz wheel.



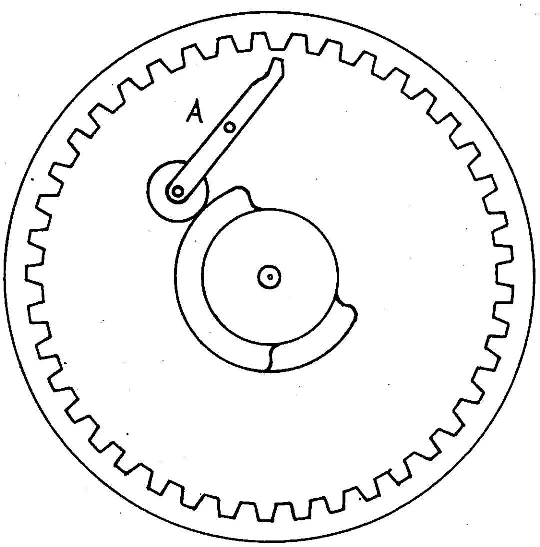
The Ohdner wheel is a similar feed, only here we have retractable teeth instead of teeth of different length. The number of teeth which protrude is set before addition. This, of course, is considerably more compact than the previous arrangement. (Cf. Galle, *loc. cit.*, pp. 33 and 34, figures 33 and 34.)



Another feed similar to the Leibnitz wheel is the feed of the "Gauss" machine. (Cf. Galle, *loc. cit.*, pp. 34, 35, 36.) Here we have a disk instead of a cylinder which has a different number of teeth at varying distances from the center. If the disk is revolved once, each feed wheel comes in contact with a varying number of teeth depending upon its position.

An ingenuous variation on the wheel type feed is represented by the Haman machine. (Cf. Meyer zur Cappelen. *Mathematische Instrumente*. Leipzig: 1944. Reprinted Ann Arbor, 1947. Pp. 93-95.) Let us consider two cylinders side by side. Each cylinder has a depression on its circumference. Thus the combination has a depression for its entire length, which depends upon the relative position of the two cylinders. One of these cylinders is fixed; the other is connected to an input lever. Thus the position of this input lever determines the length of depression common to both cylinders.

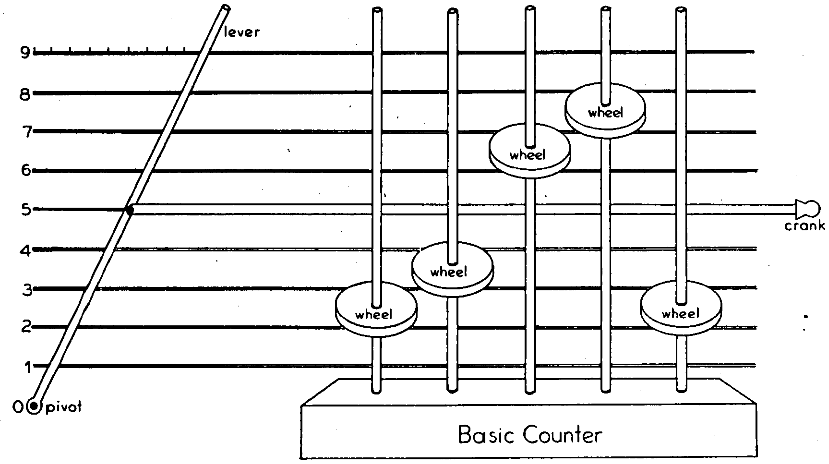
Concentric with the pair of cylinders is a gear wheel with teeth on its inner circumference. There is also a lever which is pivoted on a point A. One end of this lever is a roller which rests on the pair of cylinders, the other end of this lever is a single gear tooth. As the point A revolves the gear tooth engages the outer gear wheel only when the roller is in the depression and thus a variable feed is accomplished.



4. In the Mercedes Euclid machine, a variable feed is obtained by means of a lever. Instead of the Leibnitz wheel, we have nine racks (i.e., linear strips with gear teeth on them). These racks are in parallel grooves. One end of each rack has a pin which fits in a slit on the lever. Thus, moving the lever causes the rack to move in the groove. Each rack corresponds to a digit. The pivot for the lever is located so that when the lever is turned through a fixed angle θ, each rack is displaced in its groove an amount proportional to the corresponding digit by a theorem on similar triangles.

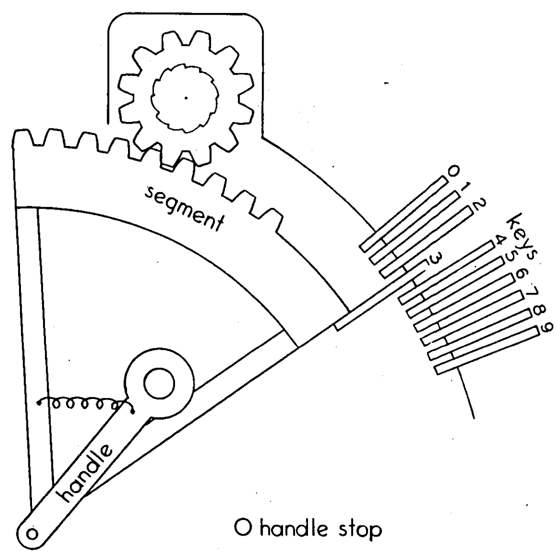
For each decimal place we have a small wheel. The axle for this wheel is perpendicular to grooves for the racks and hence if we wish to feed the digit α into a given place, we may displace the wheel along its axis until it is over the α rack. We then turn a crank which turns the level through θ and hence moves the α rack, α units. Since the wheel and the α rack are now in contact, the wheel turns a corresponding amount. (Cf. Galle, *loc. cit.*, pp. 30-32.)

In view of the advantages of the keyboard input, the actual positioning of the wheel which connects the racks with the rods which go into the fundamental counter is by means of a keyboard arrangement. On each rod there are five wheels and each wheel is normally between two racks. Pressing a key moves one of the wheels onto the appropriate rack.



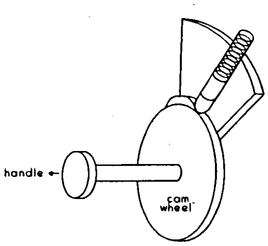
5. All the previous feeds were similar in that a choice of possible motion is made in order to obtain the proper feed. A somewhat different principle is used as the basis of keyboard machines, for instance, the comptometer, an American invention. Here the various feeds are obtained by stopping a motion at different places. To illustrate the principle, we introduce an extremely simplified version of such a feed.

In this version, the feed is actuated by pulling on the crank marked "handle" until it reaches the stop. A segment of a wheel is connected to the handle by means of a spring. The segment has the same axle as the crank and pulling on the crank



will cause the segment to revolve until it is stopped by whichever one of the ten keys is pressed down. Thus the amount of rotation of the segment is determined by pressing down the key. A portion of the rim of the segment is toothed and gears into a feed wheel which has a rachet connection with the dial of the corresponding counter. When the handle is released, a restoring spring returns the segment to its original position. This does not affect the counter due to the rachet connection. Another example of a key feed is given in Dyck, *op. cit.*, p. 147, No. 25.

It is desirable that the original handle be permitted to make a complete revolution in an operation like the above. Also that the feed for different decimal places occur at different times both to distribute the load and also to avoid certain difficulties with the tens transmission which we will discuss below. This can be accomplished by making a cam connection between the segment and the handle shaft. The crank is rigidly connected to a wheel with a bump on it. On the segment is a cylinder in which is a spring and a plunger which presses against the rim of the wheel connected to the handle. When the wheel begins to rotate, the plunger slides over the wheel until the bump comes in contact with the plunger. Then the segment moves with the cam wheel until the segment is stopped by the pressed down key. The bump now presses the plunger into the cylinder and passes by. Thus one turn of the crank results in one feed. For different decimal places, the bump is at a different position, relative to the handle, so that each plane is affected at a different portion of the revolution of the crank.



The above is an extremely simplified version of an adding machine based on the original comptometer principle. In practice for various reasons, the stops are not immediately associated with the keys but instead are connected to them by means of levers. There are precautionary measures to insure that no two keys corresponding to the same place can be pressed down at once and that normally the zero stop is down. Also the problem of timing as between different decimal places is treated in a more complicated but also more efficient manner. And finally, of course, all modern machines are designed for a multitude of functions, which tend to complicate the original situation.

6. There is one variation of the keyboard machine, which may well be mentioned at this point. The normal keyboard has a bank of keys, i.e., ten rows of ten keys each. There is a touch system for such a board similar to the touch system in typing so that the operator need not look at the board but only at the work. However, in order to simplify this touch system, ten key machines have been developed in which one enters a number 389 by successively pressing the 3 key, the 8 key and the 9 key.

In the machine, there is for each decimal place available a set of feed controls or feed limiters. These feed limiters can shift relative to the actual feed and initially the feed limiter on the left-hand side of the set is connected to the keyboard and it can also be thought of as being to the right of all the feeds. When the 3 key is pressed, this left-most limiter is set at the value 3 and shifted onto the feed unit position which, of course, is furtherest to the right of all the feeds. The next left-most limiter or feed control is now connected to the board and when the 8 key is pressed, this control is set at 8 and the set of limiters shifted. Thus when the 9 key is finally pressed, the first limiter controls the hundreds' position, the second the tens' position and the third the units' position. The machine can then feed 389.

Again there are certain variations of this device. For instance, there are double zero and triple zero keys, which make it possible to put a number like 6,700,000 in four steps.

7. The magnetic clutch counter can also be used as a feed for an adder. One recalls that in this type of machine, the counter wheel has attached to it a revolving brush which makes different contacts in order to indicate the digit. The machines are based on a time cycle with a fundamental period. This fundamental period is divided into fifteen subperiods. The first ten of these are numbered 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, and by means of a commutator arrangement a pulse is sent through the corresponding contact for the counter wheel brush during each of these subperiods. Thus if a counter wheel stands at 4, its brush will receive a pulse at the 4 subinterval. Suppose this brush is connected to another counter wheel which is to receive the feed of 4. The 4 pulse activates a holding relay which in turn activates the magnetic clutch. This clutch connects the counter wheel of the receiving counter to a revolving shaft. At zero time, the hold relay is disconnected and the clutch releases. Thus the amount of turn is four units and corresponds to the time of the initial pulse.

Instead of the initial counter and commutator arrangement, one can substitute a punched card and thus add a number coded on the card. This card has a hole punched in it for each decimal place and moves between two rollers which are synchronized with the fundamental cycle. At the appropriate subinterval, a contact is made through the hole in the card. One thus has a pulse properly timed to yield the corresponding digit of the addend and this is sent to the receiving counter. (Cf. *A Manual of Operation for the Automatic Sequence Controlled Calculator*. Cambridge: Harvard University Press, 1946.)

8. In our description of the above devices, we have always supposed that the counter is like those described previously and that the various feeds for different places occur successively. For the type of counter we have described up to now, this is essential, since otherwise there will be interference between the feed and the tens transmission. In the case of ten-to-one gear connection

tens transmission, feeding different places at the same time will cause jamming. In the case of the partially stripped gear tens transmission, if we have simultaneous feeding, the tens transmission may have no effect. Thus if we add 66 to 66, we will obtain 122. For, due to the feed, both the tens wheel and the unit wheel are moving. The partially stripped gear then just moves with the tens wheel and does not increase the amount of rotation of the latter.

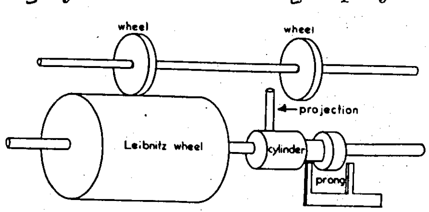
For this reason most calculating machines use a different arrangement. The basic counter does not possess a tens transmission. Instead, the machine possesses an auxiliary feeding device for this purpose. When the counter wheel passes from 9 to 0, no transmission takes place, but instead the auxiliary feed is set up to add a one later. After the original feed has been put in, then these auxiliary feeds are applied successively.

In general, the addition of a number corresponds to a complete revolution of a certain shaft. During the initial part of this rotation, the original feeds are entered, and, if any carrying is involved, then the auxiliary feed is set up for this purpose. Then the auxiliary feeds are introduced successively with the units first then tens, etc. The auxiliary feeds are then cleared.

To see this, one can operate such a machine very slowly and watch the numbers change on the main counter. Suppose we have 666 on the counter and wish to add 334. The successive positions of the main dial are:

0 6 6 6	Initial
0 7 7 7 0 8 8 8 0 9 9 9	Original feed
0 9 9 0	Here the tens transmission has been set between the last two places.
0 9 0 0	Here one has been added to the tens place and the auxiliary feed for the hundred place set up.
0 0 0 0	Here one has been added to the hundreds place and auxiliary feed for the 1000 place set up.
1 0 0 0	Here all the auxiliary feeds have been used. After this the auxiliary feeds are cleared.

The machines based on the Leibnitz wheel or its derivatives (except the "Gauss" which feeds the places successively) in general have the auxiliary feed on the same general plan as the original feed. Thus the Leibnitz wheel has an extension on which is a sliding cylinder with a single projection.



A prong determines the position of this cylinder. Thus when the unit wheel passes from nine to zero, it moves the prong which moves the cylinder on the tens wheel. In the new position, the single projection on the cylinder will eventually engage a feed wheel and move it a tenth of a revolution. This feed wheel is on the same axle as the feed wheel over the major part of Leibnitz wheel. Hence the effect of the above is to increase the tens by one. At the end of the cycle all the prongs are moved back to the neutral position.

The situation in the key board machines is analogous. Here we have feed based on a stop principle. Normally, the feed is stopped at zero, but when the unit wheel passes from nine to zero, it removes the zero stop on the extra feed at the tens place. The zero stop is then retained at its new position until the end of the cycle.

A somewhat different type of carry is employed in the contact type counter. There are two tens carry contacts. One of these is made and held if during the feed the counter wheel passes from nine to zero. During a latter part of the operational cycle, a pulse is transmitted to the next highest counter. This pulse operates a hold relay which in turn causes the magnetic clutch to engage for a unit time after which the hold relay is released. The other contact is made if the counter wheel stands at nine after the feed and it transmits a carry impulse received from below to the next highest counter. The impulse also turns the given counter from nine to zero. Both carry contacts are knocked off at the end of the cycle.

9. Owing to the nature of their feeds and tens transmissions, many of the above devices cannot be operated in a reverse direction in order to subtract. Subtraction must be, in general, carried out by a different method. This method utilizes the fact that if our counter contains n places then the number on the counter is not necessarily the actual total but rather the total modulo 10^n.

For example, suppose n = 5 and we have 00946 registered on the counter. Furthermore, suppose we wish to subtract 325. To do this, we add 100,000 - 325 = 99675. The actual total then will be 100,621. However, since our counter has only five figures, the first one will not appear and the counter will register 00621, the desired result.

The rule for passing from the number to be subtracted to the number which is to be added is a particularly simple one in general. In each place except that of the last non-zero digit, one changes the given digit α to $9 - \alpha$. However, the exception is a little troublesome and the rule which is utilized mechanically is the following: Let A be a given number. Suppose 9999 is the largest number which can be registered on the counter. Then add 99,999 - A and 1. The number 99,999 - A is immediately obtainable from A by changing every digit α in A to $9 - \alpha$. After this, one is added.

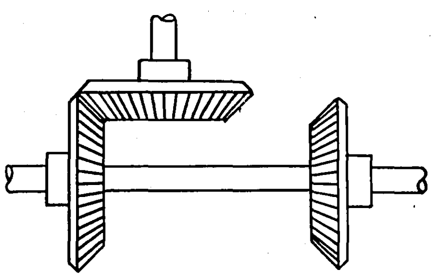
In many machines, provision is made for subtraction by given two values for each possible feed, one for addition and one for subtraction. Thus, in the original rachet dial feeds, at each position on the dial, we find two digits, one in black and one in red, whose sum is nine. To add, the black digits are used; to subtract, one must use the red. This scheme is applicable to the Leibnitz wheel machines in general.

In the Mercedes-Euclid machine, the analogous result is obtained very simply by having another possible pivot for the lever. The subtraction pivot for the lever is located on the nine line. If the lever rotates around this point, then the linear displacement of the α rack is $9 - \alpha$ units.

In certain of the key board machines, a key controls different stops, depending on whether one is to add or subtract.

However, there is a method of reversing certain of the machines. In these, there is in general a doubling of the tens transmission auxiliary feed. With two such auxiliary feeds, one can assure that there is a tens transmission feed which follows the regular feed in the cycle. If there is nothing on this feed, of course, nothing happens and the operation is not affected.

Another method for getting the same result is to utilize the gear train connecting the feeds to the basic counter to reverse the relative sense of rotation. For instance, we can pass from a train with an odd number of gears to one with an even number of gears. Or one can use a bevel gear arrangement in which two bevel gears, facing each other, are mounted on a collar keyed to the input shaft. The bevel gear on the output is mounted between these. The relative direction of rotation between input and output is reversed by moving the output so that it engages the other input bevel gear. The central position for the output is "neutral." The tens carry trip from the counter to the feed must be such that one gets the same result passing from 0 to 9 as from 9 to 0.



III. Digital Multipliers

1. The relationship of multiplication to addition is analogous to the relationship of addition to the operation of taking the successor, i.e., the fundamental operation of counting. We recall that we add 5 to 4 by first taking 4 and then passing to the successor five times, i.e., we count the number of times we pass to the successor. Now if we wish to take 5 times 4, we add 4's five times, i.e., we count the number of steps of addition.

Thus an adder can be used as a multiplier provided it can repeatedly add a quantity without resetting, and it has a counter to count the number of times the addition has been performed. If resetting is necessary, as in the elementary rachet dial system, the process is far too tedious to be practical. However, resetting is clearly not necessary in either the Leibnitz wheel or the key board machine. Even the revolving wheel and rachet dial combination can be arranged so that resetting is not necessary. Instead of setting the dial, one sets up a stop at the feed value. One has a spring on each dial so that after a feed has been made and the gearing on the center wheel has passed on, then the dial turns until it is held by the stop. Thus, after every revolution of the center wheel, the dials are automatically reset. It is clear that this type of machine is basically similar to the key board machines.

It is, of course, desirable to take advantage of the decimal representation to lessen the number of turns. This is readily accomplished by means of a shift of the feed relative to the basic counter. For if the feed is shifted one place to the left relative to the basic counter, the effect of the feed is precisely ten times as great.

Suppose now we have an adder with an auxiliary counter and with a feed which may be shifted. We give an example which shows how such a device may be used as a multiplier. Let us multiply 365 by 132. To do this, we begin with the feed in the normal position and set it for 365. Then, we apply it as an adder twice. 730 will now appear on the basic

```
           | 0 0 3 6 5 |  Feed
                 ↓ ↓ ↓
| 0 0 0 |  | 0 0 0 0 0 |  Basic Counter
Auxiliary
   Counter
```

counter. The auxiliary counter has a unit feed in the units place. Since we have added twice, this extra feed has been applied twice to the units place of the auxiliary counter and hence it reads 2. We now shift the feeding mechanism relative to the counters.

```
        | 0 0 3 6 5 |  Feed
              ↓ ↓ ↓
| 0 0 2 |  | 0 0 7 3 0 |  Basic Counter
Auxiliary
   Counter
```

We next apply the feed mechanism three times. This adds 10950 to the basic counter and changes the auxiliary counter by three in the tens place. We again shift the feeding mechanism.

```
      | 0 0 3 6 5 |     Feeding Mechanism
          ↓ ↓ ↓
| 0 3 2 |  | 1 1 6 8 0 |  Basic Counter
```

We apply the feeding mechanism once and the answer is obtained. Both factors appear and

```
     | 0 0 3 6 5 |
         ↓ ↓ ↓
| 1 3 2 |  | 4 8 1 8 0 |
```

can be checked. We have duplicated mechanically the elementary method of multiplication

$$\begin{array}{r} 365 \\ \times 132 \\ \hline 730 \\ 1095 \\ 365 \\ \hline 48180 \end{array}$$

More precisely, we have duplicated the operations

$$\begin{array}{r} 365 \\ 365 \\ 3650 \\ 3650 \\ 3650 \\ 36500 \\ \hline 48180 \end{array}$$

In the simplest type of electrically driven adding machine, which is used as a multiplier, one has simply a counter in addition to the fundamental addition mechanism. The multiplicand is entered into the latter in the same way as an addend but a lever is set so that the board will not be cleared after each addition. Then the multiplicand is added repeatedly. The number of additions appears on the extra counter. When this number reaches the unit digits of the multiplier, the operator shifts the two registers one place to the right. The addition of the multiplicand is then repeated until the tens digit of the multiplier appears in the second place on the extra counter. One then shifts and this process is continued until the multiplier appears on the extra counter.

This process can be shortened by the use of "short multiplication." Short multiplication is simply the use of complements when this is favorable. Thus to multiply by 29 one multiplies by 30 - 1, to multiply by 98 one multiplies by 100 - 2.

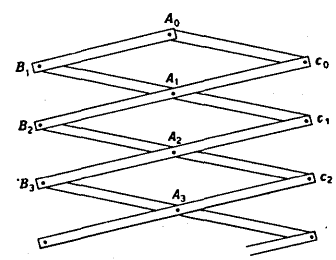
When the addition mechanism is also arranged for subtraction, the above combination can also be used for division. The dividend is entered into the register as far to the left as possible by first putting it into the addition mechanism, then registering it and clearing the feed. The divisor is then entered into the feed and it is subtracted from the dividend a number of times which appears as the first digit of the quotient in the extra counter. If the division is manual, the operator notes when the remainder in the main register becomes less than the divisor and then shifts the registers to the left. A "shortened division" is also possible if the registers will hold negative numbers. If the usual remainder is large relative to the divisor one overshoots and then after shifting one adds for the next place, until one gets a positive remainder in the main register. This addition is counted negatively in the quotient register.

In most machines division is automatic. There is a provision by which the subtraction is permitted to overshoot so that the remainder becomes negative. At this point, the divisor is added once and the registers shifted automatically.

For automatic multiplication, in general, another register is introduced. This second register holds the multiplier and the multiplicand appears in the normal addition mechanism as before. The multiplier register acts like a negative counter and each time the multiplicand is added, one is subtracted from the units place until the units place reads zero. At this point, the shift occurs and the process is repeated for the tens place. Sometimes this automatic multiplier register is equipped with an extra input of the ten key type or even a full key bank, which, of course, can be used to retain the multiplier. In other cases, the multiplier is first entered into the regular key bank and then a special key is pressed which transfers this into the multiplier register. The latter can be modified so as to retain the multiplier.

For special bookkeeping purposes, machines have been developed with many registers and special control racks, so that a sequence of operations can be performed with a minimum amount of attention from the operator. Thus if a sequence of operations is to be performed with, say, only two numbers varied from one sequence to the next, these machines can be set up so that after the operator enters one such variable the machine will proceed automatically to the point where the second variable is to be entered.

2. There are other methods of obtaining directly the multiples 730, 1095 and 365 of the multiplicand rather than by repeated addition as above. One such method is that of the Selling machine. Here the multiplication is accomplished by means of a combination of links formed into a number of parallelograms similar to usual extensible brackets. In the accompanying diagram, it is obvious that each of the parallelograms are congruent and hence that the length $A_0A_k = k\ A_0A_1$. Thus this arrangement permits us to obtain a displacement which is an integral multiple of another displacement. This holds even



if we consider only a change in A_0A_1 and the corresponding change in A_0A_k.

In the Selling machine, we have for each place a pair of such link arrangements, with the corresponding A points connected by cross bars. At the center of each cross bar there is a hole, with a pin which can be pushed down by a key or push button. In line with these holes, there is a long bar which has holes corresponding to cross bar holes when the parallelogram arrangement is in the most contracted position.

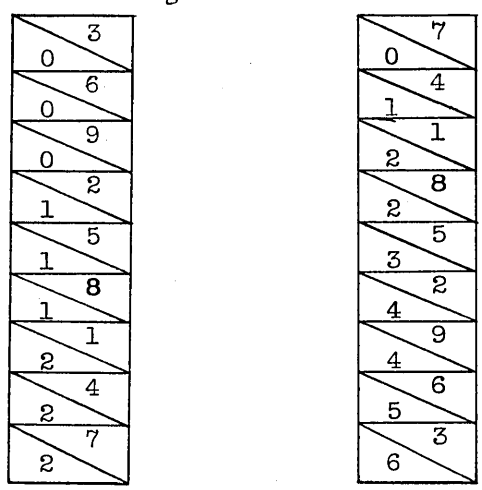
The multiplication is begun with the parallelogram arrangement in this latter position. The multiplicand is entered into the machine by pressing the key at each place for the corresponding digit.

This connects the cross bar for that digit to the underlying bar.

Now if the lowest cross bar is moved an amount β, the α cross bar is moved an amount $\alpha\beta$. If β is the first digit of the multiplier, then the lowest cross bars are all moved an amount β. Then if the i'th digit of the multiplicand is α_i, the underlying bar in the i'th case is moved $\beta\alpha_i$. This underlying bar has a rack extension which turns a wheel connected to a counter. This counter has a rather special method of tens transmission, which involves the simultaneous addition from two sources and we shall discuss it later.

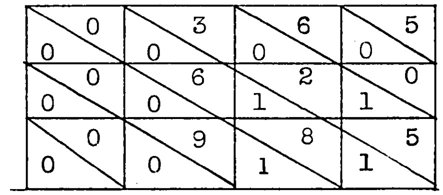
After the first multiple of the multiplicand is fed into the counter, the counter shifts relative to the feed mechanism and the second multiple is fed in. The rest of the multiplication then is similar to the previous process.

3. Another method of obtaining the multiples can be traced back to the "Napier Bones" and it will be interesting to consider this device.



In one form, we have strips of ten different kinds, a kind for each digit α. Let us describe an α strip. It has nine squares arranged one below the other along it and each square is divided by the diagonal of negative slope into two triangles. If we go down to the β square, we find two digits, one, γ, in the lower left triangle and the other δ in the upper right-hand triangle. γ is the digit in the tens place of the product of α and β and δ is the digit in the units' place.

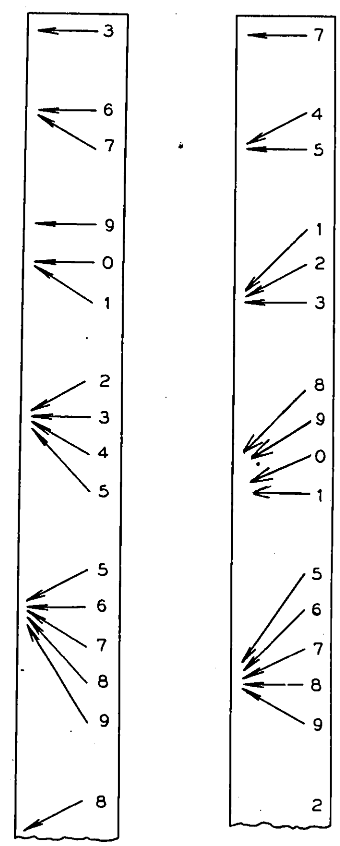
The various multiples of the multiplicand are obtained as follows: For each digit of the multiplicand, we take a strip of the corresponding kind and lay them side by side in the order in which they appear in the multiplicand. The product of the multiplicand and β can now be read from the β row. The units digit is obtained from the upper right-hand triangle on the extreme right of the β row. We get the tens digit by adding the digits in the next pair of triangles in the β row and the hundreds digit from the next pair, adding one if there is anything to carry from the tens' place. Thus, in our example, we would have



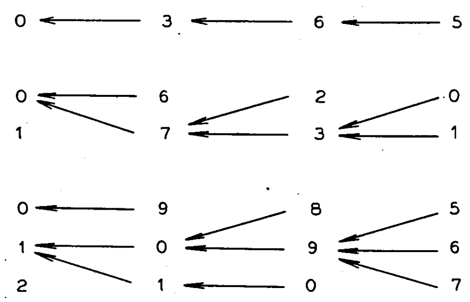
It is evident that the method described when applied to the third row will yield

$$\begin{array}{r} 15 \\ 18 \\ \underline{9} \\ 1095 \end{array}$$

There is another form of this device which facilitates the addition and carrying. Here again we have strips of ten different kinds, but on the α strips for each β we have a sequence of digits, written one under the other, beginning with δ and going down to $\delta + \beta - 1$, with of course 0, 1, 2, etc., instead of 10, 11, 12, etc. These numbers are written along the right-hand side of the strip. From each number, there is an arrow which points, in general, to one of two places on the left edge of the strip. The upper one of these two places is down γ lines below the uppermost digit of the set and integers of the set preceding 0 have their arrows pointing at this place. The other place is one line lower and the integers beginning with zero have their arrows pointing at this lower place.



The multiples are obtained as follows: For each digit of the multiplicand, a corresponding strip is chosen and the strips are laid side by side in the order of the digits of the multiplicand. Each multiple is obtained by starting at the right hand of the corresponding row and following the arrows. Due to the slant of the arrows, this procedure accomplishes the equivalent of the addition of pairs of digits of the previous method and also any carrying that might be necessary.



4. The above process of "splitting" the multiplication table is the basis of the "Millionaire" machine, as proposed by the French inventor, Bollee. We describe the essential idea in terms of our previous example. (Our description differs slightly from the precise situation.) Again, we wish to multiply 365 by 132. The machine has three feeds, one for the units in the multiplication table, one for the tens in the multiplication table and one for the tens transmission.

We begin by entering the multiplicand, 365, into the machine. We then select 2 as the last digit of the multiplier and enter it into the machine at the proper place. We then turn the handle once. During this last operation, the machine proceeds as follows: Firstly, using the units feed, it enters 620 into the fundamental counter. (Note: 2 x 3 = 6, 2 x 6 = 12, 2 x 5 = 10. The digits 620 are the unit digits in these products.) Next it applies its tens transmission feed. In the third step, the fundamental counter is shifted. Then, using the tens feed, it enters 0 1 1. The result

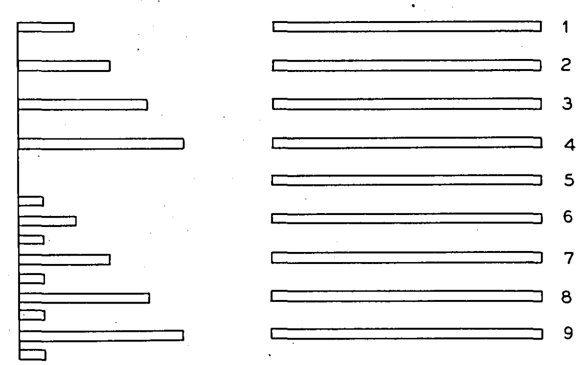
Feed	0 0 3 6 5	0 0 3 6 5
Counter	0 0 6 2 0	0 0 6 2 0
	Result of Steps 1 and 2	Result of Step 3

is 7 3 0. Then the tens transmission feed is used again.

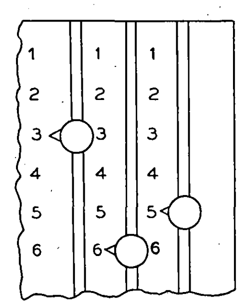
Feed	0 0 3 6 5
Counter	0 0 7 3 0
	Result of Steps 4 and 5

Thus we have entered the first multiple of 3, 6, 5 and the fundamental counter has also been shifted to a position suitable for entering the second multiple. We now move the multiplier indicator from 2 to 3, the second digit of the multiplier 132 and turn the crank. The machine repeats the previous five steps with three instead of two. Firstly, it feeds in 985 by means of the units feed, yielding 9580. Then the tens transmission is applied, yielding 10,580. Next the shift is made. Then from the tens feed, 011 is entered. This yields 11,680. Finally, the tens transmission is again applied.

We are now in a position to add the remaining multiple, 365. We shift the multiplier indicator to 1 and turn the crank. The machine then shows 48,180. (In the above description, we have used the order 2, 3, 1. I believe in the actual machine, the order would be 1, 3, 2, and the shift works the other way. However, the above is somewhat simpler from a theoretical point of view and we retain it.)



The part of the machine into which the multiplicand is entered, is analogous to that found in the Euclid machine. We have nine racks, one for each digit, mounted so that they can only move lengthwise and parallel to each other. For each place in the multiplicand, we have a pinion, mounted on an axle perpendicular to the lines of the racks. This axle has a square cross section and the pinion can be displaced along its axle. The number 3, 6, 5 is entered by moving the pinion in the units place, until it meshes with the rack for the digit five.



Similarly, we move the pinion in the tens place until it meshes with the rack for the digit 6 and the pinion in the hundreds place until it meshes with the rack for the digit 3. Externally, these operations are performed by moving a knob which projects through a slot. The slot is parallel to the axle of the pinion and the pinion itself is moved by certain projections on the lower and unseen portions of the knob. Thus, we have entered the multiplicand 365.

The multiplication feed involves a plate which is perpendicular to the line of racks and has on it lines of projecting rods of various lengths. When the multiplier selector is moved to the number 2, the second line of projecting rods is moved (up or down) to the same level as the racks. There are in all 18 rods projecting from this second line. (Some are of zero length.) At the beginning of the revolution of the crank, the rod, in the same line as the rack for the digit α, has a length proportional to the units digit of the product of 2 and α. Thus the rod in line with the rack for the digit 3 has length 6. The rod in line with the rack for the digit 6 has the length 2, and that in line with the rack for the digit 5 has length 0.

The units feed is now achieved by moving the plate parallel to itself until it reaches the line containing the left-hand end points of the racks in their undisturbed position. During this process, each unit's rod displaces the rack in line with it, by an amount equal to its own length, and the rack turns whichever pinions are set on it. It is easily seen that this gives the correct units feed.

After the units feed has been made, as the handle turns, the tens transmission is applied and the counter shifted. As the counter is shifted, the plate is also shifted forward so that the second line still remains in the horizontal plane of the line of racks, but now the alternate rods are in line with the different racks. These, of course, have their length proportional to the tens digit of the product of two and the digit of the corresponding rack. Then the tens feed is made in a manner analogous to the units feed.

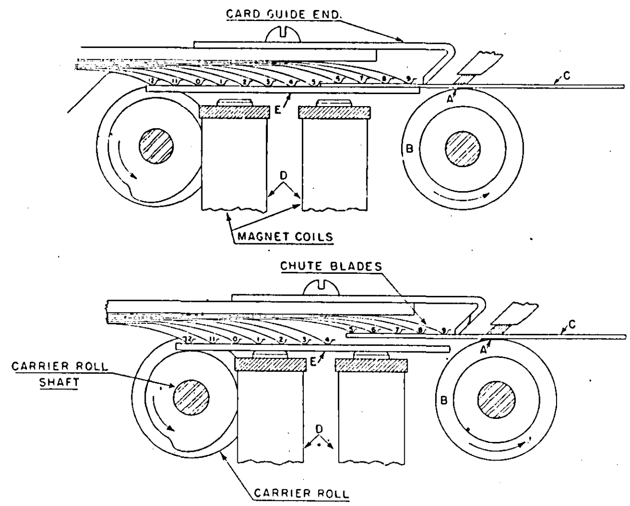
IV. The Punch Card Machine

There is another type of machine, the punch card machine which has received wide commercial exploitation. The punch card machine is particularly suitable for calculations which involve many numerical quantities in such a way that each numerical quantity is used a number of times. Of course, for use in these machines, each number must be punched on a card, an operation more laborious than entering a number into an ordinary calculating machine. However, once the card has been cut, it may be used repeatedly.

The first Hollerith machine was devised for the census of 1880. This machine was a sorter. It divides a given pack of cards into various packs, each pack containing only cards having a specified property. In the census referred to, a card was made out for each individual in the country. The cards contained a number of columns. Each column referred to a trait, concerning which the census has information. In each such column a hole was punched and the position of this hole corresponded to the information obtained. There were ten positions for the hole, not all of which would be used in every case. The modern cards have twelve positions or rows.

For instance, there is an "age" column. If a hole is punched in the first position, the person would be less than ten years of age, if the hole is in the second position, the person would be between 10 and 20, and so forth. Naturally, by the use of two columns, the precise age in years could be entered.

The machine acts by means of a relay which is activated by a contact made through the hole in the card. The principle of the action is illustrated in the following simplified diagram which is reproduced by permission of the International Business Machines Corporation.



The card passes into the machine between the brush contact A and a roller B. If this diagram were to extend far enough to the left it would show thirteen bins. From above each bin there extends a pair of parallel guides which end in the numbered chute blades shown. Suppose that the card is punched "4." It will travel under the 9, 8, 7, 6, and 5 chute blades before contact is made. When contact is made, the magnet D attracts the armature E and the chute blades, 12, 11, 0, 1, 2, 3, and 4 follow down with the armature E. This creates an opening between the 5 and 4 chute blades through which the card is conveyed by carrier rolls to the bin. There is an extra bin for unpunched cards.

For example, suppose that we wanted to count the number of individuals in the above census whose age was thirty-eight. We first set the sorter for the column corresponding to the division into ten-year age groups. The machine then sorts the cards into packs, one of which corresponds to the ages 30 to 39. We then sort this pack according to the last digit in the age.

The twelve positions permit one to make alphabetical entries onto a card. The alphabet is divided into three groups, A to I, J to R, and S to Z. Two holes in a column will specify a letter. The upper hole which is in either the 12, 11 or 0 position determines the group and the second hole in a position 1, 2, to 9 determines the alphabetical order within a group. The machine can be set to sort relative to either set of holes. It is desirable to sort relative to the 1, ... 9 set first.

The above illustrates the fundamental principles of the punch card machine. The combination of punched card and electrical contact is a timing or spatial control device which permits one to control a desired operation in extent.

It is clear that this can be used as a feed in an adder. When the card enters the roller a feeding device begins to turn and continues until it is stopped by a relay activated by a contact made through a hole in the card. Actually the high end of the card enters the rollers first, so the above describes the subtraction process. The addition feeder is started by the hole and ends when the card has passed. I believe multiplication is by a split table method.

In general, the principles of the calculating portions of the punch card machines are analogous to those of other digital machines, except in one respect. It is clearly preferable to use electrical impulses rather than gear connections between different portions of such a machine, including the tens transmission, since this permits greater flexibility in connections.

An important additional possibility with the punch card machines is "keying." Thus if the machine contains two pairs of rollers, a certain column can be used to control what is done with some other columns. Thus the machine can be set up to add columns 2, 3 and 4, provided we have a 3 in the first column.

Keying is based on a "digital selector" which consists, essentially, of a revolving brush which makes various contacts at different times in the fundamental cycle. Thus if a pulse which occurs at a certain time in the fundamental cycle is fed onto the brush, it will appear on and only on the contact corresponding to this time.

These contacts and the brush are connected to jacks on a special board which also has jacks for the various columns on a card and for holding relays. Thus if a connection is made between the jack for a certain column and the brush of the selector and then the 4 contact on the selector is connected to a holding relay, the holding relay will be operated if and only if there is 4 punched in the card column. The holding relay can then control a counter to add or subtract by means of further connections which are also made on the board. Of course, this means that the keying column has to be read first.

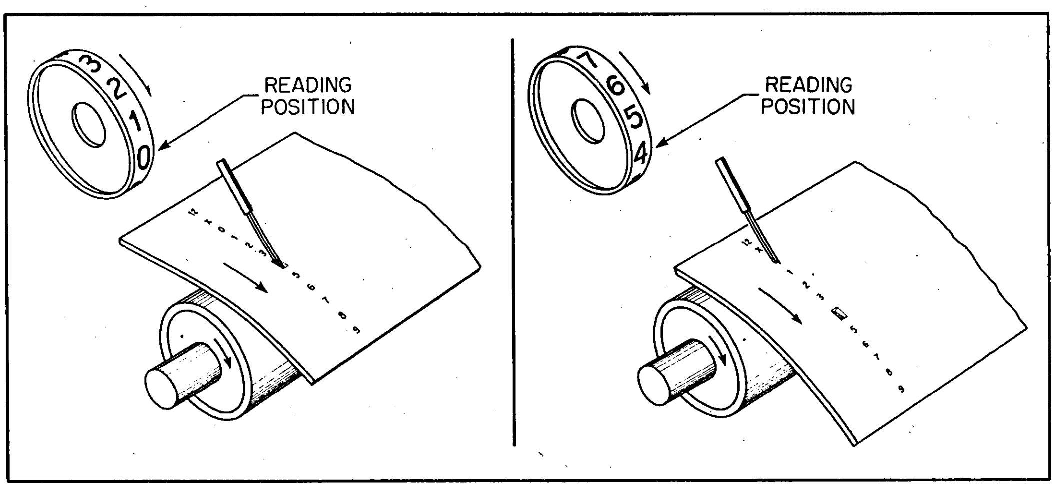
Since we have contacts on the selector for each column, we may arrange each digit to control a relay or a number of relays. Thus any function defined for digits can be realized or with sufficient relays contact any number of functions. This is utilized in various schemes involving the distributive law for addition and multiplication and can also be used to control the punching of cards.

Instead of cards, tapes may be used. Complicated devices of this sort have been constructed, in which one tape is a "master tape" which controls what different parts of the machine will do and the order of operations.

Summary

The basic digital operation is counting. Counters in general are based on the decimal system and have a cylinder or disk for each place. The tens transmission is based on either the partly toothed gear carry or a continuous ten to one gear ratio.

Adding machines involve a basic register and a set of feeds one for each decimal place. A variety of feeds exist, some are based on the partly toothed gear principle; for instance, the Leibnitz and Ohdner wheels, some on a stop principle



MOVEMENT OF A COUNTER WHEEL IN ADDITION

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like the Comptometer. A time interval feed is used in the electromechanical clutch adder, electrical pulses are used in electronic digital computers. A lever and rack feed and a cam type feed have also been described.

The tens transmission for the register in an adder must not jam the addition feed. Four systems have been developed. One of these consists of a regular counter carry and the successive feeding of each decimal place. Another involves a simultaneous main feed for each decimal place and an auxiliary carry feed which is successive. A third system involves a simultaneous main feed followed by a doubled carry system which functions simultaneously. A fourth system is based on the free addition of the differential which simultaneously adds two inputs.

Subtraction can be obtained in an adder in some cases by merely reversing the feed; for instance, in the case of a successive main addition feed or the free addition carry. To reverse direction with the other systems a double carry system is introduced. If nine complements are used it is not necessary to reverse the feed. One can also obtain subtraction by reversing the connection between feed and register.

Multiplication proceeds by steps involving the successive digits of the multiplier and hence a shifting feed or register is necessary. The digital multiples of the multiplicand are obtained by either repeated addition or by direct geometrical multiplication or by splitting the multiplication table.

Division involves a reversal of the operations involved in multiplication. It is readily obtained automatically. Automatic multiplication involves an additional counter which contains initially the multiplier and controls the repeated additions and shifting.

A punch card set-up has in addition to the above arithmetrical abilities, a memory in the form of cards. Keying permits a variety of logical and digital functions to be introduced.

PART TWO: CONTINUOUS OPERATORS

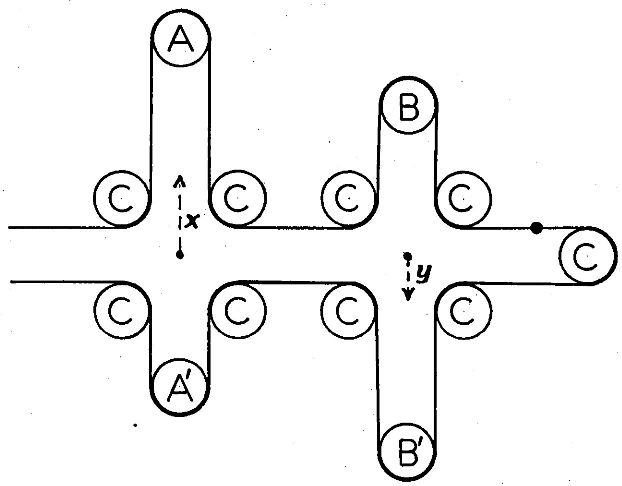
I. Adders

In our previous discussion, we have considered digital machines. However, numerical quantities can also be represented by magnitudes. In general, the magnitudes used in calculating devices are 1) linear displacements, 2) rotations, 3) direct current values, 4) direct current voltages, 5) resistances, 6) the amplitude of alternating currents, 7) the phase angle of alternating currents. Theoretically, it seems possible to use many other physical quantities.

Devices using a continuous method to represent numerical quantities are not in general as accurate for the same size as digital devices but they are much simpler if the input is a variable quantity.

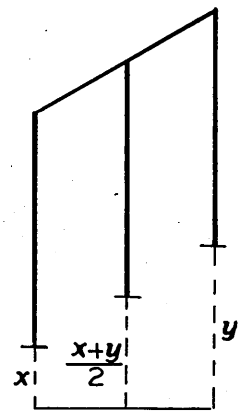
In this part of the course, we shall deal with devices for operating on magnitudes, considering successively adders, multipliers, the representation of functions, amplification devices, integrators and differentiators.

1. Addition of Lengths or Linear Displacements. Suppose our quantities are represented by the displacement of certain rods (from fixed initial positions) in the apparatus along themselves. We wish to obtain a displacement corresponding to the sum of two such displacements. There is a simple arrangement by which one uses an endless chain or tape to add displacements. The chain passes around sprocket wheels A, A^1, B, B^1 and C. The C sprocket

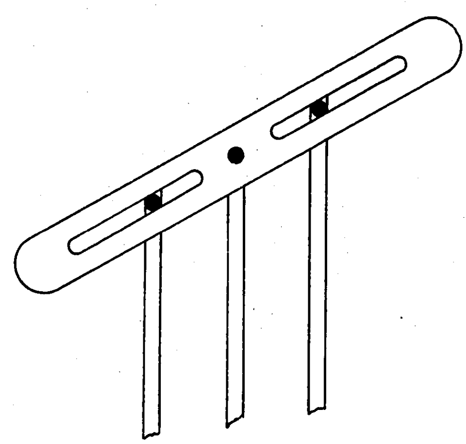


wheels are fixed in position, the wheels A, A^1 are a fixed distance apart and similarly the pair B, B^1. If we move the pair A A^1 along their line of centers an amount x and the pair B B^1 an amount y, then a point R on the chain will move an amount 2(x+y). It is clear that this can be applied to any number of addends. In the drawing, y is negative.

Another way to do this would be to have the three rods parallel and to have connection joining the three, in such a way that three points, one fixed on each rod, remain colinear. Then if one outside rod is displaced an amount x and the other outside rod is displaced an amount y, then the middle rod is displaced an amount $\frac{x+y}{2}$.

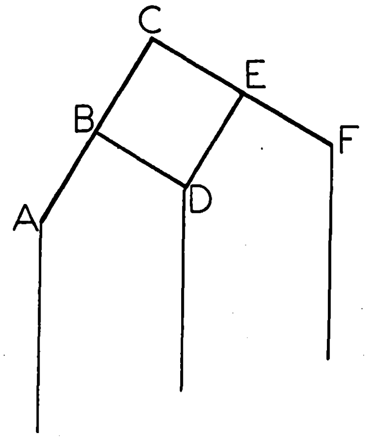


There are a number of ways in which the three rods can be connected so that the specified three points will remain colinear. One may have a cross bar pivoted upon the central bar, This cross bar has a slot on each side in which a pin is fixed on the corresponding rod slides. The pins and the pivot are constrained then to be colinear.

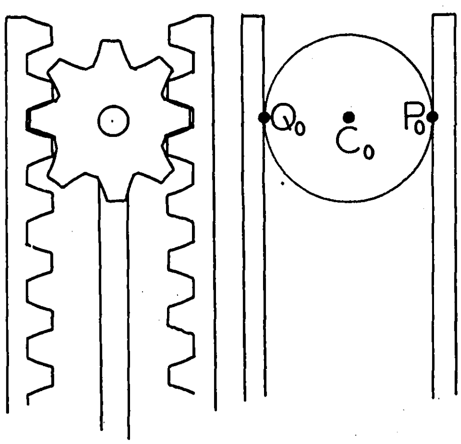


This cross bar arrangement can be replaced by a pantagraph. The bars AC and CF are rigid and equal with midpoints B and E respectively. DE = BC and BD = CE.

AC and CF are hinged at C, AC and BD at B, BD and DE at D, DE and CF at E. Owing to the equality of opposite sides, BCED is a parallelogram. Hence, BD and CF are parallel. Since B is the midpoint of AC and BD is one half CF in length, this implies that D is the midpoint of the line segment AF and hence, A, D and F are always colinear. There is also a "lazy tong" parallelogram arrangement, which we will discuss later.



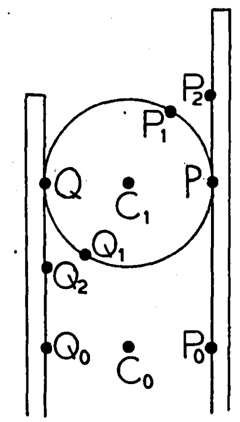
There is still a third way of accomplishing the same objective. Two racks are used instead of the outer rods and a pinion is mounted on the middle bar. Again the output is $\frac{1}{2}(x+y)$. The gear teeth are constructed in such a fashion that the movement of such a system is strictly similar to the movement of a pair of rods and a wheel in contact with them and which does not slip relative to the rods. The motion of the latter system can be easily specified. Let C_0, P_0, Q_0 denote a reference position



for the center of the wheel and its points of contact to the two bars. Let C_1, P_1, Q_1 refer to the corresponding points fixed on the circle P_2, Q_2 the corresponding points fixed on the bars. Now if the system is moved to a new position in which P and Q are the new points of contact, we see that since the wheel did not slip relative to the bars that $P_2P = \overset{\frown}{P_1P} = \overset{\frown}{Q_1Q} = Q_2Q$. Since $C_1P = C_1Q$, this yields that P_2, C_1 and Q_2 are colinear.

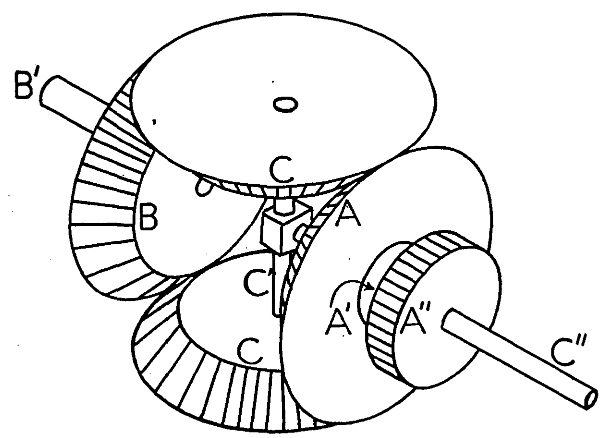
Note that the displacement of the P rod has the value $y = P_2P_0 = P_2P + PP_0 = P_2P + C_1C_0 = C_1C_0 + r\theta$ where $\theta = \angle P_1C_1P$ in radians and $r = C_1P$. Similarly the displacement of the Q rod is given by the expression $x = C_1C_0 - r\theta$. Hence, $x + y = 2C_1C_0$, i.e., the displacement of the center of the wheel is the average of the other displacements.

This combination will conveniently add displacements of a considerable size but normally there will be backlash between the pinion and the two racks.

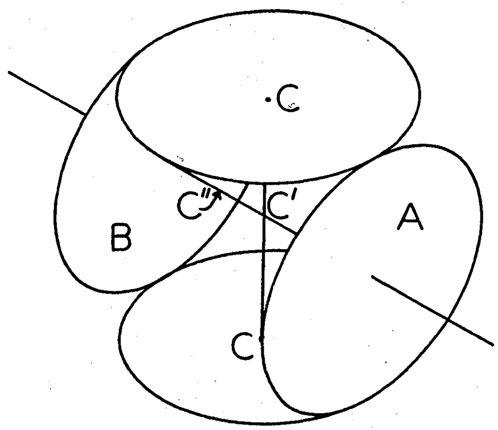


2. The customary method of adding rotations is by means of a differential. This device is analogous to the rack and pinion adder of the previous section with the translations, however, replaced by rotations.

There are two types of differential, 1) the bevel gear differential and 2) the spur gear differential. The accompanying diagram illustrates

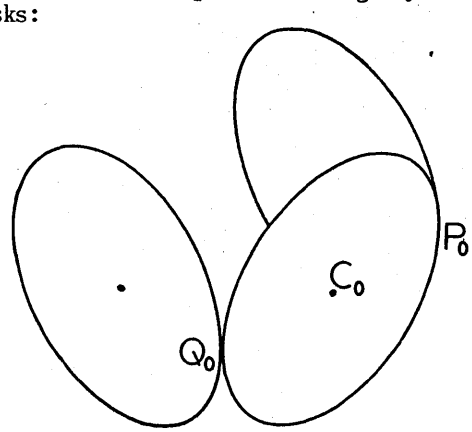


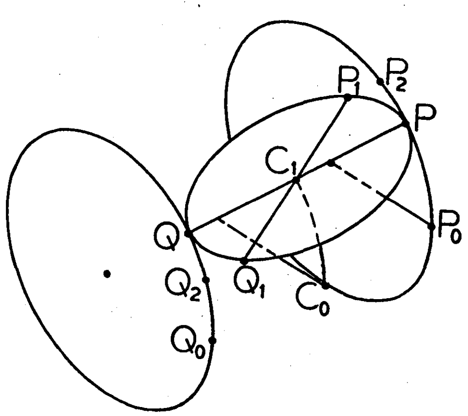
the arrangement of gears in a bevel gear differential. The bevel gear A, the input spur gear A" and the connecting collar A' are rigidly connected but are free to rotate around the axle C". The rotation of the combination A, A', A" constitutes the input rotation x, which is applied through a gear meshing with A". The shaft B' is rigidly connected to the bevel gear B and the rotation of this combination constitutes the input y. The bevel gears C are free to rotate around the axle C'. However, the combination of C and C' is connected to the axle C" which is perpendicular to C', so that this entire combination may rotate around the axis of the axle C". The rotation of the shaft C" is the output $\frac{x+y}{2}$ of the combination.



Geometrically, the motion of the bevel gears is equivalent to the motion of non-slipping right circular cones or frustums of cones. We can even replace each gear by a disk contingent to other disks, the disk representing a cross-section of the cone perpendicular to the axis. These disks must rotate without slipping.

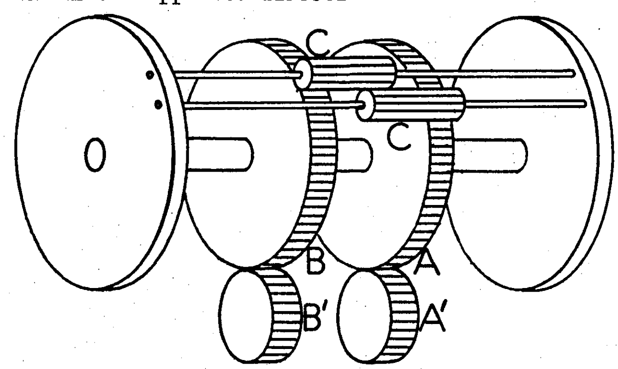
Let us consider only one disk C and let C_0 denote the center of this disk and P_0 and Q_0 denote the points of tangency with the other disks in some reference position for the system. Let C_1, P_1, Q_1 denote the corresponding points fixed on the C disk, P_2, Q_2 the points fixed on the other disks. Let us suppose the system moves so that P and Q are the new points of tangency between the disks:





The rotation of the disk A can be measured by the arc P_2P_0, i.e., $x = P_2P_0 = P_2P + PP_0$. Let z denote the rotation of the axis C' around C". This is measured by the arc $C_1C_0 = PP_0$. Thus $x = z + P_2P$. Since the disks C and A move without slipping, $P_2P = P_1P = r\theta$ where $r = P_1C_1$ and θ is the radian measure of the angle $P_1C_1P = Q_1C_1Q$. Hence $x = z + r\theta$. A similar discussion will show that the rotation y of B is $z - r\theta$. Hence, $x + y = 2z$.

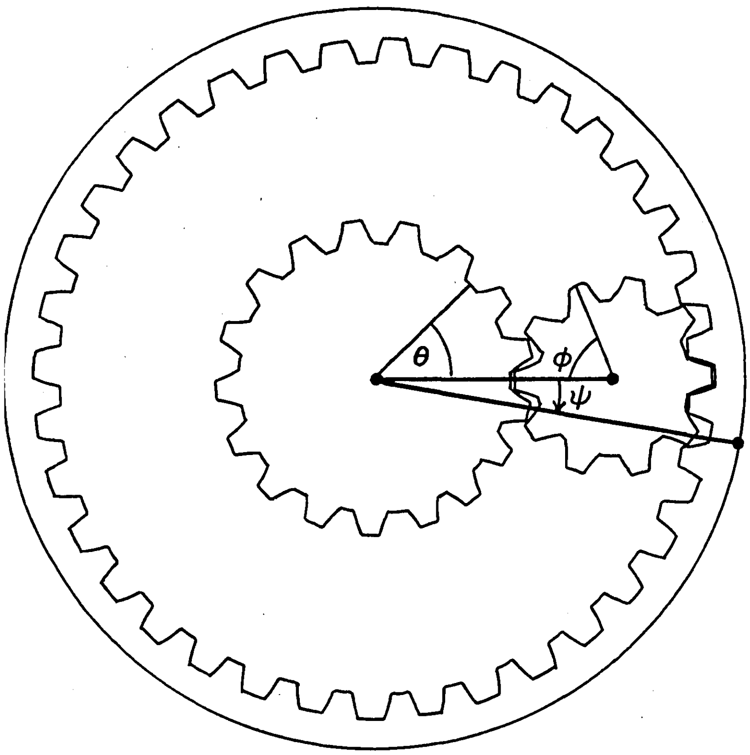
It is clear that the purpose of the gear C is to insure that if the x input rotates an amount θ relative to the output z, the y input will rotate precisely the same amount relative to z but in the opposite direction. This purpose can also be accomplished by a spur gear combination. In the accompanying diagram, the output combination involves the two outside disks, which form a mounting for two meshing spur gears C. One of these mesh with the gear A, the other with the gear B, which mesh with input gears A' and B'. When the mounting for the C gears is held stationary and the A gear rotated by A', then the B gear will turn equally and in the opposite direction.



Thus if the inputs x and y are applied to A and B, and z is the rotation of the C gear mounting, then the relative motion of A to C, i.e., $x - z$ must equal the relative motion of C to B, i.e., $z - y$. Hence, $x - z = z - y$ or $x + y = 2z$.

There is another form of the differential which consists of an epicycloidal gear train. (See following diagram.) Here we have three gears in the same plane, the outer one being an "annular gear," i.e., one in which the teeth are on the inside of gear. The intermediate gear is mounted on an axle which in turn is attached to an arm which revolves around the common axis of the other two. It is the rotation of this arm which is the output. For simplicity we shall suppose that the radius of the central gear is equal to the diameter 2r of the intermediate gear.

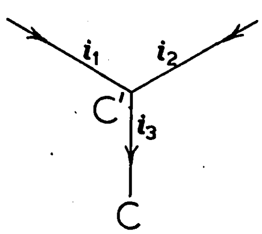
Let us now consider what happens in a motion of the system. For the purpose of this discussion, gears can be considered as circles. Let us suppose that the output arm has rotated an amount z and the central gear an amount x, so that $x - z = \theta$ is the relative motion of these two. Relative to the arm, the intermediate gear will turn an amount φ, which has the same arc length on the intermediate gear as θ has on the central gear or $r\varphi = 2r\theta$. Similarly, we see that the outer gear has a rotation by an amount ψ relative to the output arm, oppositely directed to θ, for which $r\varphi = 4r\psi$. Hence, $\psi = 2\theta$. If y is the total



rotation of the outer gear, we have $y = z - \psi$ or $\psi = z - y$. Substituting for ψ and θ in the equation $\psi = 2\theta$, we obtain $3z = 2x+y$.

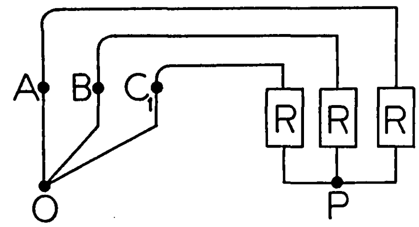
Since the differential permits the simultaneous addition of two quantities, it can be used in an adding machine to combine the regular feed and the tens transmission from a lower place. This avoids the difficulties represented by the additional tens transmission feed. This device was used by Selling in his multiplication machine.

3. The Addition of Currents and Voltages. Kirchhoff's Law on the Currents at a point of junction can be used for electrical addition. In the first place if we have two currents flowing to a common point in an electrical network, we get a current which is the sum of the two given currents if we join them.



Kirchhoff's Law states that the sum of the currents flowing to a junction point is zero. Here C was the original junction point, C' is the new one added to obtain the addition. At C', we have $i_1 + i_2 - i_3 = 0$.

Kirchhoff's Law also permits the addition of a number of voltages, which are measured from a common point. Suppose in an electrical network we have a number of points, A, B, C, etc., and let x, y, z, etc., denote the potentials of these points relative to a fixed point O. Let us suppose



that each of the points A, B, C, etc., is connected through a large resistance of value R to a point P. Suppose that P has a potential w relative to the origin O. Let i_1, i_2, i_3, etc., denote the current flowing from A, B, C, etc., to P. Then

$$i_1 = \frac{x-w}{R},\ i_2 = \frac{y-w}{R},\ i_3 = \frac{z-w}{R},\ \text{etc.}$$

Since by Kirchhoff's Law, the sum of these currents is zero we have

$$0 = i_1 + i_2 + i_3 + \ldots = \frac{x-w}{R} + \frac{y-w}{R} + \frac{z-w}{R} + \ldots$$

or

$$x + y + z + \ldots = nw$$

if n is the number of terms on the left. In passing, we might mention that we can easily obtain a linear combination of x, y, z, etc., by using different values of R in the connections rather than a fixed value. However, then the coefficient of w depends upon these values of R.

Of course, we would like either to measure w or use it as an input for another circuit. In the first case, let us suppose we use a voltmeter whose resistance is $\frac{R}{\lambda}$. This we would connect between P and O and hence there would be an additional current of $-\frac{\lambda w}{R}$ flowing from O to P. The equation for w then becomes

$$x + y + z = (m + \lambda)\, w$$

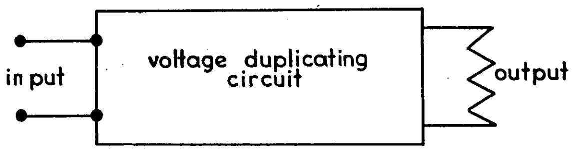
Thus the effect of introducing the voltmeter is just to change the constant or proportionality for w.

If we wish to use the voltage w as an input in other portions of the circuit, it is essential that this does not disturb the original equation to too great an extent. This can be done by means of a "voltage duplication" vacuum tube circuit. This circuit reproduces the input voltage w, as a potential drop due to current flowing through a resistance but with a relatively large current. The two input leads of this circuit are applied to the original circuit in the same way as the leads of a voltmeter and relative to the original circuit; it behaves like a voltmeter of rather high resistance, for instance of 1 megohm.*

The purpose of the "voltage duplication cir-

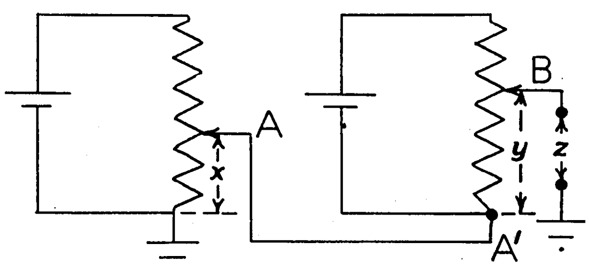
* A megohm is 10^6 ohms.

cuit is power amplification. For instance, if the original current flowing from P to 0 through R is i_1, then $i_1 = w/R = w \cdot 10^{-6}$ and the power is $w \cdot i_1 = w^2 \cdot 10^{-6}$. If the output resistance is, say, 10^3 ohms, the output power is $w^2 \cdot 10^{-3}$ and hence the power has been amplified a thousand times.



Actually even better power outputs are possible and also voltage amplification can be obtained if desired.

Voltages from independent circuits can be added by connecting the circuits in a proper fashion. For example, if we have two batteries with potentiometers across them and connected as in the diagram below, the voltage z = x + y. For the

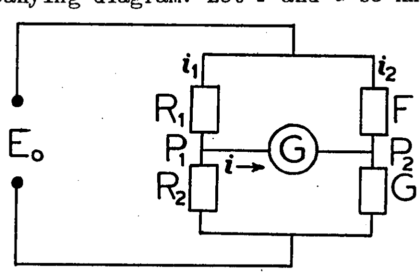


points A and A' are x volts above ground and the point B is y volts above these. Hence, B is x + y volts above ground. The trouble with this game is that it can be played only once. After one such connection the circuits are no longer independent. However, the batteries can be replaced by the secondary coils of transformers.

4. When two resistances are in series, the resulting resistance is the sum of the two. Inductances in series or capacities in parallel have the same property (provided there is no interaction in the first case).

Resistances are compared by means of the Wheatstone bridge. Although the addition of resistances is only a relatively minor application of this device, we will discuss it here in the most general form since the differences between the special and general case are relatively trivial.

A Wheatstone bridge for resistances consists of two parallel systems of resistances, with a galvanometer connected between the circuits as in the accompanying diagram. Let F and G be known resistances. G may be variable but its value is known. R_1 and R_2 are resistances to be compared. Let i_1 denote the current through R_1 i_2 through F and i, the current through the galvanometer, which we suppose flows from P_1 to P_2. Let E be the applied voltage. The current through $R_2 = i_1 - i$ and hence



$$E_0 = R_1 i_1 + R_2(i_1 - i) = (R_1+R_2) i_1 - R_2 i$$

Similarly, the current through $G = i_2 + i$ and

$$E_0 = (F+G) i_2 + G i$$

Let the galvanometer have internal resistance r, then the voltage across the galvanometer is

$$F i_2 - R_1 i_1 = r i$$

If we eliminate i_1 and i_2 between these equations we get

$$\left(\frac{R_2}{R_1+R_2} - \frac{G}{F+G}\right) E_o = i\left(r + \frac{R_1R_2}{R_1+R_2} + \frac{FG}{F+G}\right) = \text{(say) } Hi$$

where

$$H = \left(r + \frac{R_1R_2}{R_1+R_2} + \frac{FG}{F+G}\right).$$

Thus to add two resistances by means of a Wheatstone bridge, we may take G = F and use the two given resistances R' and R'' in series as R_1. We then vary R_2 until i is zero. In this case we have

$$\frac{R_2}{R_2+R'+R''} - \tfrac{1}{2} = 0 \quad \text{or } R_2 = R' + R''$$

However, our formula will also give the sensitivity of our experiment. For we may write the equation in the form

$$\frac{(R_2 - [R'+R''])}{(R_2 + R'+R'')} E_0 = 2 H i$$

In the denominator and in H we may write R' + R'' = R_2 and thus obtain

$$\frac{R_2 - (R'+R'')}{R_2} E_0 = 4(r + \tfrac{1}{2}[R_2+F])i$$

or if $\delta R = R_2 - (R'+R'')$

$$E_0 \cdot \frac{\delta R}{R_2} = 2 (F+R_2+2r)i$$

Hence, a fractional error $\delta R/R_2$ will yield a current i of this amount through the galvanometer. Knowing the smallest current that we can detect with the galvanometer and the quantities E_0 and F, we can find the largest possible fractional error in our result.

Example. Suppose in a given experiment, 5 microamperes is detectable. Then if F has the value 10^2, R, 10^2 and r, 4×10^2 ohms and E, 3 volts then

$$3 \cdot e = 2{,}000 \times 5 \times 10^{-6}$$
$$e = 3.3 \times 10^{-3}$$

Thus the error in the determination of R is about a third of an ohm.

Ohm's Law can also be used to determine resistances. For since I R = E, if we apply a known voltage E and read I on an ammeter, we can determine R. However, the linearity of a good ammeter is seldom as good as 1 percent, and this limits the accuracy of this method.

It should be mentioned that when we are measuring the resistances of wires by means of bridge methods, we are actually measuring lengths. For the resistances may change with temperature but if we use the same material in all parts of the bridge, this effect would not change the balance point provided the change in temperature is uniform in all parts of the bridge. In a sense then, this is another method of adding lengths, and frequently more convenient and accurate than any direct method.

5. We describe briefly certain elementary notions in electrical circuit theory. In many interesting cases the differential equations which describe the behavior of a mathematical device are identical with those of an electrical circuit. This fact is very useful since the most important problems, such as stability, have been carefully investigated for circuits.

An electrical circuit consists of points called "nodes" and "branches" connecting the nodes. In each branch AB, we have a current i_{AB}, which depends upon certain voltages. In each branch we can conceive of a generator, say a voltage generator, producing a voltage e(t). The current i_{AB} produces counter voltages depending upon the "lumped" impedances, while the currents in the other branches may also induce voltages in this branch. Therefore, in general, for each branch we have an equation

$$-\Sigma M\frac{di_{A'B'}}{dt} + L\frac{di_{AB}}{dt} + R\, i_{AB} + \frac{\int i_{AB}dt}{C} + e(t) = e_{AB}$$

where e_{AB} is the total voltage drop from A to B, M refers to the various mutual inductances between branches, L is the inductance, R the resistance and C the capacitance in the given branch These last, of course, contain the internal impedances of the generator.

This system of equations relates the set of quantities $i_{A,B}$ with the set $e_{A,B}$. Usually we are given the various constants M, L, R and C and the e(t) and we endeavor to find all the $i_{A,B}$'s or all the $e_{A,B}$'s. If we have found the $e_{A,B}$'s, we can invert the above system to find the $i_{A,B}$'s as linear operators on the $e_{A,B}$'s. The effect of e(t) is then similar to a current generator.

Now for the quantities $i_{A,B}$, we have at each node an equation on the currents flowing from A,

$$\Sigma_{B \text{ connected to } A}\, i_{A,B} = 0$$

(Kirchhoff's law on currents.) There is a suitable modification of this to incorporate the effect of vacuum tubes. For the $e_{A,B}$, there is the loop voltage law. If $P_0, P_1, \ldots, P_k$ is any set of nodes such that P_i is connected by a branch to P_{i+1} and P_k to P_0 then

$$\Sigma_{i=0}^{k-1}\, e_{P_i\, P_{i+1}} + e_{P_k\, P_0} = 0.$$

It is evident that these linear relations are not independent. Thus if we add the equation for all the nodes, we will get zero, since each current i_{AB} will appear as flowing away from A in the A node equation, and toward B in the B node equation.

Using our branch relations, we can express these equations with either the i_{AB}'s or the e_{AB}'s as unknowns. However, in order to deal conveniently with the question of linear dependence, it is customary to introduce new unknowns in terms of which the $i_{A,B}$'s or the $e_{A,B}$'s can be expressed.

When we are seeking the $i_{A,B}$'s the new unknowns are associated with various loops and are called mesh currents. We give a method of obtaining mesh currents which determine all the $i_{A,B}$'s.

We deal with the case in which every node can be connected to any other node by a chain of branches. If the circuit breaks up into more than one connected part, each part can be treated in an analogous manner.

We specify certain points and branches as follows: Let a node A_0 be chosen. Let A_1 be a point which is directly connected to A_0 by a branch. We choose one of the branches connecting A_1 to A_0 and call it the return branch for A_1. If A_0, $A_1, \ldots, A_{k-1}$ have been enumerated, let A_k be a point directly connected to one of these and let the return branch for A_k be such a connecting branch. It seems advantageous to use a return branch $A_k A_{k'}$ with k' as low as possible.

For the remaining branches, we introduce loops in the following way Suppose we have a branch connecting A_k with an $A_{k'}$ with lower subscript. We regard this as the first branch of the loop. Another branch is the return branch $A_k A_{k''}$ for the point A_k. We continue the loop from $A_{k'}$ and $A_{k''}$ by return branches to points of lower subscripts until we reach a common point, whose subscript, of course, will be the lowest subscript of the nodes on the loop. The loop has two parts, the down part which starts with the given branch $A_k A_{k'}$ and continues down to the point of minimum subscript and the remaining or up part.

The mesh current for this loop, we take to be the current in the given branch $A_k A_{k'}$. We consider it as flowing completely around the loop.

Kirchhoff's law on current permits us to express the current in any return branch $A_k A_l$ in terms of the mesh currents. The current in $A_k A_l$ is the sum of the mesh currents for which $A_k A_l$ is on the down part of the mesh, minus the sum of the mesh currents for which $A_k A_l$ is on the up part. This is readily seen to hold for A_n and it can be shown in general for A_{n-p} by an induction on p.

Thus we have expressed the current in each branch in terms of the mesh currents in such a way that Kirchhoff's law on current holds. Furthermore for each loop we can write the voltage equations around the loop and if we express the currents in the return branches as in terms of the mesh currents, we will have as many equations on these last as we have unknowns. These are differential equations in the mesh currents and since our operators involve only differentiation and constants, we know that we can treat such a system as if the differentiation operator were a numerical quantity. Mathematically, these equations are independent, since in each mesh equation, we find an arbitrary impedance which does not appear in any other equation, i.e., the impedance of the branch which determines the mesh in the above construction.

We can solve immediately for the current in any branch which is not a return branch

$$\Delta\ i = f$$

where Δ is the determinant of the system and f is a linear combination of the generator voltages of the system. A similar formula holds for the current in a return branch, since the current in a return branch is a linear combination of the mesh currents with ± 1 as coefficients.

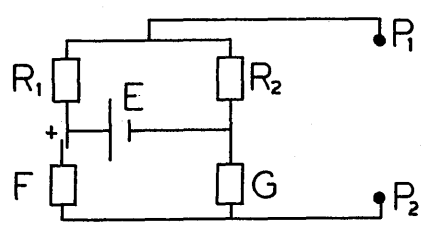
Now Δ is linear in the impedance z for the branch in which i flows; in fact it is linear in any impedance. Let $\Delta = \Delta' + \Delta'' z$ and let us divide by Δ''. We then get

$$(z' + z)\ i = f/\Delta'' = E' \qquad (T)$$

where $z' = \Delta'/\Delta''$. If z is very large, this is approximately $z\ i = E'$ and E' which is a linear combination of the generated voltages is called the open circuit voltage for the branch. z' is called the internal impedance.

Sometimes this form (T) permits a simplification of the circuit analysis when one's interest is confined to one branch. This is particularly true if we have only resistances in the network and constant voltages since then z' and E' are constants. Suppose, for instance, that we can insert various resistors in this one branch and measure the corresponding currents in this branch. Then even if we know nothing else about the rest of the circuit, two such measurements will suffice to determine z' and E' and hence the behavior of the current in this branch.

For example, let us return to the Wheatstone bridge. Removing the galvanometer let us consider



it as a two-terminal network. The open circuit voltage across $P_1\ P_2$ is clearly:

$$E' = \left(\frac{R_2}{R_1+R_2} - \frac{G}{F+G}\right)E.$$

The internal impedance z' does not depend upon E and we can obtain it most simply in the case $E = 0$. The usual rules on parallel and series resistance, then yield

$$z' = \left(\frac{R_1\ R_2}{R_1+R_2} + \frac{FG}{F+G}\right)$$

Thus if r is the resistance in the cross piece, we have the following formula for the current

$$\left(r + \frac{R_1\ R_2}{R_1+R_2} + \frac{FG}{F+G}\right) i = \left(\frac{R_2}{R_1+R_2} - \frac{G}{F+G}\right) E.$$

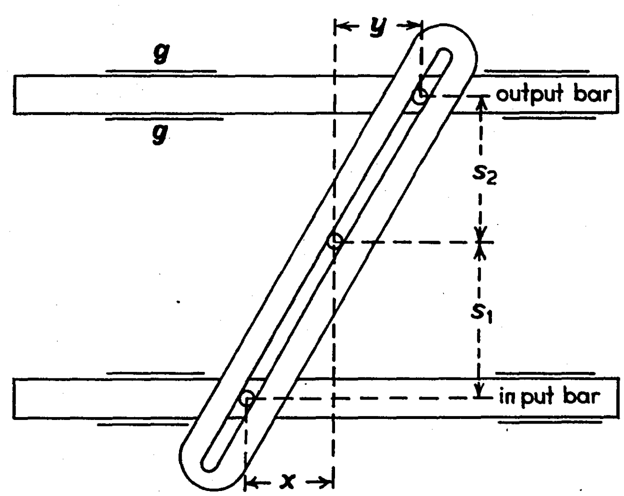
which can be written in the form $Hi = E_0$.

If it is relatively simple to solve the e_{AB} equations for the i_{AB}'s, we may introduce the nodal voltages e_A, relative to a fixed node A_0 as unknowns. Since $e_{AB} = e_A - e_B$, the loop voltage equations are all satisfied. For every node other than A_0, we have Kirchhoff's current law, in which every i_{AB} has been expressed in terms of $e_{AB} = e_A - e_B$. This gives as many equations as unknowns. When the circuit also contains vacuum tubes, these nodal equations are frequently preferred. The vacuum tube is considered simply as a current generator, with internal impedance, of course, connected between the cathode and plate.

II. Multipliers

It is far easier to multiply quantities if one of of them is constant. We discuss this method first, for displacements, rotations, the multiplication of a voltage by a constant, and the multiplication of an alternating current voltage by a constant. For the multiplication of two variable displacements, there are a number of ways based on similar triangles. The product of two rotations can be obtained by either converting to displacements or directly by means of "difference of squares" multiplication. The use of logarithmic cams will reduce either problem to the addition problem. Another method for multiplication is based on the Wheatstone Bridge and servo motors. Servo motors can also be used for logarithmic resistance multipliers. For linear combinations in which addition as well as multiplication is desired, a "difference of squares" resistance multiplier is possible. Integrators also offer methods of multiplication. Later on we shall discuss certain other methods of multiplication involving electronic methods.

1. Multiplication of a Displacement by a Constant. This can be theoretically accomplished by means of the lever principle. Let us suppose that our input

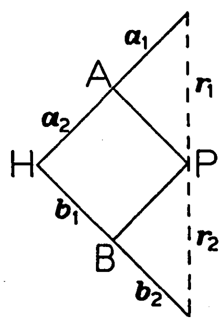


and output displacements apply to bars which are constrained to move parallel to each other, for instance, in grooves or guides. On each bar, we have a pin which fits in a slot in a cross-bar. We also have a pivot in the slot which may be fixed in any position along a line perpendicular to the two parallel bars. When the position of the pivot is determined and one of the parallel bars is moved the other is displaced a proportional amount. For by congruent triangles

$$x/y = s_1/s_2.$$

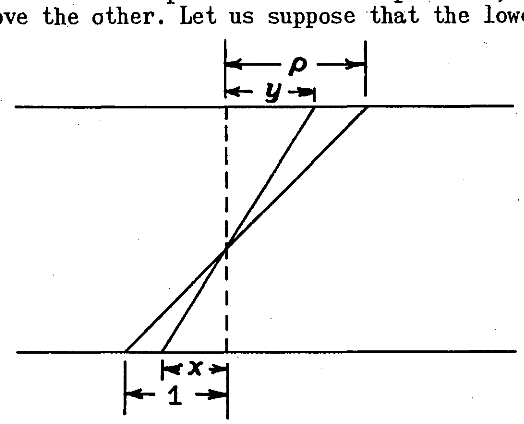
This arrangement even takes care of the sign, for if the pivot is not between the two parallel bars the displacement is in the same direction, the equivalent of multiplying by a positive factor, while the displacements are in opposite directions when the pivot is between the parallel bars.

It is possible to replace the cross bar by a pantagraph arrangement. This gives better mechanical results but four adjustments have to be made to set up the smaller parallelogram.



If the desired ratio is r_1/r_2, we must first choose A so that $a_1/a_2 = r_1/r_2$ and B, so that $b_1/b_2 = r_1/r_2$. We then must fix the lengths of the remaining arms so that $AP = b_1$ and $BP = a_2$.

The above is not suitable for a continuous input of the ratio $\rho = s_1/s_2$, because the scale for ρ is not linear. However, we can position the pivot by an arrangement, whose input is linear in ρ as follows. Let us suppose we have two such arrangements in parallel planes, with a common pivot which may move in a direction perpendicular to the bars. We assume that the pairs of bars are parallel, one set above the other. Let us suppose that the lower



arrangement is analogous to our previous one and that the upper arrangement will be used to position the pivot. Suppose we move the upper bar over the x bar to the unit position and the one over the y bar an amount ρ. This will determine the position of the common pivot so that $s_1/s_2 = \rho$. Hence the output y will have the value $y = \rho x$ where ρ and x can be variable.

2. Multiplication of a Rotation by a Constant. If the constant is to have only one value, then a pair of gears can be used to obtain the product. We wish to discuss briefly certain parts of the theory of gears in this section. We begin by describing the motion of a lamina.

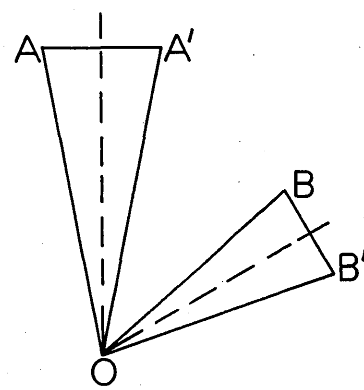
Consider a right cylinder, i.e., one whose directrix is arbitrary but whose elements are perpendicular to the plane of the base. This is the geometrical description of a lamina. We will only be concerned with rigid motions which do not change the plane of the base, so we can neglect the height of the cylinder. Let us call such a motion a permitted motion.

Lemma. Any permitted motion is equivalent to either a rotation or a translation.

Proof. We first show that two points determine the motion. For suppose A moves to A', B to B'. Let C be any other point, we wish to show that the position of C' is determined. For since AB = A'B', AC = A'C', BC = B'C', we see that the triangle A'B'C' is congruent to ABC. (There is an obvious special case neglected in which A, B, C are collinear.) This shows that C' can have at most one of two possible positions and one of these is ruled out by the requirement that the motion be in a plane.

Thus the rigid motion can be specified by the condition that $A \rightarrow A'$, $B \rightarrow B'$. Let us then construct the perpendicular bisectors of the line AA' and BB'. There are three possibilities.

Case 1. Suppose these bisectors intersect at a point O. Then AO = A'O, BO = B'O by the equidistant



property of the bisectors. We also have AB = A'B' since the motion is rigid. Thus the triangles AOB and A'OB' are congruent. Since the rigid motion brings AB into A'B' and is orientation preserving, it must bring one of the triangles onto the other. Hence the point O is fixed under the motion which is therefore a rotation around O.

Case 2. These bisectors are parallel. Thus AA' is is parallel to BB'. We also have AB = A'B'. This implies that sin < AA'B' = sin < A'AB

A ———— A'

B ———— B'

and the quadrilateral AA'B'B is either a parallelogram or a equilateral trapezoid. In the latter case, the bisectors would coincide. Thus the quadrilateral is a parallelogram. Hence, AB has been moved parallel to itself, i.e., we have a translation.

Case 3. The bisectors coincide. Under these circumstances it is easily seen that if we continue the lines AB and A'B', they are either parallel or they intersect at a point O which is on the bisector. In the first case an argument like Case 2 is applicable. In the second case, we can readily show that the point O is taken into itself, by considering line segments.

Instantaneously then every motion of a lamina can be considered as a rotation whose center is termed the instantaneous center of motion.

Let us now consider two lamina in contact and with a common base plane. Let us consider only the bases. At the point of contact, the edges must have a common tangent. This is also sufficient to permit contact provided no other point of the bases coincide. It is possible to express the sufficiency conditions in the small for contact in terms of the expression for the curvature of the edge. The reader is urged to look at these conditions.

Let us suppose now that one lamina is fixed and that the other moves on it without slipping. Let us take the point of contact as the origin and the common tangent as the x-axis. Let the equations of the laminas be respectively:

$$y_1 = a_2 x^2 + a_3 x^3 + \ldots$$
$$y_2 = b_2 x^2 + b_3 x^3 + \ldots$$

Now if the point (x_1, y_1) on the first curve is to be in contact with the point (x_2, y_2) on the second curve we must have

$$\int_0^{x_1} \sqrt{1+y_1'^2}\, dx = \int_0^{x_2} \sqrt{1+y_2'^2}\, dx$$

This equation becomes

$$x_1 + \tfrac{2}{3} a_2^2 x_1^3 + \ldots = x_2 + \tfrac{2}{3} b_2^2 x_2^3 + \ldots$$

which can be regarded as defining x_2 as a function of x_1. By implicit differentiation and the Taylor's expansion we obtain

$$x_2 = x_1 + 2\,(a_2^2 - b_2^2)\, x_1^3 + \ldots$$

Thus while the change in y, $([b_2 - a_2]\, x^2 + \ldots)$, is in general of the second order, the change of x is of the third order. Hence for small values of x, the displacement $x_1 > x_2$, which occurs as the upper lamina rolls in such a way that its point of contact changes from the origin to x_2, is nearly perpendicular to the x-axis. It is clear that the point on the moving lamina which was originally at the origin must have a similar motion only upward instead of downwards. It is easy to see that the rotation which describes the motion of the lamina must have its center somewhere between these two displacements, which are approximately parallel but in opposite directions.

This is true for every motion of this sort, no matter how small. Consequently, in the limit, we have:

The instantaneous center of motion of one lamina rolling upon another is at the point of contact.

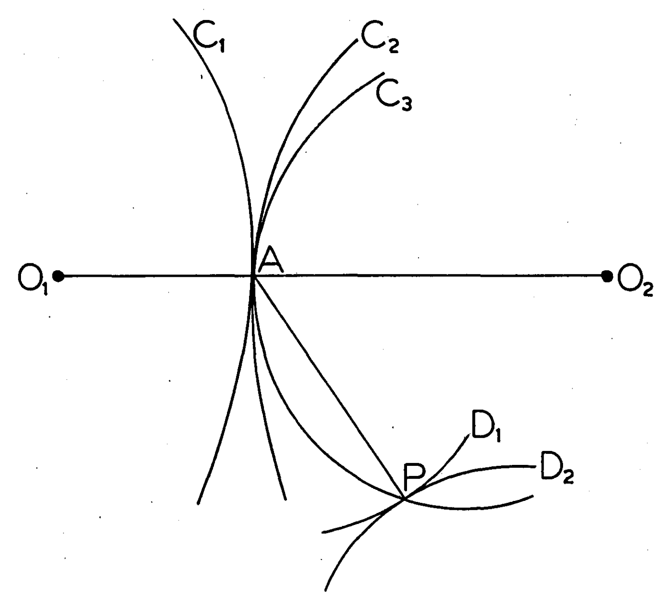
This has the following consequence:

Let P be fixed on the rolling lamina and let D be the path of P. The normal to D at P goes through the point of contact between the laminas.

We are now in a position where we can describe the possible points of contact between two laminas, each of which revolve around an axis perpendicular to the base.

Let O_1 and O_2 be the centers of rotation of the bases of the laminas. Let A be a point on the line joining the two centers and let C_1 and C_2 be two curves which are cotangent at A. Let C_3 be still another curve with the same tangent at A. Let P be a point fixed on C_3.

Let C_3 roll on C_1 and then suppose it to roll on C_2. Let D_1 be the path of P in the first case and D_2 the path of P in the second case.



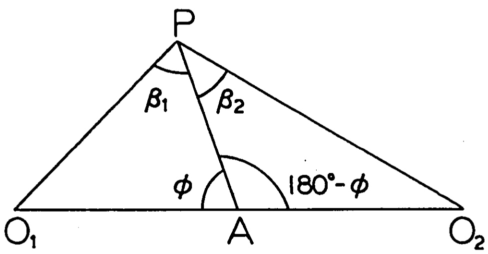
The point P is common to the curves D_1 and D_2, since this is a position corresponding to contact of C_3 with C_1 and C_2 at A. Since C_3 is rolling on the other curves in each case D_1 and D_2 have a common normal AP at P from the above. Thus, D_1 and D_2

are cotangent at P. It follows that if we have a lamina with edge D_1 and another with edge D_2, then P is a possible point of contact between these laminas. It will be convenient to designate a lamina by its edge.

Now suppose the curves C_1 and C_2 are such that if the corresponding laminas are rotated around O_1 and O_2 respectively, then it is possible for them to remain in rolling contact, without slipping, along the line O_1O_2. C_3 can then also roll on these without slipping and with the common point of contact. Of course, in this process the point P moves in space, but if we consider the lamina D_1 as rigidly connected to the lamina C_1 and D_2 to C_2, then P is still a possible point of contact between D_1 and D_2, since the cotangency property is preserved during this operation.

Suppose now that we have two laminas D_1 and D_2 which are free to rotate around O_1 and O_2. Let us suppose now that we have D_1 and D_2 in contact at P and that D_1 moves through a small angle θ_1. Let $\beta_1 = \angle APO_1$, $\beta_2 = \angle APO_2$.

The point P_1 fixed on D_1 which correspond to P initially, will be displaced an amount $O_1 P \theta_1$ (approximately) and the component normal to D_1 will be approximately $O_1 P \sin \beta_1 \theta_1$.



The lamina D_2 will rotate an amount θ_2. At P, the normal component of the motion must be the same for each lamina if they are to remain in contact. (In general, the laminas will slip relative to each other, but this corresponds to a difference in the tangential components of the motion of the laminas at P.) Thus, θ_2 is determined approximately by the equation

$$O_1 P \theta_1 \sin \beta_1 = O_2 P \theta_2 \sin \beta_2.$$

Let $\varphi = \angle PAO_1$. Then $\angle PAO_2 = 180 - \varphi$ and by the law of sines, we obtain $O_1P \sin \beta_1 = O_1A \sin \varphi$ $O_2P \sin \beta_2 = O_2A \sin (180 - \varphi) = O_2A \sin \varphi$. Substituting in our previous equation and dividing by $\sin \varphi$ we obtain

$$\theta_1/\theta_2 = O_2A/O_1A \qquad (1)$$

In the limit, therefore, we have

$$\frac{d\theta_2}{d\theta_1} = O_1A/O_2A. \qquad (2)$$

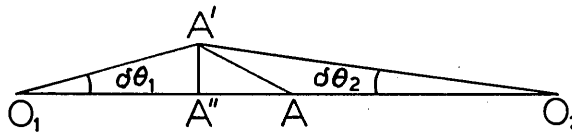
We have established this equation under the hypothesis that D_1 and D_2 rotate in contact at P.

The above is essentially a sufficiency discussion which proves that if two laminae D_1 and D_2 have been constructed by means of curves C_1, C_2, and C_3 they can rotate around O_1 and O_2 in contact respectively so that (2) holds.

But these arguments can also be used in the reverse direction. Suppose we take two arbitrary laminae D_1 and D_2 mounted so that they can rotate around O_1 and O_2 respectively, and suppose that they rotate in contact. If P denotes the point of contact and AP, A on O_1O_2 is the common normal to D_1 and D_2 at P, then the above argument shows that (2) holds for the motion.

Now let C_1 and C_2 be defined as the locus of A attached to D_1 and D_2 respectively. We can readily show that the laminae D_1 and D_2 move as if C_1 and C_2 were in non-slipping rolling contact, since (2) holds.

For suppose (2) is satisfied and laminae D_1 and D_2 rotate an amount $\delta\theta_1$ and $\delta\theta_2$ respectively as indicated in the accompanying diagram. We can suppose that in this rotation the point A moves to a



new point A" along the line O_1O_2. The new point A" corresponds to two points A', one on C_1, the other on C_2. The condition (2) is readily seen to be equivalent to the statement that before the motion takes place A'A" is the same in each case to first order differentials. Thus A'A and ds is the same for each curve and thus the motion of C_1 and C_2 is a non-slipping roll.

We have shown then that the general rotational motion of two laminae D_1 and D_2 which are in sliding contact can be referred back to a non-slipping motion of two laminae C_1 and C_2 attached to D_1 and D_2 respectively.

3. The result of the previous section is the basis for the theory of gear teeth. The objective in shaping a gear tooth is that the rotation of one shaft should be a constant multiple of the rotation of the other. Two circles C_1 and C_2 which are in non-slipping contact will rotate in this fashion if their diameters have the appropriate ratio.

But if one shaft is to drive the other, it is clearly undesirable that the contacts be purely frictional. Instead one introduces the laminae D_1 and D_2 constructed as in the above so that the motion will have a component normal to the surfaces of contact. This component is the driving element and, while friction is now present due to the slip between D_1 and D_2, it no longer affects the relative motion but just the energy necessary to accomplish the result.

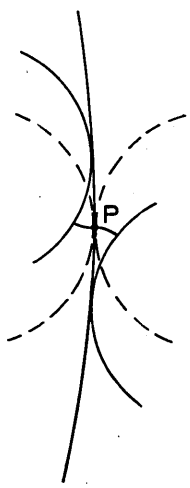
The curves C_1 and C_2 are specified by the requirement on the relative motion and are referred to as pitch circles. C_3 remains arbitrary to a considerable extent. When C_3 is also a circle it it is referred to as the tooth circle. Choosing C_1 and C_2 will determine the shape of D_1. For a small part of its circumference, D_1 is traced out by a point of C_3 as it rolls on C_1. The latter, of course, is fixed.

If C_3 is considered as rolling on the outside of C_1 for D_1, it must, of course, be considered as rolling on the inside of C_2 when determining the shape of D_2. However, a cycloidal curve such as the path of P will have cusp points at which one can reverse these relations. Thus before the cusp point one can regard C_3 as rolling outside C_1 and inside C_2 and afterwards as inside C_1 and outside C_2.

The situation can be clarified perhaps by a simple example. Suppose one wishes to design two gears with pitch circles C_1 and C_2 with diameters 2 and 3 units respectively and 200 and 300 teeth respectively. Suppose we wish a tooth circle of 1 unit diameter.

It is clear that while two specified teeth are in contact, the two pitch circles will move in contact an amount on their circumference of $\pi/100$ units. Thus to shape the gear we begin by dividing the circumference of each pitch circle into arcs of length $\pi/100$. The tooth circle C_3 is also divided in this fashion.

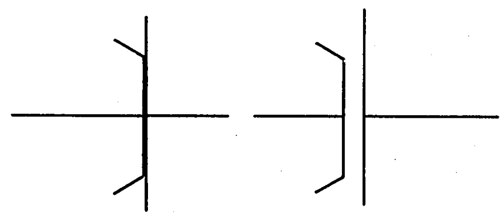
Now we want the midpoint of each interval on the pitch circles to correspond to a cusp point or



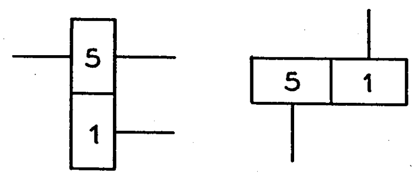
change over point as described above. So the face of a tooth is determined by placing C_3 outside C_1 in contact at P and then rolling C_3 down an amount corresponding to half an interval. The locus of P, fixed on C_3, then gives half the face. The other half is obtained by placing C_3 inside C_1, in contact at P and rolling C_3 up. This determines one face of each tooth and if the motion were always in one direction this would be adequate provided the remaining parts of the two gears did not interfere.

To provide for motion in the other direction, the other side of each tooth must be shaped in a similar fashion, with P taken at some other point P_0 in the interval. To avoid binding the width P_0P of each tooth at the pitch circle must not exceed one half an interval if it is the same for each gear. It should not be too small since this leads to backlash when the direction of motion is changed. The indentation between teeth is always deeper than the shaped faces, the outside of each gear tooth, i.e., the part between the faces is finished off on a circle concentric with pitch circle, in general.

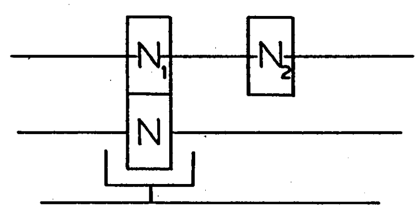
4. The use of gears permit one to multiply a rotation by one fixed ratio. If it is desired to have a number of ratios at one's command, it is customary to use a clutch and shift arrangement. The details of such an arrangement is not of great interest to us; however, we will define a clutch as an arrangement to disconnect a shaft and use the symbols below for it, the first indicating that it is normally closed or closed in the situation described.



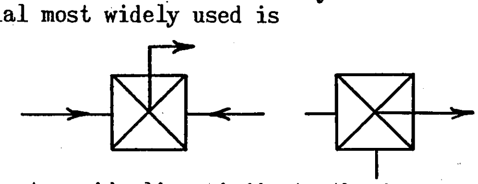
For a pair of gears we shall use the notation



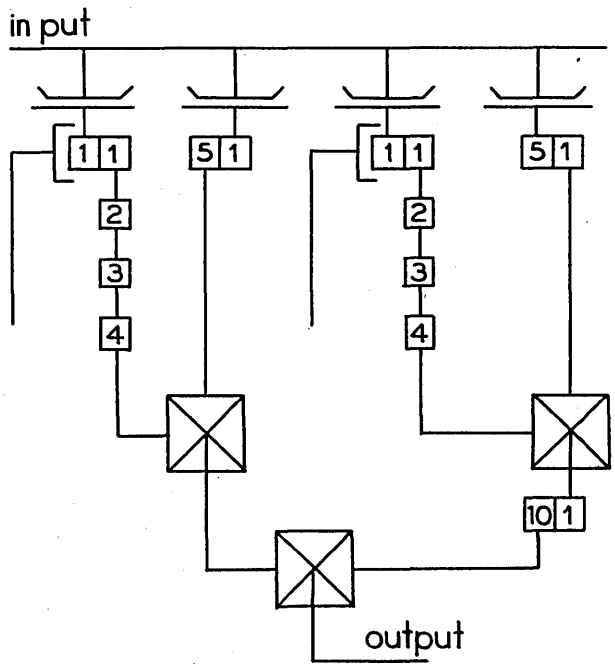
The ratio of rotation of corresponding shafts will be indicated in the boxes by a number, not by the size of the boxes. The possibility of shifting will be indicated as follows:



The full possibilities of shift and clutch gear arrangements can only be realized when the differential is introduced. The symbol for a differential most widely used is



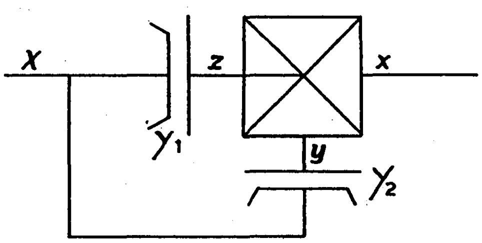
The two side lines indicate the inputs. They may be put in different positions. The output is connected to the middle. The following is a gear box rangement which would give a large number of ratios from relatively few gears.



(There is little practical difficulty in arranging the indicated gear shift.) Notice that the output can be any multiple from 0 to 99 of the input.)

Clutches and differentials can be used together for the same purpose without gears. However, in this case the ratio of input to output is expressed with radix two. In what follows, γ is to stand for a "clutch function," i.e., its independent variable is the clutch condition and it has the value of 1 when the clutch is engaged, zero otherwise.

Let us consider a single differential with two clutches. Let x be the rotation of one side wheel



of the differential, y the negative of the other and z that of the center wheel. Then

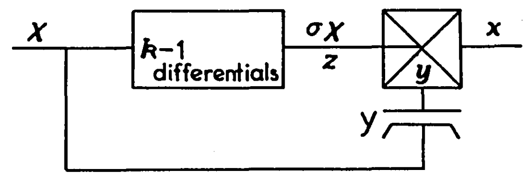
$$x - y = 2z \quad \text{or} \quad x = y + 2z.$$

Let γ_1 and γ_2 be the clutch functions of the z and y clutches respectively, X the input. Then the output

$$0 = x = (2\,\gamma_1 + \gamma_2)\,X.$$

Thus the possible ratios are 0, 1, 2, 3.

Now suppose that k-1 differentials can yield ratios 0, 1, ..., s_{k-1} where s_{k-1} is an odd number. We will show that k differentials will yield ratios from 0 to s_k where $s_k = 2s_{k-1} + 1$. For consider the arrangement. Here again $x = 2\,z + y$;



and $z = \sigma\,X$ where σ is a possible ratio from the k-1 differentials. Hence,

$$x = (2\,\sigma + \gamma)\,X$$

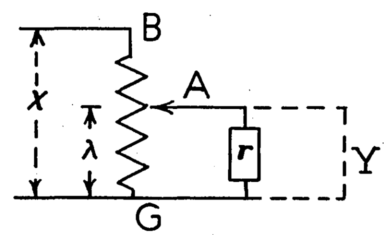
where γ is the clutch function for the y clutch. It is clear that if σ can take on the values from 0 to s_{k-1} $2\sigma + \gamma$ can assume all values from 0 to $2s_{k-1} + 1$. Since $s_1 = 3$, we can conclude from this that

$$s_k = 2^{k+1} - 1.$$

Proper and improper fractions can be obtained in somewhat a similar manner. One can locate the "decimal" point by means of gears. The electrical equivalent is interesting.

In closing this section, we wish to point out that the variable speed drive which will be discussed as an integrator in a future chapter will also permit one to multiply a rotation by a constant. The limitations on this will also be discussed.

5. A potentiometer permits one to multiply a voltage by a constant with the output a voltage. However, unless one has an infinite resistance in the take-off circuit, this process will disturb the voltage drop along the potentiometer and in general even the original voltage.*



Let X be the input voltage, Y the output voltage across the resistance r (which may be an impedance of a more general character). Let P be the total resistance in the potentiometer, λ the fraction of P in parallel with R from A to G. The resistance from A to G is that of the two parallel resistances λP and r, i.e.,

$$R_{AG} = \frac{r\,\lambda\,P}{r + \lambda\,P}$$

* It is true that both these effects can be compensated for, but we wish to study them first without the compensation.

while the resistance from B to A is

$$R_{BA} = (1-\lambda)\ P.$$

Thus,

$$Y = \frac{R_{AG}}{R_{AG} + R_{BA}} X = \frac{r\lambda}{r\lambda + (r + P\lambda)(1-\lambda)} X$$

Now if we let $\mu = P/r$ and simplify we get

$$Y = \frac{\lambda}{1 + \mu\ (\lambda - \lambda^2)} X$$

Thus, if μ is small, i.e., r is large in relation to P, then Y depends almost linearly on λ. Let ρ be the output ratio. Then

$$\rho = \frac{\lambda}{1 + \mu\ (\lambda - \lambda^2)}.$$

The term $\mu(\lambda - \lambda^2)$ measures the fractional departure from linearity and has a maximum at $\lambda = \frac{1}{2}$ with value $\frac{1}{4}\mu$.

We also note

$$\frac{\partial\rho}{\partial\mu} = - \frac{\lambda^2\ (1-\lambda)}{(1 + \mu\ [\lambda - \lambda^2])^2}.$$

The latter indicates how the ρ will vary with the load. The maximum $\frac{\partial\rho}{\partial\mu}$ is approximately obtained for $\lambda = \frac{2}{3}$ and has a value of about .15 (except for relatively large values of μ). If σ is the fractional variation in r, $\frac{\partial\mu}{\partial\sigma} = \mu\sigma^2$. Taking $\sigma = 1$, we get the approximate formula

$$\frac{\partial\rho}{\partial\sigma} = -.15\mu.$$

Hence, if $\mu = .1$, a variation of 10 percent in σ will vary ρ by .0015.

The above discussion is quite conservative. Notice that λ fixed $\frac{\partial\rho}{\partial\mu}$ increases as $\mu \rightarrow 0$. Hence the value $.15 > 4/27$ = maximum value at $\mu = 0$, is greater than any possible value for $\frac{\partial\rho}{\partial\mu}$.

Let x be a measure for the deflection of the pointer on the potentiometer scale and let us suppose that x runs from 0 to 1. For a linear wound potentiometer $\lambda = x$, but, in general, λ can be any monotonically increasing function of x. The derivative $\frac{d\lambda}{dx}$ gives the increase in resistance with x and by interposing the correct resistance between taps, it can take on any positive value. (There is, of course, a discreteness difficulty here but for carbon resistance potentiometers even this is not important. However, in the latter there are some very interesting mathematical questions.)

The derivative $\frac{d\rho}{dx}$ gives a measure of the ease and correctness with which one can set in a given ratio. For an error in x will be multiplied by this factor when it appears in ρ. In the case, λ=x we have

$$\frac{d\rho}{dx} = \frac{1 + x^2\mu}{(1 + \mu\ [x-x^2])^2}$$

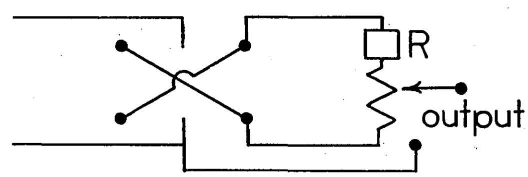
For x between 0 and 1, the numerator is increasing and the denominator has its least values at the end points. For this interval, $\frac{d\rho}{dx}$ is a maximum where the numerator is greatest and denominator is least, i.e., for x = 1. The value of the maximum is $1 + \mu$. This will give little difficulty in the matter of setting if $\mu \leq .25$.

If r and hence μ is constant, then it is possible to wind the potentiometer so that $\rho = x$. This, of course, would permit ρ to be fed in as a continuous input. We must determine λ as a function of x so that

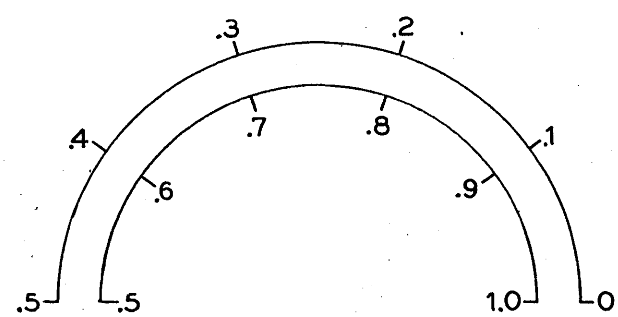
$$\frac{\lambda}{1 + \mu\ (\lambda - \lambda^2)} = \rho = x.$$

Since this is quadratic in λ, one can solve for λ in terms of x. However, especially wound potentiometers are quite expensive and it is customary to use linear potentiometers and make μ small. Indeed it is possible to get the effect of a very small μ by using vacuum tubes, for instance, as "cathode followers."

One can double the range of a potentiometer by the use of a double pole, double throw switch and an extra resistance. The extra resistance has pre-

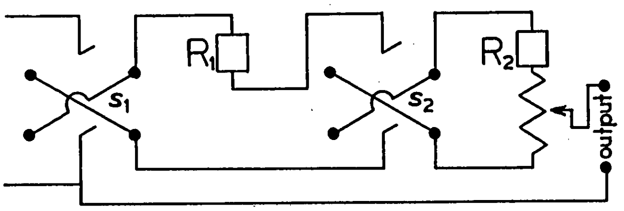


cisely the resistance of the potentiometer P. This has the effect of "folding the scale." The potentiometer has now two scales, one for the right-hand position of the switch, the other for the left-hand. The scale would look like this if the total deflection was 180° and

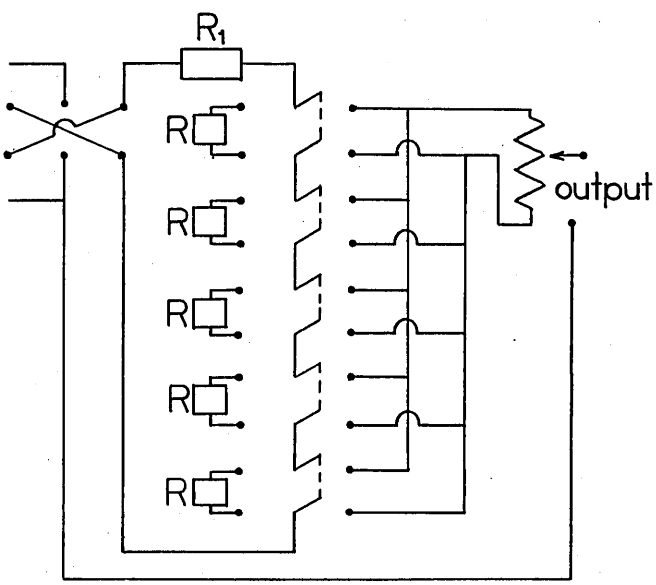


μ was so small as to be negligible. The upper scale corresponds to the right-hand position for the switch. Practically the effect is to double the scale. For instance, if the switch is in the right-hand position, then for positions between 0 and 5 the effect is similar to that of a potentiometer with twice the scale length and twice the resistance. The right-hand position of the switch corresponds to the range between 0 to .5 for λ on this equivalent large potentiometer, the other position that for .5 to 1.

It is possible to repeatedly fold the scale by such means. For instance, here the scale is folded

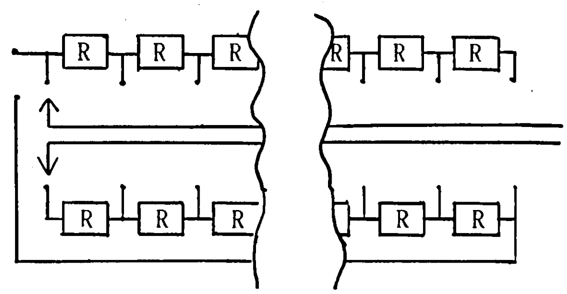


twice, and R_1 has the resistance 2 P. The lowest end of the scale of the equivalent larger potentiometer is obtained by setting both switches to the right. Let us call this the r_1, r_2 switch position. The reader can verify that successive switch positions are r_1 l_2, l_1 l_2 and l_1 r_2, i.e., to pass from one position to the next, only one switch is thrown. If only a few such switches are used, all scales can be placed on the potentiometer and at each turn in the scale, the corresponding switch can be indicated. Ten such switches would give a multiplication of 1024. It is, of course, possible to construct a purely dyadic potentiometer. It is also possible to use six d.p.d.t. switches to give an almost directly reading multiplication by ten.

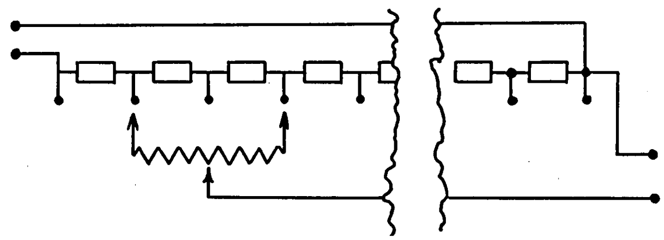


The resistances R have the value P, the value of R_1 is 5 P. The first d.p.d.t. switch, of course, distinguishes between the λ range, 0 to .5 and .5 to 1.0. Only one of the remaining switches should be thrown to the right. This will insert the potentiometer into the circuit in series with four of the smaller R's in same order.

Many such switching arrangements are possible. In fact, the usual method of setting up a decimal potential is based on a double selector switch. The output has the resistance of one of the R's which are all equal.

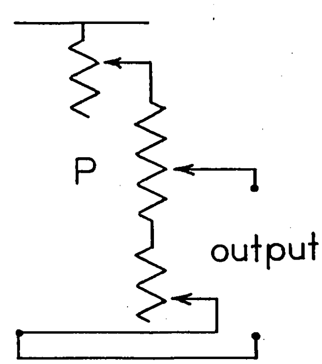


A more efficient method of decimal setting is to have eleven resistors in series and have a



double selector which shunts two of these by a potentiometer having 2R resistance. Of course, the potentiometer could be itself a decimal arrangement of this sort.

In the case of continuous resistance potentiometers and indeed in other cases also, to realize the greatest possible accuracy in setting, the potentiometer may be set by a Wheatstone bridge. When one has a large number of such potentiometers, this would eliminate the necessity for a careful calibration of each and save dial space and so on. In this case, the addition of two rheostats in series with the potentiometer, will give a "vernier" setting for the resistance.



Let P denote resistance of the potentiometer. Now if we have added $\delta_1 P$ from the lower rheostat and $\delta_2 P$ from the upper rheostat, we have the equivalent of replacing the potentiometer P with setting λ_0, by a larger potentiometer, in which the total resistance is $(1 + \delta_1 + \delta_2)$ P and the setting is

$$\lambda = \frac{\lambda_0 + \delta_1}{1 + \delta_1 + \delta_2}$$

while $\mu = \mu_0\ (1 + \delta_1 + \delta_2)$. Now at $\delta_1 = \delta_2 = 0$, we have the following partial differential relationships:

$$\frac{\partial\lambda}{\partial\delta_1} = 1 - \lambda_0,\ \frac{\partial\lambda}{\partial\delta_2} = -\lambda_0,\ \frac{\partial\mu}{\partial\delta_1} = \mu_0 = \frac{\partial\mu}{\partial\delta_2}$$

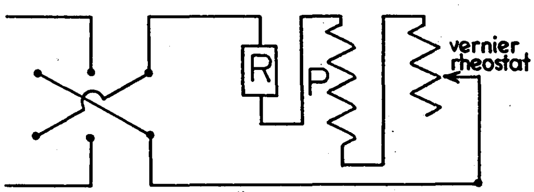
and since $\rho = \dfrac{\lambda}{1 + \mu(\lambda - \lambda^2)}$ and $\dfrac{\partial\rho}{\partial\delta_i} = \dfrac{\partial\rho}{\partial\lambda}\dfrac{\partial\lambda}{\partial\delta_i} + \dfrac{\partial\rho}{\partial\mu}\dfrac{\partial\mu}{\partial\delta_i}$

we have

$$\frac{\partial\rho}{\partial\delta_1} = \frac{1 - \lambda_0}{(1 + \mu[\lambda - \lambda^2])^2},\ \frac{\partial\rho}{\partial\delta_2} = -\lambda_0\frac{(1 + \mu\lambda)}{(1 + \mu[\lambda - \lambda^2])^2}$$

Since the ratio of scale displacement on the rheostats to δ_i can be made large, we see that for $\lambda < \frac{1}{2}$, varying δ_1, will give a vernier action and for $\lambda > \frac{1}{2}$, we may vary δ_2. Of course by having the neutral setting for the rheostat in the middle and counting half its resistance with P in each case, we can arrange to have negative δ_i's.

Combinations of the above may often be effectively used. For instance, in the following circuit the d.p.d.t. switch permits one to omit one vernier rheostat and also have the scale doubling associated with the switch and resistance.

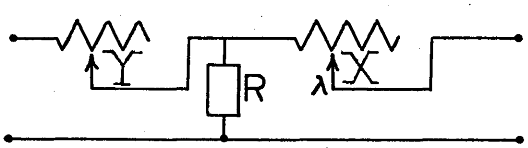


In general, it is not easy to synchronize two potentiometers. However, it is possible to use a second potentiometer to adjust for the variation of total resistance due to the different settings of the potentiometer provided the load is specified. Commercial combinations of this type exist.

If we return to the beginning of this section, we see that the total resistance of a potentiometer is

$$R_{AB} + R_{AG} = P\ (1 - \lambda + \frac{\lambda}{1+\mu\lambda}) = P\ (1 - \frac{\mu\lambda^2}{1+\mu\lambda})$$

It is, of course, possible to have another variable resistance with total value $P\mu\lambda^2/(1+\mu\lambda)$ which is cut in as λ increases to counteract the effect of the second term. Another arrangement of this sort is the following:



Here x and y are variable resistors connected so that $x = R/\lambda - 2R$ and $y = R\lambda$. It is readily seen that for every value of $\lambda < \frac{1}{2}$, the total resistance is R if the output has resistance R. Also that for $\lambda < \frac{1}{2}$, the output voltage $X = \lambda Y$, where Y is the input voltage. A chain of these can be used without load errors.

6. If the quantity used is the amplitude of an alternating current, this amplitude may be changed by a constant in a transformer. A transformer in general consists of an iron core and a number of coils wound around the core. We will confine our attention to the case in which there are two coils. One coil will be referred to as the primary, the other as the secondary. Approximately the voltages across these coils are in the same ratio as the number of turns.

Let us consider such a transformer. A current flowing in a coil produces a "magnetic force" H on the core, which may be measured in ampere turns, i.e., it is proportional to the number of turns and to the current in the coil. Thus we have

$$H = n_p\ i_p + n_s\ i_s \qquad (1)$$

where n is the number of turns and i is the current in the primary or secondary coil, depending on the subscript.

The magnetic force H results in a magnetic flux φ in the core, where

$$\varphi = f(H) \qquad (2)$$

In the case of an air core transformer, φ is a linear function of H, $\varphi = kH$ where k depends upon geometrical considerations.

However for an iron core φ is a much larger function of H for a given area but not a linear one; in fact it depends upon the entire past history of the core.

If the magnetic flux inside a coil changes, a voltage is impressed on the coil, which is proportional to the number of turns. Thus if E is the voltage applied to a coil and R the resistance

$$E_p = R_p\ i_p + n_p\frac{d\varphi}{dt} \qquad (3)$$

$$E_s = R_s\ i_s + n_s\frac{d\varphi}{dt} \qquad (4)$$

$$\varphi = f\ (n_p\ i_p + n_s\ i_s) \qquad (5)$$

For definiteness let us assume that $E_s = -R'\ i_s$ (i.e., it is due to an external resistance drop) and let $r_s = R_s + R'$. Let us also make the assumption that $f(H) = kH$. This, of course, is false in iron cases where the rate of change of φ will depend on φ as well as H. We then have

$$E_p = R_p\ i_p + n_p\frac{d\varphi}{dt}$$

$$0 = r_s\ i_s + n_s\frac{d\varphi}{dt}$$

$$\frac{d\varphi}{dt} = k\ (n_p\frac{di_p}{dt} + n_s\frac{di_s}{dt})$$

If we eliminate $\frac{d\varphi}{dt}$ between the first two equations we can express i_p in terms of i_s. This can be used to eliminate i_p and $\frac{di_p}{ds}$, and we obtain

$$-n_p n_s \frac{dE}{dt} = \frac{r_s R_p}{k} i_s + (n_p^2 r_s + n_s^2 R_p) \frac{di_s}{dt}$$

If we suppose that $E = A \sin wt$ ($w = 2\pi f$ where f is the frequency in cycles per second) then the steady state solution of this equation is $i_s = -a \sin (wt + \gamma)$ where

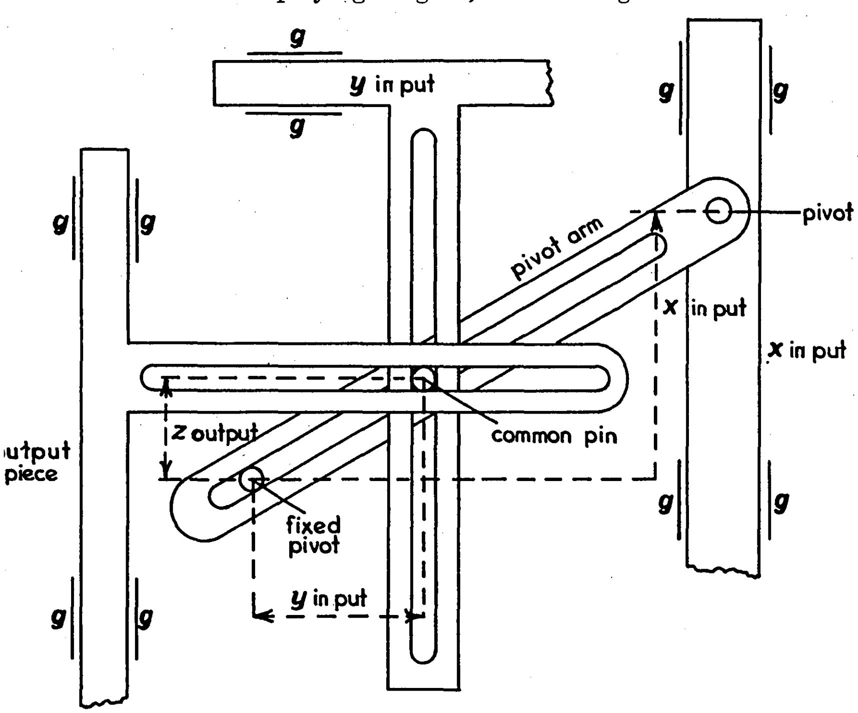
$$\tan \gamma = \frac{R_p r_s}{w k (n_p^2 r_s + n_s^2 R_p)}$$

and

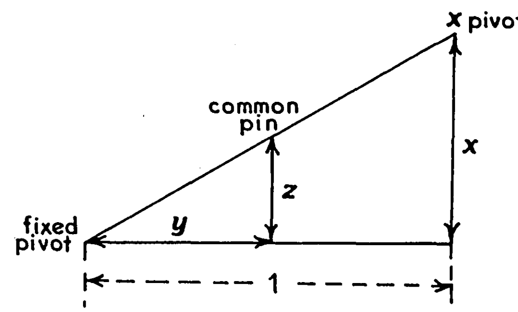
$$a = \frac{A n_p n_s}{\sqrt{\left(\frac{r_s R_p}{k w}\right)^2 + (n_p^2 r_s + n_s^2 R_p)^2}}$$

Hence, for an air core transformer, a depends linearly on A. This is not true for the iron core transformer because the relationship between φ and H is not linear.

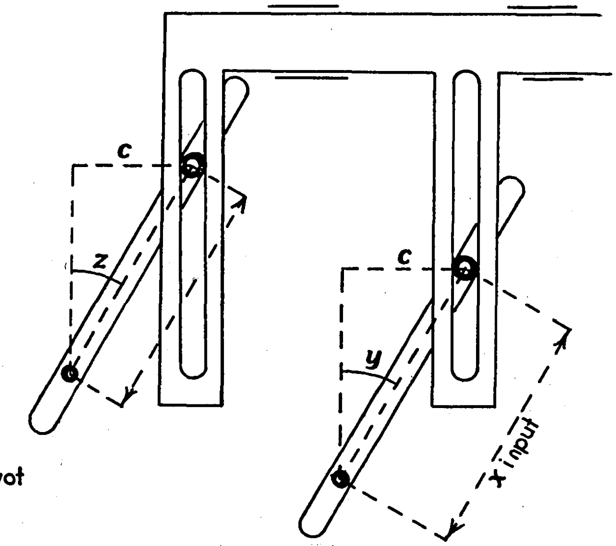
7. In Section 1 of the present chapter we described a similar triangle multiplier in which the central pivot was movable. In the present section we describe a device in which the central pivot is fixed. In the accompanying diagram, the lines g



stand for grooves and indicate that the element contained between them (or the part of an element) can only move parallel to itself. The output element and one input element are essentially similar but at right angles to each other. The other input element has a pivot arm which links the fixed pivot with the common pin of all three elements. It is easy to see that the two inputs and the output are related as in the triangle. By similar triangle we see that $\frac{z}{y} = \frac{x}{1}$ or $z = xy$.

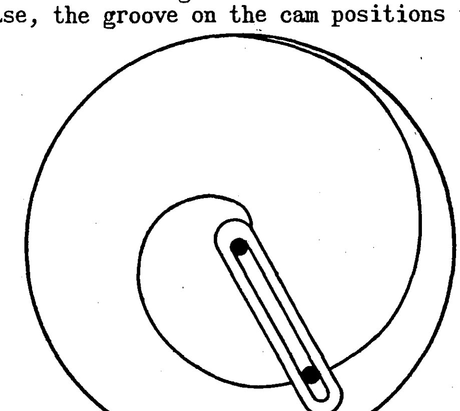


There is another kind of multiplier which while it is only approximate and is limited in range has been very often used because of its simple positive action. This is what is frequently referred to as the "links multiplier." The principle can be understood from the following diagram, where how-



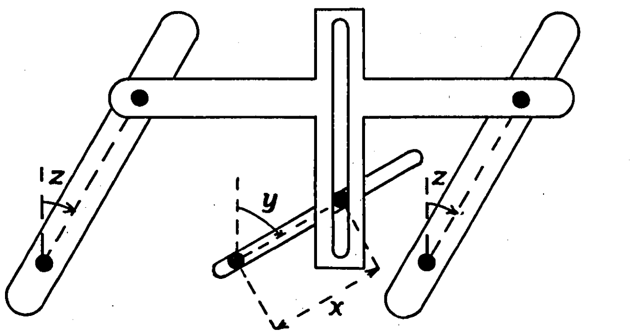
ever the method for putting in the x input is not indicated. The y input and the output z are angles. The cross piece is constrained to move parallel to itself. The x input is a length. The cross piece causes the two triangles pictured to have a common side c. For this common side we have $c = \sin z$, $c = x \sin y$, or $\sin z = x \sin y$ which is approximately $z = xy$.

The x input is usually entered by means of a screw or a groove cam. In the latter case, the groove on the cam positions the



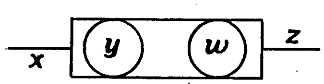
x pivot along the y bar, according to the amount of relative motion between cam and bar. Hence, if the bar is turned an amount y, we must turn the cam an amount y + x. This can be accomplished by means of a differential which adds the two inputs x and y. In the screw case the situation is similar.

The above described device differs from the ordinary "links multiplier" in the way in which the cross piece or link is constrained to move parallel to itself. Usually this is accomplished by the use of another pivot arm similar to the output arm. The result is a parallelogram as indicated in the accompanying figure.



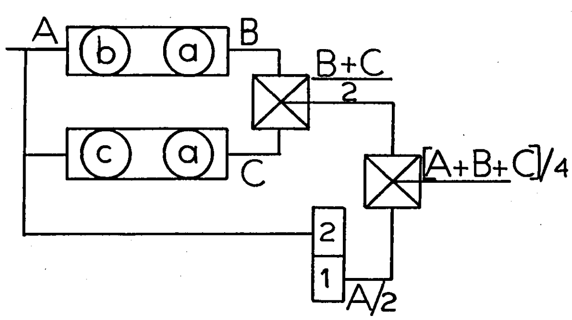
The equation sin z = x sin y may itself be very useful, especially in connection with the law of sines in trigonometry. When one has our previous arrangement, i.e., the one with the link in a groove, we may take the z arm also as a variable and get the equation W sin z = x sin y.

We can construct a device on this basis for solving a triangle in which three sides are given. Let the symbol



denote a links multiplier which realizes the equation x sin y = w sin z.

Suppose three sides, a, b, c, of a triangle are given. Let a be the largest of these so that angle A is the largest. Consequently B and C are acute. We now use two links multipliers as follows:



In this device, we continue to feed in A, starting (say) with A = 0. The outputs of the two links multipliers will be the acute angles B and C for which b sin A = a sin B and c sin A = a sin C respectively. These angles are added and A also. The actual output is $\frac{1}{4}$ (A+B+C) and one continues to increase A until this quantity has the value 45°.

8. We have seen in our discussion of gears that we can shape two laminas D_1 and D_2 in such a way that when they revolve in contact, we get the same motion as that in which the two laminas C_1 and C_2 revolve in rolling contact, along the line joining the centers of rotation. We now consider this for the purpose of cams.

Let two such laminas C_1 and C_2 be given. Let us consider some initial contact position for the two and let us give the equations in polar coordinates for the pair of edges, $\rho = f(\theta)$, $\sigma = g(\varphi)$. If we rotate the first through an angle θ and φ is the corresponding value of the second coordinate we have

$$f(\theta) + g(\varphi) = d \tag{1}$$

where d is the distance between the centers of rotation. The tangency requirement is that

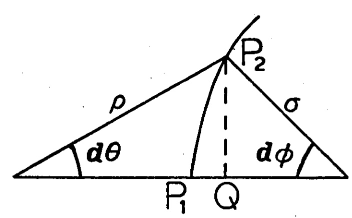
$$\frac{1}{\rho}\frac{df}{d\theta} + \frac{1}{\sigma}\frac{dg}{d\varphi} = 0 \tag{2}$$

Suppose that φ is to be a specified function of θ. We have found in Section 2 above, that the condition

$$\frac{d\varphi}{d\theta} = \frac{\rho}{\sigma} \tag{3}$$

determines the relative motion since A the point of contact is on the line of centers.

(3) also indicates that the motion is non slipping. Consider the diagram. Since $\rho d\theta \simeq P_2Q \simeq \sigma d\varphi$,



P_1 and P_2 are (to the first order) points of contact. The arc length on each C curve is the same $P_1 P_2$ and hence the motion is non-slipping.

Equation (1) determines g when f is known. It also yields

$$\frac{dg}{d\varphi}\frac{d\varphi}{d\theta} + \frac{df}{d\theta} = 0 \tag{4}$$

which makes (2) and (3) equivalent to

$$\frac{d\varphi}{d\theta} = \frac{f(\theta)}{d - f(\theta)} \tag{5}$$

Let us take d = 1. We have

$$f(\theta) = \frac{\frac{d\varphi}{d\theta}}{1 + \frac{d\varphi}{d\theta}} \tag{6}$$

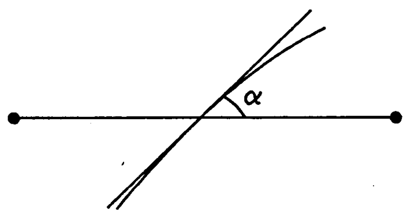
Since φ is a given function of θ, this determines f.

(1) and (6) then determine C_1 and C_2. In particular (1) becomes

$$g\ (\varphi\ [\theta]) = \frac{1}{1 + \frac{d\varphi}{d\theta}}$$

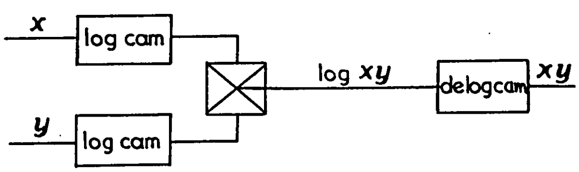
Of course, $\frac{d\varphi}{d\theta}$ must be positive.

Let α be the angle between the tangent at the point of contact and the line of centers. If a



unit torque is applied to the first cam, the force between the surfaces is proportional to sec α. If α is close to 90°, this will introduce numerous friction difficulties. It is better under these circumstances to pass to laminas D_1 and D_2. One should notice that an epicycloidal curve has a cusp at the point where it meets the base curve. Its tangent is perpendicular to the base curves and hence the new angles α for the D curves differ by 90° from the previous ones. This can be best be used by using a number of D curves, i.e., properly shaped gear teeth.

Thus the theory of cams permits us to mechanize any function φ of one variable θ for which $\frac{d\varphi}{d\theta} > 0$. For instance, we may take $\varphi = \log \theta$ and by the use of three such cams and a differential we may multiply two positive quantities. One of them is to be operated with φ as input.

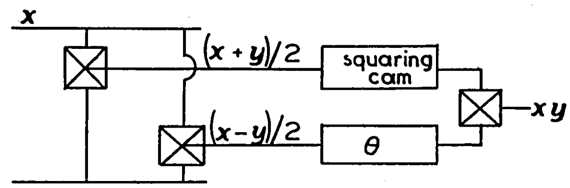


There are other types of cams, as well as those described in previous sections. In Section 7, we have mentioned the groove cam. A groove cam in general must have the angle of rotation as the input. The output is the displacement of a pin which slides along a fixed radius of the disk. The output, of course, is described by the equation in polar coordinates of the groove.

9. Logarithmic multiplication does not permit a change of sign. An alternate method of multiplying is by the use of squares, since

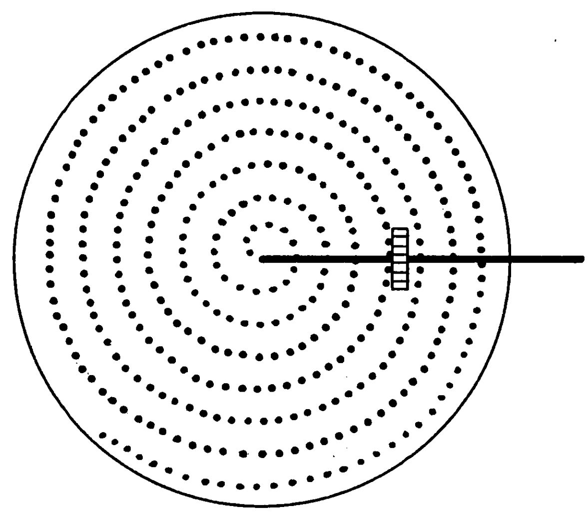
$$xy = \left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2$$

Here three differentials but only two cams are used (see diagram at top of next column).

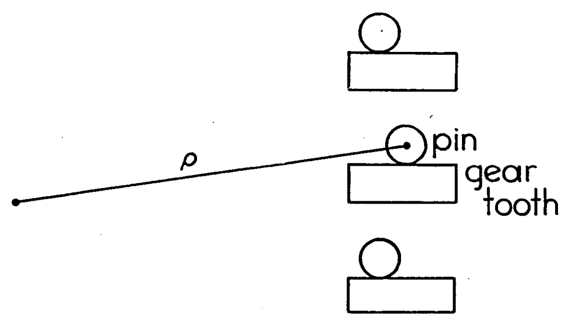


In the usual form of this device it is customary to use a pin cam.

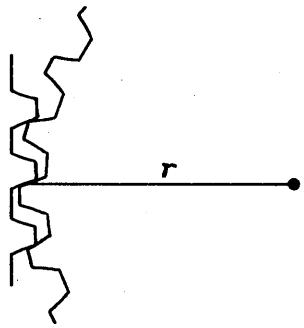
A pin cam is a disk with pins mounted on it in a spiral. As the disk revolves these pins successively push a wheel which is mounted on an axis parallel to the disk. In order to permit more than one turn of the main disk provision must be made to move the little wheel parallel to itself. This motion is not to appear in the output, only the rotation of the little wheel. We will return to this point.



As a pin passes the little wheel it causes the wheel to revolve. Let the pin be at a distance ρ from the center of the disk. The relative motion of disk and wheel are such as if the wheel and spiral were turning in contact with same component of motion in the direction which is parallel to the disk and the plane of the wheel at their point of contact. We can see this if we take a cross-section of the pins and gears at the region of contact. Let θ be the angle of rotation of the disk, β that of



the little wheel and r its "pitch radius." From the



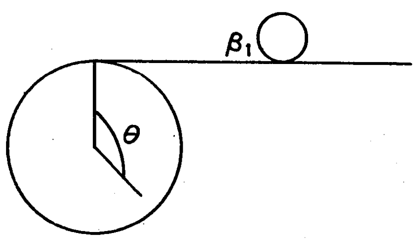
contact, we see that $\rho\frac{d\theta}{dt} = \frac{d\beta}{dt}$ or $\frac{d\beta}{d\theta} = \frac{\rho}{r}$

In the squaring pin cam, $\rho = k\theta$. Hence,

$$\frac{d\beta}{d\theta} = k\,\theta \quad \text{or} \quad \beta = k'\,\theta^2 + c.$$

As we have mentioned, it is necessary for the little wheel to move along its axle as the disk turns. The axle, of course, has a polygonal cross-section and the pinion is displaced along it by means of prongs which are mounted on a screw which turns with θ. The linear displacement then given by the screw is, of course, proportional to θ. This is effective in the case of squaring cams since ρ is essentially equal to θ. Presumably in the case of pin cams with a different spiral, some sort of guiding groove on the disk itself would be necessary.

Another method by which squares may be obtained is by wrapping a tape around itself. The length of the tape is proportional to the square of the angle turned by the spool.



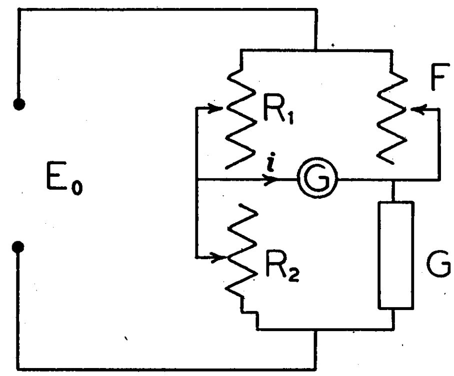
Let us consider such a tape on a spool whose angle of rotation we will call θ. The length of the tape can be measured by causing it to turn a cylinder an amount β. If r is the radius of the cylinder $r d\beta = \rho d\theta$. We see that if d is the thickness of the tape, $\frac{\Delta\theta}{2\pi}$ is the number of revolutions for a change Δθ and $\Delta\rho = d\frac{\Delta\theta}{2\pi}$. Thus we may assume $d\beta = k\theta d\theta$.

We shall see later, electrical methods for squaring, based on a rectifier. In a later section, we will discuss methods for getting square value resistances.

10. It is also possible to base multiplication of two positive quantities on the Wheatstone bridge principle. (Cf. Section 4 of the previous chapter.)

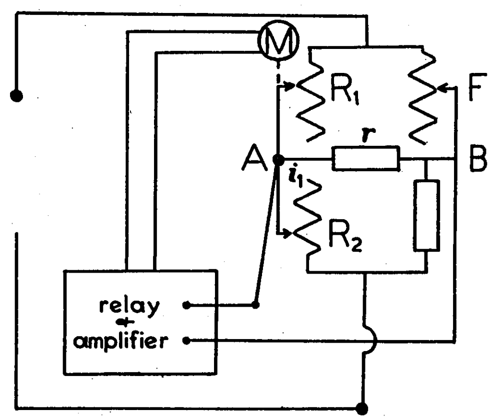
$$\frac{R_2}{G} - \frac{R_1}{F} = k\ i \quad \text{or} \quad \frac{FR_2}{G} - R_1 = F\,ki.$$

We may consider the resistances F and R_2 as inputs, G as a constant (or as a quantity, we are dividing by) and R_1 as the output. To multiply, we put in F and R_2 by means of linear rheostats and then vary R_1 until i, as read on the galvanometer is zero.



Since F and R_2 are linear rheostats, the inputs can be considered as a rotation. The output can also be obtained as a rotation if R_1 is a linear rheostat turned by a "servo motor."

A servo motor is an electrical (or hydraulic) motor which can be controlled to turn in either direction depending upon a signal. In this case, the signal would be the current i. In our given bridge, let us replace the galvanometer G by a resistance r. Then the voltage drop A B has the



same sign as i. If i is positive R_1 is too small and hence should be increased. If A B is positive, a voltage amplifier and a relay circuit will cause the motor to turn so as to increase R. On the other hand, if i is negative R_1 is too large. Since the voltage drop AB is negative, the relay circuit will cause the motor to turn in the opposite direction and reduce R_1.

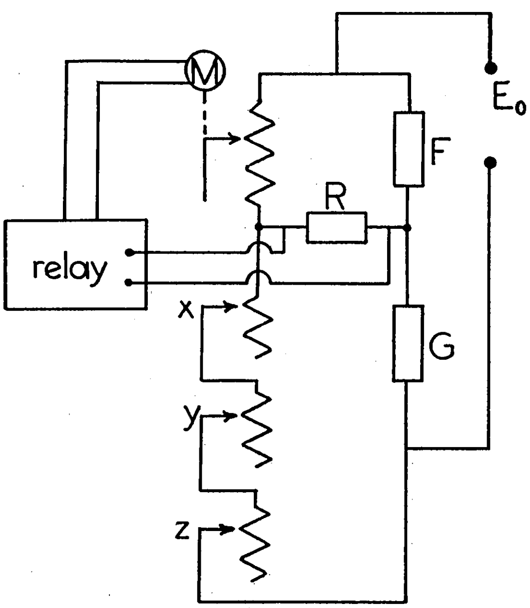
Notice that the inputs and output of this device are rotations. There is another advantage in this device which is, however, not immediately obvious. The output comes from a motor in such a way that the load on the inputs is precisely uniform. In most devices for multiplication of a mechanical sort, the load on the inputs depends on the value of the other input and the output. This does not occur here. On the other hand, one must admit that the use of servo motors has certain disadvantages, which we will discuss later.

11. The device mentioned in the preceding section does not, of course, take advantage of all the possibilities inherent in the use of a servo motor and a Wheatstone bridge to force the equality of two resistances. We mention two other uses for this combination; one in this section.

In the first case, we will consider a method of obtaining the product of any number of positive quantities by the use of logarithmic potentiometers. In the simplest case, we will consider all the quantities as greater than 1. Of course, the quantities have to be bounded in any case and if they are also bounded away from zero, we can satisfy this restriction by the use of a scale factor. We will discuss later the most effective way of doing this, but for the moment, let us suppose that our factors are all greater than 1. For instance, suppose we want the product $w = xyz$, and let us suppose for definiteness that x varies between 1 and 1000, y and z between 1 and 100.

We can use a Wheatstone bridge and a servo motor to force the equality

$$\log w = \log x + \log y + \log z$$



If R_0 is the value chosen as unit resistance, the maximum resistance for x should be $3R_0$, that for y and z $2R_0$ and for W, $7R_0$. This illustrates the convenience of logarithms when one has to deal with greatly varying quantities. However, this has its difficulties too as we shall see.

Resistances must be positive and one way of dealing with the case in which x, y and z are less than one is to multiply the original equation by a power of 10 in such a way that the quantities entered into the machine are all greater than one. For instance, if x varies between .01 and 10, y between .1 and 10 and z between 1 and 100, then $10^3 w = (10^2x)(10y)\cdot z$ and we may equate $\log (10^3w) = \log (10^2x) + \log (10y) + \log z$.

It is interesting to study the question of accuracy for these devices. Let us suppose that we can obtain logarithmic rheostats whose total resistance is a multiple of R_0, the resistance whose value is log 10 in the above equation. One of these is to yield value $\log 10^2x$. Hence, its resistance must vary between 0 and $3R_0$. Let us imagine the scale as divided into 1000 parts. The contact is to be connected to x by a gear so that as x goes between .01 and 10, the contact goes from 0 to 1000 on the scale.

Notice that on the scale the first mark out of a thousand must contain $\log 2R_0 = .301 R_0$ resistance. This is one-tenth of the total resistance in about one thousand of the scale. The next mark will contain $.477 R_0$ total or $.176 R_0$ more. Thus the increase in resistance between the 0 and the first mark is more than 50 percent greater than that between the first and second mark. This could not, of course, be given by a potentiometer with a wire winding of the usual sort, with one winding besides the next. Some kind of special construction would be necessary in this case, as, for instance, an extra form for winding the lower part of the scale. This is feasible because the resistance here is relatively large. Mechanical devices to put in extra resistance at the lower end of the scale could also be used.

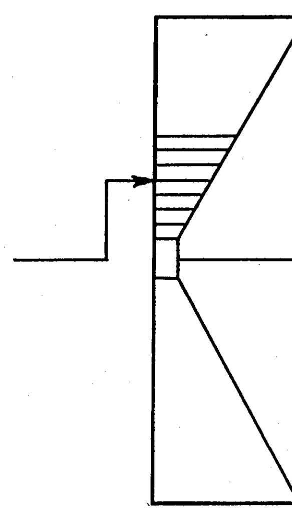
The situation, of course, relative to W, would be worse.

12. Frequently, it is desirable to use multiplication by squares, rather than by logarithms. This is particularly true when it is desired to obtain a linear combination of variable inputs as for instance

$$y = \Sigma_{i=1}^{n} a_i x_i + b$$

There are two relatively simple ways of obtaining a resistance which is the square of an input which may be either a displacement or a rotation.

One method involves a K wound potentiometer card.



Let us suppose for simplicity in our explanation that our contact is to be displaced linearly. The K is made up of two right triangles, with a common apex and with a leg of one along an extension of the leg of the other. The common line of these two legs is the line of motion of the contact and the wire is wound perpendicular to this edge. The resistance from the center point is proportional to the length of wire which in turn is proportional to the area of the K between the center point and the line on the triangle perpendicular to the contact edge at the point of contact. At the apex, one must replace the windings by a solid bar for physical reasons. If the beginning of the winding has a proper resistance between it and the center point, this does not introduce any error in any other portion of the scale. Let Δx be the length of one-half this solid contact region. Then for a contact in a neighborhood of the zero, the absolute error in the resistance is $(\Delta x)^2$.

By displacing the contact in one direction by an amount a and the K winding in the opposite direction by an amount x, we can obtain a resistance with value $(a+x)^2$. Similarly, if they are both displaced in the same direction, we can obtain a resistance $(a-x)^2$.

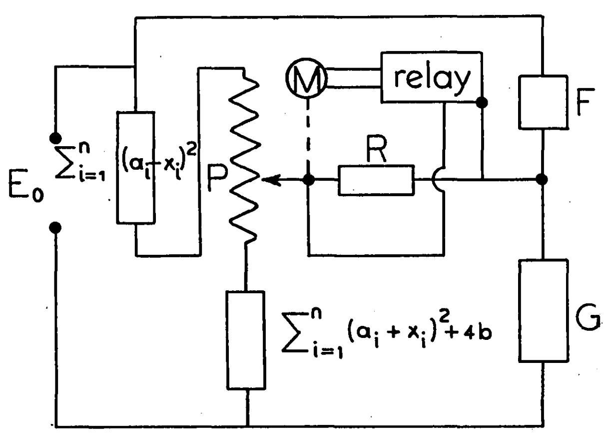
This can be used to obtain a linear combination by means of a Wheatstone Bridge. For the equation

$$y = \Sigma_{i=1}^{n} a_i x_i + b$$

can be written in the form

$$R + 4y + \Sigma_{i=1}^{n} (a_i - x_i)^2 = \Sigma_{i=1}^{n} (a_i + x_i)^2 + 4b + R$$

and we can obtain the value of y from the following circuit

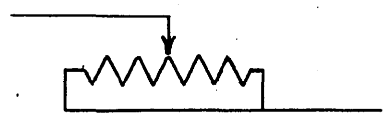


The potentiometer P is linear and if y corresponds to the displacement of its contact from the midpoint, the resistance of it in one circuit is R - 2y and in the other R + 2y. One sees then from our equation that for equality we must have

$$y = \Sigma_{i=1}^{n} a_i x_i + b.$$

In general, one can hope that the error due to the center piece in the K cards will not occur. In any case, one can by proper design insure that the error of the amount Δx^2 is negligible relative to the other quantities involved.

However, there is still another way of obtaining a square by means of resistances, which can be used. This involves only a linearly wound potentiometer and hence does not have any of the difficulties associated with the center piece. Let us suppose we

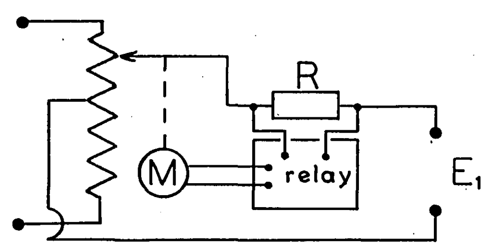


have a potentiometer whose ends are joined. Let x denote the displacement of the contact from the center point. Then the resistance between the contacts and the common ends consists of two resistances in parallel. One has value $R(1-x)$, the other $R(1+x)$ and hence the total has the value $R(1-x^2)/2 = R_0 - R_0 x^2$ or in suitable units $1 - x^2$.

As before, we can obtain $1 - (a_i+x_i)^2$ and $1 - (a_i-x_i)^2$ and then our original equation can be written in the form

$$R + 4y + \Sigma_{i=1}^{n} (1 - (a_i+x_i)^2) = \Sigma_{i=1}^{n} (1-(a_i-x_i)^2) + 4b + \dot{R}$$ which can be realized as before.

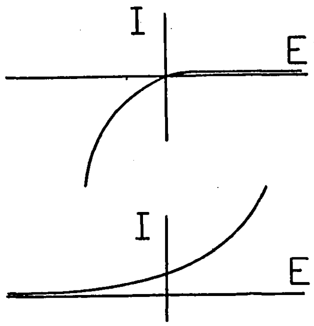
13. In the foregoing we have not discussed multiplying in the case in which both inputs are variable and electrical in nature. Of course, one way in which this can be done is to convert the electrical input into a geometrical one by a servo arrangement and then use one of the above methods. For instance a rotation or a displacement proportional to a voltage may be obtained by means of a servo and linear potentiometer. The purpose of the servo is to place



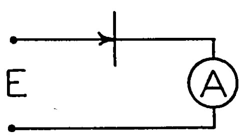
the contact so that the displacement from the center of the scale is proportional to the input voltage. By using the voltage across the potentiometer as another variable, a quotient can be obtained. The center tap on the potentiometer is used to permit the signs of the various quantities to be free.

However, we will see later on certain ways in which vacuum tubes can be used to multiply. One of these is reasonably direct although the range of variables is limited. There is a way also in which logarithms are used. However, it seems to me that the most effective method of electrical multiplication is by squares. This is because any rectifying device can be used as a squarer. For if we apply an alternating voltage whose amplitude is not too great across a rectifier, the direct current component of the resulting current, in general, will be proportional to the square of the amplitude of the applied voltage.

Let us examine this more carefully. A rectifier can be defined as a device in which the current I is a function f (E) of the applied voltage in such a way that f (E) does not equal -f (E). Sometimes rectifying devices are used in which I depends upon the previous history of the device as well as E but these would not be very useful for our present discussion. Two possible such functions f (E) are illustrated. We can imagine the rectifier in



series with an ammeter and a voltage applied. The above graphs and the functions f (I) are obtained by plotting I against E for different fixed values



of the latter. We assume that I depends only on E. This would mean that the above graphs are valid even when E is not fixed but a function of the time.

Let us suppose now that a voltage in the form $e_o + e_1 \sin wt$ is applied. Let us apply Taylor's Theorem with the remainder to f (E) around the point e_o, i.e., with $x - a = e_1 \sin wt$. Then

$$I = f(e_o) + e_1 f'(e_o) \sin wt + \frac{e_1^2}{2!} f''(e_o) \sin^2 wt + \frac{e_1^3}{3!} f'''(e_o) \sin^3 wt + \frac{e_1^4}{4!} f^{(iv)}(e^1) \sin^4 wt$$

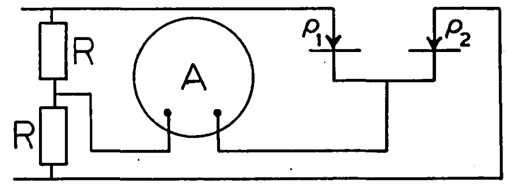
where e^1 is a function of e_o, e_1 and t.

Now an ordinary direct current ammeter will yield the average value of I, provided w is not too small. To find this average value, we integrate I from say 0 to $2\pi/w$ and multiply by $w/2\pi$. The average of sin wt and $\sin^3$ wt is of course zero. Hence the result is

$$I_a = f(e_o) + \frac{e_1^2 f''(e_o)}{4!} + e_1^4 H$$

where $H = \frac{w}{4!2\pi}\int_0^{2\pi/w} f^{(iv)}(e^1) \sin^4 wt dt$. We regard e_o as fixed and e_1 as the input. We may disregard f (e_o). Of course, H depends upon e_o and e_1 but we will get a square output if e_1^2 H is negligible compared with $f''(e_o)/2$. Later on we will discuss methods of improving this situation.

A "full wave" rectifier can be used to give a similar result, provided we have voltage $-E_1$ as well as $+E_1$. This is true even for direct current. Let us suppose we wish to square the value of a current i. We have two similar rectifiers and suppose we operate them relative to 0 voltage. It is



clear that if the voltage across the rectifier ρ is e_1, that across the rectifier ρ_2 is $-e_1$. (We neglect the current drawn by the rectifiers here.) Consequently the current through ρ_1 is

$$i_1 = f(o) + f'(o)e_1 + \frac{f''(o)e_1^2}{2!} + \frac{f'''(o)e_1^3}{3!} + \frac{f^{(iv)}(e^1)e_1^4}{4!},$$

while that through ρ_2 is

$$i_2 = f(o) - f^1(o)e_1 + \frac{f''(o)e_1^2}{2!} - \frac{f'''(o)e_1^3}{3!} + \frac{f^{(iv)}(e'')e_1^4}{4!}$$

The ammeter measures the sum of these two in which the odd powers cancel and of course this is essentially a square. The difficulty with this circuit is of course matching the rectifiers and in general a rather elaborate biasing arrangement would be necessary.

III. Integrators and Differentiators

A variable input is, of course, a function of time or it may be considered as a function of another variable. We have seen in the foregoing, methods by which we could add or multiply two such functions. The remaining two operations which we would like to consider are integration and differentiation.

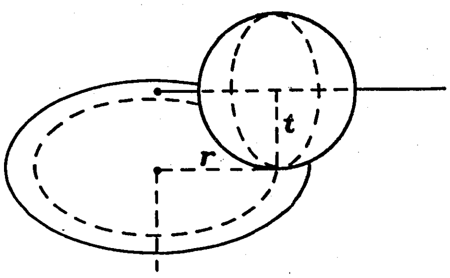
For these a relatively large number of methods are known. For the integration of a rotation or displacement we have a variable speed drive. This can be used as a differentiator also by the use of a suitable servo arrangement. There are a large number of instruments which are essentially differentiators, for instance, a speedometer or tachometer.

Electrically a watt hour meter is an integrator. Also a condenser can be used as an integrator and with a resistance as a differentiator. Theoretically an inductance can be used for either purpose also.

A word of caution should be inserted concerning the objectives of the present chapter, which is also applicable to a certain extent to our previous discussion. We are dealing with the principles which may be used to perform the indicated operations, our discussions are still inadequate to describe practical devices. In the present chapter in particular, we are forced to assume that our outputs have zero or practically zero loads. Consequently, to complete the discussion here to the point where the principles can be incorporated into actual devices, we must utilize the theory of amplifiers as given in the next chapter.

To a certain extent this was true of the earlier devices but in the previous cases a certain amount of load could be tolerated without introducing essential errors. In the theory of integrators, almost no load is permitted. Our reason for discussing integrators and differentiators before the amplifiers is that the theory of the latter is in general more readily understood if the principles of the integrators and differentiators are available.

1. The standard method for the integration of a displacement or rotation is based on a principle which can be illustrated by the simple consideration of a sphere rotating in contact with a disk.



The sphere is mounted on an axle which intersects the axis of the rotating disk. If the contact between the disk and the sphere is a non-slipping one and if for the moment we consider the relative positions of the center of the sphere and disk as fixed, then a rotation $\Delta\ \alpha$ of the disk will cause a rotation $\Delta\ \beta$ of the sphere, such that $r\Delta\alpha = t\Delta\beta$ where t is the radius of the sphere and where r is the distance of the center of the sphere from the axis of the disk. Let us suppose $t = 1$. Then $\Delta\beta = r\Delta\alpha$.

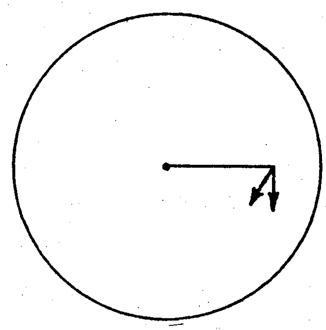
For a brief introductory discussion, let us first suppose that the contact between the two occurs at a point. Let us suppose also that there is another mechanism not shown, which moves r in such a way that r and α are both functions of a variable τ. Since the motion of r is along the axis of rotation of the sphere, it does not contribute to the rotation of the sphere and hence we have $d\beta = rd\alpha$ or β is an integral of the differential $rd\alpha$. However this requires that the point of contact move with perfect slip along the radii of the disk but with perfect non-slipping contact on the circles with the same center as the disk.

Very many variations of the above have been introduced using, say, instead of a disk and a sphere, a cone and a sphere or two spheres or cone and ellipsoid! However, they are all subject to the difficulty given above, when no more than two elements are used.

The modern "ball cage variable speed drive" uses essentially four elements and is not subject to either the difficulty mentioned above or to a certain extent to the difficulty represented by the fact that the point of contact of the disk and sphere is not a point.

To understand this device, let us consider the rotation of a sphere which is placed on a rotating disk and which is being shoved by an apparatus that does not interfere with its rotation, along a radial line of the disk. At first let us suppose that the contact is at a point.

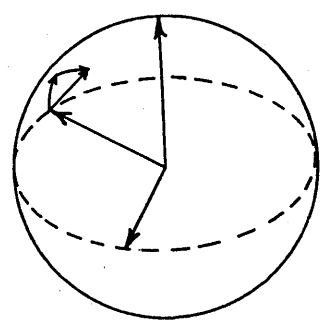
It is immediately apparent that the point of contact moves on the disk, a vectorial amount which



is the sum of two mutually perpendicular components, one of which has the value Δr and the other $r\Delta\alpha$. If we suppose that contact is non-slipping then a point of the sphere also has this motion and one can see that the instantaneous rotation of sphere is about an axis in the plane parallel to the plane of the disk.

This was, of course, on the assumption of contact at a point. Actually, we have an area of contact with a line of no relative motion and areas in which the relative motion is opposing. This gives a tendency to rotate around an axis perpendicular to the plane of the disk. The actual instantaneous rotation then must be compounded from this instantaneous rotation and the instantaneous rotation of the previous paragraph.

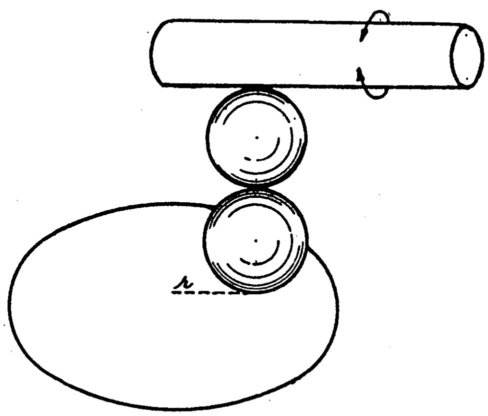
This can be done by considering the motion of a point on the end of a radius perpendicular to both axes of rotation. It is easy to see that the actual instantaneous axis of rotation is in the plane



given by the other two axes of rotation. In fact, we have shown that the instantaneous rotation, we are interested in, can be considered as made up of three components, each of which is a rotation around one of three mutually perpendicular axes and, of course, this is true for any rotatory motion.

Let us now consider the H. Ford variable speed drive. The essential elements are a disk, two spheres in a cage, one on the disk, the other on the first, and a cylinder, with axis parallel to the face of the disk and in contact with the uppermost sphere.

Let us consider such an arrangement and see what happens to each of the three components of the rotation. The component due to the rotation of the disk is around an axis of the sphere which passes through the axis of the disk. This component is



easily seen to be transmitted to the upper ball and the latter in turn transmits it to the cylinder and if β is the amount of rotation of the cylinder, α that of the disk and r the displacement of the point of contact from the center of the disk, then $d\beta = r d\alpha$.

The component of the rotation which is due to changing r is about an axis perpendicular to the previous one but parallel to the disk. The upper sphere receives a rotation about a parallel axis but in the opposite direction. If this sphere is in non-slipping contact with the cylinder, this rotation will cause it to roll along an element of the cylinder without transmitting any motion to the cylinder and of course this is the desired result.

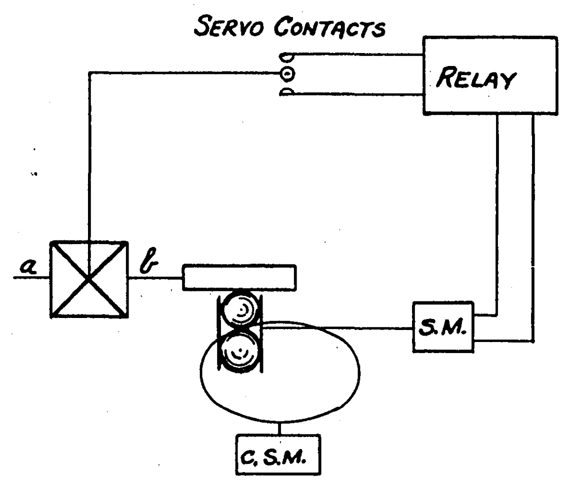
Thus in these two cases, as long as we have non-slipping contact, the desired effects occur. The situation relative to the third component is not so happy. For as we have seen if there is an area of contact between the disk and the sphere, we must have a rotation around the axis perpendicular to the disk if we are to have non-slipping contact. On the other hand, under the same conditions for the contact of cylinder and upper sphere, we must have no rotation about this axis. Consequently there must be a certain amount of slipping at the three points of contact if the device is to move at all. These devices are constructed so that there are large pressures on the contact points. Presumably the last effects are not negligible.

Two integrators can be used as a multiplier. Thus $uv = \int u\,dv + \int v\,du$. In general, however, slipping limits the application of this formula to the case where one uses the formula

$$\frac{d(u\cdot v)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

2. A ball cage variable speed drive can be used as a differentiator by the use of a servo hook-up. Suppose we have a shaft whose rate of rotation is to be measured. The idea is to cause another shaft to turn with the same rate by means of a variable speed drive. The displacement of the ball cage in the variable speed drive shows the rate of turn of the latter provided the disk is turned by a constant speed motor. Since the servo hook-up is supposed to match the rates, the displacement of the ball cage is the answer.

But one should be cautioned against trying to match the rotations themselves in such a set up when the desired matching is between rates. Let us take the following simple example of a set up that



will not work. Here a is the shaft whose rate of rotation is to be measured, b is the shaft whose output is to match the other. The output of the differential is the difference of the shafts a and b. The servo contact arrangement is such that when a-b is positive, the servo motor tends to increase $\dot{b}$ $(=\frac{db}{dt})$ and conversely when a-b is negative, the arrangement tends to decrease $\dot{b}$.

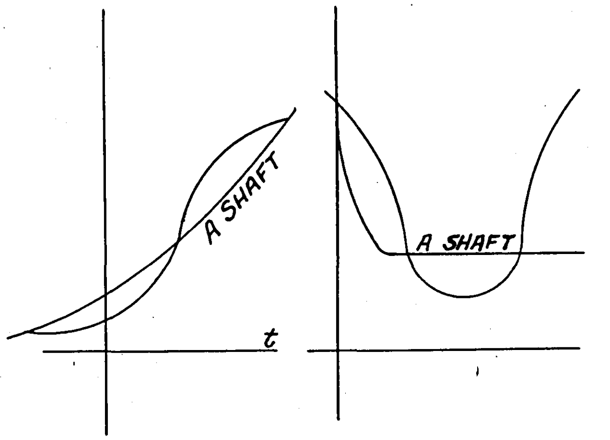
Let $\Delta(\sigma) = +1$ for $\sigma > 0$, $\Delta(\sigma) = -1$ for $\sigma < 0$. For simplicity in our discussion, let us neglect the acceleration of the motor, i.e., let us suppose that it runs at two speeds, + v and - v. Then the differential equation for the motion of the b shaft is simply

$$\ddot{b} = \Delta(a-b)\ v.$$

Hence b is a parabolic function of t.

Let us describe the situation more precisely in terms of various possible motions for the a shaft.

The curve which expresses a as a function of time divides the t-b plane into two parts. Above this



curve the b curves are parabolas bent down, below they are parabolas bent up. When a parabola crosses the boundary curve the continuity of the first derivative determines the path below the curve.

It is clear that the result is an oscillating motion for the difference which may even increase in magnitude. The reader is urged to consider the possibilities in which the velocity of the servo motor V ($=\dot{b}$) is governed by different laws, for instance

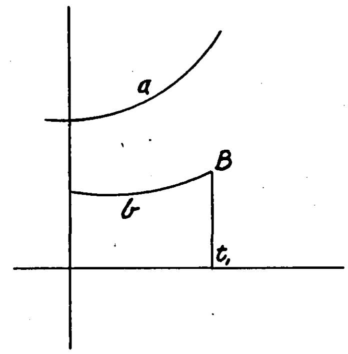
$$V_0 \, \Delta(a-b) = V + \beta V'$$

where β is positive and about 100 in size or say has different constant values above or below the a curve.

It is clear that even when the follower approximates the a curve, its rate has an added oscillation.

The correct method for matching rates then must avoid the difficulties inherent in the effort to match the rotations themselves. It is clear that what is desired is to make the difference a constant not necessarily zero. Let us mount our servo contacts then on a shaft which is positioned so that normally neither contact is made but that a slight shift will close one or the other contact and reach the limits of motion of this shaft. Let us now couple this contact shaft to the difference shaft by a frictional connection which will permit slipping when the contact shaft is at the end positions. Then when a-b is increasing, one contact will be closed and this can be used to increase b through the servo arrangement. Similarly, by the use of the other contact, we can insure that b decreases when a-b is decreasing.

In the simplified case for the motor velocities, i.e., if v can have only three values $+v_0$, 0 and $-v_0$, it is readily seen that the b follow curve will be parabolic until its tangent is parallel to the tangent of the a curve for some value of t. The situation is clearly indicated by the accompanying diagram. Here we suppose that $\dot{a}$ is positive, b is zero initially. Since $\frac{d}{dt}(a-b) = \dot{a} - \dot{b}$



is positive, a - b is increasing and the servo contacts cause $\dot{b}$ to increase. Presumably, the motor is effective enough to cause $\dot{b}$ to overtake $\dot{a}$ and hence we eventually get to a situation where $\dot{a} = \dot{b}$. It is clear that any device which registers the sign of $\dot{a} - \dot{b}$ can be used instead of the friction coupling indicated.

At this point in the simplified assumption, we can suppose that if $\dot{a}$ is a constant, the motor would stop and $\dot{b}$ would remain equal to $\dot{a}$. Actually what does happen in general is that the value of $\dot{b}$ oscillates about $\dot{a}$. The nature of these oscillations is of great importance and we shall discuss them but it is hoped that they should be relatively small. However, the time it takes for the initial equality of $\dot{a}$ and $\dot{b}$ is frequently of great importance especially in evaluating the operation of devices whose value depends upon their speed of obtaining the solution. Let us call it the transient period. During the transient period, $\dot{b}$ gives no information concerning $\dot{a}$.

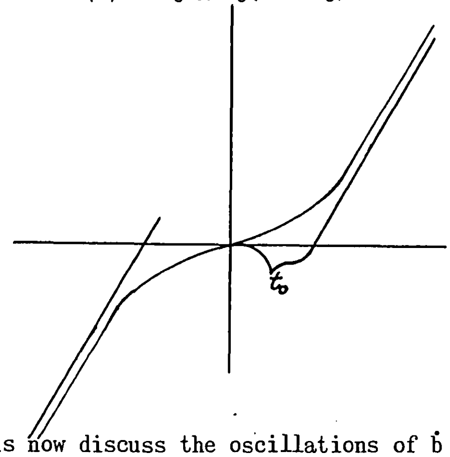
When the time needed to accelerate the motor can be ignored, the length of this transient period can be obtained by solving the equation

$$\dot{a}(t) = \dot{b}_0 \pm v_0 t$$

where the sign of the $v_0 t$ term depends upon the initial difference of $\dot{a}$ and $\dot{b}$. In many applications $\dot{a}$ can be predicted to within certain parameters and hence the possible values of t determined. For instance, the reader is urged to consider the case in which say $a(t) = \alpha_0 + \alpha_1 t$ where α_0 may take on values between -1 and +1, α, between 0 and 1. Let $\dot{b}_0 = 0$ and suppose that we can vary v between 3 and 5. The possible speeds are limited by the oscillations which occur after the transient phase.

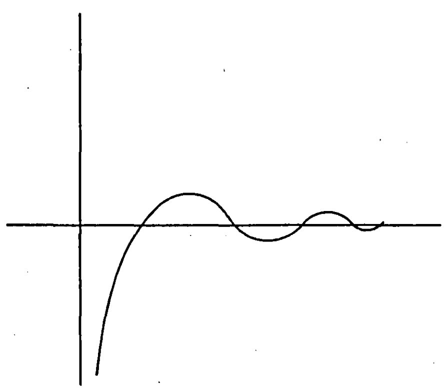
In the above discussion the change in $\dot{b}$ was taken to be $\pm v_0 t$ and the time necessary for acceleration ignored. In general, however, it is quite easy to take the acceleration time into account for the purpose of finding the transient period. For the output of most motors can in general be considered as asymptotic to a line, $\Delta\dot{b} = \pm v_0(t - t_0)$. For the purpose of finding the transient period, these asymptotic expressions can be used, i.e.,

$$\dot{a}(t) = b_0 \pm v_0(t - t_0).$$



Let us now discuss the oscillations of $\dot{b}$ about $\dot{a}$ which occur after the transient period. The specific answer in a particular case must of course depend upon the type of servo connection and the motor characteristics. However, we can point out certain aspects of the situation which are relatively general.

Let us consider the case in which $\dot{b}$ is originally too small. In general $\dot{b}$ will increase until it overshoots $\dot{a}$. The servo mechanism will, of course, bring it back and in general it will overshoot in the other direction. In a well designed mechanism; this overshooting should decrease in magnitude rapidly. Thus $\dot{b} - \dot{a}$ should look somewhat like this:

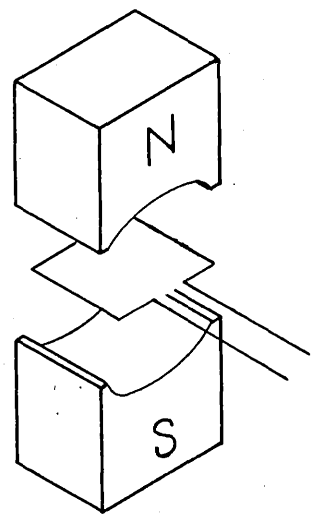


After the first maximum, the curve can be compared essentially with $e^{-\alpha t} \sin wt$. If the ratio of successive maxima is fairly constant this can be considered as a measure of damping. If the times between zeros are relatively constant they can be termed half periods.

The knowledge of the nature of such oscillations is necessary in order to analyze how such a mechanism would behave as part of a more complicated device. Oscillations of one sort or another are almost always present in a calculating machine and in the inputs also. They are called "noise." The behaviour of a device relative to noise is very important in evaluating it. This subject is somewhat obscure and the reader is referred to: Moulin, E. B. *Spontaneous Fluctuations of Voltages*. Oxford: Clarendon Press, 1938.

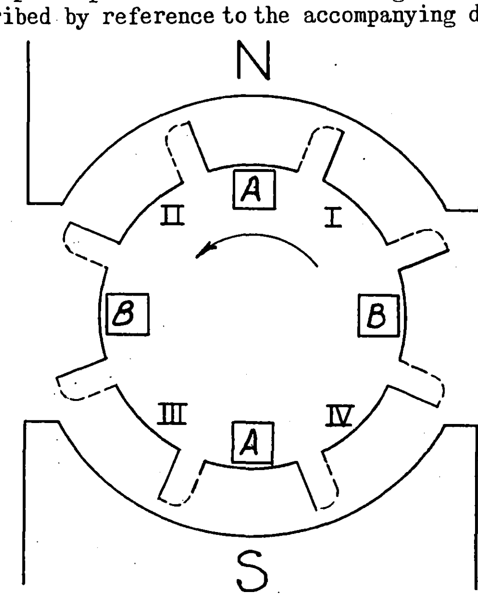
For the general problem of designing servo followers for various purposes, based upon the theory of feedback amplifiers, the following reference is given: MacColl, L. A. *Servo Mechanisms*. New York: D. Van Nostrand Co., 1945.

3. The usual speedometer is an instrument for measuring the rate of rotation of a shaft. If we have a wire moving in an electromagnetic field, a voltage is induced which is proportional to the rate of increase of flux within the loop formed by the wire. In order to be more specific, let us as-



sume that the field is uniform and that the wire loop lies in a plane. If θ is the angle of rotation of the loop from a fixed position of maximum flux. then $\varphi = A \cos \theta$ and $e = k\dot{\varphi} = -kA \sin \theta \frac{d\theta}{dt}$.

The principle of a direct current generator can be described by reference to the accompanying diagram.



Here a permanent field is given by the two poles of a magnet. Within the poles is an armature whose iron core we ignore. On this armature we have mounted various wire loops which are represented in the diagram by a solid radial line and a dotted loop which is suppose to indicate that the wire goes all the way around the armature, in fact may be a coil which loops the armature a number of times. The solid portions of circle are the commutator bars, A and B are the take-off brushes which are fixed in space. (We have simplified the following explanation by doubling the brushes. The actual windings used vary in different ways depending on the voltage and current characteristics desired and the number of poles, but those aspects of the situation which are of interest to us are precisely the same.)

It is clear that the coils near the A brushes have minimum flux across them since they are almost parallel to the field and those near the B brushes have maximum flux through them. Thus the flux is increasing for those coils we have marked II and IV, decreasing for those marked I and III. Thus in all four paths going from A to B we have a voltage rise.

This voltage rise depends upon the position of the coils as well as the rate of rotation. Thus the voltage rise is not simply proportional to the rate of rotation, $\frac{d\theta}{dt}$, but contains a varying component. In the diagram, however, it is clear that the geometric situation repeats itself every 45° and we could make this repetition occur at 360°/n by taking n coils instead of 8.

Thus the direct current generator is a differentiator. Notice that the difficulties mentioned are of relatively little importance if all that is desired is the sign of the derivative, as when the rotation of the difference shaft in previous section is considered. A generator could be used instead of the friction arrangement to control the relay. The characteristics would be somewhat different though, the generator might permit a certain accumulation of difference if the rate were slow, i.e., if a - b is not too great in absolute value.

An alternating current generator differs from a direct current generator in that the rotating coils are in series (except where it is desired to increase the current at the expense of the voltage) and electrically the position of the coils relative to the slip rings is fixed. In the following diagram, the solid radial lines represent wire loops, the dotted lines simply represent connections. The two circles are slip rings.

For this diagram the total flux through coils 1, 2 and 3 is

$$A\cos\theta + A\cos(\theta+45°) + A\cos(\theta+90°)$$

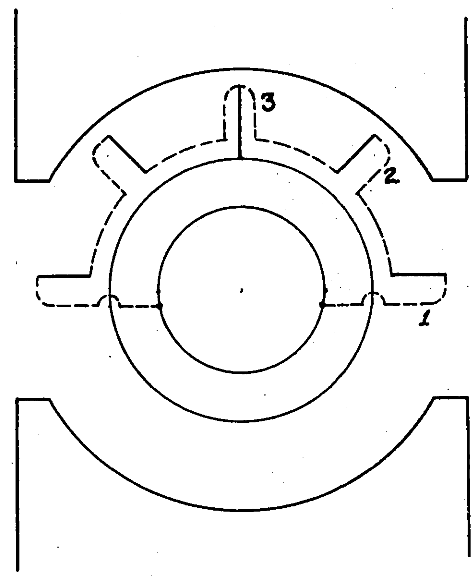
where θ is the amount of rotation from the position shown. (The other circuit is analogous.) Consequently the voltage rise is

$$e = \frac{d\varphi}{dt} = -A[\sin\theta + \sin(\theta+45°) + \sin(\theta+90°)]\,\dot\theta$$

$$= -A\,[\sin\theta\,(1+\cos 45° + \cos 90°) + \cos\theta\,(1+\sin 45° + \sin 90°)]\;\dot\theta$$

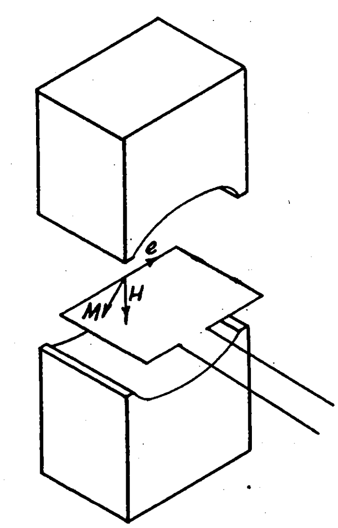
$$= -A'\,[\sin(\theta+\gamma)]\;\dot\theta$$

where $\tan\gamma = \frac{1+\cos 45° + \cos 90°}{1+\sin 45° + \sin 90°}$.



Now if $\dot\theta$ is relatively constant, we see that the output is an alternating current which is modulated both in magnitude and frequency by $\dot\theta$. This can be used in a number of ways. The above discussion generalizes to any number of coils.

It may be well at this point to indicate the relationship between generators and motors. In the accompanying diagram, let us consider the ac-



tion of the magnetic field $\mathbb{H}$ on an element dl of length of the wire. Let $\mathbb{m}$ denote the motion of the element. Then the voltage rise e is a vector which is given by the equation $e = \mathbb{m} \times \mathbb{H}\,dl$, i.e., e is perpendicular to both $\mathbb{m}$ and $\mathbb{H}$ and has size $mH \sin\theta' dl$ where θ' is the angle between $\mathbb{m}$ and $\mathbb{H}$.

On the other hand, if we had a current i through the wire and $\mathbb{H}$ is the same, then there would be a force f on the wire element $\mathbb{f} = \mathbb{H} \times i$. If we had two such elements symmetrically placed relative to the axis of the rotating conductor, we would get a torque with value $2rf \sin\theta$, where r is the radius of the conducting loop.

Considering the sum of such elements, we get the result that if we put a current through the conductor, instead of rotating it, we will get a torque instead of a voltage rise. This, of course, is the principle of the electric motor.

In the usual ammeter, this is the method used to measure currents. The current produces a torque which is proportional to it. This torque is measured by observing how far a spring is compressed.

In the wall galvanometers, an amount q of charge is measured by discharging it through such a device. Here $\frac{dq}{dt} = i$ and hence the torque T is proportional to $\frac{dq}{dt}$. The current i causes a turning impulse $= \int T dt$ which is proportional to the charge. The turning impulse gives a corresponding change in momentum, from rest say to a velocity v. With the velocity v is associated a certain kinetic energy. The above discharge occurs almost instantly and the rotating part is then turning with a certain energy. This energy is then gradually transformed into the potential energy of a delicate torsion spring. One obtains a measure of this energy by observing how far the spring is wound up when the rotating part is first brought to rest. (The above, of course, is a qualitative, not a quantitative description. A quantitative description would have to consider back e.m.f.'s in the discharge process which are due to the motion of the rotating part, i.e., its generator action and the influence of damping in the winding up process.)

4. A watt hour meter is also used as an integrator. Essentially this is an electric motor with a special type of load which permits it to represent

$$\int_{t_1}^{t_2} IE\,dt.$$

Consider a motor whose field is obtained from an electromagnet and hence is proportional to the current I in the coil. The current I for the load to be measured goes through this coil while the armature current is obtained through a resistance shunted across the load and hence is proportional to E. From the preceding section we see that the torque is proportional to IE.

Let w be the rate of rotation of the armature. Let m denote the moment of inertia of the armature and the associated rotating parts. Then we have the torque equation

$$IE = T = m\frac{dw}{dt} + lw$$

where lw is the load torque.

Solving for w, we obtain

$$w = \int_{-\infty}^{t} \frac{1}{m} e^{\frac{l}{m}(\tau - t)}\, IE\,d\tau + Ce^{-\frac{l}{m}t}.$$

Let us ignore the transient, i.e., the last term and write this:

$$w = \int_{-\infty}^{t} \frac{l}{m} e^{\frac{l}{m}(\tau - t)} \left(\frac{1}{l}IE\right) d\tau.$$

This is a weighted average of the function $\frac{1}{l}IE$ over the interval from $-\infty$ to t. Changing the variable of integration we have

$$w = \int_{-\infty}^{0} \frac{l}{m} e^{\frac{l}{m}\tau} \frac{1}{l} IE(t + \tau)\,d\tau$$

and

$$\frac{l}{m}\int_{-\infty}^{0} e^{\frac{l}{m}\tau}\,d\tau = 1.$$

It is customary to refer to this average as a time delay and indeed if $\frac{l}{m}$ is sufficiently large and IE does not charge too rapidly, the integral may be approximated as

$$w = IE(t - \tau_0)$$

where τ_0 is the value such that the total weight from $-\infty$ to $-\tau_0$ is $\frac{1}{2}$. For this we have

$$\frac{1}{2} = \int_{-\infty}^{-\tau_0} \frac{l}{m} e^{\frac{l}{m}\tau}\,dt = e^{-\frac{l}{m}\tau_0}$$

or

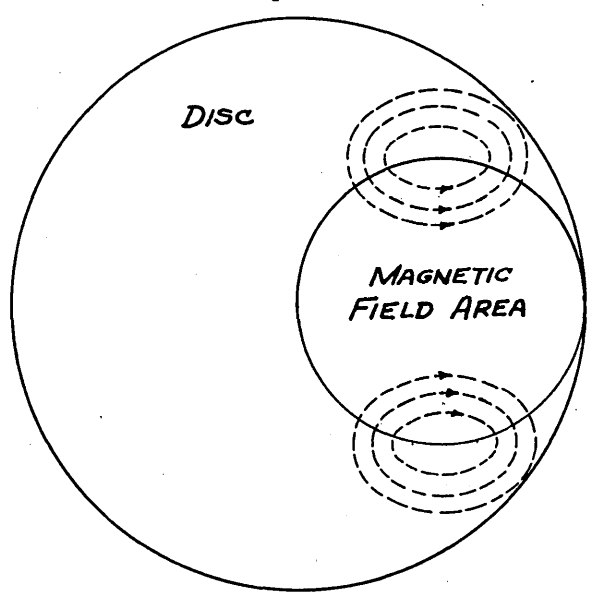
$$\tau_0 = \frac{m}{l}\log_e 2 = .7\,\frac{m}{l}.$$

Since $w = \frac{d\theta}{dt}$, we see that

$$\theta_1 - \theta_2 = \int_{t_2}^{t_1} IE(t - \tau_0)\,dt.$$

Thus the value of the integral is given by the rotation of the shaft.

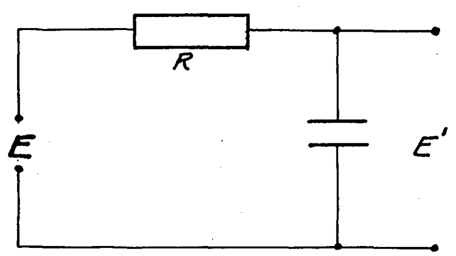
Theoretically the load torque lw could be obtained as follows: Let us turn a generator with a velocity w. The generated potential E = kw. Let us apply E across a resistance R. Then the current is E/R = k'w. Since there is now a current in the armature of the generator we have a counter torque, i.e., the generator begins to act like a motor. In turning the armature we do work against this torque and this work is the energy dissipated in the resistance. This counter torque is proportional to the current and hence equal to lw.



We must then hook our meter motor to a dissipating generator, remembering to keep m low. This is accomplished in the meter by introducing an aluminum disk between two poles of a magnet. The meter motor turns the disk. The magnetic field is located on one side of the center of the disk. The rotation of the disk sets up a voltage rise from the center to the edge of the disk on this side and hence a current flows in the disk whose energy is dissipated by the resistance of the disk. In the diagram (foot of foregoing page) the current is indicated by dotted lines.

Such currents are called eddy currents and will occur in any conductor rotating in a magnetic field. They are very objectionable in electrical machinery where they introduce a dissipating torque proportional to the speed. To minimize these losses the iron in such machines is laminated so that the resistance to these currents is as high as possible.

5. The simplest type of an electrical integrator is a condenser. Let us consider the following simple circuit. If q is the charge on the condenser,



then the voltage E' across the condenser is q/C where C is the capacity of the condenser. The current i through the resistance is $\frac{dq}{dt}$ and thus the input voltage

$$E = R\frac{dq}{dt} + \frac{1}{C}q$$

$$\frac{E}{R} = \frac{dq}{dt} + \frac{1}{CR}q$$

$$\frac{Ee^{t/CR}}{R} = e^{t/CR}\frac{dq}{dt} + e^{t/CR}\,(1/CR)\,q$$

$$= \frac{d}{dt}\,(e^{t/CR}q).$$

$$\int_{t_1}^{t_2} E\frac{e^{\tau/CR}}{R}\,d\tau = e^{t_2/CR}\,q_2 - e^{t_1/CR}\,q_1.$$

Dividing by $Ce^{t_2/CR}$ we get

$$\frac{1}{CR}\int_{t_1}^{t_2} Ee^{(\tau-t_2)/CR}\,d\tau = E_2' - e^{\frac{(t_1-t_2)}{CR}}\,E_1'.$$

Now if 1/CR is small, i.e., if CR is large, then we see that we have

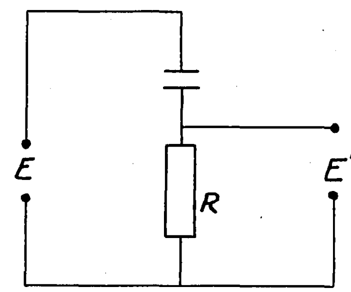
$$\frac{1}{CR}\int_{t_1}^{t_2} Ed\tau = E_2' - E_1'.$$

The factor 1/CR is not particularly troublesome since we can amplify the output E'.

For this purpose a mica condenser or an oil or paper condenser should be used. An electrolytic condenser is not suitable because to be effective the polarity must be maintained.

If we take R = 10 megohms and C = 10 microfarads we get a time base of a second in which the errors due to ignoring the factors $e^{(\tau-t_2)/CR}$ and $e^{(t_1-t_2)/CR}$ is less than 1 percent.

A similar circuit but with different values of the constants can be used to differentiate E. Let us put the output across the resistance. We will



have again the equation

$$E = R\frac{dq}{dt} + (1/C)q.$$

Differentiating and dividing by R, we get

$$\frac{1}{R}\frac{dE}{dt} = \frac{di}{dt} + (1/CR)\ i.$$

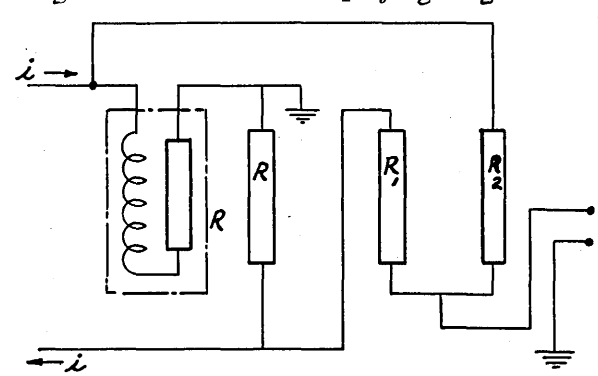
We may solve this for i

$$i = i_1\ e^{(t_1-t)/CR} + C\int_{t_1}^{t} e^{(\tau-t)/CR}\,\frac{1}{CR}\,\frac{dE}{dt}\,dt.$$

If at the time t_1, $i_1 = 0$ and if 1/CR is relatively large, then the integral on the right represents a time delayed value of $C\frac{dE}{dt}$.

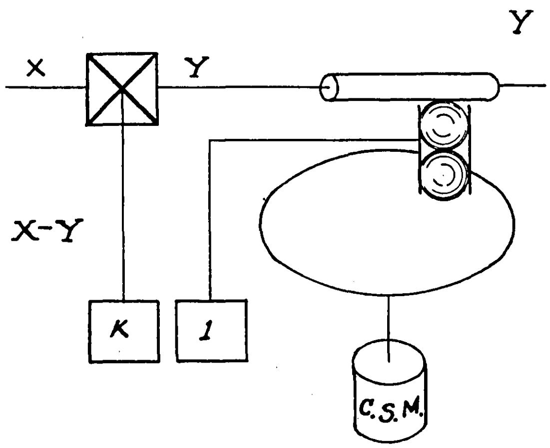
The derivative of a current can be obtained in an analogous way, at least theoretically from a linear inductance. The difficulty, of course, is that most inductances have a good deal of resistance associated with them. However, the latter can be compensated for by means of a certain circuit.

Let us suppose we have a current i flowing through a circuit. The accompanying diagram shows



part of this circuit and we suppose that the variations in voltage in the part shown is negligible as far as i concerned. If i is steady, the voltage drop in the box equals that across R and since the effect of the resistances $R_1 = R_2$ is to average the end voltages we see that the output voltage is zero. On the other hand, if i varies the voltage across the box is $L\frac{di}{dt} + Ri$ and across the resistance R, it is Ri. The output voltage is then an average of $L\frac{di}{dt} + Ri$ and $-Ri$ and proportional to $\frac{di}{dt}$.

In closing this section on the condenser integrator, we wish to point out that if a variable speed drive is connected with a differential as in the accompanying diagram, then we have the



mechanical equivalent of a condenser circuit. The output X - Y of the differential is applied through gearing to the rate input of the ball cage variable speed so that the equation

$$Y' = \frac{1}{k}(X - Y)$$

is obtained or

$$X = Y + kY'.$$

As before Y can be considered as lagging value for X or depending on the size of k as an integrating or differentiating circuit.

6. In the previous discussion, we have seen that the output of various devices contains besides the correct answer a noise term. Relatively, the noise term is oscillating while the signal is a constant. In the present section we wish to discuss filters, which will permit the relatively constant part to go through (with a time delay) but will cut down the oscillating part. More generally we will see that it is possible to arrange circuits which will discriminate in favor of some frequencies and against others.

Let us first consider the reaction of an elementary two-terminal network to a voltage which contains various frequencies.

$$E = e_0 + e_1 \sin(w_1 t + \alpha_1) + e_2 \sin(w_2 t + \alpha_2) + \ldots + e_k \sin(w_k t + \alpha_k).$$

If the circuit has inductance L, resistance R and capacity C, the voltage equation is

$$E = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q$$

in terms of the charge on the condenser. Differentiating, we get the current equation

$$\frac{dE}{dt} = L\frac{d^2i}{dt^2} + R\frac{di}{dt} + (1/C)\,i.$$

This assumes that the resistance, inductance and capacity are in series.

Now if we solve the above equation for i and neglect the transient terms, we obtain a term for each frequency that is present in the applied voltage E. Since the network equations are linear, this additivity property extends to any network made up of such elementary networks. For each frequency present in the circuit, we may solve the system of equations which we obtain by writing all other components of the various voltages as zero. In each case we will get a current of the same frequency. The current in any individual elementary network is the sum of the currents in this network thus obtained for every frequency present. This procedure, of course, ignores the transients.

For the component of the voltage $e \sin(wt + \alpha)$, solving the above equations yields

$$i = \frac{e}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}} \sin(wt + \alpha + \gamma')$$

where $\tan \gamma' = \frac{(\frac{1}{wC} - wL)}{R}$.

Now it is customary to consider an alternating voltage or current as a complex quantity when one wishes to consider only a single frequency. The real part of this complex quantity is the usual value for the quantity. Thus if e is the voltage with value $e \sin(wt + \alpha)$

$\varepsilon = R\,[e \sin(wt + \alpha) - i\,e \cos(wt + \alpha)]$

$= R\,[e(-i)(\cos(wt + \alpha) + i \sin(wt + \alpha))]$

$= R\,[e \exp(i(wt + \alpha - \pi/2)].$

The voltage can be represented by the complex quantity $\varepsilon = e \exp[i(wt + \alpha - \pi/2)]$.

Similarly the current can be represented by

$$\iota = \frac{e}{\sqrt{R^2 + (wL - \frac{1}{wC})^2}} \exp[i(wt + \alpha + \gamma' - \pi/2)].$$

Furthermore our result shows that if Z_w is the complex quantity $\sqrt{R^2 + (wL - 1/wC)^2} \exp(-i\gamma')$ then the equation

$$\iota Z_w = \varepsilon$$

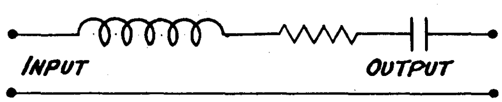
holds. This is readily seen to be a generalization of Ohm's Law.

Another way of considering the above is to note that if we use a complex value for the voltage in the differential equation we will get a complex value for the current. Since, however, the coefficients in the differential equation are real, the same derivative relationship holds between the real parts of these complex quantities as between the quantities themselves. Thus for a fixed frequency, the differential equation for a given elementary circuit can be replaced by an algebraic equation with a complex constant Z_w.

The resulting equation is a generalization of Ohm's Law. Z_w is the generalization of the notion of resistance. It is called the complex impedance, while $|Z_w| = \sqrt{R^2 + (wL - \frac{1}{wC})^2}$ is generally called the impedance. $-\gamma'$ is called the phase shift. Note that $|Z_w|$ has a minimum for

$$w^2 = \frac{1}{LC} \quad \text{or} \quad w = \frac{1}{\sqrt{LC}}$$

If v is the frequency of the voltage corresponding to w, then $w = 2\pi v$ and the frequency which gives the minimum impedance is $v = \frac{1}{2\pi\sqrt{LC}}$. Thus if we wish only frequencies near a fixed value V_0 to be present in a certain part of a circuit, we may insert in the input an elementary two-terminal network of whose impedance is a minimum for the desired frequency. Thus if the output z is small

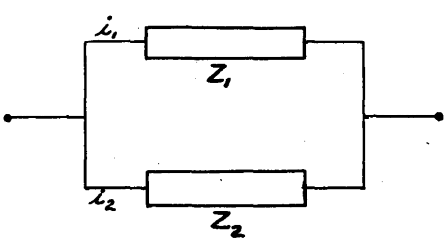


relative to Z_w for the undesired frequency, then if a voltage ε of this frequency is applied only a small part of it will appear across the output since

$$\varepsilon_0 = z\, i = \frac{\varepsilon\, z}{Z_w + z}$$

Here ε_0 is the output voltage of this frequency and i is the current. Since the voltage across the interposed circuit is $z_w i$, we see that the total voltage $\varepsilon = Z_w i + zi = (Z_w + z)\, i$ which determines the current i. Incidently this shows that impedances in series yield an impedance equal to the sum of the given impedances.

It is also possible to get an impedance which discriminates against one fixed frequency. Consider for a moment, two elementary circuits in parallel. Let ε denote the applied voltage. The $\varepsilon = z_1 \iota_1$.



The total current is

$$\iota = \iota_1 + \iota_2 = \left(\frac{1}{z_1} + \frac{1}{z_2}\right) \varepsilon$$

or

$$\iota = \left(\frac{z_1 + z_2}{z_1\, z_2}\right) \varepsilon$$

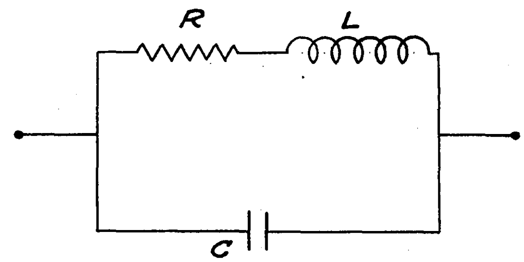
or

$$\varepsilon = \frac{z_1\, z_2}{z_1 + z_2}\, \iota.$$

Let us now recall that $z_0 = \sqrt{R^2 + (wL - \frac{1}{wC})^2}\, e^{-i\gamma}$ where $\tan\gamma = \frac{(\frac{1}{wC} - wL)}{R}$. Hence

$$z_0 = \sqrt{R^2 + (wL - \frac{1}{wC})^2}\,(\cos\gamma - i\sin\gamma)$$
$$= R + i\,(wL - \frac{1}{wC})$$

The customary choking circuit has two parts in this form. Hence, the above discussion applies to



it with $z_1 = R + i\,w\,L$, $z_2 = -\,i/Cw$. The complex impedance is

$$z = \frac{z_1\, z_2}{z_1 + z_2} = \frac{\frac{L}{C} - i\,\frac{R}{Cw}}{R + i\,(wL - \frac{1}{Cw})}$$

and the impedance in the usual sense is

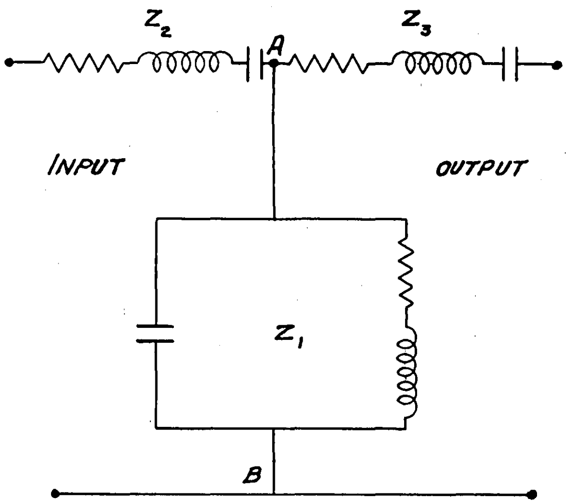
$$|z| = \frac{\sqrt{\frac{L^2}{C^2} + \frac{R^2}{C^2 w^2}}}{\sqrt{R^2 + (\frac{1}{Cw} - Lw)^2}}$$

$$|z| = \frac{\sqrt{L^2 w^2 + R^2}}{C\, w\sqrt{R^2 + (\frac{1}{Cw} - Lw)^2}}$$

$$|z| = \frac{L}{C}\,\frac{1}{\sqrt{R^2 + (\frac{1}{Cw} - Lw)^2}}\sqrt{1 + \frac{R^2}{L^2 w^2}}$$

In general, these circuits are considered for the case in which R is small relative to Lw. Hence the last factor can be considered one and z clearly has a maximum for $w^2 = \frac{1}{LC}$ Thus by the use of two such circuits we can pick out a fixed frequency out of any applied voltage and let it come through. Thus the usual discriminating filter with a single stage can be set up in the fashion as shown in the diagram at the top of the next page. Suppose the impedance of the crossing circuit is z_1, that of the inserted circuits z_2 and z_3 and the load z. From our previous results, we see that the total impedance must be

$$z_2 + \frac{z_1\,(z_3 + z)}{z_1 + z_3 + z}.$$



Hence the voltage drop ε_{AB} from A to B is

$$\frac{\varepsilon\, z_1\ (z_3 + z)}{z_1 + z_3 + z} \Big/ \left(z_2 + \frac{z_1\ (z_3 + z)}{z_1 + z_3 + z}\right)$$

$$= \varepsilon\, z_1\ (z_3 + z)\ /\ z_2\ (z_1 + z_3 + z) + z_1\ (z_3 + z)$$

$$= \varepsilon\, z_1\ (z_3 + z)\ /\ [z_2\, z_1 + (z_1 + z_2)(z_3 + z)].$$

The output voltage drop is $\varepsilon_{AB}\, z/(z + z_3)$ =

$$\varepsilon\, z_1\, z/[z_2\, z_1 + (z_1 + z_2)(z_3 + z)]$$

$$= \frac{\varepsilon\, z_1\, z}{z_2\, z_1 + (z_1 + z_2)(z_3 + z)}$$

$$= \frac{\varepsilon\, z_1\, z}{z_1\ (z_2 + z_3 + z) + z_2\ (z_3 + z)}.$$

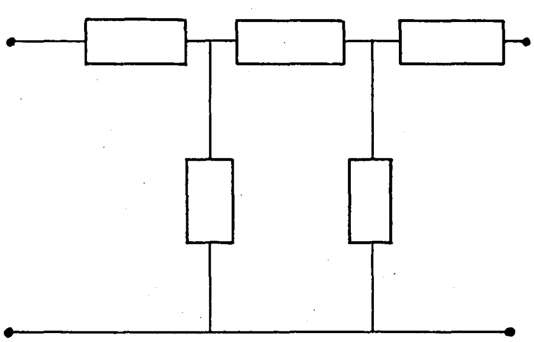
Now if z_1 is to be a maximum at a given frequency and z_2 and z_3 minima for this frequency, then for the given frequency the output ratio is essentially

$$\frac{z}{z_2 + z_3 + z}$$

if z_2/z_1 is small. On the other hand, for frequencies where z_1 is small and z_2 and z_3 are large, the output ratio is essentially

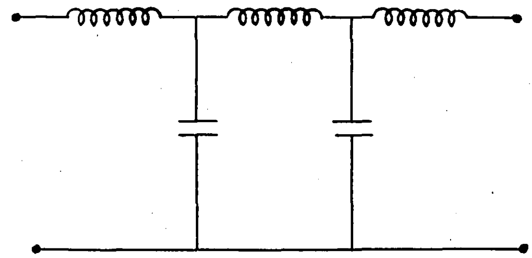
$$\frac{z_1\, z}{z_2\ (z_3 + z)}.$$

One could of course use a number of stages in succession, for instance



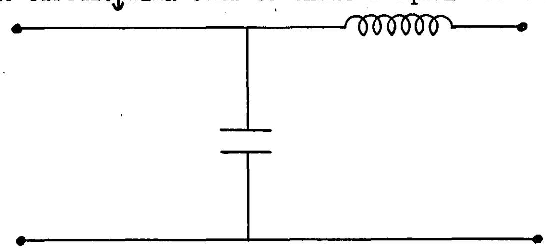
with the shorting circuits having a high impedance for the desired frequency and the series circuits having low impedance. It is clear that this shunts the undesired frequencies through the shorting circuits, while the low impedance of the series circuits favors the desired frequency.

For constant current filters, the situation is essentially simpler. Thus the circuit has the prop-



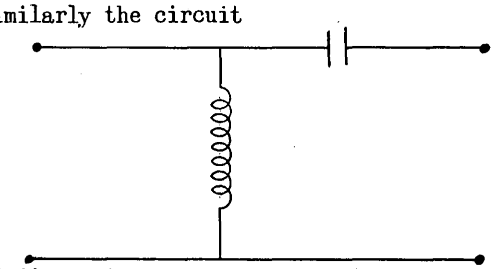
erty that it tends to shunt the non-constant currents but offers little resistance (just that necessarily associated with the inductances) to the direct current.

A band pass filter is somewhat analogous. Thus the circuit will tend to shunt frequencies for

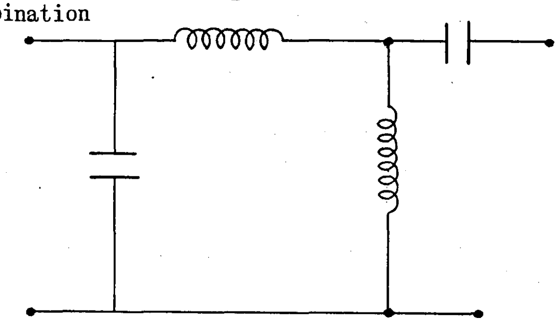


which $1/Cw << wL$. For the condenser offers a path with impedance $1/Cw$ for these frequencies, while the impedance of the other path contains the term wL. (There is a case where this won't work. The reader is urged to consider the possibilities for the output impedance.)

Similarly the circuit



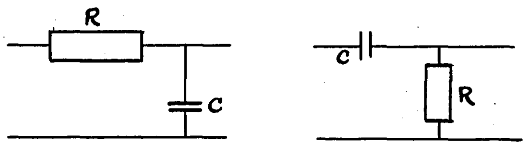
will discriminate against frequencies for which $Lw << 1/Cw$, i.e., against low frequencies. A combination



can be used to discriminate against all frequencies, which do not lie in a certain interval. The first part discriminates against frequencies which are higher than a certain number, the second against those which are lower than another number.

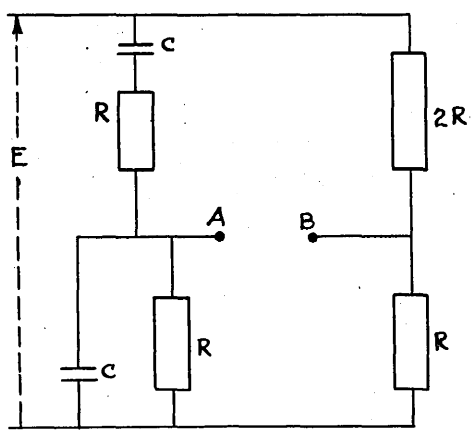
A sequence of such filter sections can be used to obtain as sharp a discrimination as desired.

For audio frequencies, 20 to 20,000 cycles per second, the use of inductances is inconvenient in general and resistor capacity combinations are preferred. Thus the first of these circuits favors the the low frequencies, the second favors the higher.



These can be combined as before in series and even parallel arrangement of series.

Another type of filter element which can be used to discriminate against a prescribed frequency is based on the use of an alternating current bridge which is balanced just at a prescribed frequency. For instance, in the circuit



the open circuit voltage between A and B is

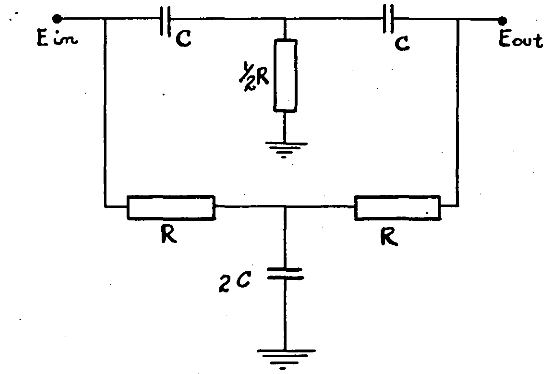
$$E_{A,B} \left(\frac{z_2}{z_1 + z_2} - \frac{1}{3}\right) E_o$$

where $z_1 = R + \frac{1}{iwC}$, $Z_2 = R/(1 + iRwC)$. Substituting yields

$$E_{AB} = \frac{1}{3}\left(\frac{i\,(RwC - \frac{1}{RwC})}{2 + i\;(RwC - \frac{1}{RwC})}\right)E_o$$

E_{AB} is zero when $w = 1/RC$ or when the frequency $f = 1/2\pi RC$.

One well known combination of these ideas which is used to discriminate against a fixed frequency is the "bridged T" circuit.

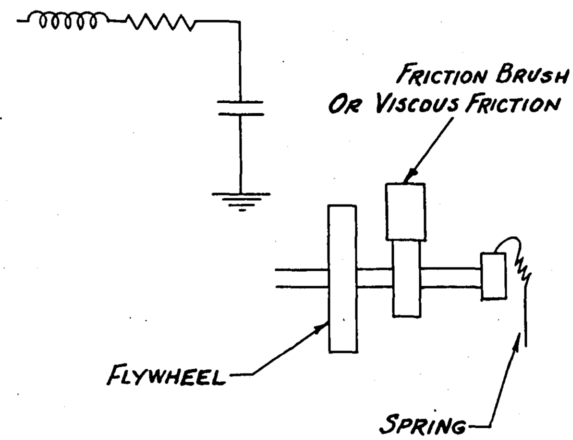


The discriminated frequency is again given by $f_o = 1/2\pi RC$. The nodal equations readily yield for any frequency f, if $\rho = f/f_o$, that $E_{out} = E_{in}\; i(\rho - \frac{1}{\rho})/[4 + 2R/R_L + i(\rho - \frac{1}{\rho}[(1 + 2R/R_L])]$ if R_L is the resistance connected across E_{out}.

The importance of the bridged T circuit lies in the fact that it provides a "bridge effect," i.e., discrimination against a fixed frequency in a circuit in which one side of input and output is grounded. This is very useful in coupling circuits involving vacuum tubes.

If the theory of calculating machines develops in certain directions, the use of different frequencies will permit the same electrical unit to be used for a number of calculations at the same time. Such filters as those discussed above will then become of great importance.

7. The above filtering and smoothing electrical circuits have their mechanical analogues. For instance, let us consider the following:



The second combination of flywheel, viscous friction drag and spring is analogous to the electric circuit if we assume that the total charge passing into the circuit is analogous to the rotation of the input shaft. The torque applied to the mechanical circuit is analogous to the applied voltage, the moment of inertia of the flywheel is analogous to the inductance and the spring whose torque is proportional to the angle turned is analogous to the condenser. All this is immediately apparent if we compare the voltage equation

$$E = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C}$$

with the torque equation

$$T = M\frac{d^2\theta}{dt^2} + R\frac{d\theta}{dt} + k\theta.$$

(The first term is the torque used in acceleration, the second is the torque used to overcome the viscous friction, and the third term corresponds to the torque of the spring.)

It is readily seen that this analogy extends to more complicated circuits. Hence, we can have mechanical filters and smoothers as well as electrical ones. In general, the frequencies are smaller in the mechanical case. Unfortunately, in general, the resistance coefficients are unknown and highly varying which is a difficulty with designing mechanical circuits.

There is another way in which the analogy can be drawn which is particularly suitable for electromagnetic devices. Here the notions of force, mass, velocity and the compliance of a spring correspond to the electrical notions of current, capacity voltage and inductance. Thus the impulse equation

$$\int_{t_1}^{t_2} F\,dt = m(v_2-v_1)$$

is compared with that for the flow of current into a condenser

$$\int_{t_1}^{t_2} i\,dt = C\,(e_2-e_1)$$

and the compliance relation for a spring

$$k\,F = x$$

where x is the displacement with the inductance relation

$$L\,i = \int e\,dt.$$

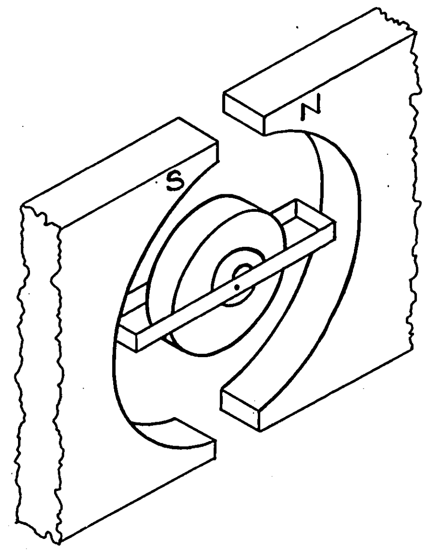
To illustrate this, let us consider a coil of resistance R, pivoted to turn so that it cuts a constant magnetic field. Let us suppose that the coil has moment of inertia I and is subject to springs which tend to keep it in a position $\theta = 0$. (The D'Arsonal movement is, of course, a good example.)

Let k be defined by the equation $k\,F_S = \theta$. That is, F_S is the force exerted by the spring when the coil has been turned an amount θ.

If there is a current i in the coil, the magnetic field will exert a force F_H on the coil which is proportional to i.

$$h\,F_H = i.$$

If F_H does work there must also be a counter e.m.f. e' and the power $F_H\,r\,\frac{d\theta}{dt}$ must equal i e'. Since



$h\,F_H = i$, we must have $\frac{r}{h}\frac{d\theta}{dt} = e'$. The equations of motion

$$F_H = I\frac{d^2\theta}{dt^2} + F_S = I\frac{d^2\theta}{dt^2} + \frac{\theta}{k}$$

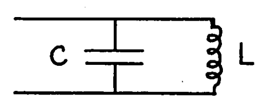
become

$$\frac{i}{h} = I\,\frac{h}{r}\frac{de'}{dt} + \frac{h}{kr}\int e'\,dt$$

or

$$i = (I\,\frac{h^2}{r})\,\frac{de'}{dt} + \frac{h^2}{kr}\int e'\,dt.$$

Thus the current voltage relationship is the same as that of circuit

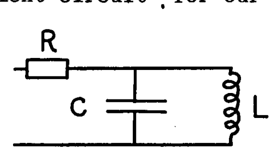


i.e.,

$$i = C\frac{de}{dt} + \frac{1}{L}\int e\,dt$$

provided $C = I\,\frac{h^2}{r}$, $L = kr/h^2$.

Power considerations also show that if $e'' = R\,i$ is the resistance voltage drop in the coil, then the total voltage drop is $e'' + e'$. Consequently the "equivalent circuit" for our coil is



where, of course, R, C and L have the values indicated above. This means that as far as the electrical circuit connected to the coil is concerned, we can get the same effect by substituting this circuit instead of the coil. A shunt resistance will indicate the effect of viscous friction.

The use of equivalent circuits in this fashion is a very effective method of determining the interplay of electrical and mechanical elements, which are connected magnetically. It is interesting to note that

$$LC = Ik$$

i.e., the resonance frequency of the mechanical circuit and its electrical equivalent are the same. When one has expressed the mechanical elements in a device by their electrical equivalents, circuit theory can be used to describe the action of the device in time and also to discuss such questions as stability.

This conversion of mechanical elements to their electrical equivalent is quite customary in the design of sound reproducing devices. Cf.: Mason, Warren P. *The Application of Electromechanical Impedance Elements in Transducers and Wave Filters*. New York: D. Van Nostrand Co., 1942.

The analogy between electrical circuits and hydraulic ones is reasonably well known. The amount of fluid corresponds to the amount of charge, the pressure difference corresponds to voltage drop, the inductance corresponds to the mass of the fluid, viscous friction corresponds to the resistance and a storage cylinder corresponds to a condenser. (In the latter either gravity or a piston pushed by a spring can be considered as providing the pressure proportional to amount of fluid in the cylinder.)

For current of a fixed frequency, the analogue of a set of gears is a transformer which increases the voltage in the same way as the gears will increase the torque. The equivalent of backlash is the hysteresis. (It may be recalled that backlash is the amount of play between gears when the driver reverses direction.)

Normally every effort is made to eliminate backlash. However, sometimes it is deliberately inserted because it makes the device insensitive to vibrations of less than a certain size. For if we have two teeth moving together, of course, with a certain momentum in the driven gear, then if the driver has a high frequency vibration of amplitude less than one-half the backlash, the backlash permits the driver to move back and forth, touching the driven wheel only at one point in each cycle of the vibration. Of course, the driven gear receives a number of impulses but its moment of inertia will smooth these out considerably. (The reader is urged to study the differential equations for the above process and also the electrical analogues.)

In most cases, as I have said before, the effort is made to eliminate backlash. Thus the driven gear may be split in two parts, which may move relative to each other but which are fastened together by a spring in such a way that the acting tooth of the driver is pressed on both sides, one by each half of the split gear. Consequently when the driver reverses direction, it reverses the half of the split gear it presses on but it does not move relative to the split gear.

There are other arrangements also for eliminating backlash. For instance in the above, the pressure of the spring must be large enough to stand the pressure between wheel and driver and is itself another pressure on the gear tooth faces of the gear. To eliminate this, it has been proposed that the split gears be kept apart by a wedge.

Consider the following patented arrangement which is due to C. W. Nieman. Again we have a split gear on the driven shaft but each half is free to rotate around the driven shaft. On each half, there is a pin which protrudes from the face of the gears and I believe that one passes through a slit on the other. On the driven shaft there is a rod perpendicular to the axis. On this rod there is a stud which slides up and down but which is normally pressed between the two pins on the halves of the split gear by a spring. (See diagram, top of page II - 36.)

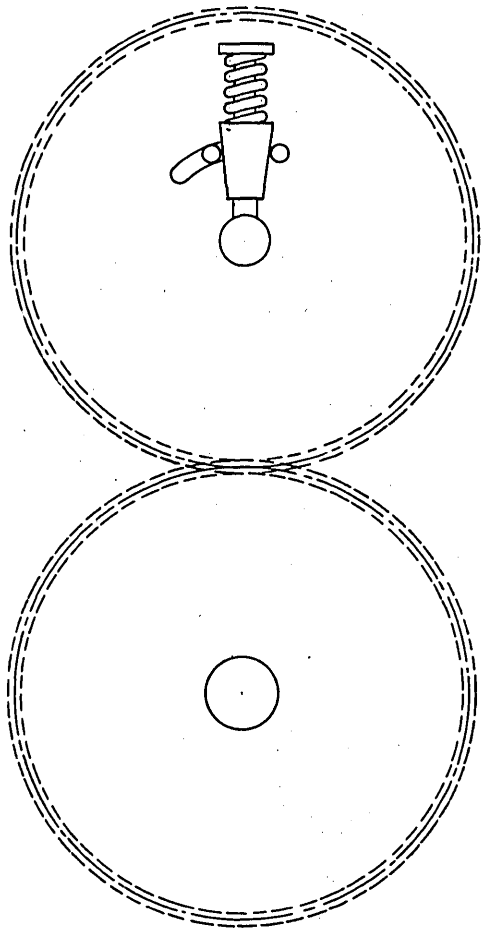
With no load the spring presses the stud onto the pins and this pressure causes the two halves of the split gear to press against the teeth of the driven gear. When the driver applies a force, this force is transmitted by one-half of the split gear and one pin to one side of the stud. Since this force is not balanced by a like force from the pin on the opposite side of the stud, the stud presses on its axial rod. Thus friction prevents the stud from moving up on the rod and the torque is transmitted by the rod to the driven shaft.

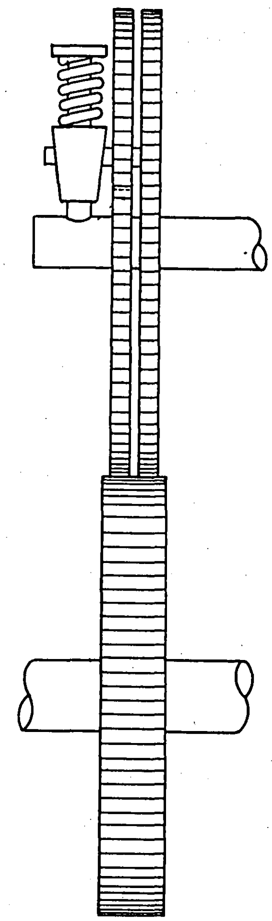
This arrangement has the advantage that the pressure by the spring on the gear teeth may be relatively light. It is quite important in many precision devices.

8. We have already discussed a number of methods for finding a rate of rotation. Besides these, there are stroboscopic methods which while very accurate have not as far as I know been adapted for calculating purposes; there are methods involving the drag of viscous fluids, there are electrical methods involving charging and discharging of condensers and finally there is the gyroscope principal of the airplane turn indicator. We wish to consider the latter in the present section.

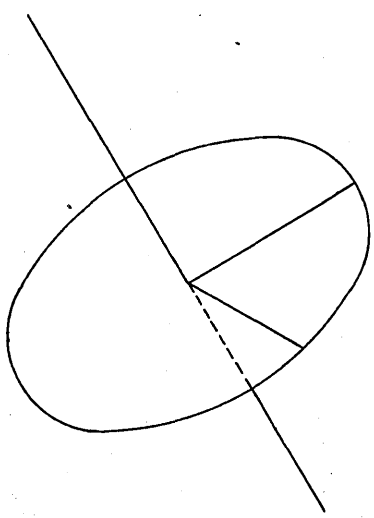
A gyroscope is a solid body having rotational symmetry around an axis which is called the spin axis. The moment of inertia around this axis will be denoted by A. If we take the moment of inertia around any axis perpendicular to this one and through the center of gravity, the result B will be the same and in the usual gyroscopes B is smaller than A.

It is customary to regard the gyroscope as rotat-

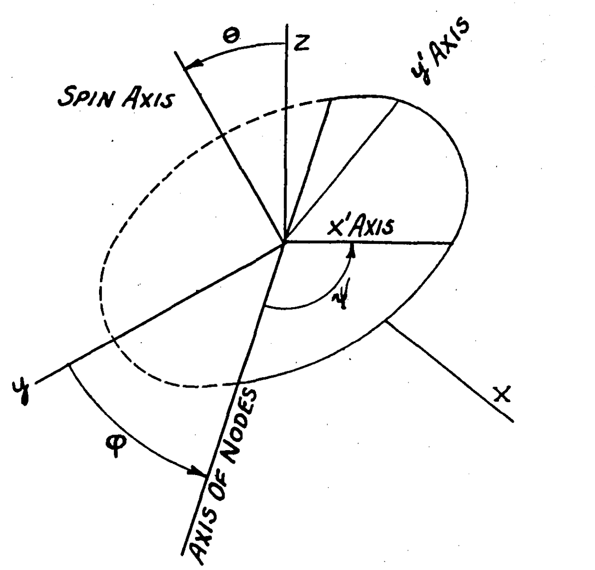




ing with a motion which has a large component around the spin axis. The gyroscope is considered as mounted in such a manner that the center of gravity remains fixed. The spin axis is generally drawn with a considerable vertical component. The intersection of the plane perpendicular to the spin axis with the horizontal plane is then called the axis of nodes. We also consider a third axis, perpendicular to these two.



It is also customary to introduce two other sets of axes. One of these is fixed in space, with the z axis extending positively upward, the x axis extending positively to the right and the y axis positively toward us. The other axes are considered as fixed in the gyroscope. The z' axis of this set



coincides with the spin axis. Consequently the x' and y' axes for this body fixed set remain in the plane determined by the axis of nodes and the third axis of our previous discussion.

The equations of motion for the gyroscope were discovered by Euler, who introduced the three angles which are known as the Eulerian angles. θ is the angle between the spin axis and the fixed z axis, φ is the angle between the fixed y axis and the axis of nodes and ψ is the angle between the axis of nodes and the body fixed x axis. These are related to the usual manner of mounting the gyroscope in gimbals. The gimbals are a set of concentric rings, each pivoted in the next outermost one. Let us number them from the outside in: The outermost is fixed, the next inner or second ring is pivoted so as to permit a rotation around a vertical axis. The third ring is pivoted so that it can rotate around a horizontal axis and the gyroscope itself is mounted on the third ring with the spin axis perpendicular to the axis of rotation of the third ring. Notice that the fixed point is also the center of the mass.

In this gimbal mounting the axis of nodes is the pivot axis for the third ring. Consequently θ is the angle of rotation of this ring around its pivot. φ is the angle of rotation of the second ring around its pivot and ψ is the angle of rotation of the gyroscope relative to its spin axis.

[It is suggested that the reader prove that the direction cosines of the spin axis are $(-\sin\theta \cos\varphi, \sin\theta\sin\varphi, \cos\theta)$, of the axis of nodes are $(\sin\varphi, \cos\varphi, 0)$ and the third axis $(\cos\theta \cos\varphi, -\cos\theta\sin\varphi, \sin\theta)$.]

We now wish to consider the motion of a rigid body in general, with one point fixed. Let us consider any transformation

$$\begin{aligned} x' &= f\,(x, y, z)\\ y' &= y\,(x, y, z)\\ z' &= h\,(x, y, z)\end{aligned} \qquad (1)$$

of a coordinate system OX, OY, OZ into one OX', OY', OZ' which preserves distances and has O as a fixed point. If A is the vector (x, y, z) and A' is the transform (x', y', z'), we write $A' = TA$. The distance preserving property yields that if A and B are any two vectors and α is any real number that

$$T\,(\alpha A + (1-\alpha)B) = \alpha TA + (1-\alpha)TB$$

also
$$T(kA) = kTA$$

(Since a straight line is the shortest distance between two points colinearity must be preserved.) Consequently the transformation is linear, i.e.,

$$T\,(aA + bB) = aTA + bTB$$

for any two vectors A and B and any two numbers a and b. Consequently the transformation equations may be written

$$\begin{aligned} x' &= a_{1,1}\, x_1 + a_{1,2}\, y + a_{1,3}\, z\\ y' &= a_{2,1}\, x_1 + a_{2,2}\, y + a_{2,3}\, z\\ z' &= a_{3,1}\, x_1 + a_{3,2}\, y + a_{3,3}\, z\end{aligned}$$

Let us now consider the condition that a vector A be such that it is taken into a multiple of itself, i.e., $TA = \lambda A$.

For such an A we must have

$$\begin{aligned} 0 &= (a_{1,1}-\lambda)x + a_{1,2}\, y + a_{1,3}\, z\\ 0 &= a_{2,1}\, x + (a_{2,2}-\lambda)\, y + a_{2,3}\, z\\ 0 &= a_{3,1}\, x + a_{3,2}\, y + (a_{3,3}-\lambda)z\end{aligned}$$

The necessary and sufficient condition that a λ satisfying these equations will exist is that the determinant of the coefficients be zero. This determinant is a cubic in λ and hence must have a real root λ.

Since $TA = \lambda A$ and the lengths must be preserved, this real root must have the values one or minus one. Since the latter involves an inversion, it does not correspond to any possible motion of a rigid body.

We may then assume,

If a motion of a rigid body leaves one point fixed, it leaves every point on a line unchanged.

The planes perpendicular to this fixed line are taken into themselves, i.e., each is rotated through the same angle.

Let us now consider a moving body. During the small interval of time Δt, the body will rotate an amount $\Delta\alpha$ about a line, with unit vector $\bar{u}$. The displacement of a vector A fixed in the moving body is approximately given by $A \times \bar{u}\Delta\alpha$ and hence $\frac{dA}{dt} = A \times w$, where w is the vectorial rotation $w = \bar{u}\frac{d\alpha}{dt}$. If the vector A is moving relative to the body with a motion A', then the total motion is

$$\frac{dA}{dt} = A' + A \times w$$

Since w is a vector, we can add it vectorially. Thus if we suppose that there is an angular velocity $\dot{\psi}$ around the spin axis, $\dot{\theta}$ around the axis of nodes and φ is changing with the amount $\dot{\varphi}$, then

$$\begin{aligned} w &= \dot{\psi}\,(-\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\\ &\quad + \dot{\theta}\,(\sin\varphi, \cos\varphi, 0)\\ &\quad + \dot{\varphi}\,(0, 0, 1)\\ &= (-\dot{\psi}\sin\varphi\cos\varphi + \dot{\theta}\sin\varphi,\\ &\qquad \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \dot{\psi}\cos\theta + \dot{\varphi})\end{aligned}$$

We may consider a set of axes which is to be fixed in space but instantaneously coincides with the spin axis, axes of nodes and third axis. Let $\bar{s}$, $\bar{n}$ and $\bar{t}$ denote unit vectors along these. Then

$$w = (\dot{\psi} + \dot{\varphi}\cos\theta)\,\bar{s} + \dot{\theta}\,\bar{n} + \dot{\varphi}\sin\theta\,\bar{t}$$

For a rigid body with a fixed point, in order to apply Newton's Laws, it is necessary to consider the angular momentum $\bar{p}$ instead of the usual momentum considerations. This is particularly easy to see in the case where we have an axis of symmetry and the fixed point is center of mass. For if we have two equal masses symmetrically placed relative to an axis, it is clear that we must apply a torque around the axis to effect the rate of rotation around this axis. It can be shown that in general Newton's Laws of Motion become

$$\frac{d\bar{p}}{dt} = T$$

where T is applied torque. The torque itself can be resolved into three components

$$T = T_s\bar{s} + T_n\bar{n} + T_t\bar{t}$$

On the other hand, from the expression for w, we see that

$$\bar{p} = A(\dot{\psi}+\dot{\varphi}\cos\theta)\,\bar{s} + B\dot{\theta}\bar{n} + B\dot{\varphi}\sin\theta\,\bar{t}$$

These equations, of course, remain valid only if we permit $\bar{s}$ and $\bar{n}$ to move. It is clear that the set $\bar{s}$, $\bar{n}$, $\bar{t}$ revolve with a vector w*, which is obtained from w by letting $\dot{\psi} = 0$, i.e.,

$$w^* = (\dot{\varphi}\cos\theta,\ \dot{\theta},\ \dot{\varphi}\sin\theta)$$

Consequently we have the formulas

$$\frac{d\bar{s}}{dt} = \bar{s} \times w^* = (0,\ -\dot{\varphi}\sin\theta,\ \dot{\theta})$$

$$\frac{d\bar{n}}{dt} = \bar{n} \times w^* = (\dot{\varphi}\sin\theta,\ 0,\ -\dot{\varphi}\cos\theta)$$

$$\frac{d\bar{t}}{dt} = \bar{t} \times w^* = (-\dot{\theta},\ \dot{\varphi}\cos\theta,\ 0)$$

These formulas permit us to substitute in the equation of motion and obtain

$$A(\ddot{\psi} + \ddot{\varphi}\cos\theta - \dot{\varphi}\,\dot{\theta}\sin\theta) = T_s$$

$$B\ddot{\theta} - A\dot{\psi}\,\dot{\varphi}\cos\theta\sin\theta - \tfrac{1}{2}(A-B)\,\dot{\varphi}^2\sin 2\theta = T_n$$

$$B\ddot{\varphi}\sin\theta + B\dot{\varphi}\dot{\theta}\cos\theta + A\dot{\theta}\dot{\psi} - (B-A)\,\dot{\psi}\dot{\theta}\cos\theta = T_t$$

In the applications of the gyroscope as a differentiator the angle φ is kept zero, $\dot{\theta}$ is the quantity to be measured. From the last equation, we see that if $\dot{\psi}$ is kept constant T_t is proportional to $\dot{\theta}$. $\dot{\psi}$ is generally kept constant and rather large by a governor controlled electric motor.

IV. Amplifiers

In the present section we wish to discuss the theory of amplifiers. Before considering details, we wish to make certain general remarks concerning the nature of calculating devices. Each such device will have a number of inputs and an output. A purely calculating device can be defined as one which has no source of energy between the input and output. For such a device the outputs must do all the work. In general it is also true that the results will be accurate only when the output does no work. Indeed the order of magnitude of the error is closely associated with the ratio of output work to input work.

It follows therefore that if we wish to use the mathematical output of such a device as an input to a similar device, we must provide an energy source. The present chapter discusses such energy sources. Since a relationship must be maintained between the input and output, we must discuss the very important question of the accuracy of such a device.

The ratio of the output energies to input energies of available amplifiers is very important in considering the physical principles upon which a device is to be based. Fortunately this ratio is quite large for electronic amplifiers and indeed the latter are frequently used as part of other amplifiers for this reason.

The reader is cautioned, however, that it is not this ratio but rather the square root of it that represents the available amplification. The reason for this is the possibility of error in the amplifier itself. We shall see that in general accuracy in amplification can be obtained only at the expense of amplification.

[The customary way in which amplifiers are described is in terms of decibels. If σ is the ratio of output power to input power, the decibel rating is

$$d = 10\log_{10}\sigma.$$

This can be used very conveniently if one remembers that $\log_{10} 10 = 1$ and that $\log_{10} 2$ is very close to .3. Furthermore the use of successive amplifiers will yield an amplifier with a decibel rating equal to the sum of the ratings of the components.]

We will discuss the various methods by which servo motors can be used as power sources, the actual torque amplifiers and then the various uses for the electronic amplifiers. We discuss the general theory of the amplifiers, the use of "negative feedback" which is essential in calculating devices, the use of rectifiers or detectors, and oscillators.

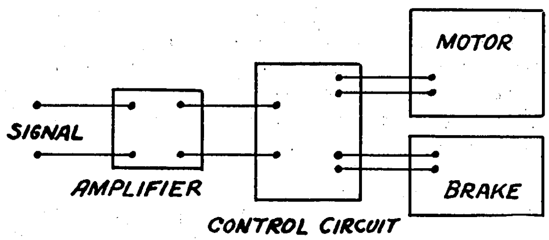
It is true, of course, that energy considerations limit the accuracy of ordinary continuous machines. However, it is also true that it is possible to combine continuous devices in such a way that digital results may be obtained, also iterative processes may be used to supplement the continuous devices.

1. In Sections 10, 11 and 12 of Chapter II above we have indicated devices in which the output is driven by a servo motor. With a proper control setup, these probably correspond to the best method of inserting power into a mechanical device. It corresponds to a number of uses of negative feedback amplification in electrical circuit and has a certain flexibility which we will discuss later.

The difficulty with mechanical following devices is, of course, the possibility of overshooting when the equilibrium point is reached. If the leading motion is intermittent, i.e., of such a nature that it consists of changes followed by periods of no motion, the following motion can be obtained by means of a system which involves braking when the equilibrium point is reached.

For example, let us consider a servo system in which the motor is to go in one direction when the input signal is sufficiently positive, remain at rest for signals in a certain small region about zero and go in the opposite direction for signals which are sufficiently negative.

Ordinarily such a circuit consists of an electronic amplifier to give voltage and power amplification of the signal and a control circuit. Thus the control circuit would normally receive the amplified signal on two "trigger circuits," whose precise character we will discuss later. A

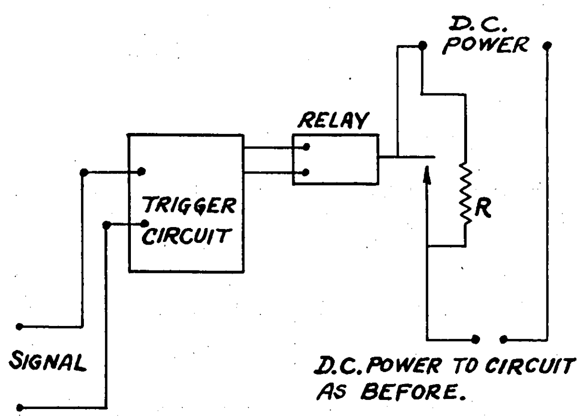


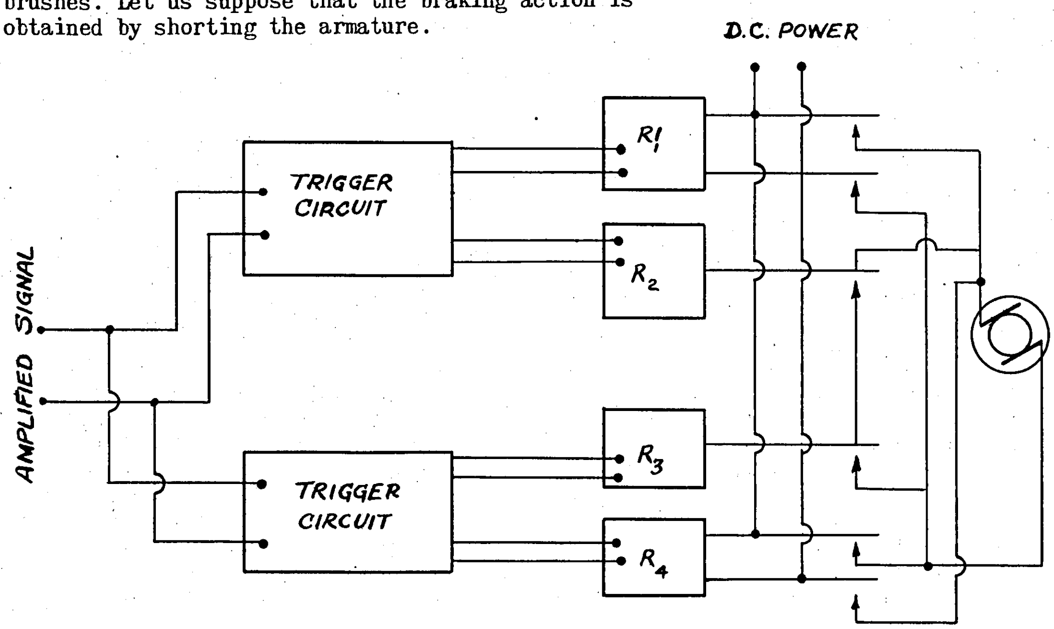
trigger circuit has two outputs. There will be a current through one output if the signal voltage is greater than a certain value and none in the other. If the voltage is less than the given value, the first output will have no current and the second will receive the current. These circuits can be used in suitable relays.

Let us suppose now that each trigger circuit has two relays. Let us number the relays R_1, R_2, R_3 and R_4. Let us suppose that the first trigger circuit is adjusted so that if the signal voltage σ exceeds ε_0 then relay R_1 works, otherwise R_2 works. Let the second trigger circuit be adjusted so that if the signal voltage exceeds $-\varepsilon_0$, then R_3 operates, otherwise R_4. For definiteness let us suppose that the motor is a direct current motor with constant field, so that the direction of rotation is determined by the polarity of the distributor brushes. Let us suppose that the braking action is obtained by shorting the armature.

From the above, it is seen that if $\sigma > \varepsilon_0$, R_1 and R_3 operate, for σ between ε_0 and $-\varepsilon_0$, R_2 and R_3 operate and for $\sigma < -\varepsilon_0$, R_2 and R_4 operate. In the case $\sigma > \varepsilon_0$, in the circuit illustrated R_1 applies the D.C. voltage to the motor, so as to turn it in one direction. R_3 makes no difference since R_2 is open. In the case $\varepsilon_0 > \sigma > -\varepsilon_0$, both R_1 and R_4 are open, so that no power is applied to the motor and the armature is shorted through R_2 and R_3. The case $\sigma < -\varepsilon_0$ is analogous to the case $\sigma > \varepsilon_0$ with R_4 instead of R_1 and R_3 instead of R_2 acting.

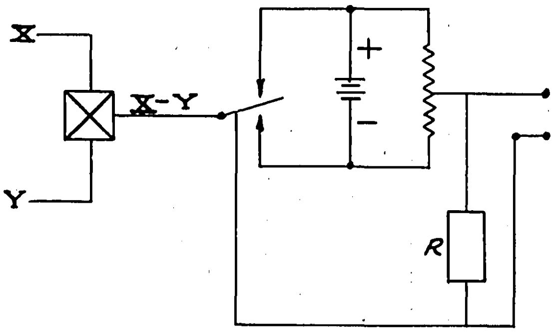
Of course, the two relays R_1 and R_2 could be replaced by a single relay operating off the upper output of the first circuit, provided it had two normally open and one normally closed contacts. (The "normal" is the non-operating condition of the relay.) But two trigger circuits are needed if we are to have three distinct regions of response to a signal. Indeed it should be clear that a trigger circuit specifies a point between regions of response.



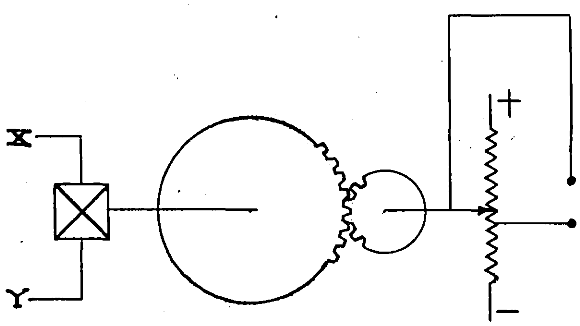


It is possible using servo systems with more trigger circuits to get varying motor speeds. Thus if we wish to have two motor speeds corresponding to positive signals, we might introduce another trigger circuit with a critical voltage ε_1. The rest of the circuit is the same as before. Normally the resistance R is in the power lead, but if the critical voltage ε_1 is exceeded, the relay shorts this resistance and a higher voltage is applied to the motor. If one wanted two speeds for the negative signal, one has another trigger circuit with critical voltage, $-\varepsilon_1$, whose lower output is also capable of shunting the resistance R. Thus R is shorted if $\sigma > \varepsilon_1$ or $\varepsilon < -\varepsilon_1$.

We wish to point out ways in which the requisite signal can be obtained in the case of two shafts

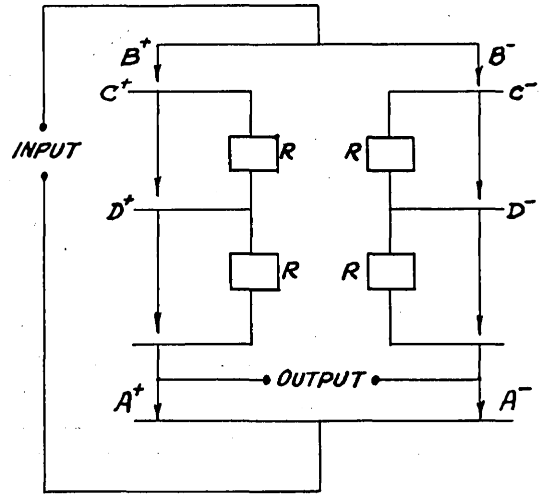


X and Y, such that Y is supposed to be driven by the servo arrangement so that its rotation matches X. The difference between X and Y is obtained by means of a differential. If X - Y can exert a certain force without injuring X, then we could apply spring contacts on X - Y so that X - Y positive would press on one contact and X - Y negative would press on the other. It is clear from the circuit illustrated that these would give voltages of different signs across the output. Of course, a potentiometer could be used to give voltages proportional to X - Y.



If the potentiometer is connected to X - Y by a gear arrangement with a partly toothed gear on the X - Y shaft, then the motion of the X - Y shaft is not restricted as much as by the previous arrangements. However, if a straight potentiometer connection is used one has a voltage proportional to the difference.

By the use of a number of trigger circuits varying speeds for the servo motor are obtainable, so that even in the case of a continuously changing motion one could keep the speed of the follower approximately equal to that of the leader. In the case of a small motor which requires little current, this can be done by means of contact arrangement on the X - Y shaft. For instance, suppose we have a contact arrangement such that if X - Y is positive, then contact A_+ is broken and B_+ is made. Suppose that C_+, D_+, etc., will be made in succession when X - Y becomes more and more positive. Suppose we have analogous contacts for X - Y negative. It is clear then that output of the circuit can be used to run a small motor at varying speeds or as a signal to a servo system for the same purpose. One could, of course, replace C^+, D^+ and the corresponding resistances by a rheostat in which the resistance is cut out as X - Y increases. However, the rheostat would have to be driven by a partly toothed gear so that it remains unchanged when X - Y is negative.

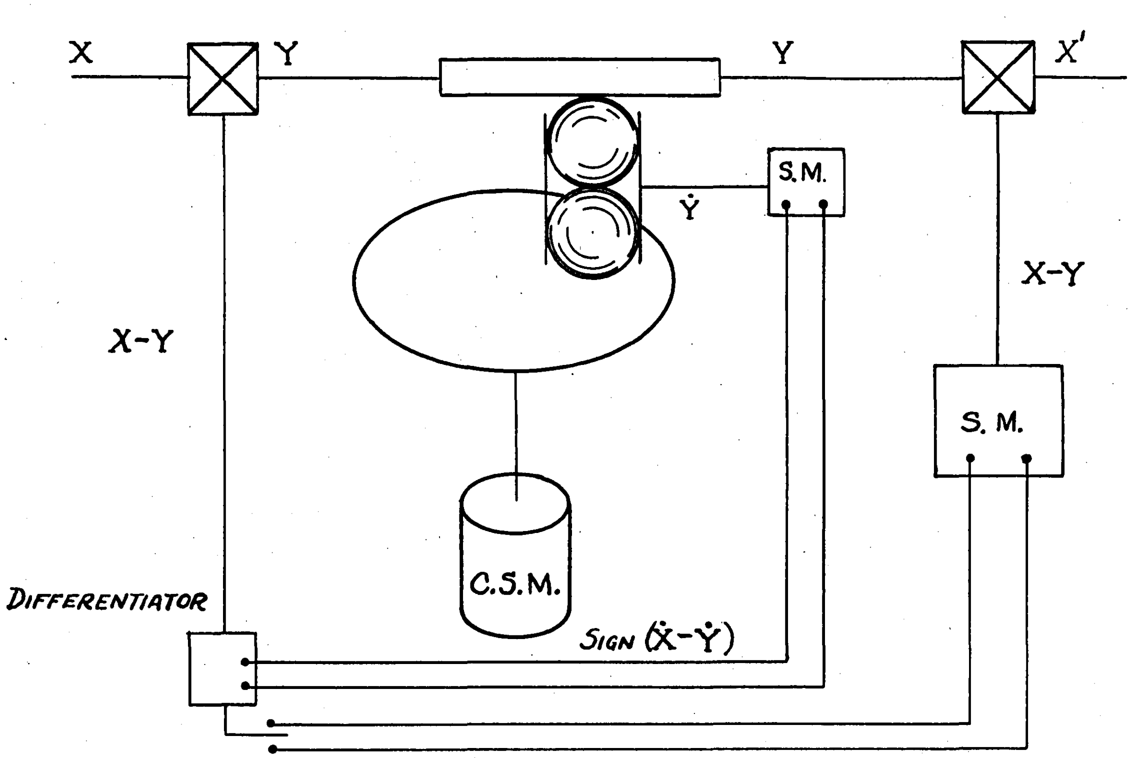


However, in the case of a continuously changing motion, the rotation matching device of Chapter III, Section 2, can be used with a supplementary device to take care of the difference X - Y which is lost in the matching process. The difference X - Y is to be put in much more slowly than Y so that it will not overshoot. As a consequence Y will lag behind X in time. The difference X - Y is used to control the supplementary system and the sign of its derivative is used to control Y. (See diagram, p. II-41.)

At the end of Section 4 we have given a time delay connection for a ball cage variable speed drive, which is often used to obtain both torque amplification and smoothing.

One exceedingly effective method of making a servo control connection is based on the fact that a polarizing disc will only permit the component of light polarized in a certain direction to pass through it. Thus if a beam of light passes through a disc A and then through another which makes an angle θ with A, the amount of light transmitted is proportional to cos θ. This is utilized to make a servo connection in the following manner.

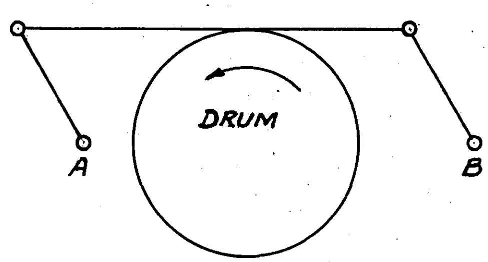
The disc A is attached to the input shaft. Two beams of light obtained by mirrors from the same source are directed through this disc. Thus as the shaft rotates, the direction of polarization of the "extraordinary component" of these beams which alone is transmitted through A rotates also. These two beams pass through separate discs B and B' which are mounted on the output shaft and then to separate phototubes. The polarizing planes for B and B' are at right angles and hence there is only one position in each quadrant at which equal



amounts of light enter both phototubes. A variation from this position by the output shaft will favor one or the other phototube and hence can be used to control a servo. (Cf. Berry, T. M., *Trans. Amer. Inst. Electrical Eng.*, LXIII (1944), p. 195).

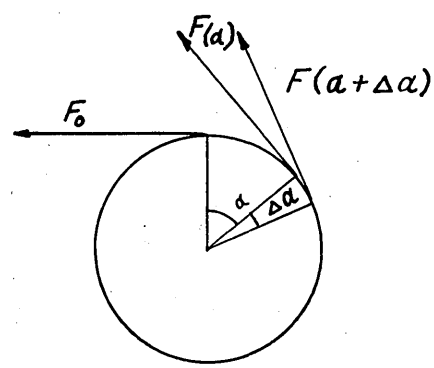
A general theory for servo mechanisms is developed in MacColl, L. A., *Fundamental Theory of Servo Mechanisms*. New York: D. Van Nostrand Co, 1945. This theory is based on the general notion of a feedback amplifier.

2. The customary method for torque amplification in the case of rotations is by means of a band on a rotating drum. The principle is illustrated in the accompanying diagram. From the lever A we have



a band, which passes once around the drum and then to B. If now we pull on A, then the band tightens on the drum and the lever B is pulled by a force which is augmented by the pull of friction between the band and the drum.

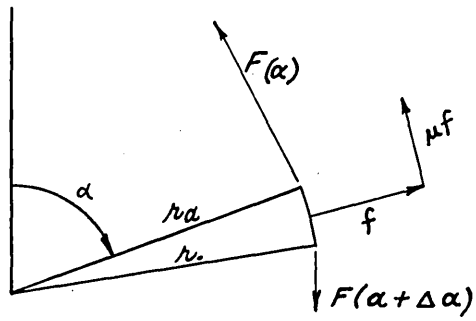
In the case in which the drum moves faster than the band, we can present the following somewhat simplified discussion of the situation. Let us consider the longitudinal tension in the band. At the point where the band touches the drum on the A side, this has a value of F_0 equal to the pull from A. Let α denote the angle between the radius



for this point of first contact and an arbitrary radius. Then the longitudinal tension in the band is a function $F(\alpha)$ of this angle.

To determine $F(\alpha)$, let us consider the piece of band between α and $\alpha + \Delta\alpha$. For this we will have the two tensions, $F(\alpha)$ and $F(\alpha+\Delta\alpha)$, pulling at the end of the piece, we will have the effect of the drum on the piece and, of course, inertial effects. Let v denote the linear speed of the band. Let f denote the resultant of the perpendicular forces of the drum on the disk. It seems reasonable to assume that f makes an angle $\frac{1}{2}\Delta\alpha$ with the radius vector for α. We can also assume that the effect of the drum perpendicular to f is μf where μ is a coefficient of friction. (This will be justified in the limit.) Let ρ denote the amount of mass of the band per unit length.

We suppose that the band is inelastic. One can easily see that the inertial resultant has a component $\rho\Delta\alpha v$, directed inwardly along r and a component $\rho\Delta\alpha\dot{v}$ along $F(\alpha)$. If we take components of the forces perpendicular to the radius vector for α, Newton's Law becomes



$$F(\alpha) - F(\alpha+\Delta\alpha)\cos\Delta\alpha + \mu f\cos\frac{\Delta\alpha}{2} - f\sin\frac{\Delta\alpha}{2} = \rho\Delta\alpha\dot{v}$$

while the components parallel to the radius vector yield

$$F(\alpha+\Delta\alpha)\sin\Delta\alpha - f\cos\frac{\Delta\alpha}{2} + \mu f\,\frac{\sin\Delta\alpha}{2} = +\Delta\alpha v$$

or

$$F(\alpha+\Delta\alpha)\sin\Delta\alpha - \rho\Delta\alpha v = [\cos\frac{\Delta\alpha}{2} + \mu\sin(\frac{\Delta\alpha}{2})]\ f.$$

This shows that f is an infinitesimal of the order of $\Delta\alpha$. Neglecting infinitesimals of higher than first order, we may eliminate f between the equations. Hence,

$$F(\alpha) - F(\alpha+\Delta\alpha) + \mu F(\alpha)\sin\Delta\alpha = \rho\Delta\alpha(\dot{v}+\mu v).$$

Dividing by $\Delta\alpha$ and passing to the limit we get

$$-\frac{dF}{d\alpha} + \mu F = \rho(\dot{v}+\mu v).$$

In general the mass of the band may be neglected and this equation simplifies to

$$\frac{dF}{d\alpha} = \mu F.$$

The solution of this equation which is F_0 for $\alpha = 0$ is

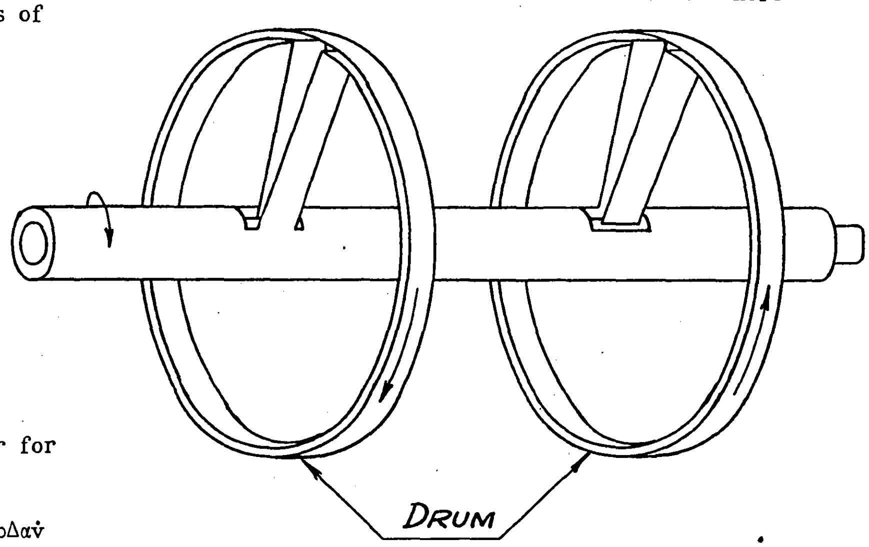
$$F = F_0\, e^{\mu\alpha}.$$

Consequently for a single loop around the drum we have the output tension

$$F_1 = F_0\, e^{2\pi\mu}.$$

Of course, this is on the assumption that the drum revolves faster than the band. If the band and drum move together, the assumption that the friction component of the force of the drum on the band is μf is no longer valid.

In the type of torque amplifier which is suitable for calculating devices, provision must be made for torque amplification in either direction. A commercially available amplifier of this character is the following. The rotating drums are concentric with the control and driven shafts and the bands which are brake bands are interior to the drums. There



is a set of drum and band for each direction. The control shaft can be considered as a hollow cylinder surrounding the solid driven shaft. The brake bands are mounted on a steel strip of spring-like nature. One end of this strip is connected to a projection from the control shaft, the other end to a projection on the driven shaft, which passes through a slot on the control shaft.

Normally each band presses against its corresponding drum a certain amount due to the action of spring. When the shafts are moving together, the corresponding torques cancel. If, however, one shaft is turned relative to the other, then one spring is tightened on itself and the other loosened on itself. The first band is therefore pulled away from its drum and the second is permitted to press on its drum. Since the torque exerted by a brake band is proportional to the radial pressure, the resultant torque exerted by the brake drums is proportional to the angular difference between the shafts. The angular difference in turn is proportional to the torque exerted by the input shaft on the springs. Thus the added torque due to drums is proportional to the input torque and hence this is also a torque multiplier.

3. Electronic amplifiers, of course, are the most common and their use is essential to the accurate operation of calculating devices of an electrical nature. They have the two advantages of being relatively inexpensive and extremely sensitive.

From the point of view of power, there are three ranges of amplifier tubes. The most sensitive types are the "electrometer" tubes. The intermediate range includes the usual radio receiving tubes. The more powerful tubes are the transmitting tubes and the industrial control tubes.

Our discussion will be concerned mainly with receiving tubes. They are definitely the least expensive and the most widely available. In general, they provide adequate amplification range for calculating devices, being more accurate and sensitive than the remaining portions of the circuit.

Electrometer tubes can be described as more accurate and more precise versions of receiving tubes. They are used in delicate physical experiments. High frequency miniature receiving tubes, in general, are similar in principle to ordinary receiving tubes. However, certain high frequency oscillators do work on somewhat different principles which are not utilized in calculating devices as yet so far as we know.

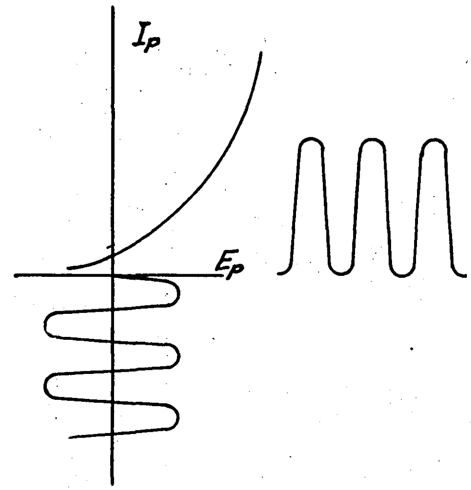
The transmitting tubes at first appear to be larger replicas of receiving tubes. But greater precision is required in their manufacture and frequently they have more desirable characteristics, even in characteristics which one might expect to become worse with size. They are naturally more expensive to purchase and to operate.

Certain tubes developed for television are also larger versions of receiving tubes (although not by any means as large as transmitting tubes). They are only slightly more expensive than receiving tubes and use the same auxiliary equipment. We will consider them as receiving tubes. This, of course, does not refer to oscilloscopic tubes which are only incidently amplifying tubes.

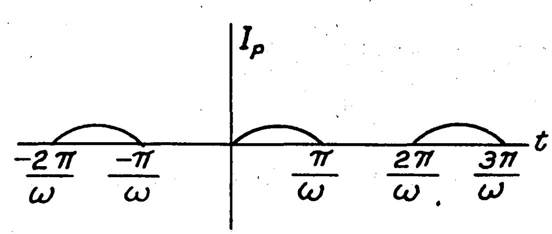
There are forms of industrial control tubes which are simply larger versions of receiving tubes. However there are in addition gaseous types which are particularly suitable for handling large currents, diodes and triodes. In addition there are photosensitive tubes. All these are exceedingly useful as auxiliary equipment in mechanical computing devices. For instance, the difference in rotation between two shafts may be measured by some photometric method which imposes almost no load on the input shaft, and the output of a photometric tube may then be amplified and used to control the other shaft.

4. In the discussion which appears in these notes, we will suppose that the reader is familiar with the *R.C.A. Receiving Tube Manual*, published by the R.C.A. Manufacturing Co. The manual describes the various types of tubes and their uses in radio receiving sets. Many of these uses are immediately applicable to calculating devices, others require a certain amount of reorientation. The manual contains the technical information necessary for the design of circuits and is a remarkable summation of the application of vacuum tubes for radio receiving purposes.

We will begin by considering the diode as a rectifying device. We have already seen how a rectifying device can be used for squaring. In the range in which it is operated, a diode has the property that a voltage drop from plate to cathode will induce a current to flow from plate to cathode, but if the voltage drop is in the opposite direction no current flows. (More precisely in the latter case, there is a minute current flowing against the voltage drop.) Consequently when an alternating voltage is applied across a diode, the current which flows is a pulsating current always in the same direction. Let E_ρ denote the plate voltage relative to the cathode, I_ρ the resulting current. The voltage-current diagram is given by



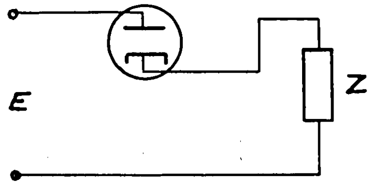
So that if we have E_ρ as function of the time, $E_\rho = A \sin wt$, then I_ρ is a function which assumes only positive values and is largest when E_ρ is largest and has minima when E_ρ has minima. It is important to notice that I_ρ is a periodic function of the time when E_ρ is and hence can be described by a Fourier series. When I_ρ is expressed as a Fourier series, it is possible to estimate the effect on various linear circuit elements. Such estimates are frequently used. Very often, in order to obtain in turn an estimate of I_ρ as a function of the time, assumptions are made concerning the nature of the functional relationship between E_ρ and I_ρ. For instance, one may assume that if $E_\rho < 0$, $I_\rho = 0$ and for $E_\rho \geq 0$, $I_\rho = kE_\rho$. Then if $E_\rho = A \sin wt$, $I_\rho = 0$ if $(2k+1)\ \pi/w \leq t \leq 2k\pi/w$, $I_\rho = k\ A \sin wt$ if $2k\pi/w \leq t \leq (2k+1)\pi/w$ for



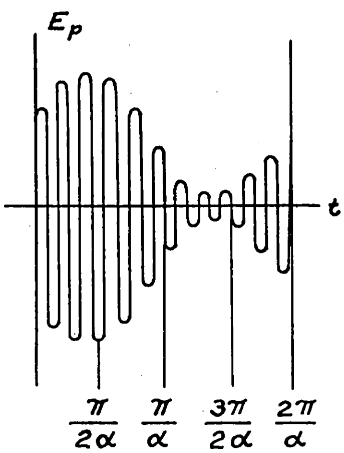
$k = 0, \pm 1, \pm 2, \ldots$. Then by the usual method of finding the Fourier series, we obtain

$$I_\rho = kA \left[\frac{1}{\pi} + \frac{1}{2} \sin wt + \frac{1}{\pi} \Sigma_{k-1}^{\infty} \frac{2}{4k^2-1} \cos (2kwt)\right].$$

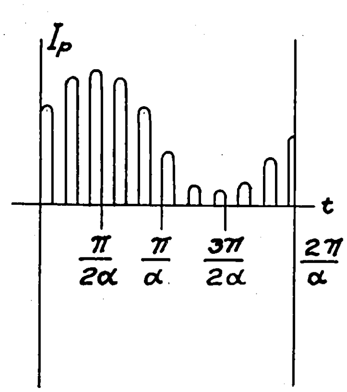
One must admit though that it would probably be better to assume that $I_\rho = k\ E_\rho^{\frac{3}{2}}$ in the case $E_\rho \geq 0$. On the other hand, simplified assumptions such as that given above can be introduced into a general circuit analysis and results on the order of magnitude of the higher frequency components obtained which are frequently very useful. The formula for I_ρ which we have obtained is used in a number of ways. For instance, if a diode is in a circuit then in some cases the assumption is made that the plate voltage is proportional to E when the latter is positive.



But probably graphical methods offer the most direct method of attack on the problem of studying the actions of vacuum tubes. Thus in the most common form of the problem of detection, we have an impressed voltage in the form $E = (A \sin \alpha t + 1) \sin wt$ where w is much larger than α. The problem is to obtain a voltage proportional to $A \sin \alpha t$. $v = w/2\pi$ is the carrier frequency, $A \sin \alpha t$ is the modulation.



If the voltage $E = (A \sin \alpha t + 1) \sin wt$ is applied to the plate of a diode we obtain a current

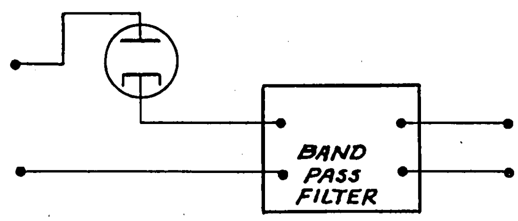


For I_ρ we have a constant component i_0 obtained by averaging. We also have a component $i_\alpha \sin \alpha t$, since it is clear that $\int_0^{\frac{2\pi}{\alpha}} I_\rho \sin \alpha t \, dt$ is not zero. (Notice that I_ρ is largest when $\sin \alpha t$ is positive and least when $\sin \alpha t$ is negative.) There are, of course, components corresponding to sin wt and its harmonics sin nwt. Hence,

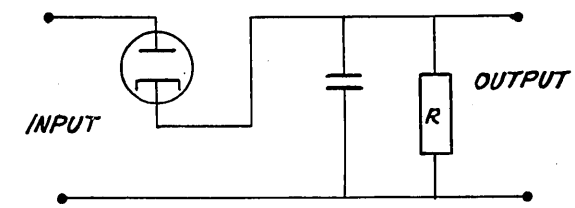
$$I_\rho = i_0 + i_\alpha \sin \alpha t + \Sigma \, i_n \sin (nut + \gamma_n).$$

If we use a band pass filter, we can obtain the component $i_\alpha \sin \alpha t$ and minimize the effect of the higher frequencies and the constant. Notice that the component $i_\alpha \sin \alpha t$ is not present in the applied voltage since $E_\rho = A \sin \alpha \sin wt = \frac{1}{2}A(\cos [w-\alpha] t - \cos [w+\alpha] t)$.

Thus the essential diode detector circuit is



In order to measure accurately the peak voltage of an alternating current, the following circuit is used



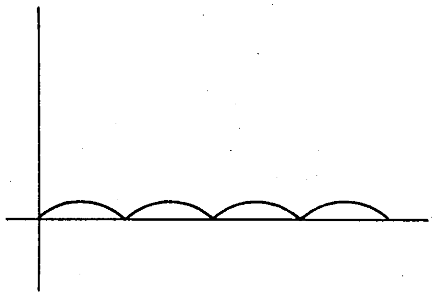
where R is supposed to be large. When an alternating current is applied to the input during the positive peaks, the condenser charges. Only a relatively small amount of the charge leaks away through the resistance. When however the condenser voltage approaches that of the positive peaks, then only as much is charged during a cycle as leaks off through the resistance. Hence, the output potential is a direct current measurement of the peak voltages. If the modulation of a variable signal is to be obtained the same circuit is used if R is large enough to prevent much of a variation during the carrier wave cycle w but not large enough to prevent the modulation cycle from appearing.

Of course, the reader should study the discussion of detection in the manual.

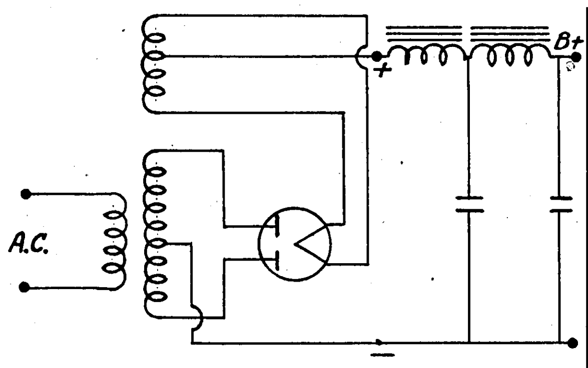
For electronic amplifiers in general, it is necessary to use high voltage direct currents. These are obtained by the use of step-up transformers and diode type rectifying tubes as explained in the tube manual. Making the same assumptions as before for the single diode, we see that the output of the full wave rectifier is in the form shown at the top of the next page and has a Fourier expansion

$$I = 2\,k\,A\,[\frac{1}{\pi} + \Sigma_{k=1}^{\infty} \frac{2}{\pi(4k^2-1)} \cos (2kwt)]$$

where, of course, now w has the value $2\pi 60$. Notice



that in the full wave rectifier the $\frac{1}{2}$ sin wt term has disappeared. It is characteristic of full wave rectifiers that the odd harmonics disappear. In order to minimize the effect of the harmonics, the output of the rectifier is sent through a filter. A typical circuit is shown below. The inductances

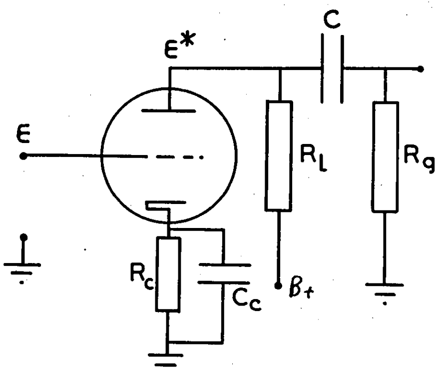


are called "filter chokes" and may have values from 8 to 30 henries, the condensers are usually electrolytic condensers of 8 to 30 microfarads. The inductances have resistances of about 200 ohms. The condensers should be of the correct "working voltage" which is, in general, the peak output voltage of the rectifier. (Sometimes it is higher if the inductances are too low.)

In the case of a choke input filter, the assumption that the output voltage of the rectifier has the form given above for I is reasonable. The reader is urged to calculate the output voltage for the various figures given under the assumption that the output load is a pure resistance of between 5 and 20 thousand ohms.

5. Essentially the two uses of a vacuum tube are 1) rectification, i.e., the process of obtaining a pulsating one directional current from an alternating voltage, and 2) amplification, the control of power by less power. We have discussed the first and we will not consider the second.

We assume that the reader is familiar with the usual discussion of the action of triodic and multielectrode vacuum tubes. We wish to point out certain aspects of amplifying circuits which are important from our point of view. Amplifiers can be roughly divided into two kinds. One type is a voltage multiplier and the other is a power multiplier proper. We begin by considering a simple resistance coupled amplifier such as those given in the tube manual. The effect of a change in value of



the input voltage, ε, causes a variation in the voltage between the grid and the cathode. This causes a change in the current flowing between plate and cathode. As far as the right side of the circuit is concerned the tube acts like a current generator, corresponding to this variation. Since the purpose of the present circuit is voltage amplification, this current variation is permitted to cause a voltage variation. When the current variation alone is considered, the left-hand portion of the circuit is a two terminal network, consisting of two parallel impedances between the plate and ground. [In the triode, at high frequencies, there are two other paths to ground. One is through the plate grid capacity, the other is through the plate-cathode capacity and cathode circuit to ground. Besides these, in the pentode there is plate-suppressor capacity path and plate-screen and screen circuit paths. Proper shielding should eliminate other paths.] The impedance between plate and ground is called the load impedance. In the circuit shown

$$Z_L = \frac{R_L\,(R_g - i/wC)}{R_L + R_g - i/wC}$$

The variation ι in the current flowing through this impedance produces a voltage $\varepsilon_\rho = Z_L\,\iota$. The right-hand side of the circuit is a voltage divider (the purpose of the condenser C is to block the d.c. plate voltage from ground). Thus the output

$$\varepsilon_o = [R_g/R_g - i/wC)]\ \varepsilon^*$$

where ε* is the voltage of the plate relative to ground.

Now let ε denote the input voltage
ε_g the variation in grid cathode voltage
ε_ρ the variation in plate cathode voltage
ε_o the output voltage, Z_L, ε* and i as above. Since we are dealing with small changes, we may write

$$\iota = \frac{\partial I_\rho}{\partial E_g}\,\varepsilon_g + \frac{\partial I_\rho}{\partial E_\rho}\,\varepsilon_\rho .$$

$\frac{\partial I_\rho}{\partial E_g} = g_m$ is the transconductance (as defined in the tube manual) and $\frac{\partial I_\rho}{\partial E_\rho}$ is the reciprocal of the plate resistance r_ρ. Hence, $\iota = g_m\,\varepsilon_g + (1/r_\rho)\,\varepsilon_\rho$.

Looking at the grid circuit we obtain

$$\varepsilon = \varepsilon_g + Z_c\,\iota, \quad \varepsilon^* = \varepsilon_\rho + Z_c\,\iota$$

where $Z_c = \dfrac{R_c}{1 - iwCR_c}$. Eliminating ε_g and ε_ρ we obtain

$$\iota\,[1 + (g_m + 1/r_\rho)\,Z_c] = g_m\,\varepsilon + (1/r_\rho)\,\varepsilon^*.$$

We also have $-\,\iota\,Z_L = \varepsilon^*$ and hence

$$-\,\varepsilon^*\,[(1 + [g_m + 1/r_\rho]\,Z_C)/Z_L + 1/r_\rho] = g_m\,\varepsilon.$$

Since we have the output voltage

$$\varepsilon_0 = [R_g/(R_g - i/wC)]\,\varepsilon^*$$

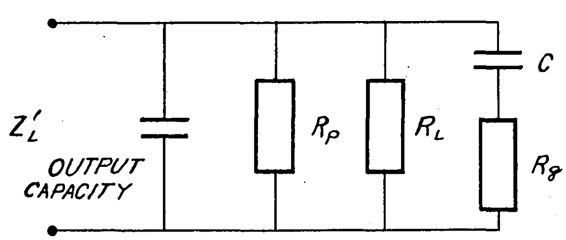
we have $\varepsilon_0 = -\alpha\varepsilon$ where

$$\alpha = (1 - i/R_g wC)\,g_m/[(1 + [g_m + 1/r_\rho]Z_C)/Z_L + 1/r_\rho].$$

Now $1/[(1 + [g_m + 1/r_\rho]Z_C)/Z_L + 1/r_\rho]$ is an impedance Z_L'. We choose the by-pass condenser C_C so large that $(g_m + 1/r_\rho)\,Z_C$ is negligible. Then we may write that

$$Z_L' = \frac{Z_L\,r_\rho}{r_\rho + Z_L}.$$

Z_L' is, of course, the inductance that one would get if one considered r_ρ as parallel to Z_L. For complete generality one could also consider the "output capacity" as parallel to Z_L and introduce it into Z_L'. Thus Z_L' is an impedance



and the formula one obtains is simply

$$\varepsilon_0 = \left[\frac{-\,g_m}{1 + \frac{1}{w^2C^2R_g^2}}\right] Z_L'\,\left(1 + \frac{i}{wCR_g}\right)\,\varepsilon$$

or

$$\varepsilon_0 = -\alpha\varepsilon$$

where

$$\alpha = \frac{g_m Z_L'}{1 + \frac{1}{w^2C^2R_g^2}}\,\left(1 + \frac{i}{wCR_g}\right)$$

is the actual amplification. This is not the "amplification factor" $\mu = r_\rho g_m$ which is a limiting value for α.

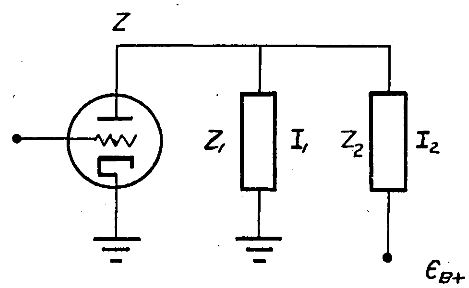
In triodes one generally chooses R_L considerably larger than r_ρ for voltage amplification but in pentodes r_ρ is larger in general. From the formula

$$\frac{1}{Z_L'} = +i\,w\,C_{\mathbf{output}} + \frac{1}{r_\rho} + \frac{1}{R_L} + \frac{1}{R_g - i/wC}$$

we notice that in general it is the smaller of r_ρ and R_L which determines the situation. R_g is generally quite large although its size may be restricted by the character of the output.

In the applications to mathematical machines, it is very frequently desirable to have α as close to being real as possible. This would incline one to take CR_g large and under certain circumstances this will work. However, one should notice that the output grid circuit of C and R_g is a time delay circuit with time delay CR_g (cf. Section 3 of the preceding chapter). If CR_g is too large the transients will persist too long.

The above discussion was carried through on the assumptions that ε_g and ε_ρ were small enough so that the differential of the plate current represents the change ι. If Z'_L is essentially real, the characteristic curves can be used to determine the amplification, as explained in the tube manual, by means of a load line. For if we consider the circuit



we see immediately that

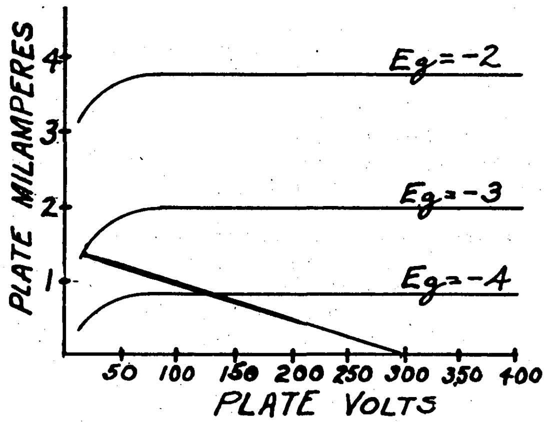
$$\begin{aligned}\varepsilon_{B+} - \varepsilon_\rho = Z_2\iota_2 &= Z_2(\iota + \iota_1) \\ &= Z_2\iota + (Z_2/Z_1)\,\varepsilon_\rho\end{aligned}$$

or

$$\varepsilon_{B+} = Z_2\iota + (1 + Z_2/Z_1)\,\varepsilon_\rho.$$

Now if we have a system of coordinates in which ε_ρ and ι are measured along the x and y axis respectively, then this corresponds to a straight line with slope $-\,(Z_1+Z_2)/Z_1Z_2 = -\,1/Z_L$ and whose x intercept is $[Z_1/(Z_1+Z_2)]\,\varepsilon_{B+}$. Now if we take a plot of the characteristic curves for different grid voltages and draw this line on it, then we know that ι and ε_ρ can only vary along this line. Thus the operating point for a specific value of ε_g is obtained by taking the intersection of this line with the characteristic curve for the value of ε_g. For example, consider the diagram for a pentode which is shown at the top of the next page.

Let $\varepsilon_{B+} = 330$ volts and the ratio $Z_1/(Z_1 + Z_2) = .9$ so that the voltage intercept is 300 volts. Suppose $Z_L' = .2$ megohms $= 2 \times 10^5$ ohms. This is, of course, reciprocal of the slope in an ampere-volt scale.

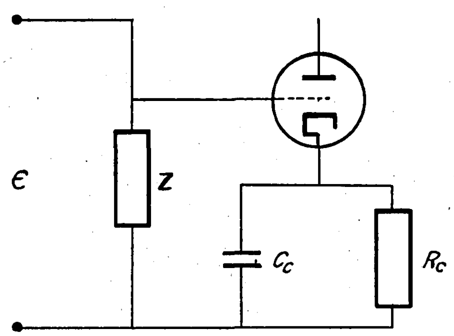


If we wish to use a system in which the unit of current is .001 amperes, then the unit of resistance is 1000 ohms. Hence in this system $Z = 2.10^2 = 200$ units. If we wish our voltage unit to be 50 volts we must give Z the value 4. (Another way of getting the above is to notice that in a milampere 50 volt system the unit of resistance is 50,000 ohms.) Consequently the slope is $\frac{1}{4}$ and we draw this line.

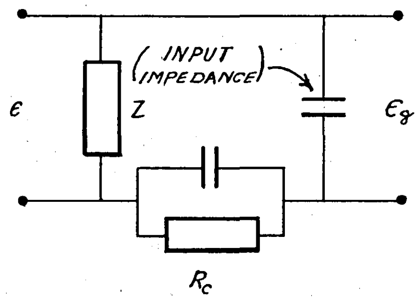
In the above discussion of voltage amplifiers we have kept in mind the situation in which the total load impedance Z_L' is mainly resistive. However if we use instead of R_L a general impedance, then α can of course be made selective; for instance, to have a maximum at a certain frequency w. Since

$$\alpha = \frac{g_m Z_L'}{1 + \frac{1}{w^2C^2R_g^2}} \quad (1 + \frac{i}{wCR_g})$$

it should be clear how this is to be done. This gives "selective amplification" which is essential in radio sets. Another place where selection may be introduced is in the grid circuit. The grid circuit may be shunted by an impedance Z. This yields

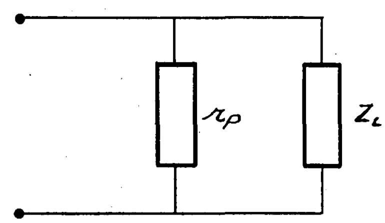


the equivalent of a filter circuit between ε and ε_g when one considers that the relationship between the grid and cathode is similar to the plates of a condenser and hence one has a capacitance connection (the "input capacity") between them. For audio



frequencies, the input capacity is negligible but very important in radio and higher frequencies. This also holds for the output capacity which we have entered into Z_L' in the above.

Of course, very high frequencies are not used in calculating devices at present. But there is a case where R_L may be replaced by a general impedance which is of interest to us and that is in power amplification. Our formula $\varepsilon = -\alpha\varepsilon$ does, of course, give the dependence of the output voltage on the input voltage in general. However in power circuits, the part of the circuit containing C and R_g is omitted and the output capacity neglected. Thus Z_L' consists simply of r_ρ and the load imped-



ance in parallel and $\alpha = g_m Z_L'$. Thus $\varepsilon_o = g_m Z_L'\varepsilon$. Thus the variation in the current through Z_L is given by $\iota = \varepsilon_o/Z_I = (g_m Z_L'/Z_L)\varepsilon$ and it is this current alone that can be used.

6. If the amplification of a circuit containing only one vacuum tube is inadequate the output can in turn be amplified by another such circuit and so on. Thus, if we have three such stages, the final output is

$$\varepsilon_o = -\alpha_1 \alpha_2 \alpha_3 \varepsilon_i.$$

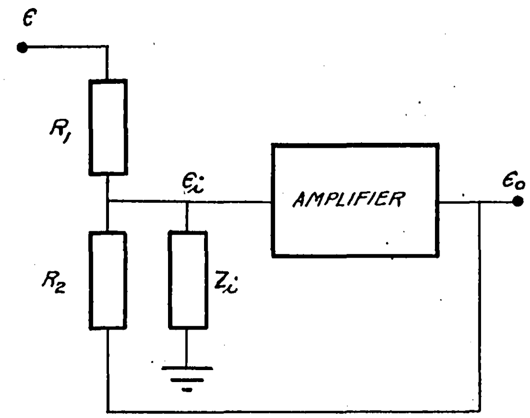
However, in a calculating device the accuracy of reproduction must be maintained in amplification. Electronic amplification is subject to three hazards: 1) Variations in characteristics of the tubes, 2) non-linearity in the characteristics of the tube, and 3) noise.

By the use of feedback amplifiers, 1 and 2 can be improved without making 3 worse. Consider the following circuit for the moment. We suppose that the amplifier is such that

$$\varepsilon_o = -\alpha \varepsilon_i.$$

We also have by Kirchhoff's Law on Currents

$$\frac{\varepsilon - \varepsilon_i}{R_1} + \frac{\varepsilon_o - \varepsilon_i}{R_2} = 0$$



if there is no input current flowing into the amplifier. Thus if we eliminate ε_i, we obtain

$$\varepsilon_0 = \frac{-\ R_2\ \alpha}{R_1\alpha + R_1 + R_2}\ \varepsilon.$$

We see then that this circuit can be regarded as an amplifier with amplification

$$\alpha' = \frac{R_2\ \alpha}{R_1\alpha + R_1 + R_2}.$$

We may write this

$$\alpha' = \frac{R_2}{R_1}\left(\frac{1}{1 + \frac{1}{\alpha}\left(1 + \frac{R_2}{R_1}\right)}\right)$$

which shows that for α large relative to $1 + R_2/R_1$, α' is essentially R_2/R_1. Thus the amplification depends upon the passive elements of the circuit under these circumstances rather than α. In fact, differentiation yields the relationship

$$\frac{d\alpha'}{\alpha'} = \frac{R_1 + R_2}{R_1\alpha + R_1 + R_2}\ \frac{d\alpha}{\alpha}$$

which shows that a percentage change in α yields a percentage change in α' which is diminished by a factor $\frac{R_1 + R_2}{R_1\alpha + R_1 + R_2}$, which is essentially the same factor as that by which the amplification has been reduced.

The tube characteristics upon which α depend can vary due to many causes; for instance, age and variations in the power supply. With ample feedback, however, the amplification does not vary greatly.

The non-linearity of the tube characteristics can be considered as varying α, making it a function of ε_i and this effect is reduced since variations in α are made less effective. Notice also that ε_i is small if α is large and this will also improve linearity.

A noise is also reduced in the same ratio as the amplification. Suppose the amplifier has an output

$$\varepsilon_0 = -\alpha\ \varepsilon_i + \zeta$$

where ζ originates in the amplifier. Eliminating ε_i as before by using the current equation, we obtain

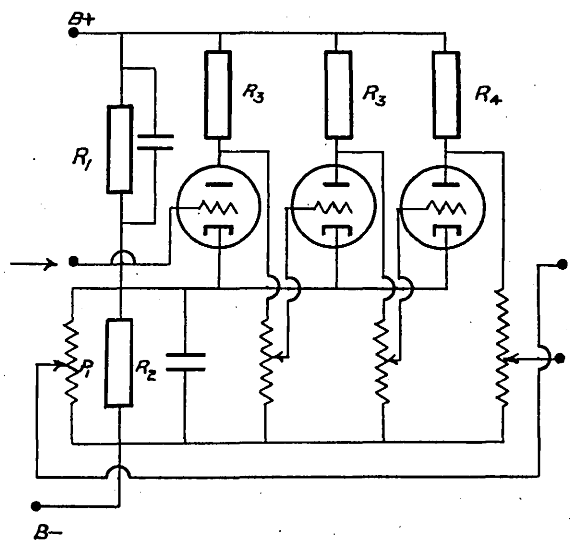
$$\varepsilon_0 = \frac{-\ R_2\ \alpha}{R_1\alpha + R_1 + R_2}\varepsilon + \frac{R_1 + R_2}{R_1\alpha + R_2 + R_1}\ \zeta.$$

Thus the new ζ' for the complete circuit is

$$\zeta' = \frac{R\alpha + R_2}{R_1\alpha + R_1 + R_2}\ \zeta.$$

Thus ζ' has been diminished by essentially the same factor as the amplification and hence the noise is not relatively any worse than before.

It is, of course, desirable in feedback amplification that the amplification factor be real. That is why that in these circuits, the output is obtained by voltage division rather than by a blocking condenser. The circuit for three triode units would look like this (for the amplifier). The first

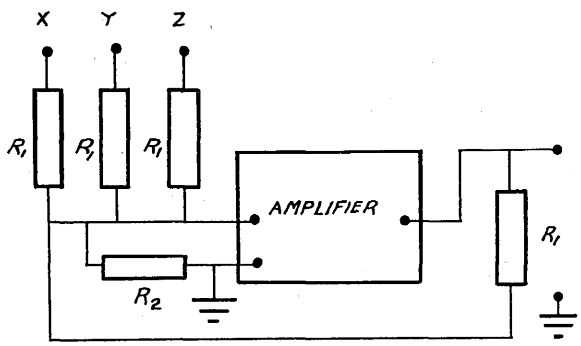


stage should be modified to minimize drift. This will also provide bias relative to ground. Confer the article by Y.P. Yu in *Electronics*, Vol. 19, No. 8 (August 1946), pp. 99-103. Relative to the output, this device is a two-terminal voltage source, whose value depends on the input voltage. The internal resistance can be calculated by considering the evident bridge circuit.

In general, however, the capacity of the tubes used in such a circuit will introduce a phase shift at each stage, and at a high enough frequency this phase shift for the three stages may add up to a regenerative feedback, which will cause such a circuit to oscillate. This can be avoided by a more sophisticated design of the stages and feedback circuit. Cf. Bode, H. W., *Network Analysis and Feedback Amplifier Design*. New York: D. Van Nostrand Company, 1945.

These circuits are used as power amplifiers when the amplification factor is one. As amplifiers, these circuits can be regarded as connecting calculating circuits. However, there has also been developed a technique in which they are used as part of calculating devices. The requisite power amplification is still obtained but the accuracy can be considerably improved.

For instance, if it is desired to add the voltages x, y and z, one has the connection



One can show as in the above that if $\varepsilon_o = \alpha\varepsilon_i + \zeta$, we have by Kirchhoff's Law of Current that

$$\frac{1}{R_1}(x+y+z - [4+R_1/R_2]\ \varepsilon_i+\varepsilon_o) = 0$$

or

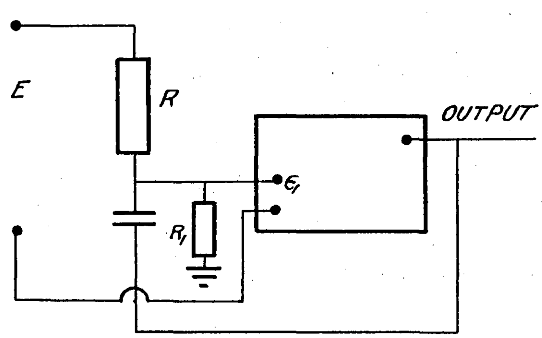
$$\varepsilon_i = \frac{1}{4+\alpha+R_1/R_2}(x+y+z+\zeta).$$

Now when we substitute in the expression for ε_o, we find that the terms in ζ almost cancel each other and hence

$$\varepsilon_o = \frac{-\alpha}{4+\alpha+R_1-R_2}(x+y+z) + \frac{4+R_1/R_2}{4+\alpha+R_1/R_2}\zeta.$$

This is typical of the feedback theory. Even when load variations occur, the output is not essentially changed. For a load variation essentially corresponds to changes in the output circuit. Hence α is changed. But as long as α is large, this has only a slight effect on the output ε_o. There is also an improvement in linearity due to the fact that the signal impressed on the grid is much smaller.

This idea can be used in many ways. For instance, we can eliminate the voltage drop across the condenser from the integrating circuit to a certain extent. Consider the following circuit:



Let J denote the voltage across the condenser. Let R_1 denote the resistance between the input and the ground in the amplifier (sometimes this may be neglected). The current i through the resistance R is easily seen to be

$$i = C\frac{dJ}{dt} + \varepsilon_1/R_1$$

where ε_1 is the input voltage relative to ground. Now

$$Ri + \varepsilon_1 = RC\frac{dJ}{dt} + \varepsilon_1\ (R + R_1)/R_1.$$

If β is the effective amplification of the circuit

$$J = (1 + \beta)\ \varepsilon_1.$$

Hence

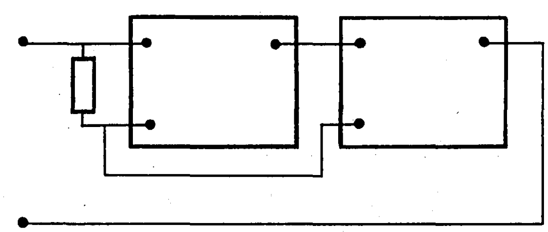
$$E = RC\frac{dJ}{dt} + \frac{R + R_1}{R_1\ (1+\beta)}J.$$

Without the amplifying circuit the equation would be

$$E = RC\frac{dJ}{dt} + J$$

The reader is urged to solve the two equations and compare the result.

It is also possible to compensate for the resistance losses in a circuit by using two feedback circuits. An extra resistance is inserted in the circuit. The voltage drop across this is a certain fraction of the total resistance voltage drop. One amplifier changes the sign, the other reverses it again so that the output voltage is proportional to the input. The gain is adjusted so that the new voltage introduced into the circuit by the amplifiers is proportional to the resistance loss.

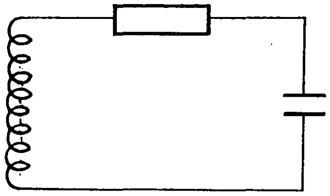


Thus we have methods for artificially changing the capacity and resistance of a circuit. The inductance can be similarly varied.

7. In conjunction with a tuned circuit, a vacuum tube may be used to produce an alternating voltage. The associated current can be amplified by means of other vacuum tube stages into larger currents capable of supplying reasonable power demands. This offers a very convenient method of producing alternating current of a wide range of frequency.

Let us consider for a moment a tuned circuit and let us suppose that at an instant t_o we have a charge q_o and a current i_o. We have, of course, the equation

$$\frac{q}{c} + Ri + L\frac{di}{dt} = 0$$



for the charge and

$$\frac{i}{c} + R\frac{di}{dt} + L\frac{d^2i}{dt^2} = 0$$

for the current. If we solved the first for q, we have that

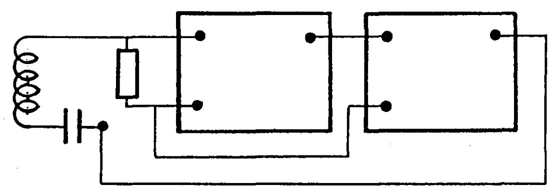
$$q = A e^{-\frac{R}{2L}t} \cos (wt + \gamma),\quad w = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and that

$$i = - A\left(\frac{R}{2L} \cos [wt+\gamma] + w \sin [wt+\gamma]\right)e^{-\frac{R}{2L}t}$$

where A and γ are determined by the conditions that $t = 0$, $i = i_o$, $q = q_o$.

Now it is clear from these expressions that the current in this circuit will die out exponentially. But we have seen in the previous section that we can change the apparent resistance of a circuit by using amplifiers. Thus if we have two stages, the circuit would be:



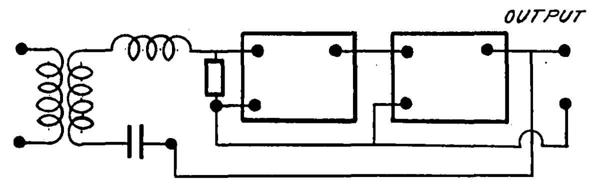
Now if the feedback were adjusted properly, the voltage drop in the circuit which is proportional to the i would be zero or positive. The resulting current would have an exponential term with a non-negative exponent and hence the current would not die out. The circuit is then said to be oscillating.

There are a great variety of oscillating circuits. For instance, a one-tube oscillator can be constructed using a transformer to reverse the direction of the output. This does not yield a perfect result but it is entirely adequate to produce oscillation. The reader is urged to study the descriptions of various audio oscillators which one finds in the *Radio Amateur's Handbook* and in the textbooks on radio.

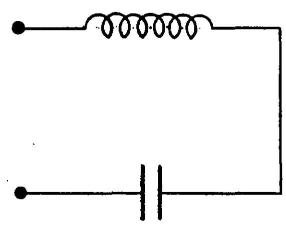
Our immediate concern is with a tuned circuit which is essentially an inductance and a capacitance. We wish to study its actions when an alternating current is impressed upon it. For definiteness let us take a circuit of the following sort where the amplifiers are supposed to compensate for the resistance and at the same time power the output. Notice that since $R = 0$, the "natural" frequency

$$w = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

and the frequency for minimum impedance $w = 1/\sqrt{LC}$ are the same.



Such a circuit can be used as a selective alternating current integrator. Theoretically it is the precise generalization of the direct current integrator of the previous section based on a condenser. Let us consider what happens when n cycles of a voltage $\alpha \sin w_1 t$ is impressed on such a circuit. We suppose that the turns ratio of the transformer is one. Thus we have a voltage $\alpha_1 \sin w_1 t$ impressed on a circuit which is essentially



Thus we have the differential equation

$$\alpha_1 \sin w_1 t = \frac{1}{C} q + L\frac{d^2q}{dt^2}$$

which may be written

$$\frac{\alpha_1}{L} \sin w_1 t = w^2 q + \frac{d^2q}{dt^2}.$$

For $w \neq w_1$, the solution which has $q = 0$, $i = \frac{dq}{dt} = 0$ for $t = 0$ is

$$q = \frac{\alpha_1}{L(w^2 - w_1^2)}[\sin w_1 t - (w_1/w) \sin wt)].$$

For this we have

$$i = \frac{\alpha_1 w_1}{L(w^2 - w_1^2)} (\cos w_1 t - \cos wt).$$

On the other hand, for $w_1 = w$ we have

$$q = \frac{\alpha_1}{2Lw}\ [- t \cos wt + (1/w) \sin wt]$$

and

$$i = \frac{\alpha_1 t}{2L} (\sin wt).$$

It is clear from the above that if $w \neq w_1$, the response of the circuit to the signal is limited in magnitude, while the response to a signal of frequency w will continue to increase in size as long as the signal is applied. After the signal is removed, the circuit will satisfy the equation

$$0 = w^2 q + \frac{d^2 q}{dt^2}.$$

Now if we have k cycles of the natural frequency imposed, the charge and current will be at the end of this time, i.e., for $t = 2k\pi/w$,

$$q = \alpha_1 k\pi/Lw^2 = \alpha_1 k\pi C$$

and $i = 0$ respectively. Let us suppose that from then on no voltage is applied to the input. The charge from then on must satisfy the new differential equation and hence

$$q = C\alpha_1 k\pi \cos wt.$$

Now let us suppose that the amplitude of the impressed voltage is slowly varying. In this case the impressed e.m.f. can be considered as made up of a sum of functions

$$E = \Sigma_{l=1}^{N} \alpha_l e_l(t)$$

where $e_l(t)$ is defined by the conditions $e_l(t) = \sin wt$ for $(l-1)2\pi/w < t \leq 2l\pi/w$ and $e_l(t) = 0$ otherwise. For each $e_l(t)$ we have a current $i_l(t)$, which is 0 for $t \leq (l-1)2\pi/w$, $i_l(t) = (\alpha_l\ [t-(l-1)2\pi/w] \sin wt)/2L$ for t between $(l-1)2\pi/w$ and $2l\pi/w$ and such that $i_l(t) = -C\alpha_l w\pi \sin wt$ for $t > 2l\pi/w$. It is easily seen that $i_l(t) = \frac{d}{dt} q_l(t)$ is such that

$$e_l(t) = \frac{1}{C} q_l + L \frac{d^2 q}{dt^2}.$$

Hence if $q(t) = \Sigma_{l=1}^{N} q_l(t)$, then

$$E(t) = \frac{1}{C} q + L \frac{dq}{dt}.$$

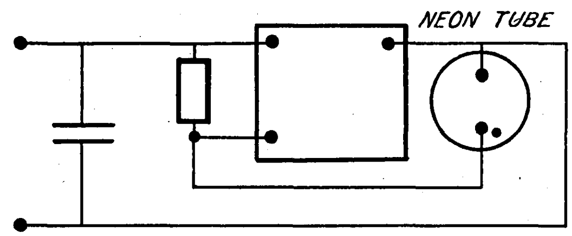
It follows that for $t \geq 2\pi N/w$, the current i has the value

$$i = - Cw\pi(\Sigma_{l=1}^{N} \alpha_l) \sin wt.$$

Hence if α_l is regarded as a function of its subscript the $\Sigma_{l=1}^{N} \alpha_l$ is approximately an integral.

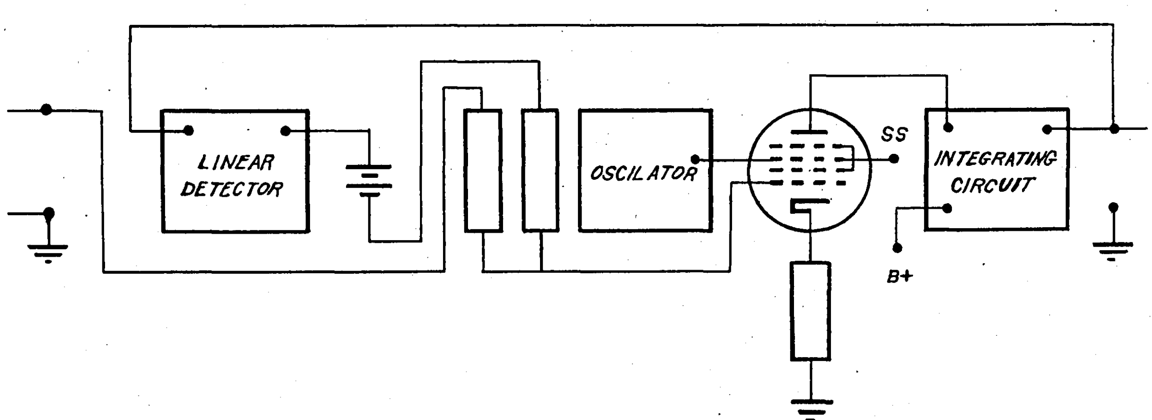
This arrangement has the advantage and disadvantage of a relatively short time base. Of course, one can never precisely compensate for the resistance of a circuit and to prevent self oscillation, a certain amount of resistance must be left in the circuit. This introduces a decay term in the output $e^{-\frac{R}{2L}t}$, which has an effect similar to the decay term in a condenser circuit.

The short time base can be compensated for by the use of an auxiliary counting circuit. This circuit has the property that whenever the integral reaches a certain value, it essentially shorts the condenser. Thus the condenser is replaced by the



above. Normally the amplifier circuit behaves just like a high resistance. But when the voltage reaches a certain value, the neon tube discharges and the circuit is briefly then a relative short across the condenser.

Alternately a "high Q" coil can be used and the capacity artificially raised (as in the previous section) to bring the frequency down to audio frequencies. (Q is the ratio of the total energy in such a circuit to the energy dissipated in a cycle. For the frequency of resonance f it is essentially 2Lf/R.)



An integrating circuit can also be used for modulation, i.e., to control the amplitude of an alternating current by a direct current (or one of lower frequency). The foregoing diagram is for the customary form of modulation in which zero corresponds to a certain value of the amplitude, larger values of the amplitude correspond to positive signals and smaller values of the amplitude correspond to negative signals. For simplicity a battery is shown to indicate the bias on the output of the linear detector.

When the input signal is zero, the integrating circuit should produce the amplitude corresponding to zero. For at this amplitude of input, the detector gives an output just adequate to maintain the other end of the battery zero. Hence, the signal grid of the mixing tube permits only a certain amount of signal to enter the circuit. This signal has a frequency determined by the oscillator. This zero signal should be considered as compensating for a certain amount of resistance in the integrating circuit.

It is clear that for every other value of the input voltage there is corresponding amplitude which will give an even output for the integrating circuit. If the output amplitude varies from the equilibrium value, a signal is sent to the grid of the mixing tube, which varies the input of the mixing tube. Thus if the output is too large, the output of the linear detector is depressed, thus lowering the signal on the signal grid of the mixer. This lowers the input to the integrating circuit which in turn eventually depresses the output.

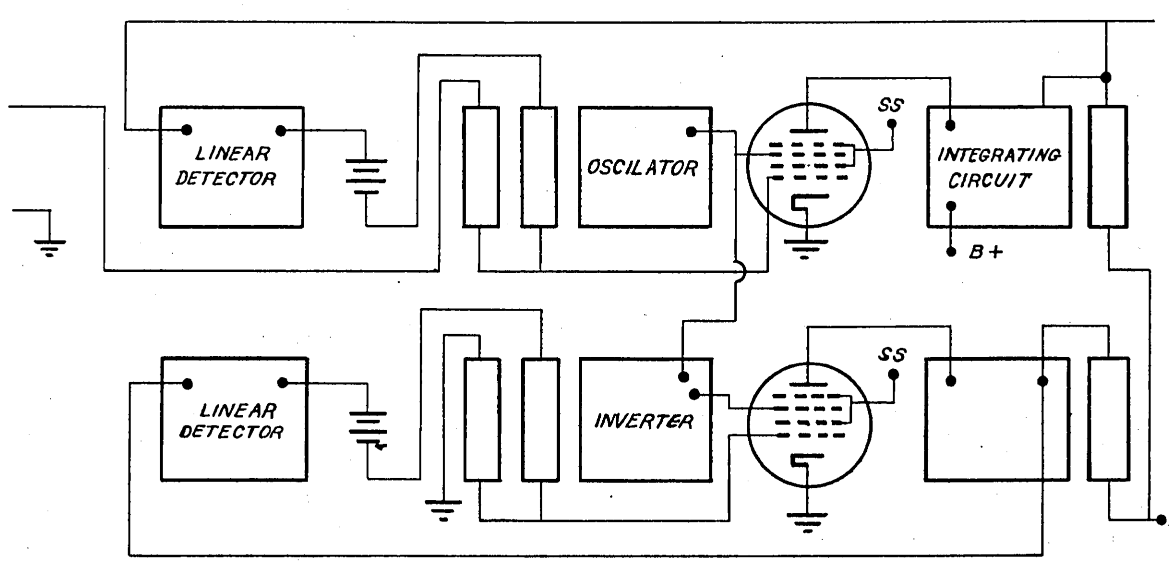
In this the resistance compensation in the integrating circuit is not critical. Much of it can be obtained from the zero input situation.

To obtain phase modulation for the amplitude of the output, i.e., if negative signals differ from positive ones by 180° in phase, it is necessary to add to the output, a signal of the same frequency differing in phase by 180° and with the amplitude corresponding to the zero signal. One way of doing this is to use two circuits similar to the diagram, bottom of page II - 51, but differing relative to the oscillator. The same oscillator is used in both circuits but the phase is inverted before being applied to the second circuit. Phase inversion is, of course, immediately obtainable from a single tube.

It should be mentioned that the above is only a suggestion. The problem of accurately modulating an alternating current so that the amplitude accurately represents a quantity is, of course, a problem of greatest importance if alternating current calculating devices are to be developed. In books on radio transmitters, the problem of modulation is discussed and a number of ways of doing this are given. Thus the plate voltage of a tube may depend upon the signal while the grid varies with the carrier frequency or the voltage on the suppressor of a pentode may depend on the signal. The use of a "mixer" tube like the 6 L 7 seems logical as suggested above. However, it seems clear that in every case, the output must be detected linearly and fed back to control the signal on the mixer.

For modulation circuits, the reader is referred to the current *Radio Amateur's Handbook* which will have sections on transmitters. (The handbook is a yearly publication of the American Radio Relay League, West Hartford, Connecticut.) Certain issues of the 1945 edition do not contain advertisements which is an unfortunate omission. The advertisements are a very important part of the book.

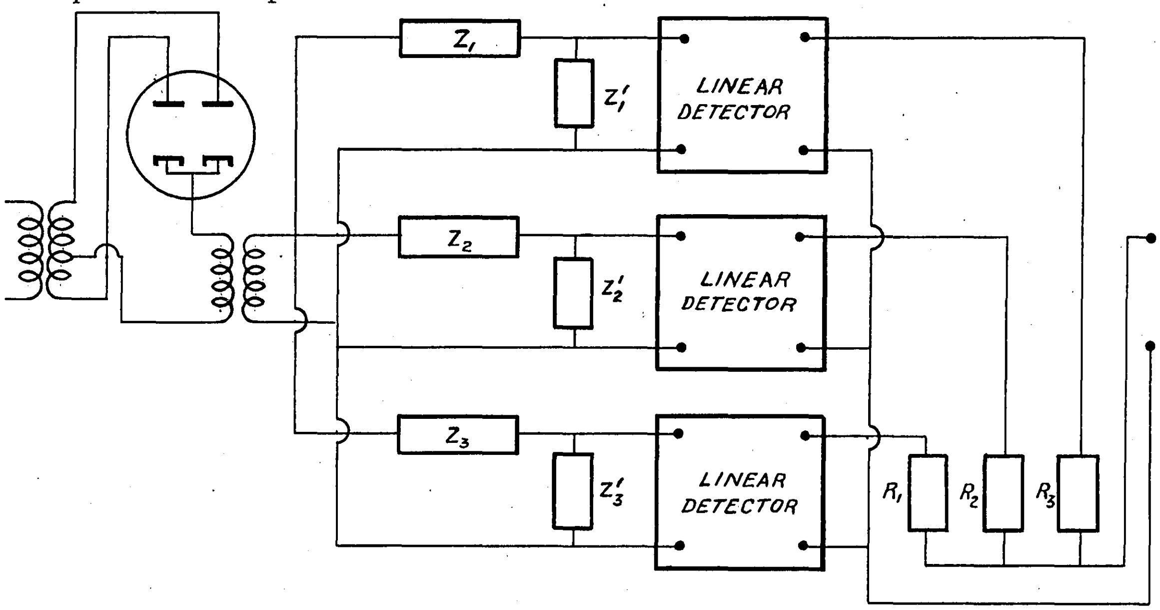
Other references are: "R.C.A. Air-Cooled Transmitting Tubes," *Technical Manual TT3*. R.C.A. Manufacturing Co., Inc., Harrison, N.J. Also by the same company: *R.C.A. Guide for Transmitting Tubes*.



8. For the purpose of electrical multiplication, it is very desirable to have a rectifier whose output is a direct current or voltage which is, to a good approximation, proportional to the square of the amplitude of the impressed alternating current. It is not particularly difficult to compensate to a certain extent for the departure of the output of a diode from a square form. We wish to discuss this question in the present section.

Notice that $A_2 - 4A_4 + 9A_6$, under the above assumptions, is a parabola and in general it would be parabolic except possibly for terms of the eighth degree.

By the use of filter circuits, it is possible to obtain such a combination. Consider the circuit where full wave rectification is used to eliminate



Let us consider again the characteristic of the diode, i.e., the curve which expresses the plate current as a function of the plate voltage.

$$I = f(E).$$

Let us suppose that within a region with which we are concerned it is possible to choose a point E_0 such that

$$I = f(E_0+h) = a_0 + a_1h + a_2h^2 + a_3h^3 + a_4h^4 + a_5h^5 + a_6h^6$$

to a sufficiently good approximation. Now if we let $h = e \sin wt$ and express the various powers of $\sin wt$ in terms of the harmonics we have

$$I = A_0 + A_1 \sin wt - A_2 \cos 2wt - A_3 \sin 3wt + A_4 \cos 4wt + A_5 \sin 5wt - A_6 \cos 6wt$$

where

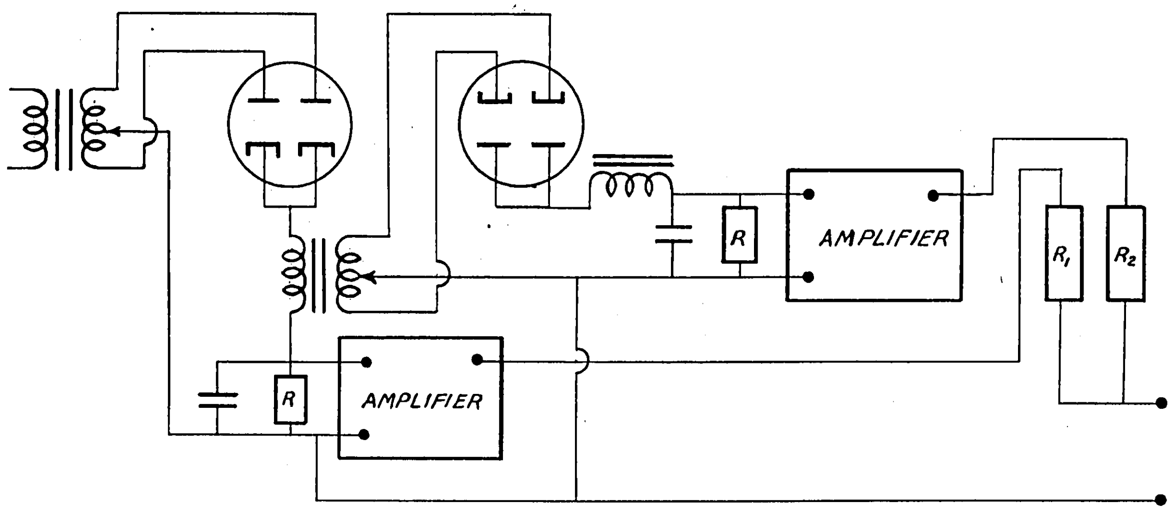
$$A_0 = (a_0 + a_2 e^2/2 + 3a_4 e^4/8 + 5a_6 e^6/16)$$
$$A_1 = (a_1 e + 3a_3 e^3/4 + 5a_5 e^5/8)$$
$$A_2 = (a_2 e^2/2 + a_4 e^4/2 + 15a_6 e^6/32)$$
$$A_3 = (a_3 e^3/4 + a_5 e^5/2)$$
$$A_4 = (a_4 e^4/8 + 3a_6 e^6/16)$$
$$A_5 = a_5 e^5/16, \quad A_6 = a_6 e^6/32.$$

Now in general for a diode, the a_i's are positive. Thus A_0 is somewhat similar to a parabola but has a tendency to increase too quickly for larger values of e. (It will be convenient to assume $a_0 = 0$ in our discussion). Notice that A_2 is similar while A_4 is also an even function beginning with e^4. A_6 is also an even function.

the odd harmonics. The combination of impedances Z_1 and Z_1' is intended to form a selective filter for the 2 w frequency, i.e., Z_1 is a minimum, Z_1' is a maximum for this frequency. Similarly Z_2 and Z_2' and Z_2 and Z_3' are selective filters for 4w and the 6w frequencies respectively. The resistances R_1, R_2 and R_3 are chosen to yield the desired linear combination of the output voltages or perhaps better, the empirical combination that works best. Presumably, the linear detector for the 4w frequency should yield a voltage opposite in direction to that of the other two. In the case of a diode detector, this is easily arranged by interchanging the plates and cathodes.

Of course, on occasion sufficient accuracy may be obtainable from two frequencies. Another possibility is to use A_0 as the main term and introduce a correction obtained by rectifying the even harmonics. (See diagram, top of next page.) The plates and cathodes are oppositely connected in the diodes. This gives a negative output. Inverting both diodes will give a positive output. Both are desirable in multiplication. In view of the transformer coupling the input can be amplified considerably before being applied to the device. The transformer can then have a considerable step-down ratio. R_1 and R_2 can be varied for best empirical results. The amplifiers can be used for resistance elimination or only one may be used on the output side of the device.

The study of accurate electrical squarers is



extremely important for the development of electrical calculators. Of course, the above refers to any rectifier. In particular, tubes of other types may be effectively used.

In connection with multipliers two further facts may be mentioned. One of these is remote cut-off tubes in general have a characteristic in the form $I = \exp(k_1 E_g + k_2)$ which can be used with feedback to obtain a logarithmic output. This offers a simple way of multiplying positive quantities.

Also in the case of pentodes inputs can be impressed on different grids and if a suitable portion of the characteristic is chosen the product can be obtained. Also in a "push-pull" yoked pair of pentodes, the difference between the screen voltages can be one input and the control grids connected to the other input. The suppressor grids can also be used instead of the screen.

V. The Representation of a Function of One Variable

1. In our previous discussion, we have pointed out a number of ways in which a function of one variable may be realized. Thus a potentiometer may be used to yield a voltage which is a function of a rotation, cf. Chapter II, Section 5, or groove cam or a cam with linear follower may be used to yield a displacement which is a function of a rotation (Chapter II, Section 7). In Section 8 of Chapter II, we have seen how a rotation can be expressed as a function of a rotation by means of geared cams and in Section 9 by means of a pin cam. Although we have not discussed it, one way of representing a function in digital machines is by a set of cards on each of which is the difference of two successive values of the function. The values on the card are added as the independent variable changes.

In connection with the new digital electronic computers, "memories" of various sorts have been introduced. These, of course, can be used to hold a function table but the serial type memory can be used to represent a function for use in devices whose output appears on an oscillograph. For instance, a magnetic wire memory consists of a wire, which is moving rapidly through a reading head, which responds to the variations in the magnetic state of the wire. To maintain a periodic function, we can impress the signal which corresponds to the value of the function on another point on the wire which will enter the reading head one period later. Such a function could be depicted on an oscillograph. If we wish to modify the function with time, as we may in an adjuster type device, we can impress the modified value on the wire. If the function does not vary in the problem, it could be placed on a closed loop of wire which runs through the machine. An acoustical delay line memory can be used in a similar way.

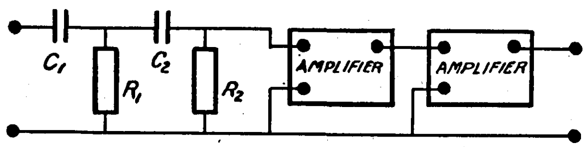
Another way of representing a function is by means of the Fourier Series. There must be a method for representing a constant function, and sin nx and cos nx (this supposes that the interval for the independent variable is $-\pi$ to $+\pi$) and a method for taking a linear combination of the output of these. In the case of a voltage, tuned circuits offer methods of representing the trigonometric functions.

Let us briefly discuss the possibilities for such a device. We consider a function on the interval $-\pi \leq x \leq \pi$. With certain reservations relative to continuity and differentiability such a function can be represented by a series

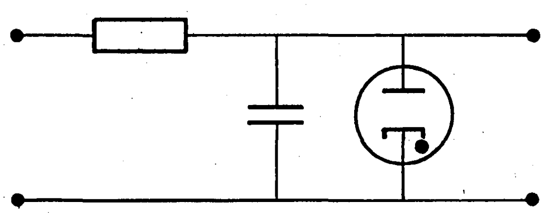
$$f(x) = a_0 + \Sigma_{n=1}^{\infty}(a_n \sin nx + b_n \cos nx).$$

In Section 7 of the previous chapter, we have seen how to produce a voltage $a_n \sin nx$, where a_n is a d.c. input and $x = wt$ where w corresponds to some convenient frequency. An analogous circuit can be used to produce $b_n \sin nx$. If we apply this output to a circuit which introduces a phase change of 90°, we obtain b cos nx.

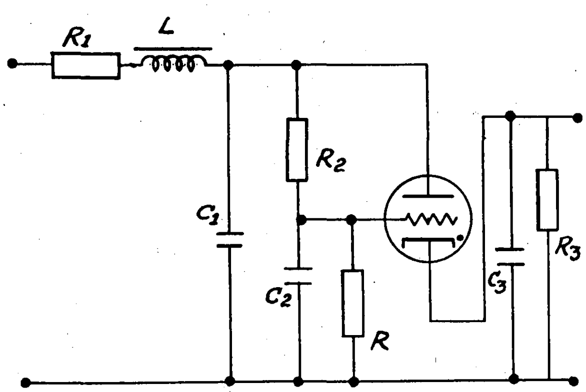
A circuit containing a condenser and a resistor can be used to change the phase. Thus if we have the circuit illustrated at the beginning of the next page, we can show that if $R_1 R_2 = \frac{1}{n^2 w^2 C_2 C_1}$, there will be a phase shift of 90° for the nw component.



We may mention that in order to synchronize the outputs one may use a common oscillator for all the integrating circuits tuned to a frequency corresponding to w, provided this oscillator is of a type which produces terms corresponding to the various higher harmonics. A relaxation oscillator has this property. A relaxation oscillator is one that is characterized by the property that there is in each cycle a period in which a condenser is slowly charged, then rapidly discharged. A number of such circuits are given in the *Radio Amateur's Handbook* but the principle can be simply illustrated by means of any gaseous tube having a breakdown characteristic. Thus in the following circuit if the input is a fixed high voltage the



condenser will slowly charge until the breakdown potential is reached, then it will discharge. Unfortunately, the current will continue even with a much lower voltage unless the plate is made negative with respect to the cathode. (This assumes that the cathode is emitting electrons.) This can be accomplished by introducing a resistance and a capacity in the cathode circuit. Thus in this cir-



cuit, the two condensers C_1 and C_2 charge until C_2 reaches the potential at which the "trigger grid" of the gaseous triode will initiate a discharge. Then C_1 will discharge through the tube and C_2 through R_2. This discharge passes partly through R_3 and partly into C_3 until the cathode becomes positive relative to the plate. (The inductance L permits the cathode to be actually positive relative to the plate.) Presumably, the output should be amplified and a condenser coupling should be used to the integrating circuit instead of the transformer coupling.

The output of these circuits are averaged by means of resistances. The result is a periodic function of the time in which the interval $-\pi/w \le t \le \pi/w$ corresponds to $-\pi \le x \le \pi$. If the original function is periodic and continuous with continuous derivatives this representation can be used effectively.

On the other hand, if it is desired to represent a function just for the interval $-\pi \le x \le \pi$, even if the function itself is continuous, the representation may have discontinuities at the end points, either in itself or its derivatives. In this case, the representation by means of trigonometric functions is neither uniformly convergent nor can term by term differentiation apply.

However, it is possible to introduce new terms which permit term by term differentiation. We do not have the time to give a precise discussion of this situation but we must content ourselves with the following outline.

It is customary to consider the orthogonal series representation of a function as representing it as a vector in an infinite dimensional function space. In this space the inner product of two vectors f(x) and g(x) is given by the formula

$$(f,\ g) = \int_{-\pi}^{\pi} f(x)\ \bar{g}(x)\ dx.$$

The functions 1, sin nx and cos nx, correspond to vectors along a set of coordinate axes. The set of coordinate axes form a "complete" set since there is no vector orthogonal to all of them. A finite sum

$$\sigma_N = a_0 + \Sigma_{n=1}^{N} (a_n \sin nx + b_n \cos nx)$$

is an approximation to f if the integral

$$\int_{-\pi}^{\pi} |f-\sigma_N|^2\ dx$$

is a minimum when regarded as a function of a_0, a_n and b_n.

In order, however, to get a series representation in which the series approximation σ_N approximates not only f but σ'_N approximates f' and say σ''_N approximates f", then we must minimize the integral

$$\int_{-\pi}^{\pi} (|f-\sigma_N|^2 + |f'-\sigma'_N|^2 + |f''-\sigma''_N|^2)\ dx.$$

To obtain a discussion analogous to the preceding one in this case, we must use a space W in which the inner product is

$$\int_{-\pi}^{\pi} (f\,\bar{g} + f'\bar{g}' + f''\bar{g}'')\,dx.$$

That there is such a space with the requisite properties is shown in the author's thesis (Trans. Amer. Math. Soc. 37, pp. 301-08 [1935]) for two independent variables. However, the discussion can be readily simplified to one variable or expanded to any number.

One can readily show that the functions 1, sin nx, cos nx are orthogonal in this space also. Thus they also determine coordinate axes. But this set is incomplete, i.e., there are functions which are orthogonal to every one of these. Let us find all functions f with this property.

Let φ denote any of the functions 1, sin nx, cos nx and suppose f is orthogonal to φ in the space W. Then

$$0 = (f,\varphi) = \int_{-\pi}^{\pi} (f\bar{\varphi} + f'\bar{\varphi}' + f''\bar{\varphi}'')\,dx$$

$$= (f'-f''')\,\bar{\varphi} + f''\bar{\varphi}'\,]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(f-f''+f^{(iv)})\,\bar{\varphi}\,dx.$$

If we let $\varphi = \sin nx$, we obtain

$$n(-1)^{n+1}\,[f''(\pi) - f''(-\pi)] = \int_{-\pi}^{\pi}[f-f''+f^{(iv)}]\,\bar{\varphi}\,dx.$$

Since the limit as $n \to \infty$ of the right-hand side is zero, we must have $f''(\pi) = f''(-\pi)$. Similarly if we let $\varphi = \cos nx$, we get

$$f'(\pi) - f'''(\pi) = f'(-\pi) - f'''(-\pi).$$

Thus we have established that the three conditions

$$f - f'' + f^{iv} = 0,\quad f''\,]_{-\pi}^{\pi} = 0,\quad f' - f'''\,]_{-\pi}^{\pi} = 0$$

are necessary if f is orthogonal to all φs in the new space W. They are clearly sufficient. (We have assumed that f has two further derivatives. This assumption can be justified but the discussion is somewhat lengthy.)

By the usual considerations of ordinary linear differential equations, it can be shown that all solutions of the equation

$$f - f'' + f^{iv} = 0$$

are given by the expression

$$f = A\,e^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + \gamma_1) + B\,e^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + \gamma_2)$$

where A, B, γ_1 and γ_2 are constants of integration.

The successive derivatives of f are easily shown to be

$$f' = Ae^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_1+\pi/6) - Be^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_2+\tfrac{5}{6}\pi)$$

$$f'' = Ae^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_1+\pi/3) + Be^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_2+\tfrac{5}{3}\pi)$$

$$f''' = Ae^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_1+\pi/2) - Be^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x+\gamma_2+\pi/2).$$

If we substitute, we find that the condition

$$f''\,]_{-\pi}^{\pi} = 0$$

becomes

$$A\sin(\gamma_1 + 5\pi/6) = B\sin(\gamma_2 - 5\pi/6)$$

and $f' - f'''\,]_{-\pi}^{\pi} = 0$,

becomes

$$A\cos(\gamma_1 + 5\pi/6) = -B\cos(\gamma_2 - 5\pi/6).$$

If we square these two equations and add, we get $A^2 = B^2$ or $A = \pm B$. $A = B$ implies $\gamma_1 + 5\pi/6 = \gamma_2 - 5\pi/6 =$ (say) α and $A = -B$ implies $\gamma_1 + 5\pi/6 = \alpha = \gamma_2 - 5\pi/6 + \pi$. However when either of these is substituted in the expression for f, it becomes

$$f = A[e^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + \alpha - 5\pi/6) + e^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + \alpha + 5\pi/6)]$$

and it can be shown that this expression satisfies the conditions for f for all values of A and α. Let us take A = 1. This expression can be written in the form

$$f = f_1\cos\alpha + f_2\sin\alpha,$$

where

$$f_1 = e^{\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x - 5\pi/6) + e^{-\frac{\sqrt{3}}{2}x}\sin(\tfrac{1}{2}x + 5\pi/6)$$

$$f_2 = e^{\frac{\sqrt{3}}{2}x}\cos(\tfrac{1}{2}x - 5\pi/6) + e^{-\frac{\sqrt{3}}{2}x}\cos(\tfrac{1}{2}x + 5\pi/6).$$

We can readily find the length of f in the new space

$$\|f\|^2 = (f,f) = \int_{-\pi}^{\pi}(|f|^2 + |f'|^2 + |f''|^2)\,dx$$

$$= (f' - f''')\,f + f''\,f'\,]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}(f - f'' + f^{(iv)})\,f\,dx$$

$$= (e^{\sqrt{3}\pi} - e^{\sqrt{3}\pi})\,(\cos^2\alpha + \sqrt{3}\sin^2\alpha)$$

$$= \text{(say) } C\,(\cos^2\alpha + \sqrt{3}\sin^2\alpha).$$

This can be used to show that $(f_1, f_2) = 0$ and

$$\|f_1\|^2 = C,\quad \|f_2\|^2 = C\sqrt{3}.$$

In the space W, the function 1 has length $\sqrt{2\pi}$, the function sin nx and cos nx have lengths $\sqrt{\pi}\sqrt{1+n^2+n^4}$. If we apply the usual method of obtaining the Fourier coefficients, we find that a function F(x) can be represented on the interval $-\pi \le x \le \pi$, by a series

$$F(x) = a_0 + d_1 f_1 + d_2 f_2 + \Sigma_{n=1}^{\infty}(a_n\sin nx + b_n\cos nx)$$

where $a_0 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi}F(x)\,dx$

$$d_1 = (1/C)\int_{-\pi}^{\pi}[F(x)\,f_1 + F'\,f_1' + F''\,f_1'']\,dx$$

$$d_2 = (1/C\sqrt{3})\int_{-\pi}^{\pi}(F\,f_2 + F'\,f_2' + F''\,f_2'')\,dx$$

while if $\varphi = \sin nx$ or $\cos nx$, a_n or b_n are equal respectively to

$$(1/\pi\,[1+n^2+n^4])\int_{-\pi}^{\pi}(F\varphi + F'\varphi' + F''\varphi'')\,dx.$$

This representation has the property that the term by term derivatives converge to the corresponding derivatives of F (at least in the mean).

Thus the problem of representing functions in this fashion can be referred to the problem of representing f_1 and f_2. These are linear combinations $e^{\frac{\sqrt{3}}{2}x} \sin \frac{1}{2}x$, $e^{\frac{\sqrt{3}}{2}x} \cos \frac{1}{2}x$, $e^{-\frac{\sqrt{3}}{2}x} \sin \frac{1}{2}x$, $e^{-\frac{\sqrt{3}}{2}x} \cos \frac{1}{2}x$. The first two of these are two linearly independent solutions of the differential equation

$$\frac{d^2g}{dx^2} - \sqrt{3}\,\frac{dg}{dx} + g = 0$$

and the second pair are solutions for

$$\frac{d^2g}{dx^2} + \sqrt{3}\,\frac{dg}{dx} + g = 0$$

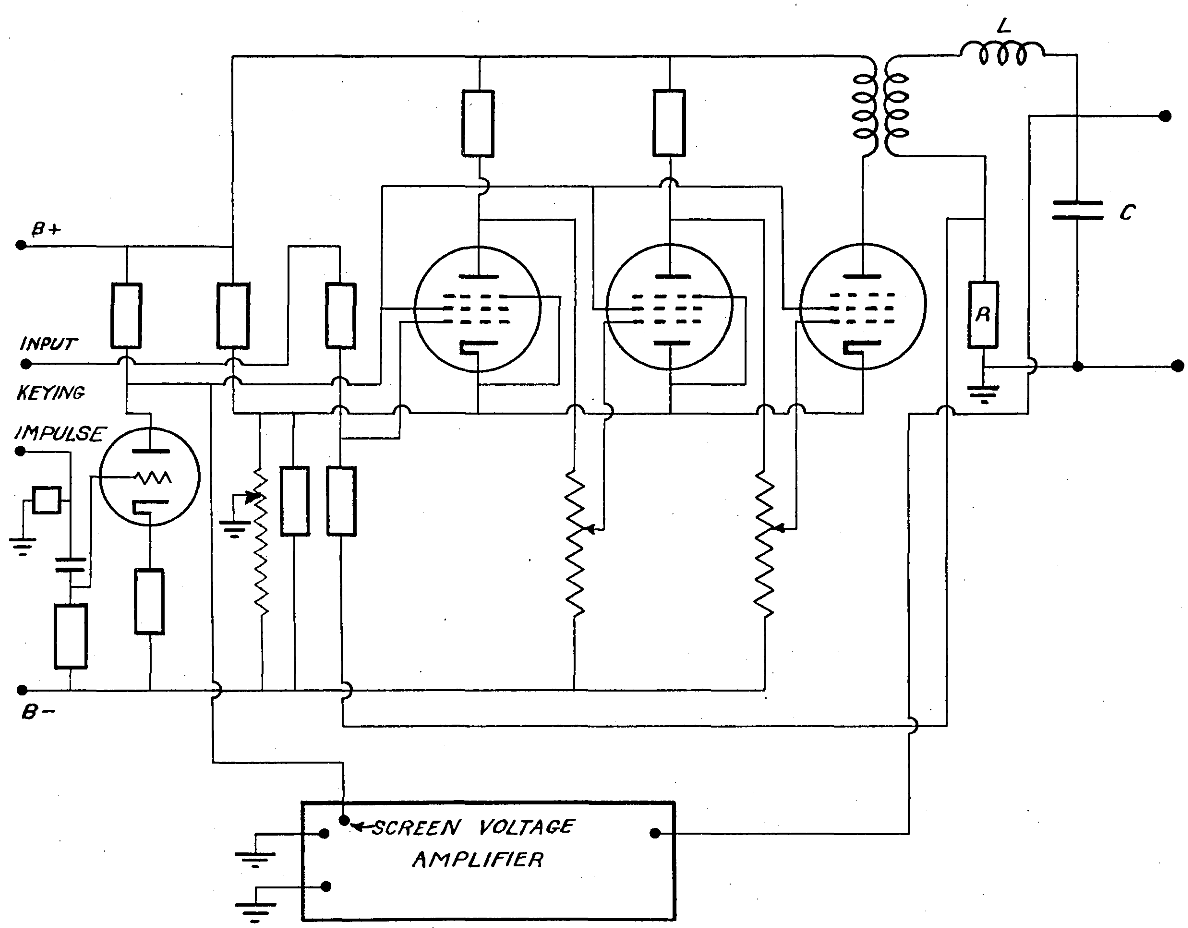
Let us consider the second equation first. It is particularly easy to obtain a function g which satisfies this equation. For if we have a circuit containing inductance resistance and capacity, the charge on the condenser will satisfy this equation if the constants are properly chosen. However, if this is to be used to yield a periodic representation of a function, we must have a means of setting the value of q and $\frac{dq}{dt}$ back to the proper initial values at the end of each cycle and the beginning of the next.

This can be done most effectively in the case in which $g = e^{-\frac{\sqrt{3}}{2}x} \cos \frac{x}{2}$ since we have $g(-\pi) = 0$, $g'(-\pi) = \frac{1}{2}e^{+\frac{\sqrt{3}}{2}\pi}$

The accompanying circuit is such that when no keying impulse is being received, the screen voltage on the pentodes is kept so low that practically no current flows through the pentodes. This also applies to the pentodes in the lower amplifier. Consequently the rest of the circuit has almost no effect on the tuned circuit of R, L and C. However when a sufficiently negative keying impulse is received the screen voltages in the pentodes become normal. The upper amplifying circuit then induces a current in the tuned circuit up to the point where the voltage drop across the resistor equals the input voltage.

Normally the lower amplifier behaves simply as a high resistance short across the condenser but when the keying impulse is received, this amplifier supplies current to make the voltage drop across the condenser zero, i.e., it becomes a low resistance short.

Various relaxation oscillators can also be used using the idea of controlling the screen voltage by an auxiliary tube.



We may use the voltage across the condenser as input. This, of course, gives $q = e^{-\sqrt{\frac{3}{2}x}}\cos\frac{x}{2}$. By taking the voltage across R as output, we obtain

$$i = \frac{dq}{dt} = -e^{-\sqrt{\frac{3}{2}x}}\ \frac{1}{2}\ (\sqrt{3}\cos x + \sin x).$$

To obtain the solutions of

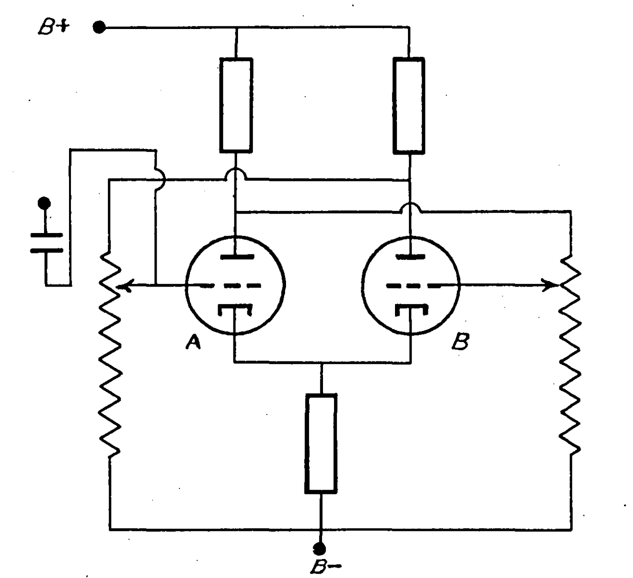
$$\frac{d^2g}{dt^2} - \sqrt{3}\,\frac{dg}{dt} + g = 0$$

it is necessary to replace the resistance R by a "negative resistance" as at the end of Section 6. We must introduce a voltage into the circuit, which is twice the drop across the resistance and opposite in value and this, of course, can be done by two amplifiers.

However in general it would seem as if a device of the above sort would be too complicated. An alternative method of obtaining repeated representations in time of a function on an interval would be to mount potentiometer contacts on a motor driven shaft and to use potentiometers which would represent the functions f_1, f_2 sin nx and cos nx. It is obvious how a function can be represented as a voltage by such a device.

2. We have previously mentioned trigger circuits in connection with servo circuits. There are two types of these of interest to us. A trigger circuit is characterized by the property that it has two states of operation. Unless perturbed it will remain in whichever state it finds itself.

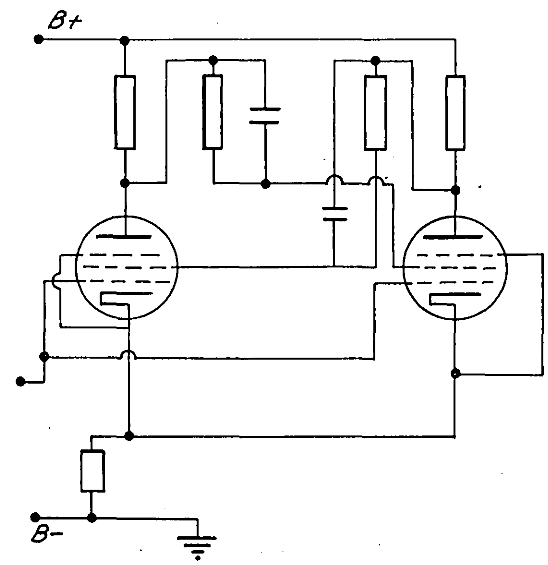
The following circuit has the property that the change depends on the voltage of input impulse. In the first state of the circuit, tube A is conducting. Since this lowers the plate voltage, the grid



of B is below cut off and the tube B is non-conducting. This is consistent with A conducting since the plate of B is high and hence also the grid of A. The second state is identical with this when A and B are interchanged.

When A is conducting a negative input impulse which drives the grid of A below that of B, will cause A to become temporarily non-conducting. This causes the A plate and B grid to rise. Thus B will become conducting and the circuit will pass over into the second state. A positive input will cause no effect while A is conducting. When A is non-conducting the impulses must be changed in sign. This type of trigger circuit is valuable for control purposes. The voltage drop across the load resistors can be utilized in a number of ways.

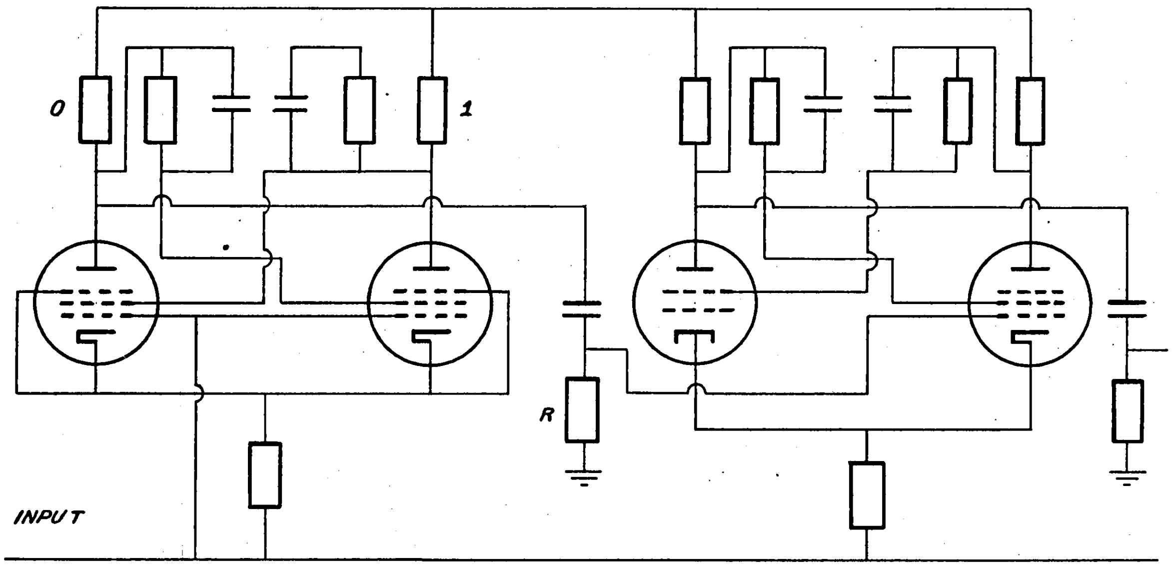
Two pentodes can be linked into a circuit which responds to negative impulse. In the accompanying diagram, when one tube is conducting, it forces



the screen of the other tube down so that the other tube is not conducting. Under these circumstances, the voltage drop across this screen resistor is low while the drop across the screen resistor leading into the conducting tube is relatively large. Now if the first grids in both tubes are brought down to cut off, both tubes become non-conducting. Now if this occurs only briefly so that the charge on the condensers do not disappear and then the first grids come back to normal, the screen grid in the tube which was formerly conducting is lower in voltage than the screen grid in the formerly non-conducting tube. This favors the flow of current through the formerly non-conducting tube so that the circuit changes state.

A number of such arrangements can be used as a binary counter. In each such circuit, one of the tubes conducting corresponds to the 0 state, the other to the digit 1. Such a system will constitute a binary counter if the plate of the 0 tube is coupled through a small condenser to the grids of the following circuit.

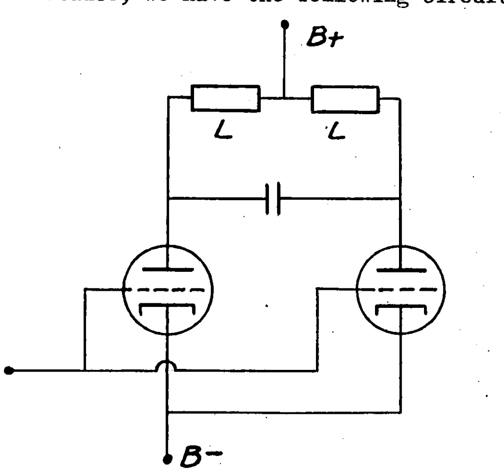
By properly choosing the value of the condenser



and R, the impulse time for the output can be regulated. The product CR for the output circuit and for the screen resistor and condenser should be equal. The situation in the counter can be conveniently indicated by an electron ray tube having two ray control electrodes as, for instance, the 6AF6-G. The target in the latter is connected to B+, the cathode is 135 volts lower and each control electrode is connected to the plate of a pentode in a pair. The shadow will indicate the conducting tube.

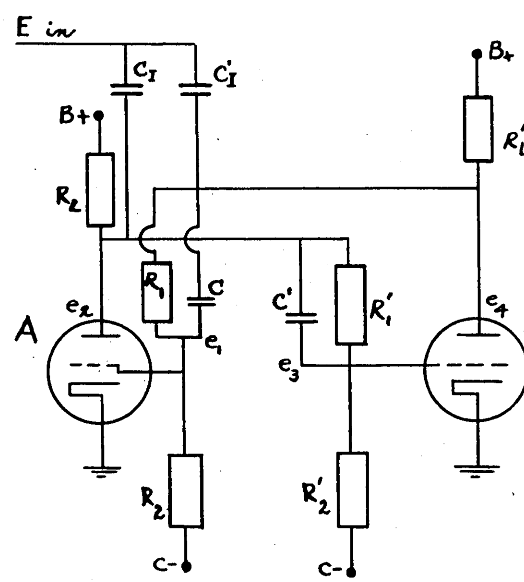
In radiation counters, trigger circuits involving gas triodes are used. The grid in the gas triode, of course, has a trigger action. If its voltage relative to the cathode is made less negative than a certain negative value a discharge occurs between the plate and cathode which will continue however independently of the voltage of the grid and must be "quenched" by lowering the voltage on the plate. This is the basis of the relatively simple circuits described in the *Electronics Engineering Manual*, New York: McGraw-Hill & Co., pp. 68-70.

For instance, we have the following circuit.



If the tube A is conducting and B is not, there will be a considerable voltage drop across the condenser. If now the grids are excited, tube B will become conducting. This lowers the voltage on the B end of the condenser and, of course, the A end is also lowered. If the voltage fall of the A plate is adequate, the A tube will be quenched.

In many counters a trigger circuit is used involving a pair of triodes, which is essentially the pentode circuit used above but with the controlling characteristics of the screen grid transferred to the grid. One can show that such a circuit has two stable states provided the circuit



constants are properly chosen. Furthermore a properly shaped input pulse of the correct magnitude will cause this circuit to go from one steady state to the other.

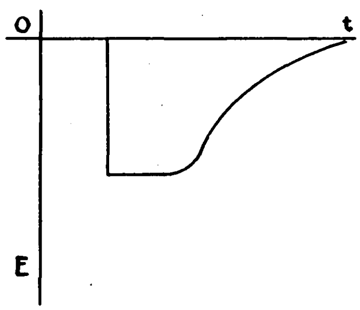
. We will refer to the state in which tube B is conducting as the normal state. In this state the current flowing through the B plate resistance R_L' causes the plate voltage of this tube to be low. Since the A grid is connected by a voltage divider tô the B plate, the A tube is biased below cut off. Consequently, the A plate takes no current and all the current in R_L for this side flows through R_1' and R_2' to C^-. In the normal state, the B grid is at ground potential which requires the relation

$$\frac{B^+}{R_L + R_1'} = \frac{-C^-}{R_2'}.$$

(C^- is a negative voltage.) The A plate voltage has a value $\frac{B^+}{R_L + R_1}$. If I_0 is the plate current for the B tube the plate voltage is $e_0 - e^*$ where $e^* = \frac{I_0 R_L' (R_1 + R_2)}{R_L' + R_1 + R_2}$. The A grid has the value

$$-ke^* = \frac{-R_2 e^*}{R_1 + R_2} = \frac{I \cdot R_L' R_2}{R_L' + R_1 + R_2}.$$

Now suppose a negative pulse is applied to the input. (We suppose that the generator of the pulse is a voltage generator with zero internal impedance.) This pulse has a steep fall initially. Both grids are capacitatively coupled to the input and thus receive the negative voltage pulse. This



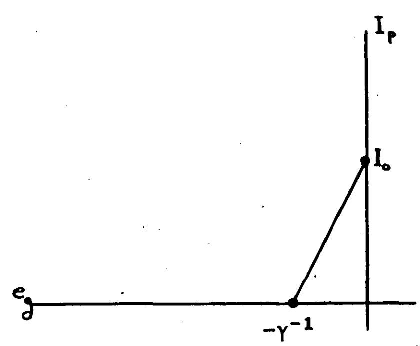
drives them both lower. Consequently the B grid is also cut off and both tubes are non-conducting. Following the steep fall the pulse has a flat portion. During this time both tubes are cut off but both grids will tend to rise to the zero voltage position. However, the circuit constants are chosen so that the non-conducting tube grid will enter the conducting region before the other grid. To see this one must appreciate that in the normal state, the voltage drop across the C_I' condenser attached to the B plate is less than that across the other C_I, since the input is normally considered to be at ground potential and, of course, the potential at the plate of the conducting tube is lower than that at the other plate. Both condensers are fed by a resistance network which will tend to bring the plate ends up to the cut off plate potential e_0. Since the B plate is further away from e_0 it will have the faster rate of rise.

On the other hand, the voltage drop across C is initially lower than that across C', by an amount $(1 - k)\, e^*$. Now the relaxation time for the C R_1 circuit is smaller than that for C_I R_L and this difference is essentially maintained during the rise of the plates.

Immediately after the initial fall E_I, the B grid was E_I volts below zero and the A grid $E_I + ke^*$ below zero. Let γ be such that $\frac{-1}{\gamma}$ is the cut off grid voltage. However, due to the faster rise of the B plate the A grid rises the amount $E_I + k\,e^* - \frac{1}{\gamma}$ before the B grid rises $E_I - \frac{1}{\gamma}$. We will discuss this in detail below. As soon as the A tube becomes conducting, the plate voltage falls abruptly and this drives the B grid down and the circuit is in the other state essentially. The charge on the C and C' condensers are permitted to adjust themselves to the new state and the E voltage rises to zero but slowly enough so that the new state is not disturbed.

A number of outputs from such a circuit are possible. The plate potentials can be directly coupled to "gate tubes" which will conduct only when the plate is high. The state of the circuit can be indicated by means of neon lamps. When a suitable resistance is in series with a typical lamp of this kind it will glow when a 90-volt potential is applied across the combination and extinguish itself when the applied voltage is less than 60 volts. Besides these two pulse outputs are obtainable by condenser coupling to the plate. Thus, when the circuit passes from the normal to the activated state, the A tube plate falls abruptly generating a negative pulse which is considerably larger than the input voltage, while the B tube will send out a positive pulse.

It is clear that the design of such a circuit involves a number of interesting mathematical questions. We will discuss mathematically a somewhat simplified situation but one which has the essential aspects of the trigger circuit. Normally, we will assume that the plate resistance of the triodes is infinite but we will also indicate how the discussion can be varied to take care of finite plate resistance. We assume that for each triode, the plate current $I_p(e_g)$ is zero for grid voltage $e_g < -1/\gamma$ and that for $-1/\gamma \le e_g \le 0$, $I_p = I_0(1 + \gamma e_g)$.



The nodal equations for this circuit are readily derived. We assume that the circuit is symmetric so that we need not write down the primes. Thus, if we add the currents flowing away from the A grid whose voltage we indicate by e_1, we have

$$\left(\frac{1}{R_1} + pC\right)(e_1 - e_4) + \frac{1}{R_2}(e_1 - C^-) = 0$$

or

$$\left(\frac{1}{R_2} + \frac{1}{R_1} + pC\right) e_1 - \left(\frac{1}{R_1} + pC\right) e_4 = \frac{1}{R_2} C^- \qquad (1)$$

For the A plate with voltage e_2, we have similarly for the sum of currents flowing away,

$$I_p(e_1) + \left(\frac{1}{R_1} + pC\right)(e_2 - e_3) + pC_I(e_2 - E_I) + \frac{1}{R_L}(e_2 - B^+) = 0$$

or

$$I_p(e_1) + \left(\frac{1}{R_1} + pC + \frac{1}{R_L} + pC_I\right) e_2 - \left(\frac{1}{R_1} + pC\right) e_3 = pC_I E_I + \frac{1}{R_L} B^+ \qquad (2)$$

Similarly we have for the B grid and plate voltages e_3 and e_4 respectively

$$-\left(\frac{1}{R_1} + pC\right) e_2 + \left(\frac{1}{R_2} + \frac{1}{R_1} + pC\right) e_3 = \frac{1}{R_2} C^- \qquad (3)$$

$$-\left(\frac{1}{R_1} + pC\right) e_1 + I_p(e_3) + \left(\frac{1}{R_1} + \frac{1}{R_L} + p[C + C_I]\right) e_4 = pC_I E_I + \frac{1}{R_2} B^+ \qquad (4)$$

Let us consider briefly the stability of the normal state. In this state, the desired values of e_1, e_2, e_3 and e_4 are respectively $-ke'$, e_0, 0 and $e_0 - e'$ respectively. We will suppose that each of these has been slightly displaced from these values but not enough to disturb the situation relative to cut off. Thus, $I_p(e_1) = 0$ and $I_p(e_3) = I_0[1 + \gamma e_3]$. Stability means that the circuit will remain in the state under these conditions and these voltages will tend to return to the correct values.

Let ε_1 denote the difference between e_1 and the desired values. Under these circumstances equations (2) and (3) become

$$\left(\frac{1}{R_1} + pC + \frac{1}{R_L} + pC_I\right)\varepsilon_2 - \left(\frac{1}{R_1} + pC\right)\varepsilon_3 = pC_I E_I \qquad (2')$$

$$-\left(\frac{1}{R_1} + pC\right)\varepsilon_2 + \left(\frac{1}{R_2} + \frac{1}{R_1} + pC\right)\varepsilon_3 = 0 \qquad (3')$$

We intend to eliminate ε_3 between these equations and obtain a second order equation on ε_2. Consequently we wish to express our initial conditions on ε_2, ε_3 as initial conditions on $\varepsilon_{2,0}$, $\varepsilon'_{2,0}$, the initial values of ε_2 and ε'_2. Initially we have

$$\left(\frac{1}{R_L} + \frac{1}{R_1}\right)\varepsilon_{2,0} + (C + C_I)\varepsilon'_{2,0} - \frac{1}{R_1}\varepsilon_{3,0} - C\varepsilon'_{3,0} = pC_I E_I\Big]_{t=0} = i_0$$

$$-\frac{1}{R_1}\varepsilon_{2,0} - C\varepsilon'_{2,0} + \left(\frac{1}{R_2} + \frac{1}{R_1}\right)\varepsilon_{3,0} + C\varepsilon'_{3,0} = 0$$

Solving for $\varepsilon'_{2,0}$, we obtain

$$\varepsilon'_{2,0} = \frac{i_0}{C_I} - \frac{\varepsilon_{2,0}}{R_L C_I} - \frac{\varepsilon_{3,0}}{R_2 C_I}$$

If we eliminate ε_3 between equations 2' and 3', and divide by CC_I, we obtain

$$\left[p^2 + \left(\frac{1}{C_I}\left(\frac{1}{R_L} + \frac{1}{R_2}\right) + \frac{1}{C}\left(\frac{1}{R_2} + \frac{1}{R_1}\right)\right) p + \left(\frac{1}{R_2R_1} + \frac{1}{R_2R_L} + \frac{1}{R_1R_L}\right)\frac{1}{CC_I}\right]\varepsilon_2$$

$$= \left[\frac{1}{C}\left(\frac{1}{R_2} + \frac{1}{R_1}\right) + p\right] p E_I$$

The coefficient of ε_3 is quadratic in p. The discriminant d is determined by the equation

$$d^2 = b^2 - 4ac = \left[\frac{1}{C_I}\left(\frac{1}{R_L} + \frac{1}{R_1}\right) - \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\right]^2 + \frac{4}{CC_I R_2^2}.$$

The coefficient of ε_2 can be expressed in the form $(p + \alpha_1)(p + \alpha_2)$ where $\alpha_1 + \alpha_2 = b$, $\alpha_1 \alpha_2 = c$. It is easy to see that

$$\alpha_1 = \tfrac{1}{2}(b - d)$$

$$\alpha_2 = \tfrac{1}{2}(b + d)$$

Now let $\omega_1 = \frac{1}{C}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$, $\omega_2 = \frac{1}{C_I}\left(\frac{1}{R_L} + \frac{1}{R_2}\right)$.
Then $b = \omega_1 + \omega_2$, $d = \sqrt{(\omega_2 - \omega_1)^2 + 4/C_I C R_2^2}$.

$$\alpha_1 = \tfrac{1}{2}\left(\omega_1 + \omega_2 - \sqrt{(\omega_2 - \omega_1)^2 + 4/C_I C R_2^2}\right) \qquad (5)$$

$$\alpha_2 = \tfrac{1}{2}\left(\omega_1 + \omega_2 + \sqrt{(\omega_2 - \omega_1)^2 + 4/C_I C R_2^2}\right) \qquad (6)$$

Both α_1 and α_2 are positive. For α_2 is obviously positive and $c = \alpha_1 \alpha_2$ is also positive.

Our equation for ε_2 becomes

$$(p + \alpha_1)(p + \alpha_2)\varepsilon_2 = (p + \omega_1) p E_I.$$

If we multiply by the obvious integrating factor $e^{\alpha_1 t}$ and integrate we get

$$e^{\alpha_1 t}(p + \alpha_2)\varepsilon_2 = \int_0^t e^{\alpha_1 \tau}(p + \omega_1) p E_I \, d\tau + \varepsilon'_{2,0} + \alpha_2 \varepsilon_{2,0}$$

If we break the integral on the right-hand side and integrate by parts, use our previous expression for $\varepsilon'_{2,0}$ and divide by $e^{\alpha_2 t}$, we obtain (since $i_0/C = pE_I\big]_{t=0}$)

$$(p + \alpha_2)\varepsilon_2 = (p + \omega_1 - \alpha_1) E_I -$$

$$\alpha_1(\omega_1 - \alpha_1)\int_0^t E_I \exp(-\alpha_1(t - \tau))\, d\tau + A e^{-\alpha_1 t}$$

where

$$A = (\alpha_1 - \omega_1) E_0 + \left(\alpha_2 - \frac{1}{R_L C_I}\right)\varepsilon_{2,0} - \frac{\varepsilon_{3,0}}{R_2 C_I}$$

and E_0 is the initial value of E_I.

We now multiply by $e^{\alpha_2 t}$ and integrate by parts to remove the various operators which modify E_I under the radical sign to obtain

$$\varepsilon_2 = E + \frac{\alpha_2\ (\omega_1 - \alpha_2)}{\alpha_2 - \alpha_1} \int_0^t \exp\ (-\ \alpha_2\ (t - \tau))\ E\, d\tau$$

$$- \frac{\alpha_1\ (\omega_1 - \alpha_1)}{\alpha_2 - \alpha_1} \int_0^t \exp\ (-\ \alpha_1\ (t - \tau))\ E\, d\tau$$

$$+ A_1\ e^{-\alpha_1 t} + A_2\ e^{-\alpha_2 t}$$

where

$$A_1 = A/(\alpha_2 - \alpha_1),\ A_2 = -\ E_o - A/(\alpha_2 - \alpha_1) + \varepsilon_{2,o}$$

Now if we return to (2') and (3') and eliminate ε_2, we will obtain

$$(p + \alpha_1)\ (p + \alpha_2)\ \varepsilon_3 = (p + \omega_3)\ p E$$

where $\omega_3 = 1/R_1C$. We now proceed to a similar discussion with ω_3 instead of ω_1. We find for $\varepsilon'_{3,o}$ the value

$$\varepsilon'_{3,o} = \frac{i_o}{C_I} + (\frac{1}{CR_1} - \frac{1}{C_IR_L})\ \varepsilon_{2,o} - (\omega_1 + \frac{1}{C_IR_2})\ \varepsilon_{3,o}$$

The constant B analogous to A obtained after one integration will be

$$B = (\alpha_1 - \omega_3)\ E_o + (\frac{1}{CR_1} - \frac{1}{C_IR_L})\ \varepsilon_{2,o} + (\alpha_2 - \omega_1 - \frac{1}{C_IR_2})\ \varepsilon_{3,o}$$

and if we let $B_1 = B/(\alpha_2 - \alpha_1)$, $B_2 = -\ E_o - B/(\alpha_2 - \alpha_1) + e_{3,o}$ we will again obtain

$$\varepsilon_3 = E + \frac{\alpha_2\ (\omega_3 - \alpha_2)}{\alpha_2 - \alpha_1} \int_0^t \exp\ (-\ \alpha_2\ (t - t))\ E\, d\tau$$

$$- \frac{\alpha_1\ (\omega_3 - \alpha_1)}{\alpha_2 - \alpha_1} \int_0^t \exp\ (-\ \alpha_1\ (t - t))\ E\, d\tau$$

$$+ B_1\ e^{-\alpha_1 t} + B_2\ e^{-\alpha_2 t}$$

It is clear then that ε_2 and ε_3 will vary like E plus some time delayed terms, plus transients. If E is not too large, these voltages vary only slightly. A precisely similar discussion holds for ε_1 and ε_4 except the $p\, C_I E_I$ is replaced by $p\, C_I - \gamma\, I_o\, \varepsilon_3$ and $\frac{1}{R}$ is replaced by $\frac{1}{R_L} + \frac{1}{r_p}$ where r_p is the plate resistance of the conducting tube.

Again the fact that α_1 and α_2 are positive insures the stability of the circuit for small changes of E.

Next let us discuss the response of the circuit to a pulse of the shape previously described. The circuit is initially in the normal state, say, when a sharp negative jump occurs in the input voltage E_I. After this jump the input voltages remains constant for a time. Initially the values of e_1, e_2, e_3 and e_4 are respectively $-ke'$, e_o, 0 and $e_o - e'$. Since these points are connected directly to the input by condensers, the immediate effect of the pulse is to change these voltages to $-\ E_o - k e'$, $-\ E_o + e_o$, $-\ E_o$, $-\ E_o + e_o - e'$. Both tubes are now cut off and the distinction between the voltage pairs e_2, e_3 and e_4, e_1 is simply in their initial values $-\ E_o + e_o$, $-\ E_o$ and $-\ E_o + e_o - e'$, $-\ E_o - k e'$. Thus if we solve our nodal equations for e_4, e_1 we can obtain the pair e_2, e_3 by simply setting $e' = 0$ in the result.

For e_4, e_1, we have the equations (4) and (1) above with $I_p(e_3) = 0$. If we subtract from e_4 the constant voltage e_o, the constant terms on the right-hand side drop off and if we further confine ourselves to an interval $t > 0$ in which E is a constant these equations become

$$-\ (\frac{1}{R_1} + p\, C)\ e_1 + (\frac{1}{R_1} + \frac{1}{R_L} + p\ (C + C_I))\ e_4 = 0$$

$$(\frac{1}{R_2} + \frac{1}{R_1} + p\, C)\ e_1 - (\frac{1}{R_1} + p\, C)\ e_4 = 0$$

where e_1 and e_4 now have initial values $-E_o - k e'$ and $-E_o - e'$ respectively. Our e_4 is now the variation from e_o.

Let $\bar{e}_1$ and $\bar{e}_2$ be the Laplace transform of e_1 and e_4. According to the usual rules, we have

$$-\ (\frac{1}{R_1} + p\, C)\ \bar{e}_1 + (\frac{1}{R_1} + \frac{1}{R_L} + p\ (C + C_I))\ \bar{e}_4 =$$

$$-\ C_I\ E_o - e'\ (C_I + (1 - k)\ C)$$

$$(\frac{1}{R_2} + \frac{1}{R_1} + p\, C)\ \bar{e}_1 - (\frac{1}{R_1} + p\, C)\ \bar{e}_4 = C\ e'\ (1 - k)$$

Now if we solve for $\bar{e}_1$ we obtain

$$(p + \alpha_1)\ (p + \alpha_2)\ \bar{e}_1 = -E_o\ (\frac{1}{CR_1} + p) + e'\ (\frac{1 - k}{C_IR_L} - \frac{1}{CR_1} - kp)$$

If we solve this for e_1; we obtain

$$e_1 = \frac{1}{\alpha_2 - \alpha_1}\ e'(1 - k)\ (\frac{1}{C_IR_L} - \frac{1}{CR_1})\ (e^{-\alpha_1 t} - e^{-\alpha_2 t})$$

$$+ \frac{1}{\alpha_2 - \alpha_1}\ (E_o + k e')\ [(\alpha_1 - \frac{1}{CR_1})\ e^{-\alpha_1 t} - (\alpha_2 - \frac{1}{CR_1})\ e^{-\alpha_2 t}].$$

The expression for e_3 is obtained from this by setting $e' = 0$. Thus

$$e_3 = \frac{E_o}{\alpha_2 - \alpha_1}\ [(\alpha_1 - \frac{1}{CR_1})\ e^{-\alpha_1 t} - (\alpha_2 - \frac{1}{CR_1})\ e^{-\alpha_2 t}]$$

Now e_3 and e_1 start from negative values $-E_o$ and $-(E_o + k e')$ respectively. In general, the circuit values are chosen so that α_1 is close to $\frac{1}{CR_1}$ and e_3 in its decay behaves approximately like $-E_o e^{-\alpha_2 t}$. The second line for e_1 behaves similarly but the first term quickly becomes positive. Consequently the grid voltage e_1 enters the conducting region first and as soon as plate current flows, e_3 is forced down and the state is determined.

The situation will be more clearly understood from a numerical example using values roughly corresponding to a known case. Suppose the grid cut-off voltage is -1, $e' = 10$, $E_o = 5$ voltage units, $C = C_I$, $R_1 = R_2 = 10\ R_2$ and let $t_o = CR_1$ be our unit of time, $t' = t/CR_1$ is the time in this unit. Using our above formulas, we find $k = \frac{1}{2}$, $\alpha_1 = 1.89/t_o$, $\alpha_2 = 11.11/t_o$ and

$$e_1 = -15.86\, e^{-11.11\, t'} + 5.86\, e^{-1.89\, t'}$$

$$e_3 = -\, 5.48\, e^{-11.11\, t'} + \;.48\, e^{-1.89\, t'}$$

(e', of course, increases both coefficients but relatively the coefficient of $e^{-1.89\, t'}$ is greatly increased.) The variation of these functions can be readily tabulated.

t'	.00	.05	.08	.09	.108
e_1	-10	-3.74	-1.45	- .87	0
e_3	- 5	-2.71	-1.84	-1.61	-1.25

Thus some time between .08 and .09 t', the previously cut off grid enters the conducting region. At this point the A plate voltage will begin to fall, driving e_3 down and establishing the new state. The remainder of the pulse is shaped so as not to disturb this new state. Thus a properly shaped pulse will change the state of the system. Our discussion has been simplified by the assumption of zero generator impedance for the input voltage.

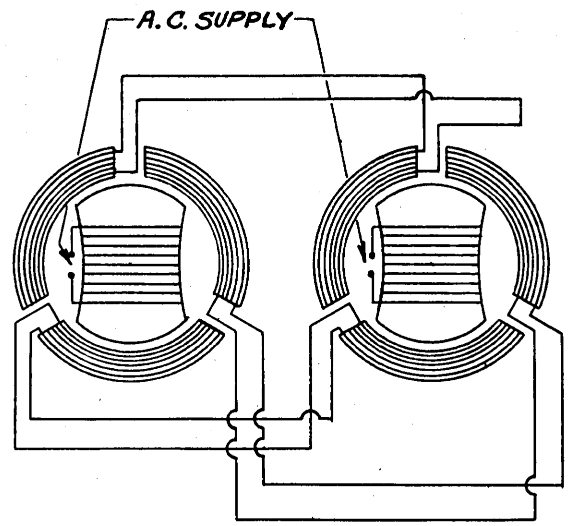
We have shown the input condensers connected to the plate. Alternately they could be connected to the grids 0. If a cathode resistor is introduced for each tube, the input condensers can be connected to the cathode, in which case a positive voltage jump will cut off both tubes and induce the change of state.

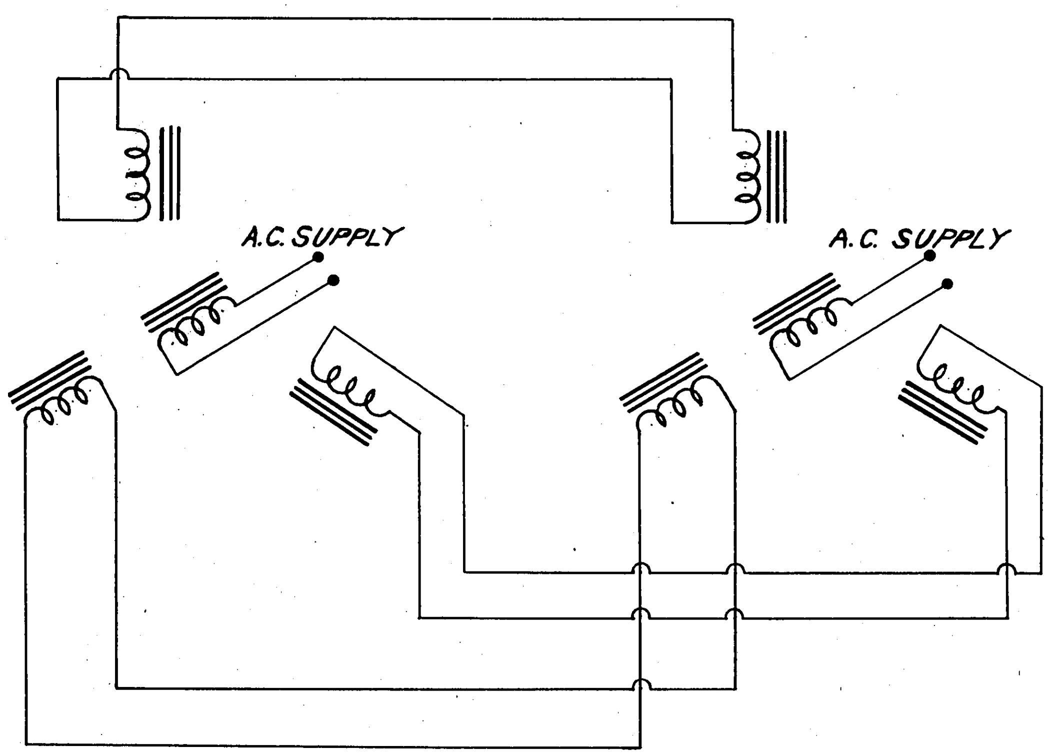
A very good discussion of circuits having various states and not used as amplifiers is contained in the reference, Puckle, O. S., *Time Bases*. New York: John Wiley & Sons.

3. A synchro system is a system set up to reproduce a rotation at a distance. The connection between the two points is purely electrical.

The simplest type of synchro system involves two parts or "units." One is a "synchro signal generator" or "synchro transformer," in which the rotation to be transmitted is set up. The other unit is the "synchro motor" which reproduces the rotation.

These units differ only in minor respects. Both units are similar in appearance to motors. In each case there is a field due to three symmetrically placed pole pieces, each having a coil. The rotor has a single coil and is an electromagnet.



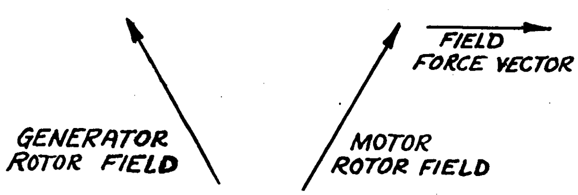


However, the units are intended to function as transformers in which the rotor coil is the primary and the field coils are the secondaries. If the signal generator rotor and the motor rotor are similarly placed relative to the field pieces, then the voltages induced in the pair of field coils are equal and opposite and no current will flow in any of the three field circuits. If, however, the rotors are not similarly placed, the voltages will not cancel and current will flow. This will set up a magnetic field which will act on both rotors, setting up a torque tending to cause the rotors to be similarly placed.

We can indicate the direction of the magnetic field by the following considerations. Let us suppose that the rotor of the motor has been removed. The effect of the induction due to the transformer rotor is, of course, to induce a current which sets up a magnetic field in the generator field poles opposite to the field of the rotor. Let us look at this first in the transformer or signal generator. Of course, there are three field poles, each with a magnetic field. But it is clear that the resulting magnetic field is opposite to the one that induced it. However, the current that flows in the generator poles also flows in the motor poles and sets up a precisely similar field there if we neglect resistance and other losses.

The effect of the motor rotor is, of course, analogous. However, actually the current that flows when both rotors are present must set up a field that corresponds to the vector difference between the two magnetic fields, one of which is opposite to the field of the generator rotor, the other opposite to the motor rotor field.

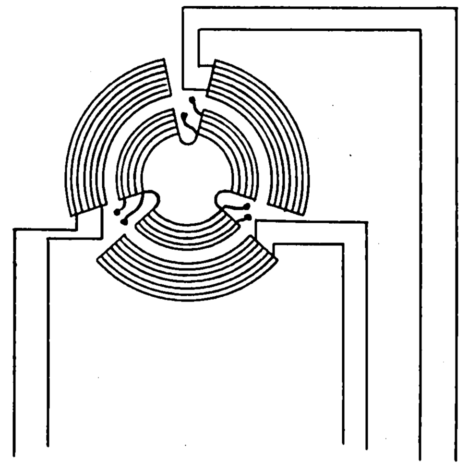
The effect of this can be readily obtained. Let us suppose that the generator rotor magnetic field and the motor field are as indicated. Let H_s denote the vector denoting the generator rotor magnetic field, H_m the corresponding vector for the motor, the field then is $k(H_m - H_s)$. The torque exerted by this field on the electromagnet H_m is given by $k(H_m - H_s) \times H_m = k\, H_m \times H_s$. This torque, of course,



is in the direction tending to bring H_m and H_s in coincidence and its size is proportional to the sine of the angle between them. The generator rotor has precisely the opposite torque exerted on it but presumably the input determines its position. Note however that this means that the input must do work if there is any load on the output.

However, torque amplification is possible if one uses an additional motor on the output shaft. The rotor is parallel to the previous motor rotor and so are the corresponding pole pieces. The current in the new motor field pole coils is controlled to be a multiple of the current in the original field coils. This is done by magnetic means.

Another selsyn unit of considerable importance is the differential selsyn. This unit is similar in appearance to the other selsyn units and the field pole pieces are the same. But the rotor has three coils and three faces precisely analogous to the field pole pieces.



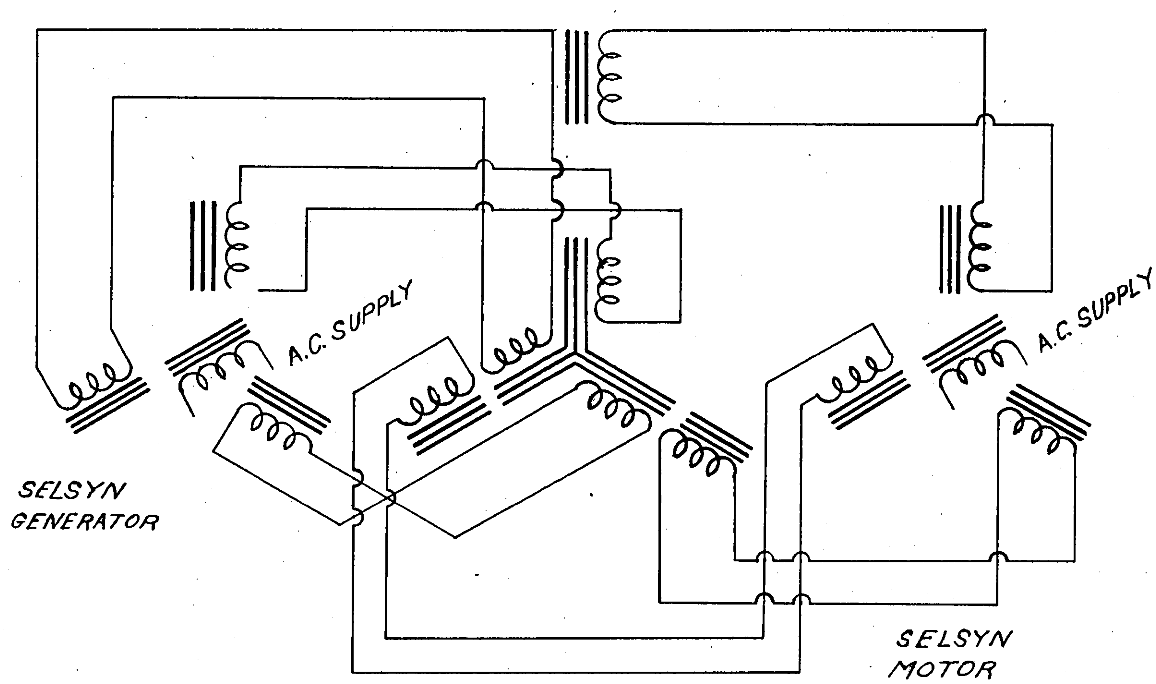
The use of the differential selsyn permits one to specify the difference between the input and the output rather than just insure equality. If we consider the effect of selsyn generator alone it is clear from the following diagram that the magnetic field set up by the rotor of the differential selsyn is a replica of that of the generator rotor relative to the field coils. Hence, if the rotor of the differential is positioned so that the rotor coils face the pole pieces, the field coils of the differential are affected the same way as those of the generator. Also rotating the rotor of the differential will change the induced currents in the field coils to those corresponding to a rotation of the same angle in the field of the generator. Thus if α is the rotation of the generator rotor, β that of the motor rotor and θ that of the differential rotor, for no torque we must have

$$\alpha + \theta = \beta.$$

The differential selsyn is particularly useful in situations where two quantities are to be added to yield a third and the three are far apart in space.

Another selsyn device of wide application is the selsyn control transformer. If a selsyn motor has no voltage impressed upon its rotor from the power source, the field itself will induce a voltage. This voltage is zero only when the receptor rotor is perpendicular to the position corresponding to the generator and its phase will show on which side of the perpendicular position the receptor rotor is. (There are, of course, two positions of zero voltage 180° apart.) This can be used to control an alternating current servo.

A selsyn system or its equivalent is valuable in



SELSYN DIFFERENTIAL

calculating devices because it permits one great freedom in connecting various units. Thus in the latest version of the differential analyzer at M.I.T. a selsyn system (based on capacity, however, rather than induction) is used to connect the units and, of course, this permits one the same freedom in the interconnecting that one would have with a purely electrical set-up.

PART THREE: THE SOLUTION OF PROBLEMS

I. Introduction

1. We have discussed in the above, various devices to perform mathematical operations. We now consider how these can be combined to solve mathematical problems.

There are a number of principles upon which such a combination can be based.

1. The Principle of Similitude. If one wishes to solve a mathematical problem, one sets up a physical situation which is governed by essentially the same equations and obtains the answer by measurements on the system.

2. Direct Calculation. This is suitable only for certain problems. In this the operational devices previously discussed are utilized and combined in a manner suitable for the calculation at hand.

3. Adjusters. In this the mathematical problem to be solved is set up as a calculational problem in which the unknowns are inputs. The values of the unknowns are then adjusted either by the operator or by an auxiliary calculating system until the requisite conditions are satisfied.

These principles are, of course, not mutually exclusive.

In the present chapter, we wish to discuss these principles in general. Our remarks are intended to be introductory in character. However, it would be advisable for the reader also to return to this chapter after he has read the remaining chapters of Part III, since certain remarks will then be clearer.

2. Let us briefly discuss the Principle of Similitude.

The devices based on this principle in general have the two advantages of being simple and relatively inexpensive. This is particularly apparent in the case of devices for the solution of partial differential equations.

In general, however, their accuracy is limited by the fact that there is no unidirectional flow of signal. In general, in such devices in an individual part one cannot clearly distinguish between inputs and outputs.

In order to make this last statement precise mathematically, let us consider an individual part F and let us suppose that it is connected to the rest of the device by connections whose state can be described by means of the variables, $x_1, \ldots, x_n$ The part F then determines a relationship between these variables, $F(x_1, \ldots, x_n) = 0$. However, this relationship will also depend upon which variables are inputs and which are outputs.

This can be easily illustrated by such a simple device as a pair of gears. For simplicity, we shall suppose that the gear ratio is one. Then if α is the amount of turn of one shaft and β that of the other, then one has the relationship

$$\alpha + \beta = C$$

or

$$\alpha + \beta = C - \delta$$

where δ is the backlash depending upon which one is driving the other.

If we consider even such a simple thing as a shaft, it is clear that the relationship between the amount of turn of the ends depends upon the driving relationship.

This variation in the mathematical relationship governing any part of the device is, of course, a limitation on the accuracy. To be effective, devices based on this principle must be simple and have relatively few parts.

3. The advantage of the other two types of problem solvers is that there is a one-directional flow of signal. For a given part, the variables are related by an equation

$$x_n = f(x_1, \ldots, x_{n-1})$$

(at least theoretically).

Each variable then has associated with it a part of which it is the output. In the complete device we have paths of signal flow and the first problem of design is the determination of the paths to be used.

One valuable result of the one directional flow of signal is the fact that digital devices may be utilized as part of the complete problem solver. For, in many instances, it is true that for a given over-all accuracy, this accuracy need only be maintained in a certain part of the circuit, i.e., in a certain signal path. In the other paths of the device, less accuracy is permissible and may even be desirable in view of the compensating speed of calculation.

II. Examples of Similitude Solvers

In the present chapter, we wish to discuss various devices, based on the principle of similitude. As we have mentioned before, it is not always easy to classify precisely a given device under one of the three principles mentioned, since a given device may be based on a number of principles. But the devices of the present chapter are characterized by a certain simplicity of the calculational parts so that a relatively complex mathematical result is obtained from few parts. In some of these, there is also a well defined flow of signal which, of course, is a great advantage.

We shall discuss devices of this sort for solving simultaneous linear equations, ordinary linear differential equations and partial differential equations. We shall point out that there is essentially two methods for attacking the latter problem. One involves the precise duplication of a situation governed by the equation. The other involves the replacement of the fundamental spatial region by a network of discrete points. The requisite finiteness obtained in this last fashion, however, can be obtained by other methods as well.

2. Linear Equation Solvers. A linear equation

$$\Sigma_{i=1}^{n} a_i x_i = b$$

can be realized in many ways. We have discussed a number of such in which the x_i's are inputs. For the present section, we must confine our attention to devices in which this restriction does not apply.

It is clear that a system of equations

$$\Sigma_{j=1}^{n} a_{ij} x_j = b_i$$

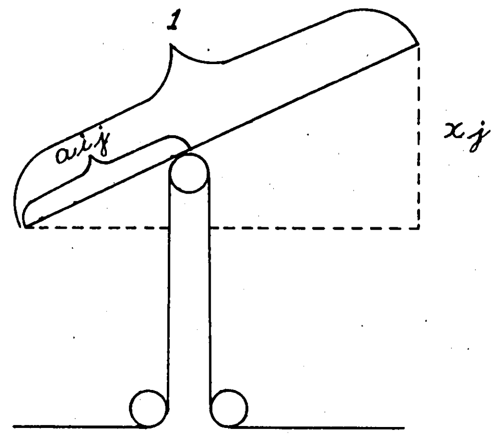
can be realized by gear boxes and differentials in such a fashion that when we set in the coefficients $a_{i,j}$ and turn the b_i's to the proper value then theoretically the values of the x_i's will be determined at the proper value. But, of course, this would be exceedingly expensive and as we see from our previous discussion the multiplicity of parts would limit the accuracy.

To use effectively the principle of similitude it is essential that the number of parts be kept as small as possible. We wish to mention two devices which have been constructed in which it certainly seems as if the number of parts is a minimum.

One device is that of Wilbur as described in the "Tech" Engineering Journal of M.I.T., vol. 16 (1935), pp. 48-49, 56, 60. Each equation is represented by a tape and each variable by the sine of the angle of rotation of a shaft. The part of the device corresponding to $a_{ij} x_j$ is illustrated above. The tape corresponding to the i equation passes over pulleys in such a fashion that the tape is shortened in length along the line corresponding to the equation by an amount $a_{ij} x_j$. This tape is part of an endless chain adder.

For each equation we have a device like this for each variable. Thus the total shortening of the tape is $\Sigma_j a_{i,j} x_j$. Now if we can permit this to equal b_i, then we will have realized the equation. The reader is referred to the discussion of the endless chain adder in Part II, Chapter I, Sec. 1.

A later version of this device is described in Wilbur, J. B. "The Mechanical Solution of Simultaneous Equations," *Journal of Franklin Institute*, Vol. 222 (1936), pp. 715-24.



More effective and definitely more expensive is the well known machine of Mallock. Here each unknown is represented by the flux in a transformer and each equation is represented by a closed circuit consisting of a series of coils, one around each variable transformer and one around the transformer corresponding to the constant term. We have then a coil for each coefficient a_{ij} and the number of turns on the coil is proportional to $a_{i,j}$.

If B_j is the total flux in the x_j transformer, then the voltage across the coil $a_{i,j}$ is

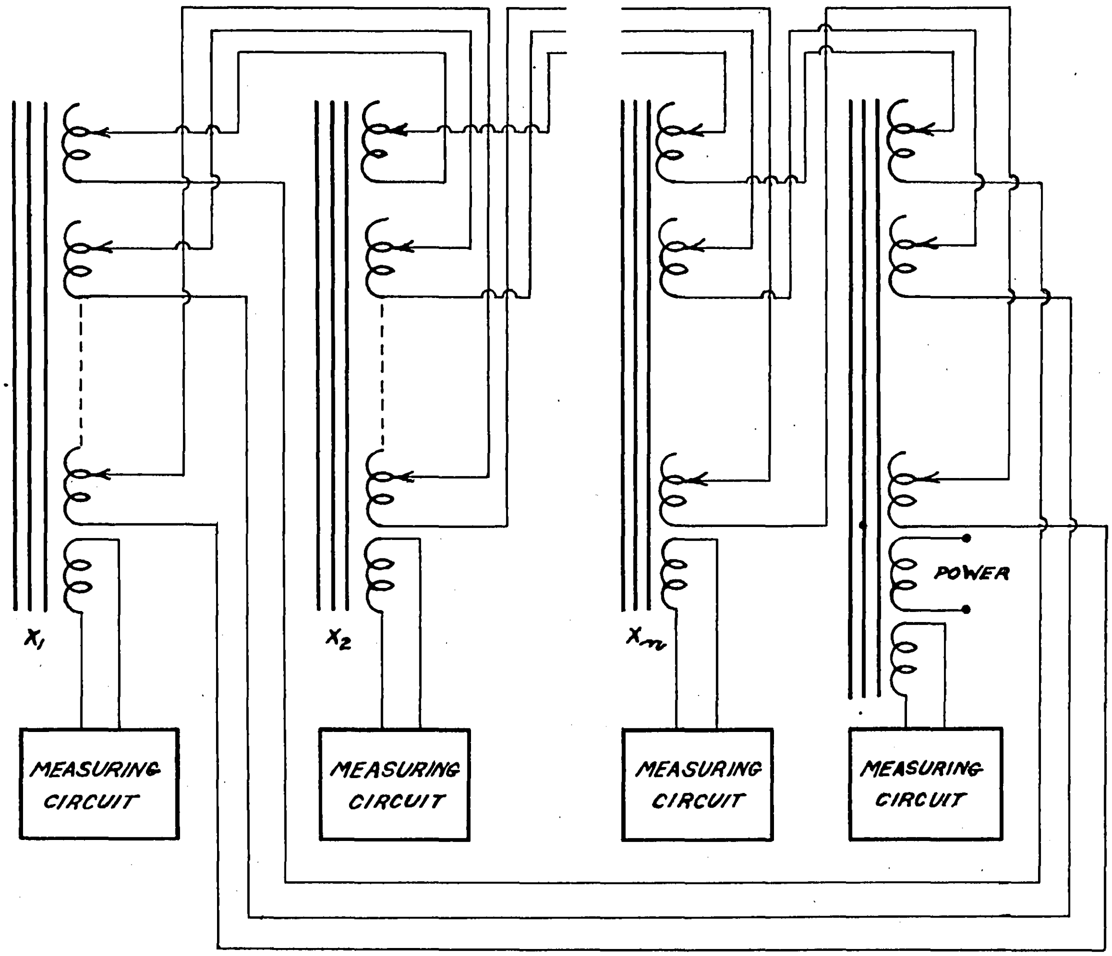
$$a_{i,j} \frac{dB_j}{dt}$$

The coils are in series and presumably we can compensate by electronic methods for the resistance loss in each circuit. (Cf. Part II, Chapter IV, Sec. 6.) Hence if we go around the circuit corresponding to the i'th equation we find

$$a_{i,1} \frac{dB_1}{dt} + a_{i,2} \frac{dB_2}{dt} + \ldots + a_{i,n} \frac{dB_n}{dt} - b_i \frac{dB_0}{dt} = 0$$

The circuit is indicated in the diagram top of p. III - 3. On the constant transformer, besides the equation coils we also have a power coil or primary and another measuring coil of a fixed number of turns. There is also a similar measuring coil on each of the variable transformers. Each measuring coil is connected across a circuit which is essentially an alternating current voltmeter, so that the flux change in each transformer is measured. An alternating current is applied to the power coil. This induces voltages in each equation circuit which in turn induce flux in the various variable transformer. Except perhaps for a brief period, the above equations are satisfied. The power is adjusted until the constant measuring circuit indicated the value 1, in which case the other measuring circuits indicate the values of the variables.

It is clear that an equation can be represented by the torques on a shaft. Here the coefficients are the distances from the axis and the variables represented by forces. The forces can be equalized



between different shafts by hydraulic methods. There was a German device of this nature about half a century ago.

A machine for finding the roots of a polynomial equation is described in the paper: Hart, H. C., and J. Travis. *Journal of Franklin Institute*, 225 (1938), pp. 63-72.

Another device for solving linear equations by mechanical means is that of Schuman, T. E. W. *Philos. Mag.* 29 (1940), pp. 258-73.

A real symmetric matrix can be represented by a network in a number of ways. Consider a simple network made up of n + 1 points, $A_0, A_1, \ldots, A_n$. At each point A_i, $i \geq 1$, we have a current generator which generates a current I_i. Suppose that each pair of nodes is connected by a conductor of conductance $Y_{i,j}$ which is the reciprocal of the impedance $Z_{i,j}$. Then the current equation for the i'th node becomes

$$- Y_{i,1}\, e_1 - Y_{i,2}\, e_2 + \ldots + (Y_{i,0} + Y_{i,1} + \ldots + Y_{i,n})\, e_i + \ldots$$

$$- Y_{i,n}\, e_n = I_i$$

Now it has been proposed to use this as a method of solving the system of linear equations

$$a_{i,1}\, x_1 + a_{i,2} + \ldots + a_{i,n}\, x_n = b_i, \quad i = 1, \ldots, n$$

where the $a_{i,j}$'s and b_i's are real and $a_{i,j} = a_{j,i}$. Obviously, we can consider the voltages e_i as the unknowns x_i. The $a_{i,j}$'s correspond to the conductances $Y_{i,j}$'s. Because of stability questions, one would prefer to use passive impedances. If resistors are used, the $Y_{i,j}$'s are all positive and $a_{i,j}$'s are clearly restricted. On the other hand, if one uses reactances $Y_{i,j} = -\sqrt{-1}/(L_{i,j}\,\omega - 1/\omega C_{i,j})$, these restrictions disappear and theoretically any matrix with real coefficients can be realized for input currents of a specified frequency.

In this case, the b_i's must be realized as current generators having a specified current output, i.e., they must be obtained from constant current generators. If one uses a normal adjustable current source with a measured output, one would have to adjust each source until all the I_i's coincide with the b_i's. However relatively constant current sources are possible. A constant current generator is one with infinite internal impedance so that variations of the external connections have no effect. This situation can be approximated by a high voltage source and a large adjustable resistance which can be set for the desired current. If this is inadequate, electronic constant current sources are available based on the high plate impedance of a pentode which becomes effectively even higher when an unbiased cathode resistor is used. Cf. Puckle, O. S., *Time Bases*. New York: John Wiley & Sons. For direct current signals the plate current itself can be used but for other frequencies, a transformer coupling would be used.

The use of inductances is not desirable because of their expense. However, by doubling the number of nodes, so that for each x_i both a voltage and its negative appears, it is possible to use only capacities, as in the paper of Many and Meiboom, *Review of Scientific Instruments*, Vol. XVIII, No. 11, pp. 831-36, or only resistors to realize a matrix. The objective of the Many and Meiboom paper is to obtain the characteristic roots of a matrix rather than to solve a system of equations but the problem of realizing the matrix is the same.

In the double node case, one has 2n + 1 nodes, $A_{-n}, A_{-n+1}, \ldots, A_0, A_1, \ldots, A_n$. The circuit is to be set up in such a way that relative to A_0 or ground, the A_{-i} voltage is to be the negative of the A_i voltage, i.e., the unknown x_i corresponds to a voltage e_i of A_i and A_{-i} has voltage $-x_i$. An equation $a_{i,1} x_1 + \ldots + a_{i,n} x_n = b_i$ is realized and also the equation obtained by multiplying both sides of -1. If $a_{i,j}$ is positive, we connect A_i and A_j by a conductance of this value and we do the same with A_{-i} and A_{-j}. On the other hand, if $a_{i,j}$ is negative, we connect A_i with A_{-j} and we connect A_{-i} with A_j. Finally at A_i and A_{-i}, we locate current sources which are such that I_i and I_{-i} are always negatives of each other. For suitable current sources, for instance, transformer output sources this can be accomplished by using two terminals of the source.

To obtain the characteristic roots of a real symmetric matrix, in the manner used by Many and Meiboom, the matrix $a_{i,j}$ is realized by a capacitator network as indicated above. For this realization, it may be necessary to add a constant term to the diagonal elements of the matrix but this term is simply added to the characteristic roots and offers no difficulty. In addition, an inductance L is connected between each node and A_0. Let $\sqrt{-1} = j$. Then the current equations become

$$j(a_{1,1}\omega - 1/L\omega)\, e_1 + j\, a_{1,2}\, \omega\, e_2 + \ldots + j\, a_{1,n}\, \omega\, e_n = I_1$$

$$j\, a_{2,1}\omega e_1 + j\,(a_{22}\omega - 1/L\omega)\, e_2 + \ldots + j\, a_{2,n}\omega e_n = I_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$j\, a_{n,1}\omega e_1 + j\, a_{n,2}\,\omega e_2 + \ldots + j\,(a_{nn}\omega - 1/L\omega)\, e_n = I_n$$

If we divide these by ω and let $\lambda = 1/L\omega^2$, we see that we have represented the system

$$(a_{1,1} - \lambda)\, e_1 + a_{1,2}\, e_2 + \ldots + a_{1,n}\, e_n = I_1/j\omega$$

$$a_{n,1}\, e_1 + \ldots + (a_{nn} - \lambda)\, e_n = I_n/j\omega.$$

Now we can solve, say, for e_1 and obtain

$$\Delta(\lambda)\, e_1 = I$$

where I is a linear combination of $I_1, \ldots, I_n$. In the theoretical case of no resistance, there are frequencies for which $\lambda = 1/L\omega^2$ yields $\Delta(\lambda) = 0$. These are the resonance frequencies of the matrix and the values of λ are the characteristic roots of the matrix. Here λ must be positive but we have already insured this when we added a constant to the diagonal terms in order to obtain a matrix which can be realized by capacitative elements.

With actual matrices of finite Q, the voltage response is humped at the resonant points and the current decreases to a minimum. But this is not used. At resonance, the input currents and the voltages are in phase and these points can be sharply detected by an oscilloscope.

Another method of locating the characteristic roots of a matrix is given in a paper of Lusternik in the *Compte Rendus (Doklady) de l'Academie des Sciences de l'U.R.S.S.*, Vol. 55, No. 7 (1947), pp. 575-78. This is based on a consideration of transients. The method of realizing a matrix is not indicated in this paper, but if a matrix is realized by a set of nodal equations and if we solve for a voltage e_1, we obtain a differential equation

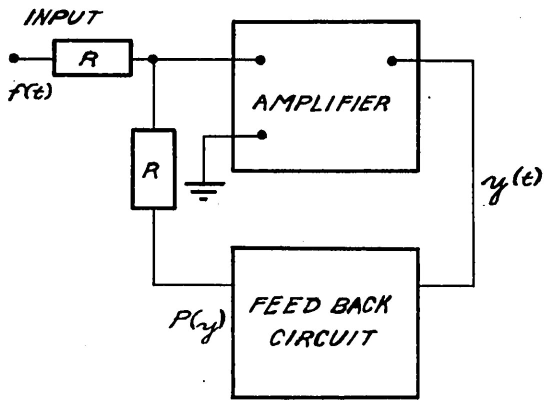
$$\Delta\, e_1 = I$$

Δ is now a differential operator since we must consider transient phenomena. The solution e_1 of this differential equation is a linear combination of exponentials $\exp(-\lambda_k x)$ where λ_k is one of the characteristic roots. The coefficients of the exponentials depend upon the choice of initial values for $e_1, \ldots, e_n$. Now the term with least λ_k will decay more slowly than any other and we would expect that after a while this term will be predominant. Thus the decay characteristics of the the transients, after a settling period, should indicate the least characteristic root.

It is desirable to represent the transients on an oscilloscope. One can blank out the part which is not appropriate. The initial voltages are impressed on the circuit, which represents the matrix by a pulse controlled circuit and the same pulse also controls the sweeps and the blanking voltages of the oscilloscope.

When the least characteristic root has been determined, the corresponding characteristic vector must also be found. Presumably the components of this vector could be obtained by comparing the transient voltages at the different nodes when the exponential terms of least λ have become predominant. To locate the next lowest characteristic root, a set of initial voltages orthogonal to the characteristic vector previously obtained is used as above, since the resulting transient will not contain an exponential term for the least characteristic root.

3. It is possible to use a feedback amplifier to invert operators of a certain kind. This idea seems to be well known now. The input is a certain function f(t), the output is y(t). The feedback circuit takes y(t) and applies an operator P(y) to



it. The resistances R are equal, consequently the input to the amplifier is the average of f(t) and P(y)

$$f(t) + P(y) = 2\varepsilon(t)$$

Ignoring noise, we see that the output $y(t) = \alpha\varepsilon(t)$ and hence

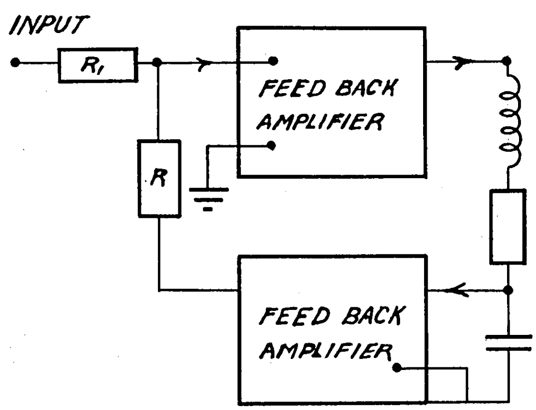
$$f(t) + P(y) = \frac{2}{\alpha} y(t)$$

Letting $Q(y) = - P(y) + (2/\alpha)\, y(t)$, we see that

$$f(t) = Q(y)$$

which is an equation which must be satisfied by the output.

This can be used to realize any second order linear differential operator in a simple way. For instance, if our feedback circuit is an ordinary linear series network as shown in diagram below,



we have that the charge q on the condenser satisfies the equation

$$y = \frac{q}{c} + R\frac{dq}{dt} + L\frac{d^2q}{dt^2}$$

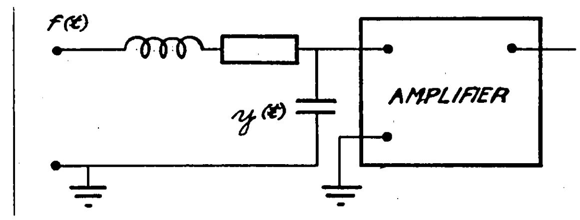
Now if our lower amplifier has a feedback ratio of unity we have

$$f(t) = \frac{q}{c}$$

if we neglect $2y/\alpha$. Eliminating q, we obtain that

$$y = f(t) + CR\frac{df}{dt} + CL\frac{d^2f}{dt^2}$$

Notice that this is precisely the opposite to the circuit whose output is the solution of the differential equation



Here if y(t) is the voltage across the condenser we have

$$f(t) = y + RC\frac{dy}{dt} + LC\frac{d^2y}{dt^2}$$

Cf. J. R. Ragazzini, R. H. Randall, F. A. Russell. *Proceedings of I.R.E.*, XXXV (1947), pp. 440 ff.

4. In a number of cases, the solution of systems of partial differential equations by analogy methods is well developed. For instance, two dimensional problems in stress or strain are mathematically analogous to the flexure problem of a thin plate. Since instruments for measuring curvature by optical methods are available, this can be used for the analysis of stress in large slabs. Cf. R. D. Mindlin. *Quarterly of Applied Mathematics*, IV (1946), pp. 279-90. This paper contains references to earlier work.

For linear partial differential equations, a change of scale is an effective method of study by analogy. For non-linear partial differential equations, the use of small scale models is particularly desirable on account of the mathematical complexity of these problems, particularly problems in fluid dynamics. The scale difficulties are partly compensated for by the use of Reynolds numbers.

The Laplacian can be solved by model systems in a number of ways. Let us suppose that we wish to obtain a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

for a particular region S, subject to certain boundary conditions.

Now the electric potential function V satisfies this equation in any conductor. It is convenient to take as our conductor, an electrolytic solution. We take a container in the shape of the region S and fill it with the solution. To approximate the boundary conditions, we place various conductors on the non-conducting walls of the container. If the boundary condition specifies that the function u has a certain value at a certain point on the boundary, then we maintain the conductor at this point on the boundary at the corresponding potential. On the other hand, if the boundary condition at a certain point specifies $\frac{\partial u}{\partial n}$, the electrical equivalent of this is to determine the current flowing across a unit area of boundary at this point. Constant current electronic devices can be set up to accomplish this. They can also be set up to establish linear relationships between u and $\frac{\partial u}{\partial n}$, which occur in certain types of boundary problems. The value of V is obtained by means of probes.

For two-dimensional Laplacian problems, a slab of a high resistance conductor can be used instead of the region S.

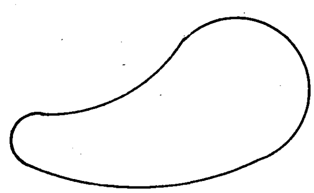
5. Another method for the solution of partial differential equations involves the replacement of the fundamental region by a network of points. This method for the analysis of physical problems actually predates the use of partial differential equations and is still very important. For instance, the recent work of Kron presents methods for the solution of Maxwell's equations and the Schrödinger equation. We present the following two recent references which contain references to previous work: *Journal of Applied Mathematics*, Amer. Soc. of Mechanical Engineers (1944), pp. A149-61. *Journal of Applied Physics*, Vol. 16 (1945), pp. 172-85.

We do not have time in our course for an adequate discussion of this method. Instead we will indicate the solution of a specific problem.

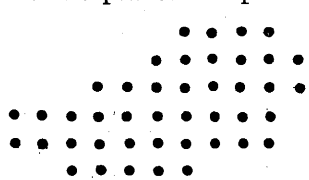
Let us try to solve the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)\ u$$

on a region S, with a specified system of boundary conditions



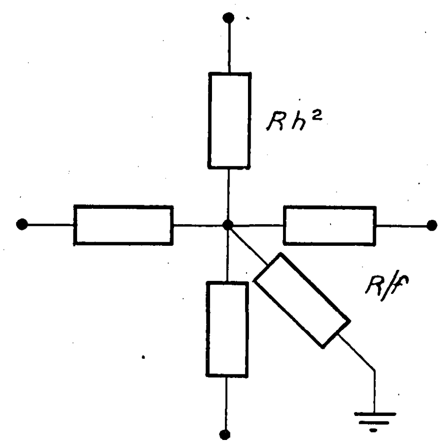
The first step in the approximate solution of this problem is to replace the given region by a network of points. The points are part of a rectangular lattice



and those points which can be connected to four points of this set correspond to interior points of the original S.

We set up an electrical network with junction points corresponding to the points of this set in which two adjacent points are connected by resistances all of which have the same value. Each junction point is also connected to the ground by a resistance inversely proportional to $f(x_{x,y})$.

Let us now consider an interior junction point



Let h denote the change in x, between two successive lines of the lattice, k that in y. Let u denote potential of the point. Then our current equation for the junction is

$$\frac{u(x+h,y) - u(x,y)}{Rh^2} + \frac{u(x-h,y) - u(x,y)}{Rh^2} + \frac{u(x,y+k) - u(x,y)}{Rh^2} + \frac{u(x,y-k) - u(x,y)}{Rh^2} - \frac{f}{R}u(x,y) = 0$$

Or

$$\frac{u(x+h,y)+u(x-h,y) - 2u(x,y)}{h^2} + \frac{u(x,y+k)+u(x,y-k)-2u(x,y)}{h^2} = f(x,y)\ u(x,y).$$

The expression on the left hand is an approximation to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and hence our u is an approximate solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)\ u(x,y)$$

For a somewhat more complete discussion the reader is referred to Gutenmacher, L., *Compte Rendus (Doklady), de l'Academie des Sciences de l'U.R.S.S. (N.S.)*, Vol. 27 (1940), pp. 198-202.

III. Direct Calculators

1. We have devices for the performance of the operations of addition, multiplication and integration. The inverses of these operations are also available as we shall see.

The present chapter is concerned with devices which are essentially combinations of such operators. Again we repeat that the classification adopted for this part is not precise. However, in each class it is possible to emphasize certain characteristics. For the present chapter, these are the use of operators designed for a single mathematical operation and the fact that the problems treated do not require an adjustment of the initial inputs. In the course of the calculation certain feedbacks and adjustments may be made but these appear only in subsidiary procedures.

2. It might be well to begin this chapter by pointing out that undoubtedly the most widespread methods for the solution of differential equations involve the use of ordinary calculating machines. These methods are such that their accuracy, in most cases, is limited only by the time available for the calculation.

We will not have time to discuss the various methods of finite differences upon which these methods are based. The basic principles, however, can be simply illustrated by an example. For this we avail ourselves of certain literature of the Marchant Calculating Machine Company (1475 Powell Street, Oakland, Calif.) M.M. 260, 261; on the Milne Method (W. E. Milne, *Amer. Math. Monthly*, 33 (1926), pp. 455-60, also 40, pp. 322-27). However, we wish to emphasize that a study of the fundamentals of finite difference theory is essential to anyone who wishes to apply calculating machines to solving differential equations. We shall see also that the existence and uniqueness theories for differential equations are necessary.

Suppose we wish to solve the system of differential equations

$$y' = f(x,y,z)$$
$$z' = g(x,y,z)$$

where the functions f and g can be evaluated by means of tables and the usual operations of calculating machines. We first change these to a system of integral equations:

$$y = y_0 + \int_{x_0}^{x} f(x,y,z)\, dx$$
$$z = z_0 + \int_{x_0}^{x} g(x,y,z)\, dx$$

One method of utilizing arithmetical calculating devices involves the replacing of these integrals by approximations. There are many types of approximations to integrals available in the theory of finite differences with different properties and the methods involve the selection and use of suitable ones of these.

The general procedure is a step-by-step process in which the values of y and z are calculated for equally spaced values of x. Let us suppose that we have

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_{n-2},\ x_{n-1}$$
$$y_0,\ y_1,\ y_2,\ \ldots,\ y_{n-2},\ y_{n-1}$$
$$z_0,\ z_1,\ z_2,\ \ldots,\ z_{n-2},\ z_{n-1}$$

and wish to continue to x_n and z_n. We first replace our integral equation by an equivalent system

$$y_n = y_{n-p} + \int_{x_{n-p}}^{x_n} f(x,y,z)\, dx$$
$$z_n = z_{n-p} + \int_{x_{n-p}}^{x_n} g(x,y,z)\, dx$$

where p depends upon the accuracy desired, the choice of intervals between the x's and the size of certain derivatives of the y's. We shall indicate in a moment the essential point in its determination.

The determination of y_n and z_n begins with an estimate $y_n^{(o)}$, $z_n^{(o)}$ based on the previously determined $y_0, y_1, \ldots, y_{n-1}$. This can be obtained by extrapolation based on the successive finite differences.

The estimate y_n^o, z_n^o is used as the first step in an iterative solution of the set of equations

$$y_n = y_{n-p} + \Sigma_{j=0}^{p} c_j\, f(x_{n-p+j}, y_{n-p+j}, z_{n-p+j})\, d$$
$$z_n = z_{n-p} + \Sigma_{j=0}^{p} c_j\, g(x_{n-p+j}, y_{n-p+j}, z_{n-p+j})\, d$$

where the finite sum represents an approximation to the previous integrals. The iterative process is the direct one, in which $y_n^{(k-1)}$, $z_n^{(k-1)}$ is substituted in the right-hand side to yield $y_n^{(k)}$, $z_n^{(k)}$.

In using a method like the above, it is necessary to use p "starting values," which must be obtained from some other method, for instance, the Taylor Series for the solution.

The essential difficulty, of course, with any finite approximation of the above sort is that errors are cumulative. It is this fact that makes the simplest approximation

$$y_n = y_{n-1} + d f(x_{n-1}, y_{n-1}, z_{n-1})$$
$$z_n = z_{n-1} + d g(x_{n-1}, y_{n-1}, z_{n-1})$$

so inefficient. In general d, the distance between two successive values of x, must be taken very small to yield comparative accuracy. The use of three terms is itself a tremendous improvement.

The accumulation of errors indicates also that to be certain of the degree of accuracy, one should also calculate the last value of the table by precise methods, say the Taylor Series as a check. Alternately a number of intermediate values of the functions may be obtained for check purposes.

In some cases, the direct use of the Picard iterative method of obtaining a solution can be used. This is effective when an approximate solution in the form of a polynomial is known and the equations are not too complicated algebraically. It is, of course, precise. The nth approximation $y^{(n)}(x)$ is obtained from the n-1 st by the formula

$$y^{(n)}(x) = y_o + \int_{x_0}^{x} f[x, y^{(n-1)}, z^{(n-1)}]\, dx$$
$$z^{(n)}(x) = z_o + \int_{x_0}^{x} g[x, y^{(n-1)}, z^{(n-1)}]\, dx$$

The convergence of this process is discussed in the existence theory for differential equations. In the practical cases the integration is carried out formally, machines being used in the various computations.

3. The differential analyzer is essentially a combination of integrators, gear boxes and differentials which is used to solve systems of ordinary differential equations. To appreciate the method of use, we will first give a theoretical discussion of the problems which may be solved by such a combination. This discussion is essentially that of the paper by C. Shannon in the *Journal of Mathematics and Physics*, M.I.T., Vol. XX (1941), pp. 337-54. (Our discussion is not as inclusive.)

Now it is clear that if we have n integrators and let W_i denote the output, U_i the linear input and V_i the angle input of the ith integrator, then we have the equations

$$(A_{0,1}) \qquad \frac{dW_i}{dx} = U_i \frac{dV_i}{dx}$$

for i = 1, ..., n. The U_i's and V_i's must be linear combinations of the other U_i's, V_i's, W_i and x. Since they must be uniquely determined, it must be possible to solve for these explicitly

$$(A_{0,2}) \qquad \begin{aligned} U_i &= a_{i,o} + a_i' x + \Sigma_{j=1}^n a_{i,j} W_j \\ V_i &= b_{i,o} + b_i' x + \Sigma_{j=1}^n b_{i,j} W_j. \end{aligned}$$

Thus the differential analyzer permits us to solve any differential system in the above form and also any differential equation system whose unknowns are a subset of the variables of a set of the above type.

We shall have to discuss the equivalence of differential equations and also a process of "expanding" the system. If we add an unknown to the system and an equation in such a fashion that if we eliminate the new unknown, we get the original system back, then we have expanded the system. The elimination process may involve an integration and a choice of the constant of integration. It is clear that if we solve the expanded system, we obtain solutions of the original system. The above form for a system is remarkably inclusive although this is not apparent. To show this, let us take what may appear at first to be a special case. Let us suppose that we have n^2 integrators, so that i=1, ..., ..., n^2 for the equation

$$\frac{dW_i}{dx} = U_i \frac{dV_i}{dx}.$$

Now we wish to replace the subscripts i = (k-1)n + l-1, j = (p-1)m + q-1 by the pairs, k,l and p,q respectively. If we do this we get

$$(A) \qquad \begin{aligned} \frac{dW_{k,l}}{dx} &= U_{k,l} \frac{dV_{k,l}}{dx} \\ U_{k,l} &= a_{k,l,o} + a_{k,l}' x + \Sigma_{p,q} a_{k,l,p,q} W_{p,q} \\ V_{k,l} &= b_{k,l,o} + b_{k,l}' x + \Sigma_{p,q} b_{k,l,p,q} W_{p,q}. \end{aligned}$$

Let

$$a_{k,l,o} = A_k, \; a_{k,l}' = A_k', \; a_{k,l,p,q} = A_{k,p,q}$$

$$b_{k,l,o} = A_l, \; b_{k,l}' = A_l', \; b_{k,l,p,q} = A_{l,p,q}.$$

Then $U_{k,l}$ does not depend upon l, $V_{k,l}$ does not depend upon k and indeed $U_{k,l} = V_{s,k}$ for every pair of l,s with each less than n. Suppose then we let $w_k = U_{k,l} = V_{s,k}$. Then the above system of equations are such that

$$(A') \qquad \frac{dW_{k,l}}{dx} = w_k \frac{dw_l}{dx}$$

$$w_k = A_k + A_k' x + \Sigma A_{k,p,q} W_{p,q}, \; k-1, \ldots, n$$

If we differentiate the last n equations and eliminate the $\frac{d}{dx}(W_{p,q})$ by means of the others we get the system

$$(C) \qquad \begin{aligned} \frac{dw_k}{dx} &= A_k' + \Sigma_{p,q} A_{k,p,q} w_p \frac{dw_q}{dx} \\ &= A_k' + \Sigma_q P_{k,q} \frac{dw_q}{dx} \end{aligned}$$

where $P_{k,q} = \Sigma_p A_{k,p} w_p$. Now if we suppose that one of the equations C is in the form $\frac{dw}{dx} = 1$, this system is clearly equivalent to the system

$$(D) \qquad \frac{dw_k}{dx} = P_{k,o} + \Sigma_{q=1}^n P_{k,q} \frac{dw_q}{dt}$$

where $P_{k,o}$ and the $P_{k,q}$'s are arbitrary linear combinations of the w's and x. We have proven:

Theorem I. Any system of differential equations in the form D can be expanded into a system in the the form A and hence can be set in the differential analyzer. Conversely every system in the form A can be contracted to the form D. (Eliminate the U's and V's.)

Notice that the equations A' give the exact connections for the analyzer. The $W_{k,l}$'s are the outputs of the integrators, the w_k's are linear combinations of these. The upper set of equations show how the integrators are to be connected, i.e., their inputs, while the lower set shows what linear combinations, w_j, are to be taken.

Again we can remark that D is more general than it appears. To show this, we first establish the general lemma.

Lemma. Let

$$(E_m) \qquad \frac{dw_i}{dx} = P_{i,o} + \Sigma_{j=1}^n P_{i,j} \frac{dw_j}{dx} \; i=1, \ldots, n$$

denote a system of differential equations in which the $P_{i,j}$'s are polynomials in x and $w_1, \ldots, w_n$ of the mth or lower degree where m is > 1. Then the system E_m can be expanded into a system E_{m-1} of the same sort in which the degree of the P's is m-1 or lower.

Proof. For simplicity in our discussion, let us add the dependent variable $w_0 = x$. Our system is then

$$(E_m) \qquad \frac{dw_i}{dx} = \Sigma_{j=0}^{n} P_{i,j} \frac{dw_j}{dx},\ i=0,\ 1,\ \ldots,\ n.$$

To prove the lemma, we append the variables $z_{i,j} = w_i w_j$, $i,j = 0, \ldots, n$ by adding the $(n+1)^2$ equations

$$(E_{m-1,0}) \qquad \frac{dz_{i,j}}{dx} = w_i \frac{dw_j}{dx} + w_j \frac{dw_i}{dx}$$

each of which is in the E_m form but with linear P's. Returning to the original system E_m, we see that in every monomial of degree 2 or larger, we may substitute for a product $w_i w_j$ a $z_{i,j}$. This process will lower every polynomial of degree greater than 1 in the set.

$$(E_{m-1}') \qquad \frac{dw_i}{dx} = \Sigma_{j=0}^{n} P_{i,j}(w,z) \frac{dw_j}{dx}.$$

The enlarged set (E_{m-1}), consisting of $(E_{m-1,0})$ and (E_{m-1}') clearly has the desired properties.

It follows that:

Theorem II. Every set E_m can be expanded into a set D and hence can be solved by the differential analyzer.

Suppose in E_m, we replace our polynomials $P_{i,j}$ by rational functions $R_{i,j} = \frac{P_{i,j}}{Q_{i,j}}$ and obtain the system

$$(F) \qquad \frac{dw_i}{dx} = \Sigma_{j=0}^{n} R_{i,j} \frac{dw_j}{dx}.$$

Theoretically, at least there is no loss in generality in assuming that all the denominators are the same and thus that (F) can be written

$$(F') \qquad \frac{dw_i}{dx} = \frac{1}{Q}\Sigma_{j=0}^{n} P_{i,j} \frac{dw_j}{dx}.$$

We wish to expand this to a system in the form E_m. We introduce the variables $v_1 = Q$, $v_2 = 1/v_2$. This leads to the expanded system

$$\frac{dw_i}{dx} = \Sigma_{j=0}^{n} v_2 P_{i,j} \frac{dw_j}{dx}$$

$$\frac{dv_1}{dx} = \Sigma_{j=0}^{n} \left(\frac{\partial Q}{\partial w_j}\right) \frac{dw_j}{dx}$$

$$\frac{dv_2}{dx} = -v_2^2 \frac{dv_1}{dx}$$

which is in the E_m form, since $\frac{\partial Q}{\partial w_j}$ is a polynomial in $w_0, \ldots, w_m$. Hence, we have:

Theorem III. Every system in the form (F) can be expanded into the form E_m and hence set up in the differential analyzer.

In the latest differential analyzers, one has a servo feedback mechanism that permits one to use the angle input as the output. Thus if U and V are inputs and W is the output, we have

$$\frac{dV}{dx} = U \frac{dW}{dx}$$

or

$$\frac{dW}{dx} = \frac{1}{U} \frac{dV}{dx}.$$

This permits a more efficient use of integrators when fractions are present.

However, our system (F) can be expressed in a simpler form

$$(G) \qquad \frac{dw_i}{dx} = R_i$$

where R_i is rational in the w's. For it is clear that the system (G) are special cases of the systems (F) and on the other hand, every system (F) is equivalent to a system in the form (G) as one can see by solving explicitly for the $\frac{dw_i}{dx}$.

The form (G) for a system of ordinary differential equations is, of course, a well known one. There is, of course, a well known process by which one can take equations of higher order than the first and express it in the form (G). Thus if we have a differential equation

$$\frac{d^k y}{dx^k} = f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{(dx)^2}, \ldots, \frac{d^{k-1} y}{dx^{k-1}}\right)$$

we may introduce unknowns $w_0 = x$, $w_1 = y$, $w_2 = \frac{dy}{dx}$, ... $w_k = \frac{d^{k-1}}{dx^{k-1}}$ and set up the system

$$\frac{dw_0}{dx} = 1$$

$$\frac{dw_1}{dx} = w_2$$

$$\frac{dw_{k-1}}{dx} = w_k$$

$$\frac{dw_k}{dx} = f(w_0, w_1, \ldots, w_k)$$

which is in the form (G) if f is rational.

However, this does not exhaust the possibilities for the differential analyzer. Consider a system

$$(H) \qquad f_i = 0 \qquad i=1, \ldots, n$$

where f_i is a polynominal in $w_1, \ldots, w_n$ and their derivatives. In general, the system (H) can be expanded into a system in the form (G).

Let us differentiate the system (H). The result will be a set of equations which are linear in the derivatives of highest order which appear. We may solve for these highest derivatives explicitly and then by introducing more variables as in the above example, obtain a system in the form (G).

We will make only one further extension. Let the system of equations on $w_1, \ldots, w_n$ and their derivatives

$$\text{(I)} \qquad f_i = 0$$

be constructed by the use of the rational operations and by functions of one variable which are themselves solutions of algebraic differential equations. Such a system can be expanded into the form (H).

Let F(x) be a non-rational function occurring in the f's, with an argument z which is a rational function of the w's. Of course, some of the functions used may themselves have non-rational functions in their arguments. But then we may go further in until we meet an F such as described. Thus, we see that there is always an F as described.

F may have a number $z_1, \ldots, z_r$ of rational arguments in different places in the system of equations. Let us consider only $F(z_1) = y$. We may use the equations

$$\frac{dF}{dz}\frac{dz_1}{dx} = \frac{dy}{dx}$$

$$\frac{d^2F}{dz^2}\left(\frac{dz_1}{dx}\right)^2 + \frac{dF}{dz}\frac{d^2z}{dx^2} = \frac{d^2y}{dx^2}$$

to express $\frac{dF}{dz_1}, \frac{d^2F}{dz_1^2}, \ldots,$ rationally in terms of $w_0, \ldots, w_n$, y, and their derivatives. Now F is determined by an equation

$$g\left(z_1, F, \frac{dF}{dz_1}, \ldots, \frac{d^nF}{dz_1^n}\right) = 0$$

by hypotheses. If we substitute for z_1, F, $\frac{dF}{dz_1}$, etc., the values given above, g will become an algebraic equation on the w's, y and their derivatives. If in the original system, wherever $F(z_1)$ occurs we now substitute y, we will have expanded our system to one which does not contain $F(z_1)$.

The above process is repeated for $F(z_2), \ldots, F(z_n)$.

It is clear that the above process can be continued until all the non-rational functions are removed.

Thus we have established:

Theorem IV. Every system of differential equations in the form (I) can be expanded into a system of the form A and hence can be set in the differential analyzer.

The above theoretical discussion is, I think, very interesting in itself, but the major reason for considering it is because it gives us a step-by-step process for finding the system (A_0) which we must have if we are to use the differential analyzer to solve a system of differential equations say in the form (I).

It is clear that the essential steps in using a differential analyzer is first to expand the given system into a system in the form (A_0) making a careful note of how the constants of integration are concerned in the expansion process and then setting up the analyzer.

Let us take a few examples:

Suppose we have a system

$$\frac{dy}{dx} = x + y\frac{dz}{dx} - z\frac{dy}{dx}$$

$$\frac{dz}{dx} = y.$$

We wish to find the solution which at $x = x_0$ has $y = y_0$ and $z = z_0$.

This system is in the form D. We rewrite it

$$\frac{dy}{dx} = x\frac{dx}{dx} + y\frac{dz}{dx} - z\frac{dy}{dx}$$

$$\frac{dz}{dx} = y\frac{dx}{dx}$$

Our discussion above shows that it will be adequate to introduce four integrators with outputs, W_1, W_2, W_3, W_4 with

$$\frac{dW_1}{dx} = x\frac{dx}{dx}$$

$$\frac{dW_2}{dx} = y\frac{dz}{dx}$$

$$\frac{dW_3}{dx} = z\frac{dy}{dx}$$

$$\frac{dW_4}{dx} = y\frac{dx}{dx}$$

and three u's which are also V's

$$U_1 = x$$

$$(y =)\ U_2 = W_1 + W_2 - W_3 + y_0$$

$$(z =)\ U_3 = W_4 + z_0.$$

(Each W has been set so as to yield $W = 0$ at $x = x_0$. It is clear from the above how the constants of integration can be entered in any problem of this type.)

Since we have the equations

$$\frac{dW_1}{dx} = U_1\frac{dU_1}{dx}$$

$$\frac{dW_2}{dx} = U_2\frac{dU_3}{dx}$$

$$\frac{dW_3}{dx} = U_3\frac{dU_2}{dx}$$

$$\frac{dW_4}{dx} = U_2\frac{dU_1}{dx}$$

the connections of the differential analyzer are completely indicated.

It is clear that the constants of integration will introduce no difficulty in the general case, provided we know the value of W_1 for $x = x_0$ for each i. It is just this that we must keep in mind in our successive expansions of the system, i.e., whenever a new variable is introduced, its initial value must be determined.

Let us consider another example:

$$\left(\frac{dy}{dx}\right)^2 = x \sin y$$

This is in the form I. Our first step is to expand this to a system in the form H. To do this we introduce $z = \sin y$ for which we have

$$\frac{d^2z}{dy^2} + z = 0$$

Now if we eliminate the derivatives of z relative to y, we obtain

$$\frac{dy}{dx}\frac{d^2z}{dx^2} - \frac{dz}{dx}\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 z = 0$$

with

$$\left(\frac{dy}{dx}\right)^2 = xz.$$

This is in the form H. However to revert to the form (G) it is desirable to differentiate only the second equation. Hence

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = x\frac{dz}{dx} + z$$

$$\frac{d^2z}{dx^2} = \frac{1}{2\left(\frac{dy}{dx}\right)^2}\left[x\left(\frac{dz}{dx}\right)^2 + z\left(\frac{dz}{dx}\right) - 2z\left(\frac{dy}{dx}\right)^4\right]$$

$$\frac{d^2y}{dx^2} = \frac{1}{2\left(\frac{dy}{dx}\right)}\left(x\frac{dz}{dx} + z\right)$$

We introduce $u = \frac{dy}{dx}$, $v = \frac{dz}{dx}$. We then have the (G) system

$$\frac{dy}{dx} = u$$

$$\frac{dz}{dx} = v$$

$$\frac{du}{dx} = \frac{1}{2u^2}(xv^2 + zv - 2zu^4)$$

$$\frac{dv}{dx} = \frac{1}{2u}(xv + z)$$

To return to an E_m system, we introduce $w = \frac{1}{u}$

$$\frac{dy}{dx} = u$$

$$\frac{dz}{dx} = v$$

$$\frac{du}{dx} = \tfrac{1}{2}(w^2 xv^2 + zvw^2) - zu^2$$

$$\frac{dv}{dx} = \tfrac{1}{2}(wxv + wz)$$

$$\frac{dw}{dx} = -w^2\frac{du}{dx}$$

This is in the form E_5. To reduce it, we introduce $p = vw$, $q = zw$, $r = w^2$, $s = zu$.

Our system then becomes

$$\frac{dy}{dx} = u$$

$$\frac{dz}{dx} = v$$

$$\frac{du}{dx} = \tfrac{1}{2}(xp^2 + pq) - su$$

$$\frac{dv}{dx} = \tfrac{1}{2}(xp + q)$$

$$\frac{dw}{dx} = -r\frac{du}{dx}$$

$$\frac{dp}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}$$

$$\frac{dq}{dx} = z\frac{dw}{dx} + w\frac{dz}{dx}$$

$$\frac{dr}{dx} = 2w\frac{dw}{dx}$$

$$\frac{ds}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

Finally we introduce $f = xp$, $g = pq$, $h = su$, $j = fp$ and obtain

$$\frac{dy}{dx} = u$$

$$\frac{dz}{dx} = v$$

$$\frac{du}{dx} = \tfrac{1}{2}(j + g) - h$$

$$\frac{dv}{dx} = \tfrac{1}{2}(f + q)$$

$$\frac{dw}{dx} = -r\frac{du}{dx}$$

$$\frac{dp}{dx} = v\frac{dw}{dx} + w\frac{dv}{dx}$$

$$\frac{dq}{dx} = z\frac{dw}{dx} + w\frac{dz}{dx}$$

$$\frac{dr}{dx} = 2w\frac{dw}{dx}$$

$$\frac{ds}{dx} = z\frac{du}{dx} + u\frac{dz}{dx}$$

$$\frac{df}{dx} = x\frac{dp}{dx} + p$$

$$\frac{dg}{dx} = p\frac{dq}{dx} + q\frac{dp}{dx}$$

$$\frac{dh}{dx} = s\frac{du}{dx} + u\frac{ds}{dx}$$

$$\frac{dj}{dx} = p\frac{df}{dx} + f\frac{dp}{dx}$$

This system is in the form (D) and can be expanded as in our above example to the form (A). For the arrangement dealing with the least number of integrators we introduce

$$V_0 = x$$

$$(y =) \quad V_1 = W_1 + y_0$$

$$(z =) \quad V_2 = W_2 + z_0$$

$$(u =) \quad V_3 = W_3 + u_0$$

$$(v =) \quad V_4 = W_4 + v_0$$

$$(w =) \quad V_5 = -W_5 + w_0$$

$$(p =) \quad V_6 = W_{12} + W_{11} + p_0$$

$$(q =) \quad V_7 = W_{13} + W_9 + q_0$$

$$(r =) \quad V_8 = 2\,W_{14} + r_0$$

$$(s =) \quad V_9 = W_7 + W_{10} + s_0$$

$$(f =) \quad V_{10} = W_{15} + W_6 + f_0$$

$$(g =) \quad V_{11} = W_{17} + W_{19} + g_0$$

$$(h =) \quad V_{12} = W_8 + W_{18} + h_0$$

$$(j =) \quad V_{13} = W_{16} + W_{20} + j_0$$

$$(\tfrac{1}{2}[j+g] - h \;=)$$

$$V_{14} = \tfrac{1}{2}\,(W_{16} + W_{17} + W_{19} + W_{20}) - (W_8 + W_{18}) + \tfrac{1}{2}\,(j_0 + g_0) - h_0$$

$$(\tfrac{1}{2}\,(f + g) \;=)$$

$$V_{15} = \tfrac{1}{2}\,(W_6 + W_{15} + W_{17} + W_{19}) + \tfrac{1}{2}\,(f_0 + g_0).$$

$$\frac{dW_1}{dx} = V_3\,\frac{dV_0}{dx}\;(= u\frac{dx}{dx})$$

$$\frac{dW_2}{dx} = V_4\,\frac{dV_0}{dx}\;(= v\frac{dx}{dx})$$

$$\frac{dW_3}{dx} = V_{14}\frac{dV_0}{dx} = [\tfrac{1}{2}(j+g)-h]\,\frac{dx}{dx}$$

$$\frac{dW_4}{dx} = V_{15}\frac{dV_0}{dx} = [\tfrac{1}{2}(f+g)\,\frac{dx}{dx}]$$

$$\frac{dW_5}{dx} = V_8\,\frac{dV_3}{dx} = (+\,r\frac{du}{dx})$$

$$\frac{dW_6}{dx} = V_6\,\frac{dV_0}{dx} = (p\frac{dx}{dx})$$

$$\frac{dW_7}{dx} = V_2\,\frac{dV_3}{dx}\;(= z\frac{du}{dx})$$

$$\frac{dW_8}{dx} = V_9\,\frac{dV_3}{dx}\;(= s\frac{du}{dx})$$

$$\frac{dW_9}{dx} = V_5\,\frac{dV_2}{dx}\;(= w\frac{dz}{dx})$$

$$\frac{dW_{10}}{dx} = V_3\,\frac{dV_2}{dx}\;(= u\frac{dz}{dx})$$

$$\frac{dW_{11}}{dx} = V_5\,\frac{dV_4}{dx}\;(= w\frac{dv}{dx})$$

$$\frac{dW_{12}}{dx} = V_4\,\frac{dV_5}{dx}\;(= v\frac{dw}{dx})$$

$$\frac{dW_{13}}{dx} = V_2\,\frac{dV_5}{dx}\;(= z\frac{dw}{dx})$$

$$\frac{dW_{14}}{dx} = V_5\,\frac{dV_5}{dx}\;(= w\frac{dw}{dx})$$

$$\frac{dW_{15}}{dx} = V_0\,\frac{dV_6}{dx}\;(= x\frac{dp}{dx})$$

$$\frac{dW_{16}}{dx} = V_{10}\frac{dV_6}{dx}\;(= f\frac{dp}{dx})$$

$$\frac{dW_{17}}{dx} = V_6\,\frac{dV_7}{dx}\;(= p\frac{dq}{dx})$$

$$\frac{dW_{18}}{dx} = V_3\,\frac{dV_9}{dx}\;(= u\frac{ds}{dx})$$

$$\frac{dW_{19}}{dx} = V_7\,\frac{dV_6}{dx}\;(= q\frac{dp}{dx})$$

$$\frac{dW_{20}}{dx} = V_6\,\frac{dV_{10}}{dx}\;(= p\frac{df}{dx})$$

4. We wish at this point to refer briefly to the literature in place of a technical discussion for which we do not have the time. The reader is probably familiar with the fact that the differential analyzer was developed to a great extent at the Massachusetts Institute of Technology. An interesting and modern version is described in the following: Vannevar Bush and S. H. Caldwell, *Journal of Franklin Institute*, Vol. 240 (1945), pp. 255-326.

A list of differential analyzers known in 1940 is given on page 127, Vol. 1, of the *Mathematical Reviews* by M. Vallarta in a review of the paper by S. Rosseland in *Naturwissenschaften*, Vol. 27 (1939), pp. 729-35, which describes a particular model.

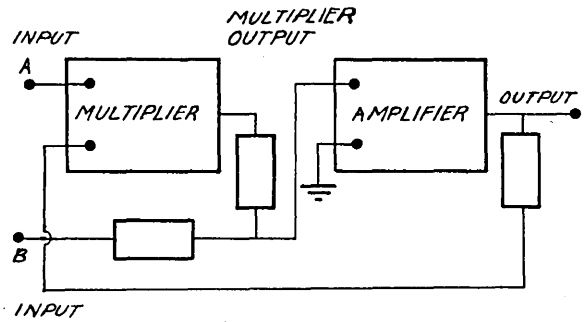
A large differential analyzer of good accuracy was constructed by General Electric and described in an article by H. P. Kuehni and H. A. Peterson, *Elect. Eng.* 63 (1944), pp. 221-28. Here the output of each integrator has a torque amplification with a polarized light connection.

A German version is described in the paper by R. Sauer and I. H. Poesh in *Engineer's Digest*, V (1944), pp. 94-96.

5. It is interesting to compare the above methods for solving differential equations. Owing to the inevitable slipping in the integrators, the last method can be compared to the situation in the first when one approximates the integral using only one term in the sum. By elaborate devices, it is possible to keep the load on the output of each integrator low which is precisely equivalent to shortening the interval. One wonders whether it might not be possible to obtain a linear combination of integrators, the outputs having varying time delays, to yield results analogous to the re-

sults obtained by means of the theory of finite differences.

It is clear that if one were to use multipliers and dividers and integrators relative to the independent variable, an analogous result can be obtained. It might be desirable at this point to indicate that a multiplying circuit can be connected as the feedback circuit of an amplifier to yield division.



6. There is a device due to O. Vierling of Germany which obtains the complex roots of a polynomial equation. An oscilloscopic tube is used to plot these as points in the complex plane. The real and imaginary parts of the polynomial are realized in terms of the modulus and argument of z as well as the real and imaginary parts of z itself. The modulus and argument of z are varying voltages. The real and imaginary parts of z are used to position the beam in the oscilloscopic tube while the parts for the polynomials are applied to trigger circuits which control the intensity of the beam in such a way that a spot shows only when both real and imaginary parts of the polynomial are zero.

7. One of the most remarkable direct calculators is the Automatic Sequence Controlled Calculator. This machine was invented and constructed by the International Business Machines Company and presented to Harvard University. A full description is published in *A Manual of Operation for the Automatic Sequence Controlled Calculator*, Cambridge, Mass.: Harvard University Press, 1946.

The calculator consists of various units whose operation is controlled by a tape. There are many adders which function as storage counters, a central multiply-divide unit and three function units which produce $\log_{10} x$, 10^x and $\sin x$. When the multiplication unit receives the multiplicand it forms and stores up a table of the nine digital multiples of the multiplicand and the multiplication is based on this table. The function units involve an ingenuous use of the power series expansion for these functions and their properties.

IV. Adjusters

1. In adjusters, the signal path is closed and the device continues to operate until the derived result is obtained. Some are intended to work in cycles, in others there is a continuous feedback.

At present, there are a number of commercially available devices for solving simultaneous linear equations by the Gauss Seidel method. We will discuss the Gauss Seidel method and point out the restrictions.

The present commercial devices are limited in the kind of system to which they may be applied although it is possible by means of a preliminary calculation to bring any system into a suitable form. The author has constructed a device which is directly applicable to any system.

We next discuss the method of making such devices vary continuously to the correct answer. We then consider the theory of adjusters in general. We indicate a mathematical form for the adjustment signal in a variety of problems. The mathematical procedure involved is always stable.

2. The Gauss Seidel method itself can be described as follows. Suppose we have a system of equations

$$\Sigma_{j=1}^{n} a_{i,j} x_j = b_i, \quad i=1, \ldots, n.$$

We begin with a relatively arbitrary assignment of values to the unknowns, $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$. Denote the corresponding point in n dimensions, $P^{(0)}$. We substitute $x_2^{(0)}, \ldots, x_n^{(0)}$ for the corresponding unknowns in the first equation, then solve for x_1. We call this $x_1^{(1)}$. Let $P_1^{(0)}$ denote the point $(x_1^{(1)}, x_2^{(0)}, x_3^{(0)}, \ldots, x_n^{(0)})$. Now substitute $x_1^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)}$ in the second equation and solve for x_2. Let $x_2^{(1)}$ denote this value and let $P_2^{(0)}$ denote the point $(x_1^{(1)}, x_2^{(1)}, x_3^{(0)}, \ldots, x_n^{(0)})$. In a similar way we obtain $P_3^{(0)}, P_4^{(0)}$, etc., until we obtain $P^{(1)}$ with coordinates $(x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)})$. We then repeat the above cycle with $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$, instead of $x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)}$ obtaining the points $P_1^{(1)}, (x_1^{(2)}, x_2^{(1)}, \ldots, x_n^{(1)})$, $P_2^{(1)}, (x_1^{(2)}, x_2^{(2)}, x_3^{(1)}, \ldots, x_n^{(1)})$ and so on up to $P^{(2)} (x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)})$. An infinite sequence of points can be obtained in this manner.

If the sequence of points

$$P^{(0)}, P_1^{(0)}, \ldots, P_{n-1}^{(0)}, P^{(1)}, \ldots, P_{n-1}^{(1)}, P^{(2)}, \ldots$$

is convergent, it converges to a solution of the system. For we have for the ith equation

$$\Sigma_{j=1}^{i} a_{i,j} x_j^{(k+1)} + \Sigma_{i+1}^{n} a_{i,j} x_j^{(k)} = b_i.$$

Hence if we have for each j, $x_j^{(k+1)} \to x_j$ as $k \to \infty$ then

$$\Sigma_{j=1}^{n} a_{i,j} x_j = b_i.$$

It is somewhat easier to consider convergence questions in the case in which the limit is zero. We confine our attention to the case in which the determinant of the coefficients is not zero. Then, we shall show that the convergence of a set of $P_i^{(k)}$ for the system

$$\Sigma_{j=1}^{n} a_{i,j} x_j = b_i$$

with starting values $x_1^{(0)}, \ldots, x_n^{(0)}$ is equivalent to the convergence of a set of points $Q_i^{(k)}$ for the system

$$\Sigma_{j=1}^n a_{i,j} x_j = 0$$

with starting values $(x_1^{(0)} - x_1, \ldots, x_n^{(0)} - x_n)$, where $x_1, \ldots, x_n$ is the solution of the first problem.

We show that $Q_i^{(k)}$ is obtained from $P_i^{(k)}$ by substracting $x_1, \ldots, x_n$ from the corresponding coordinate. This is certainly valid for P^o. Let us assume that it is true for all the predecessors of $P_i^{(k)}$. Now x_i^{k+1} is determined by the equation

$$\Sigma_{j=1}^{i-1} a_{i,j} x_j^{(k+1)} + a_{ii} x_i^{(k+1)} + \Sigma_{j=i+1}^n a_{i,j} x_j^{(k)} = b_i = \Sigma_{j=1}^n a_{i,j} x_j$$

or

$$\Sigma_{j=1}^{(i-1)} a_{i,j} (x_j^{(k+1)} - x_j) + a_{i,i} (x_i^{(k+1)} - x_i) + \Sigma_{j=i+1}^n a_{i,j} (x_j^{(k)} - x_j) = 0.$$

On the assumption of our induction, this shows that $x_i^{(k+1)} - x_i$ is the i'th coordinate of $Q_i^{(k)}$. The other coordinates have not changed from $Q_{i-1}^{(k)}$ and so the statement holds for $Q_i^{(k)}$. It is clear then that convergence for either sequence implies that of the other.

This discussion shows in particular that if a matrix $((a_{i,j}))$ is such that we have convergence for the system

$$\Sigma_{j=1}^n a_{i,j} x_j = 0$$

for every set of initial values, then we have convergence for every initial condition for the system

$$\Sigma_{j=1}^n a_{i,j} x_j = b_i$$

and every set of values of the constants $b_1, \ldots b_n$.

Not every system yields convergent sequences. For instance if we take the system

$$x - y = 0$$
$$x + y = 0$$

and start with a set of values (a, b) with $b \neq 0$, we get successively (a,b), (b,b), (b,-b), (-b,-b), (-b,b), (b,b) and from here on the cycle repeats.

It is therefore desirable to obtain sufficient conditions on a matrix so that we have convergence in the sequence $P^{(o)}, P_1^{(o)}, \ldots$, for the problem

$$\Sigma_{j=1}^n a_{i,j} x_j = 0$$

for every initial condition. The following pair of conditions are sufficient: 1) $a_{i,j} = a_{j,i}$, i.e., the matrix is symmetric; 2) the matrix is positive definite, i.e., if

$$\mu = \Sigma_{i,j} a_{i,j} x_i x_j$$

then $\mu > 0$ except when $x_1 = \ldots = x_n = 0$.

Let us consider such a μ. It is clear that the condition 1) shows that the equation

$$\Sigma_{j=1}^n a_{i,j} x_j = 0$$

can be written

$$\frac{\partial\mu}{\partial x_i} = 0$$

It follows then that the process of applying the Gauss Seidel method is a matter of finding a series of points at which μ is stationary relative to individual variables, taken in cyclic order.

However, we can show that these one-variable stationary values are minima relative to the variable involved. To show this notice that $a_{i,i} > 0$ for every i. For if in μ, we substitute $x_j = \delta_{i,j}$, we find that μ has the value $a_{i,i}$, which must be greater than zero by condition 2). Let us now consider the process in which $x_i^{(k+1)}$ is determined. The variables other than x_i are fixed. μ is then a quadratic expression in x_i, with $a_{i,i}$ as the coefficient of x_i^2 and $x_i^{(k+1)}$ is the stationary value. Thus μ can be written

$$\mu = a_{i,i} (x_i - x_i^{(k+1)})^2 + \mu_o.$$

Since $a_{i,i} > 0$, the stationary value must be a minimum.

It follows then that the Gauss Seidel process corresponds to minimizing μ relative to the different variables taken in cyclic order. Thus the values of μ are decreasing. Since they are all positive, they must approach a minimum value $m \geq 0$.

Let us now consider μ on the unit sphere S, $\Sigma x_i^2 = 1$. μ is continuous on S and since S is compact, it must assume its relative minimum λ, on some point of S. Since μ is not zero on S by condition 2, we have $\lambda_1 > 0$. Thus $\Sigma x_i^2 = 1$ implies $\mu \geq \lambda_1 > 0$.

Now if we let $x_1, \ldots, x_n$, be arbitrary, we have $\Sigma_{i=1}^n x_i^2 = k^2$ and the vector $(x_1/k, \ldots, x_n/k)$ is a unit vector. Thus

$$\mu (x_1/k, \ldots, x_n/k) \geq \lambda_1$$

or

$$\mu (x_1, \ldots, x_n) \geq \lambda_1 k^2 = \lambda_1 (\Sigma_{i=1}^n x_i^2)$$

This has two consequences: 1) Inasmuch as the values of μ are bounded, it follows that the set $P^{(o)}, P_1^{(o)}, \ldots$, is bounded, i.e., there is a number C such that $(\Sigma x_i^2)^{\frac{1}{2}} < C$ for every point in the sequence. 2) If the limit m, of the values of μ is zero, then the P_i's must converge to zero.

Hence, we must show that $m = 0$ for every set of initial conditions.

By a direct calculation, we see that

$$\mu = \tfrac{1}{2} \Sigma_{j=1}^n \frac{\partial\mu}{\partial x_j} x_j$$

Now for every point of the sequence $|x_j| < C$ and thus we have

$$|\mu| \leq \tfrac{1}{2} C \left(\Sigma_{j=1}^{n} \left|\frac{\partial\mu}{\partial x_j}\right|\right)$$

We know, of course, that the sequence of values of μ converges. Hence for $\varepsilon > 0$, we can find an integer N so that for $k \geq N$, we have $\mu(P^{(k)}) \leq m+\varepsilon$. When $x_i^{(k+1)}$ is to be determined, we write μ in the form

$$\mu = a_{i,i} (x_i - x_i^{(k+1)})^2 + \mu^1.$$

Since $m \leq \mu \leq m + \varepsilon$ for x_i between $x_i^{(k)}$ and $x_i^{(k+1)}$ we have

$$a_{i,i} (x_i^{(k+1)} - x_i^{(k)})^2 \leq \varepsilon$$

Consequently, we see that

$$\left|\tfrac{1}{2} \frac{\partial\mu}{\partial x_i} (P_{i-1}^{(k)})\right|^2 \leq a_{i,i}\, \varepsilon$$

and

$$|\Delta x_i|^2 = |x_i^{(k+1)} - x_i^{(k)}|^2 \leq \varepsilon / a_{i,i}$$

We also have that

$$\tfrac{1}{2} \frac{\partial\mu}{\partial x_i} (P^{(k)}) = \tfrac{1}{2} \frac{\partial\mu}{\partial x_i}(P_{i-1}^{(k)}) - \Sigma_{j=1}^{i-1} a_{i,j}\, \Delta x_j$$

Taking absolute values, we get

$$\left|\tfrac{1}{2} \frac{\partial\mu}{\partial x_i} (P^{(k)})\right| \leq \sqrt{\varepsilon} \left(\sqrt{a_{i,i}} + \Sigma_{j=1}^{i-1} |a_{i,j}| / \sqrt{a_{j,j}} \right)$$

$$= \text{(say)} \sqrt{\varepsilon}\, A_i$$

If we then return to our inequality on μ, we have

$$|\mu(P^{(k)})| \leq \tfrac{1}{2} C \left(\Sigma_{j=1}^{n} \left|\frac{\partial\mu}{\partial x_j}\right|\right) \leq C \sqrt{\varepsilon} \left(\Sigma_{j=1}^{n} A_j\right)$$

Since we may take ε arbitrarily small, we obtain

$$\mu(P^{(k)}) \to 0$$

This implies that $m = 0$, the desired result.

3. Conditions 1 and 2 of the previous section, of course, are restrictive. Besides, although it is easy to test 1, even the most convenient general method for testing 2 is not easier than solving the equations directly.

A symmetric quadratic form μ is positive definite if and only if

$$a_{1,1} > 0, \begin{vmatrix} a_{1,1}, & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} > 0, \ldots, \begin{vmatrix} a_{1,1}, & \ldots, & a_{1,n} \\ \cdot & \cdots & \cdot \\ a_{n,1}, & \ldots, & a_{n,n} \end{vmatrix} > 0$$

The proof is by induction on n. It is clearly true for $n = 1$. Let us suppose that the result holds in the case $n - 1$. We show it for n.

μ can be written in the form

$$a_{1,1} x_1^2 + 2 (\Sigma_{i=2}^{n} a_{1,i} x_i) x_1 + \mu_{n-1}.$$

For $x_2, \ldots, x_n$ given, this quadratic expression is positive for all values of x_1 if and only if we have

$$a_{1,1} > 0$$

and

$$a_{1,1}\, \mu_{n-1} > (\Sigma_{i=2}^{n} a_{1,i} x_i)^2$$

The last inequality can be written

$$v_{n-1} = \Sigma_{j=2}^{n} \Sigma_{i=2}^{n} (a_{1,1} a_{i,j} - a_{1,i} a_{1,j})\, x_i\, x_j > 0$$

By the hypotheses of our induction, $v_{n-1} > 0$ for all values of $x_2, \ldots, x_n$ is equivalent to the inequality

$$\begin{vmatrix} a_{1,1} a_{2,2} - a_{1,2}^2, & \ldots, & a_{1,1} a_{2,k} - a_{1,2} a_{1,k} \\ \cdot & \cdots & \cdot \\ a_{1,1} a_{2,k} - a_{1,2} a_{1,k} & \ldots, & a_{1,1} a_{k,k} - a_{1,k}^2 \end{vmatrix} > 0$$

for $k = 2, \ldots, n$. Now if we take the determinant

$$\begin{vmatrix} a_{1,1}, & \ldots, & a_{1,k} \\ \cdot & \cdots & \cdot \\ a_{1,k}, & \ldots, & a_{k,k} \end{vmatrix}$$

and multiply each column except the first by $a_{1,1}$ and then proceed by subtracting multiples of the first column from the others to obtain the form in which the first row is 1, 0, ..., 0, we can show that

$$a_{1,1}^{k-1} \begin{vmatrix} a_{1,1}, & \ldots, & a_{1,k} \\ \cdot & \cdots & \cdot \\ a_{k,1}, & \ldots, & a_{k,k} \end{vmatrix} = \begin{vmatrix} a_{1,1} a_{2,2} - a_{1,2}^2, & \ldots, & a_{1,1} a_{2,k} - a_{1,2}\, a_{2,k} \\ \cdot & \cdots & \cdot \\ a_{1,1} a_{2,k} - a_{1,2}\, a_{2,k}, & \ldots, & a_{1,1} a_{k,k} - a_{1,k}^2 \end{vmatrix}$$

Thus the combined conditions $a_{1,1} > 0$ and $v_{n-1} > 0$ are equivalent to

$$\begin{vmatrix} a_{1,1} & \ldots, & a_{1,k} \\ \cdot & \cdots & \cdot \\ a_{k,1} & \ldots, & a_{k,k} \end{vmatrix} > 0$$

for $k = 1, \ldots, n$ and we know that these are equivalent to $\mu > 0$.

But this test is just about as difficult to apply as solving the equations themselves. However, given the system of equations

$$\Sigma_{j=1}^{n} a_{i,j} x_j = b_i;\ i = 1, \ldots, n \qquad \text{(a)}$$

with a unique solution, we can find a positive definite system with the same solution. Define

$$\varepsilon_i = \Sigma_{j=1}^{n} a_{i,j} x_j - b_i.$$

The equivalent system is

$$\Sigma_{i=1}^{n} \varepsilon_i\, a_{i,k} = 0,\ k = 1, \ldots, n \qquad \text{(b)}$$

or

$$\Sigma_{j=1}^{n} (\Sigma_{i=1}^{n} a_{i,j} a_{i,k})\, x_j = \Sigma_{i=1}^{n} b_i a_{i,k};\ k=1,\ldots,n \quad \text{(c)}$$

It is clear that (a) and (b) are equivalent when the determinant of the $a_{i,j}$'s is not zero.

The matrix for the system (c) is clearly symmetric. We show that it satisfies 2). Form

$$v = \Sigma_{k=1} \Sigma_{j=1} (\Sigma_{i=1}^{n} a_{i,j} a_{i,k}) x_j x_k$$
$$= \Sigma_{i=1}^{n} (\Sigma_{j=1}^{n} a_{i,j} x_j) (\Sigma_{k=1}^{n} a_{i,k} x_k)$$
$$= \Sigma_{i=1}^{n} (\Sigma_{j=1}^{n} a_{i,j} x_j)^2.$$

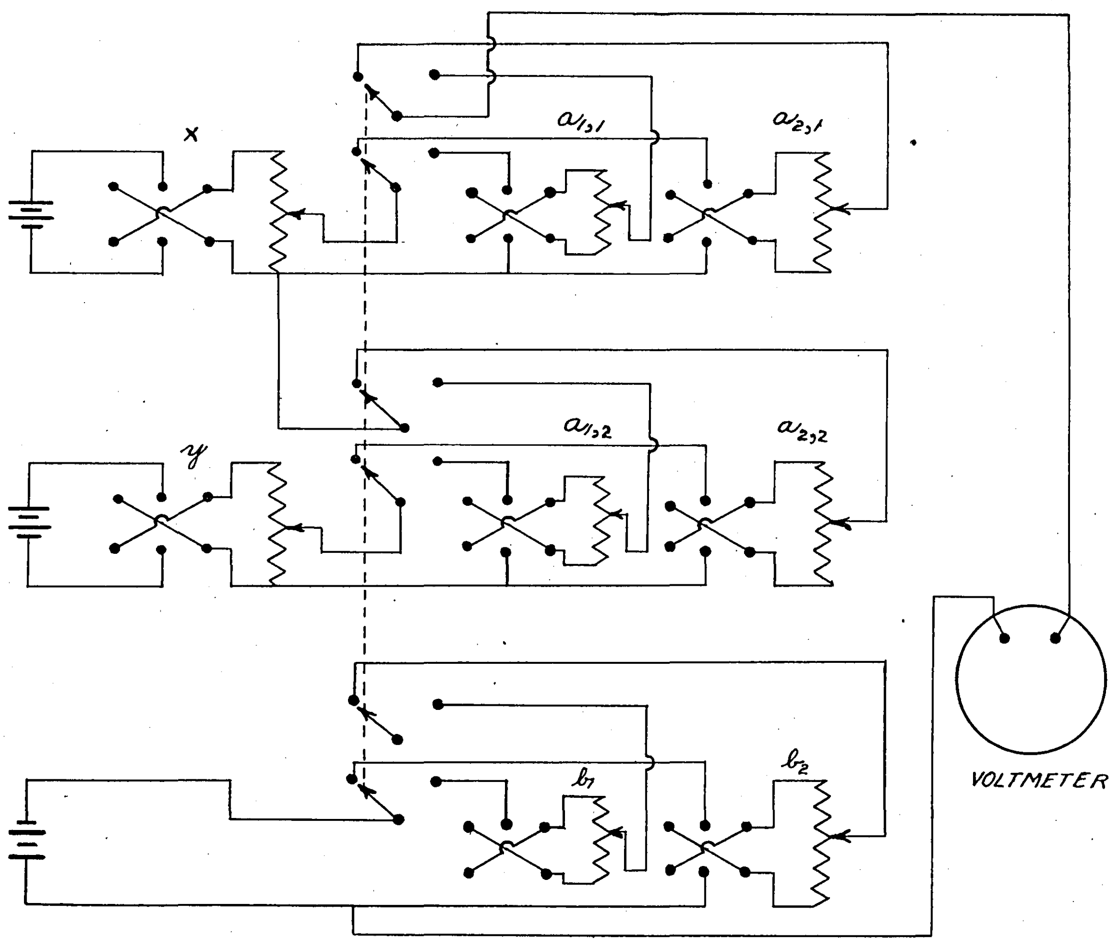
Thus $v = 0$ is equivalent to $\Sigma_{j=1} a_{i,j} x_j = 0$ for $i = 1, \ldots, n$ and since the determinant is not zero, this is equivalent to $x_1 = x_2 = \ldots = x_n = 0$.

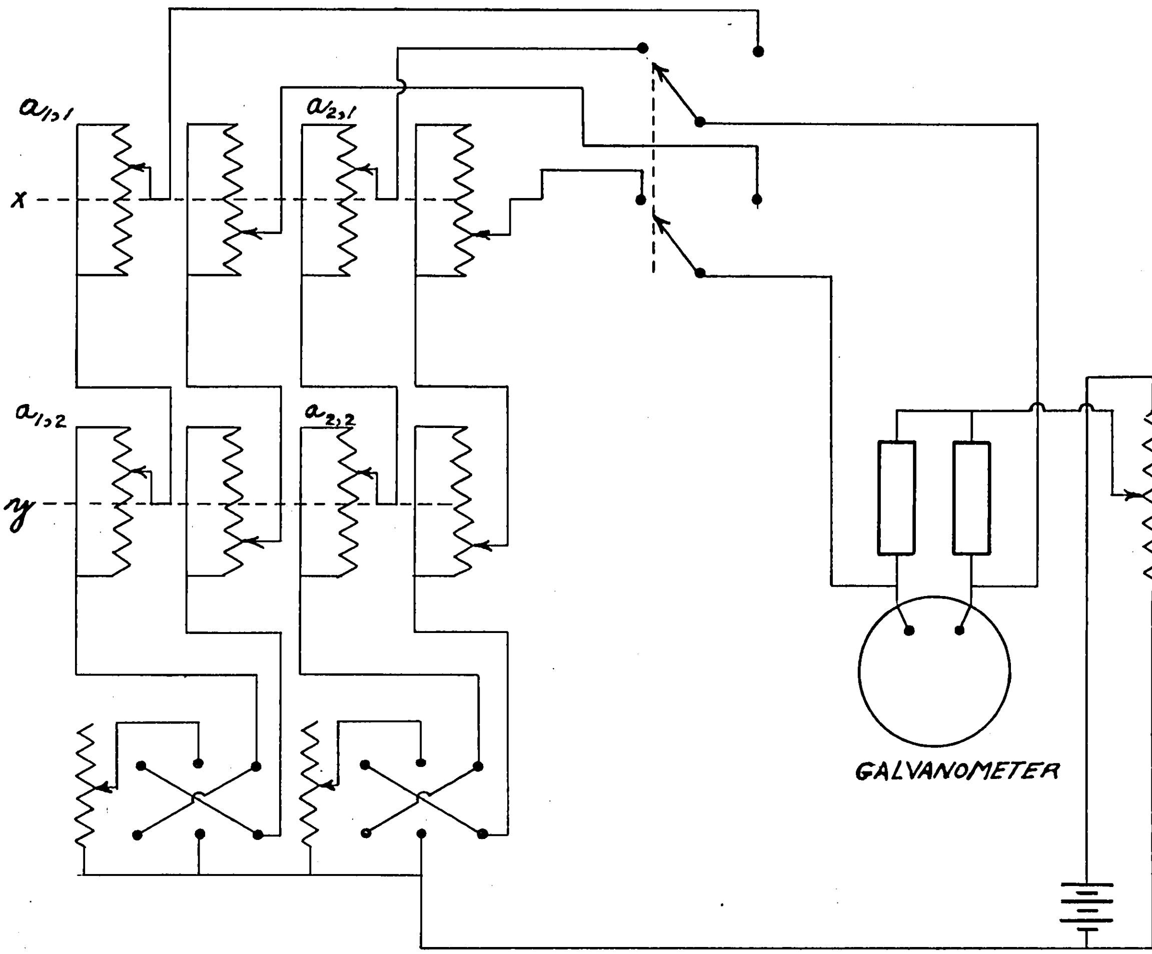
4. It is, of course, relatively simple to construct devices to solve simultaneous linear equations based on the Gauss-Seidel method and a number of commercial devices for this purpose are available. These are based on direct current methods, one equation is realized at a time, a gang switch changes the equation. The multiplication is by a potentiometer method and the addition is by means of the addition of voltages. We show a possible circuit for such a device for two equations below. The six pole switch is a double throw switch, which determines the equation. The double pole switch and potentiometer marked x, gives a voltage x. A potentiometer $a_{1,1}$ across the output gives $a_{1,1}$ x as its voltage output. We add $a_{1,1}$ x and $a_{1,2}$ y and $- b_1$ to realize the voltage

$$a_{1,1} x + a_{1,2} y - b_1$$

which is measured by the voltmeter.

Alternatively, we have seen how a linear combination can be realized as a resistance in Section 12 of Chapter II of Part II above. Using linear potentiometers we mount two potentiometers for each coefficient on each variable shaft. This, of course, can be done in a number of ways but for simplicity let us suppose that the resistance portion of the potentiometer turns with the variable shaft. Let us begin with each variable shaft in the zero position. Then each coefficient is entered by displacing the contacts from the center position a proportionate amount, the contacts going in opposite directions on the two potentiometers associated with the same coefficient. This also permits one to enter the sign of each coefficient. These contacts are now fixed in space. If then the x shaft is rotated, we see from Sec. 12, that one potentiometer will have resistance





$$R\left[1-\left(\frac{x+a_{1,1}}{2}\right)\right]^2$$

and the other

$$R\left[1-\left(\frac{x-a_{1,1}}{2}\right)\right]^2$$

(See diagram above.) For a given equation all the resistances of the first kind are connected in series and all the second kind in another series. A resistance corresponding to the constant term is connected to one or the other of these series, depending on its sign.

The two series are used as two branches of a Wheatstone bridge. When equality is obtained, we have

$$Rb_1 + R\left[1-\left(\frac{x+a_{1,1}}{2}\right)^2\right] + R\left[1-\left(\frac{y+a_{1,2}}{2}\right)^2\right] = R\left[1-\left(\frac{x-a_{1,1}}{2}\right)^2\right] + R\left[1-\left(\frac{y-a_{1,2}}{2}\right)^2\right]$$

or

$$b_1 = a_{1,1}x + a_{1,2}y$$

This circuit is due to the author. It has the disadvantage that two potentiometers are used for each coefficient, each coefficient has to be entered twice or some mechanical arrangement to accomplish this purpose has to be used. However, it has the advantage that only one voltage is used and the value of this voltage does not enter into the calculation. In fact, a potentiometer across this voltage may be used for volume control purposes. This permits one to use sensitive galvanometers. The switching arrangement is far simpler and the device can be readily augmented so as to produce the value of each equation.

The following paper may prove of interest: Berry, C. E., D. E. Wilcox, S. M. Rock, and H. W. Washburn, "A Computer for Solving Linear Simultaneous Equations," *Journal of Applied Physics*, Vol. 17, No. 4, pp. 262-72. This article describes a device for solving simultaneous linear equations by the Gauss-Seidel method. Alternating current is used for convenience. The multiplication is by successive potentiometers and the addition is by the resistance averaging method. The coefficients are set by a Wheatstone bridge method.

5. The devices of the preceding section can be used only when the equations permit one to apply the Gauss-Seidel method. In general, this would require a preliminary transformation as indicated in Sec. 3. However, the author has constructed a device which is immediately applicable to any system. (See diagram below.)

The machine produces directly

$$\mu = \Sigma_{i=1}^{n} \varepsilon_i^2 = \Sigma_{i=1}^{n} (\Sigma_{j=1}^{n} a_{i,j} x_j - b_i)^2.$$

The variables are used in rotation to minimize μ, whose value appears on a meter. This is equivalent to applying the Gauss-Seidel method to the system

$$\frac{\partial\mu}{\partial x_i} = 0,\ i = 1,\ \ldots,\ n$$

whose matrix is positive definite, as one sees when one sets $b_i = 0$ in the expression for μ.

A preliminary model has been constructed for four equations and four unknowns. However, the ideas can be completely outlined by using the two variable, two equation case. The schematic is the diagram below. The variable boxes, x and y produce an alternating voltage, and by an averaging process the voltages proportional to

$$\varepsilon_1 = a_{1,1}\,x + a_{1,2}\,y - b_1\,t$$

$$\varepsilon_2 = a_{2,1}\,x + a_{2,2}\,y - b_2\,t$$

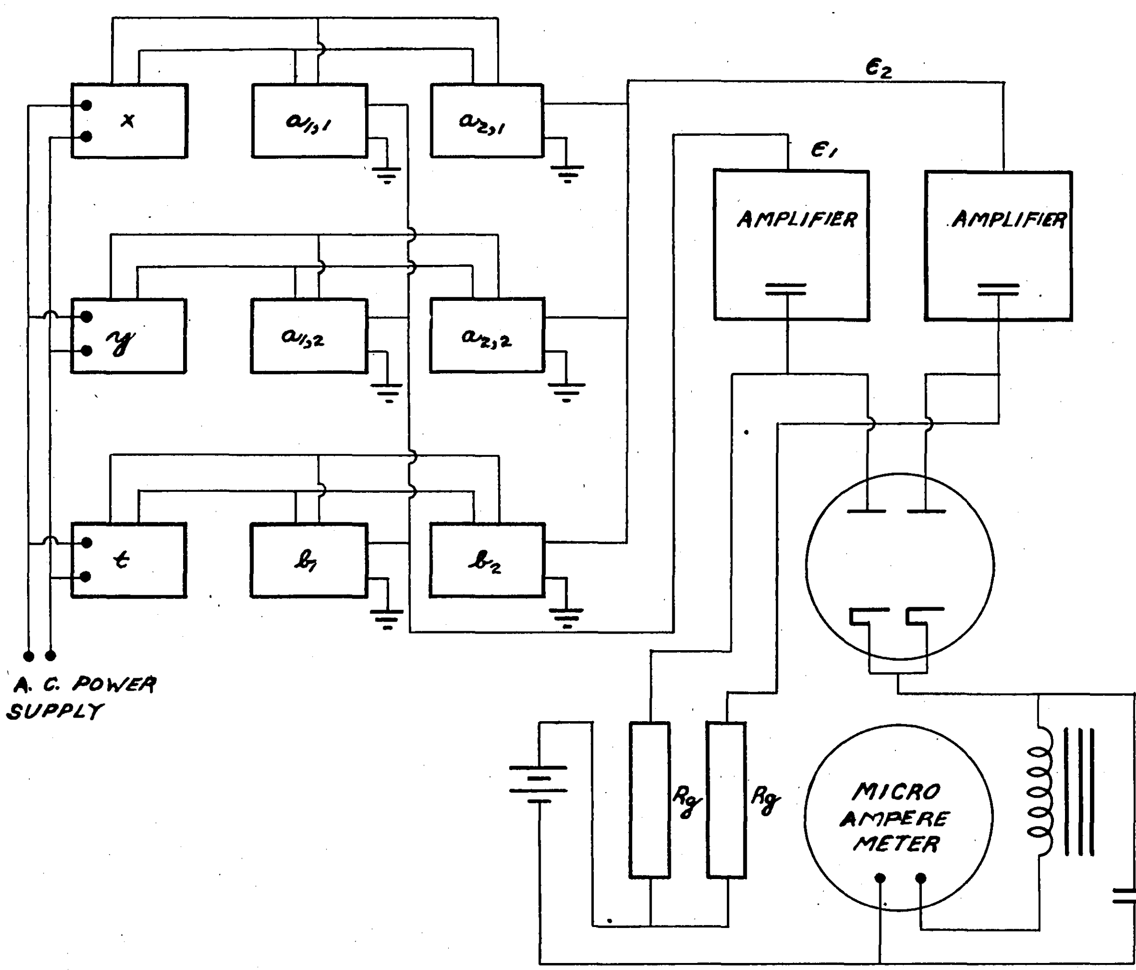
are produced. (t is a scale variable which is frequently very useful in fitting the device to the problem and for volume control purposes.)

ε_1 and ε_2 are alternating voltages and hence may be amplified by means of an audio frequency amplifier suitable for that frequency. The amplified signal is applied to the plates of a diode and by square law rectification a direct current proportional to

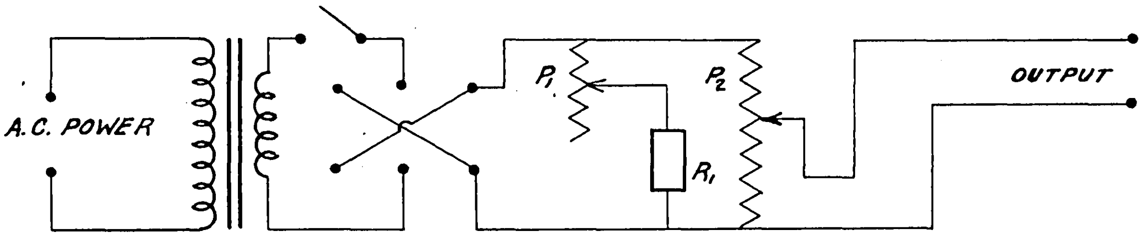
$$\mu = \varepsilon_1^2 + \varepsilon_2^2$$

is obtained and read on the meter.

The resistances R_g shown are the output grid resistances of the amplifying stages. Each is joined to the plate of the final amplifying stage by a blocking condenser. These resistances are necessary since they supply a path for the normal direct current generated by thermal emission in the diode. The voltage drop generated by this current would over bias the diode plates if the battery were not provided. A smoothing circuit is associated with the microammeter.



Each variable box has the circuit

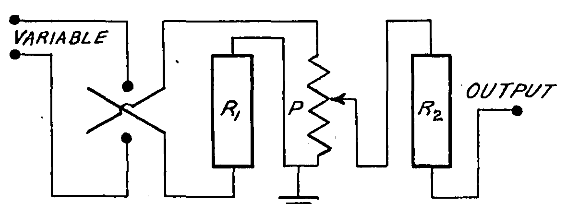


The power is obtained from a step down transformer across the line. Bell transformers are used in the model. It would be better if a single transformer having secondaries for each variable and the constants were available. The double pole switch determines the sign of x and the 400-ohm wire-wound potentiometer P_2 determines the size of x. (We give values for 4 × 4 model.) The constant R_1 of 3000 ohms and the .1 megohm volume control are used to equalize the load on the transformer with different x settings. It is not necessary that this be done with great accuracy and one adjustment when the value of x is approximately known is all that is necessary.

(The larger the value of x, the less is the total resistance of the potentiometer P_2 and its load. This is to be balanced by increasing the resistance in the shunt P_1, R.)

The single pole single throw switch is convenient for testing purposes.

The coefficient boxes with values for the 4 x 4 model are illustrated in this diagram



The double pole double throw switch is set according to the sign of $a_{i,j}$. The potentiometer P is 10,000 ohm wire wound and R_1 is a resistance matched to P. R_2 is .5 megohm and must be matched with the R_2s of the other coefficient boxes. This can be easily done by taking twice as many .25 megohm resistors evaluating each and then combining them in pairs with the proper total resistance.

The groundings in the various coefficient boxes locate the variable circuits relative to ground and hence permit one to add by averaging the voltages $a_{1,1}$ x, $a_{1,2}$ y and b_1 t through the matched resistances R_2. Adding by averaging voltages is desirable since it permits one to obtain the different equations simultaneously. But if we have negative coefficients and wish to average voltages, we must have a voltage corresponding to -x as well as one for +x. In the present device, this is accomplished by locating the x voltage in a balanced position relative to ground so that one terminal is as far below ground as the other is above.

It is seen then that the use of alternating current in this device has three advantages. One of these is the ease of positioning the variable circuits relative to ground. Two is the use of simple audio frequency amplifiers. The third one is the ease of squaring.

The model gave results accurate to about 1 percent of the largest unknown. The model was crude and better results could hardly have been expected from it. However, it should be pointed out in connection with devices for solving simultaneous linear equations that it is a very simple matter to improve the results by an iterative process. For suppose $x_1^{(1)}, \ldots, x_n^{(1)}$ is an approximation to the answer. Let Δx_i be defined by the equation

$$x_i = x_i^{(1)} + \Delta x_i .$$

The equation

$$\Sigma_{j=1}^{n} a_{i,j} x_j = b_i$$

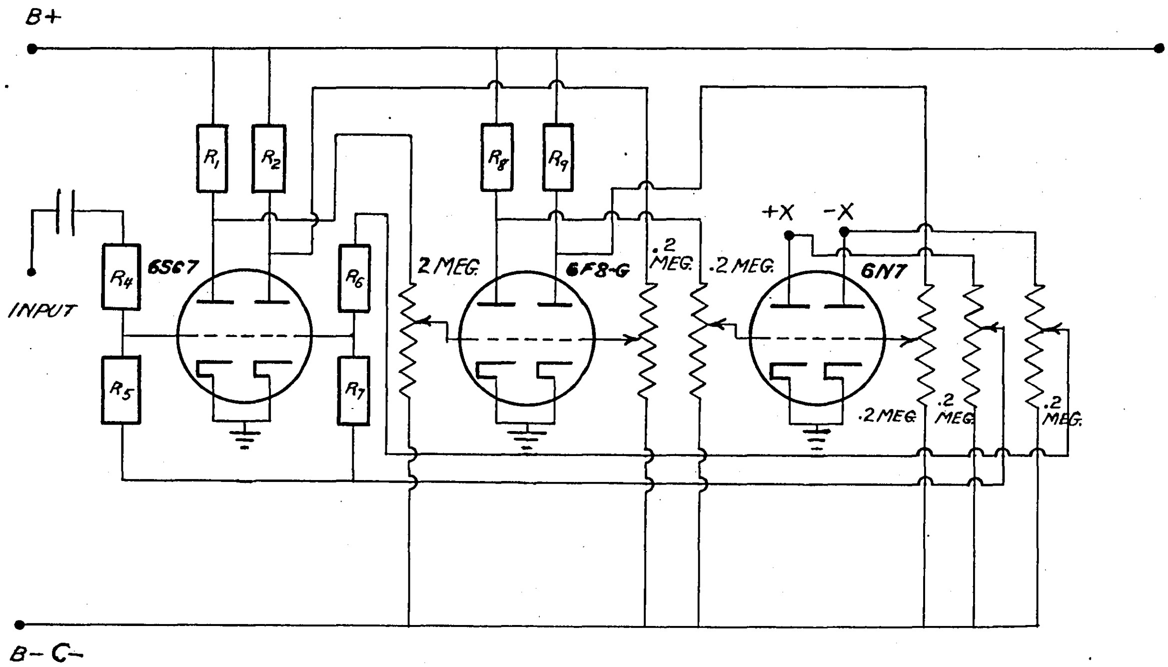
then becomes

$$\Sigma_{j=1}^{n} a_{i,j} \Delta x_j = b_i - \Sigma_{j=1}^{n} a_{i,j} x_j^{(1)} = \Delta b_i .$$

Now if the Δb_i's have only one-tenth the value of the b_i's we may introduce a scale factor for the Δx_i's, i.e., multiply the equation by 10 and solve for $10\,\Delta x_1$, $10\,\Delta x_2$, ..., $10\,\Delta x_n$. Thus as long as the accuracy is adequate to reduce the constants by a factor of 10 at each stage, we may conveniently obtain any accuracy desired. Since it is not necessary to reset the coefficients $a_{i,j}$ in the reiterative process, the labor in each stage is mainly one of calculating the errors Δb_i and resetting the constants to these values.

The use of distinct transformers introduces a phase lag between the line and the alternating voltages corresponding to the variables. In the present model, varying the load permits one to equalize the lag between the variables to the accuracy of the device. A better equalization could undoubtedly be obtained if the variable voltages were obtained from distinct secondaries on the same transformer.

However, the customary method of dealing with a situation in which one wishes to maintain a constant voltage under a variable load is to use a feedback circuit. The idea can be illustrated by means of the following diagram for a power box.



However, in using this power box in the general schematic shown on page III - 18, B+ replaces ground in the coefficient boxes and blocking condensers should be inserted at ε_1 and ε_2. The input to the power box is taken from a potentiometer which determines the absolute value of x. These potentiometers in turn can be powered by an oscillator tuned to some frequency which should in general not be a harmonic or subharmonic of the power supply a.c. frequency.

The value of the resistors depends upon the B supply available and the frequency chosen. For low frequencies and B = +300, C- = -100, one might suggest, R_1, R_2 = .5 megohm, R_8, R_9 = 40,000 ohms. We suppose that the output load is 2,500 ohms between B+ and x+ and 2,500 ohms between B+ and x-. We suppose that the B+ and C- supplies are separate and each has a grounded terminal.

The right-hand sides and the left-hand sides of the tubes each constitute three stage feedback amplifiers, the right-hand side taking its input from the output of the other. When the coefficients are set at different values the load on the terminals vary. But this is equivalent to a variation in the gain of the last stages of the feedback amplifiers and the feedback will compensate for this to a great extent. Once adjusted such a circuit should be stable except possibly at high frequencies.

For higher frequencies, the phase shift due to tube capacities and parasitic effects of the stages increases and if the feedback is too high the circuit will oscillate. On the other hand, the gain of each stage falls off too and if the signal fed back is less in amplitude than the original for the frequencies with regenerative feedback, the circuit will be stable. Inductances or capacities can also be used in the feedback to weaken it for the higher frequencies. The reader is again referred to Bode's book on feedback amplifier design.

6. It is, of course, possible to provide a feedback arrangement which is such that the adjustment leading to a solution will be performed automatically. We will consider this question in detail for there are a number of interesting variations and the discussion generalizes in many directions.

Consider again a system of simultaneous linear equations. Of course, we must minimize

$$\mu = \Sigma_{i=1}^{n} \varepsilon_i^2 = \Sigma_{i=1}^{n} (\Sigma_{j=1}^{n} a_{i,j} x_j - b_i)^2.$$

Now we obtain

$$\frac{\partial \mu}{\partial x_j} = 2 \Sigma_{i=1}^{n} \varepsilon_i a_{i,j}.$$

We recall that

$$v = \Sigma_{j=1}^{n} \left(\frac{\partial \mu}{\partial x_j}\right)^2$$

is a positive definite quadratic form in $\varepsilon_1, \ldots, \varepsilon_n$. A result of Sec. 2 above shows that there is a constant C such that

$$v \geq C (\Sigma_{j=1}^{n} \varepsilon_i^2) = C \mu.$$

In any calculating device, the x's must be functions of the time and hence μ must be also. Thus,

$$\frac{d\mu}{dt} = \Sigma_{j=1}^{n} \frac{\partial \mu}{\partial x_j} \frac{dx_j}{dt}.$$

Thus $\frac{d\mu}{dt}$ is the inner product of the gradient vector $(\frac{\partial \mu}{\partial x_1}, \frac{\partial \mu}{\partial x_2}, \ldots, \frac{\partial \mu}{\partial x_n})$ and the rate vector $(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots, \frac{dx_n}{dt})$. It is clear that to reach a solution, we must minimize μ, i.e., we want $\frac{d\mu}{dt}$ to be negative.

One method of doing this is to take $\frac{dx_j}{dt} = -\lambda\frac{\partial\mu}{\partial x_j}$. Then

$$\frac{d\mu}{dt} = -\lambda \Sigma_{j=1}^{n} (\frac{\partial\mu}{\partial x_j})^2.$$

One method of realizing the equations $\frac{dx_j}{dt} = -\lambda\frac{\partial\mu}{\partial x_j} = -\lambda \Sigma_{i=1}^{n} \varepsilon_i a_{i,j}$, is to realize the ε_i's and then form a linear combination of these. Thus what is desired is a pair of devices for forming linear combinations. The coefficient inputs can be mechanically linked so that a single shaft will determine the values of a certain coefficient in both boxes. Having formed $-\lambda\frac{\partial\mu}{\partial x_i}$, we may integrate this to obtain x, i.e., x is continuously changed in such a way that $\frac{dx}{dt} = -\lambda\frac{\partial\mu}{\partial x_i}$.

We show that such a device will converge asymptotically to the correct answer. For we have

$$|\frac{d\mu}{dt}| = \lambda \Sigma_{j=1}^{n} (\frac{\partial\mu}{\partial x})^2 \geq \lambda C \mu.$$

Thus

$$\frac{\frac{d\mu}{dt}}{\mu} = \frac{d \log \mu}{dt}$$

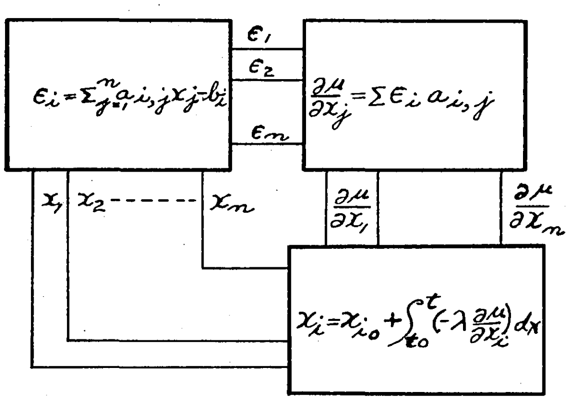
is more negative than $-\lambda C$. Hence, if we start the device at $t = 0$

$$\log \mu - \log \mu_0 < -\lambda C t$$

and we may conclude that

$$\mu < \mu_0 e^{-\lambda C t}.$$

So we see that in general a device with the block diagram (illustrated below) will converge to a position of $\mu = 0$, when the solution is unique.



Notice that we have established the *stability* of this device. An arbitrary feedback, for instance, $\frac{dx_i}{dt} = \lambda\varepsilon_i = \lambda(\Sigma a_{i,j} x_j)$ is not stable except for special matrices $[a_{i,j}]$ as one can readily see by solving this system of differential equations.

Of course, the idea used here is not confined to linear equations. For instance, if we are engaged in solving a system of equations

$$f_i(x_1, \ldots, x_n) = 0, \; i=1, \ldots, n$$

(the f's may depend upon parameters not shown) we may let $\varepsilon_i = f_i$ and form $\mu = \Sigma_{i=1}^{n} \varepsilon_i^2$ and proceed as before to obtain

$$\frac{1}{2}\frac{\partial\mu}{\partial x_j} = \Sigma_i \varepsilon_i \frac{\partial f_i}{\partial x_j}$$

and then feed in the x's so that the machine goes down the gradient, i.e.,

$$\frac{\partial x_j}{dt} = -\lambda \frac{\partial\mu}{\partial x_j}.$$

Of course, the accuracy of the device certainly depends upon the accuracy with which the equations

$$f_i = 0$$

are realized. On the other hand, we certainly are permitted a large range in percentage accuracy as far as realizing

$$\frac{dx_j}{dt} = -\lambda \Sigma_i \varepsilon_i \frac{\partial f_i}{\partial x_j}$$

is concerned. For instance, as long as $\frac{dx_j}{dt}$ and $\frac{\partial\mu}{\partial x_j}$ are opposite in sign and there is a contant $\theta > 0$, such that

$$|\frac{dx_j}{dt}| > \theta |\frac{\partial\mu}{\partial x_j}|$$

we will have the result

$$\frac{d\mu}{dt} \leq -\theta (\Sigma_{j=1}^{n}[\frac{\partial\mu}{\partial x_j}]^2) \leq -\theta C \mu$$

which will imply as in the above that

$$\mu \leq \mu_0 e^{-\theta C t}.$$

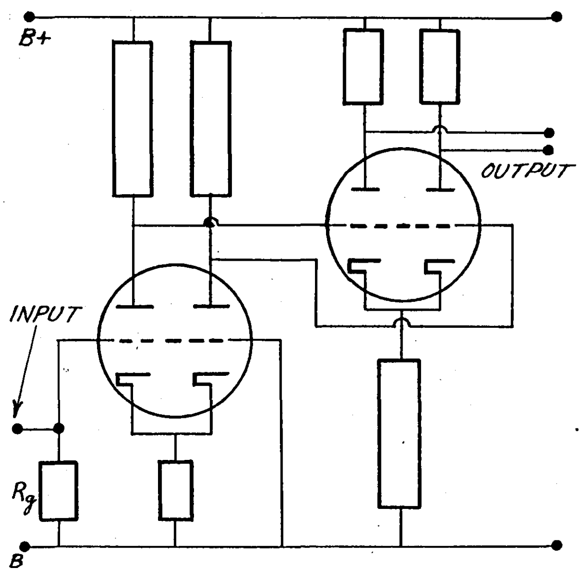
A complete feedback arrangement is always relatively expensive even if the auxiliary feedback is not as accurate as the original equations. We think that it is clear from the above that what is desired in the feedback circuit is sensitivity around the zero, while relatively large *percentage* errors can be tolerated.

This permits a number of compromises between a completely automatic set-up and a minimum of device parts. The device described in Sec. 5 was obtained originally as such a compromise, not as an effort to generalize or improve existing devices. In it, μ only is calculated. The operator determines the sign of $\frac{\partial\mu}{\partial x_i}$ for each i in succession and varies x_i accordingly.

Another compromise is readily obtainable. One can use a relatively inexpensive combination of resistances with highly amplified versions of the ε_i to obtain

$$\frac{\partial\mu}{\partial x_j} = \Sigma_{i=1}^{n} \varepsilon_i a_{i,j}$$

approximately. (The amplification should not involve any great phase lag and for this purpose yoked stages of the phase inverter type such as described in the end of Sec. 5 should be used. A simple circuit of this type is indicated. Essentially this is two stages, with the second stage grid at the voltage of the plate of the first. The yoked stages have a number of advantages. One of these is the absence of by-pass condensers. (When pentodes are used one can even yoke the screens and use no screen condenser. Of course, the voltage supply limitations should be observed.) This means that the only delaying capacities are in the tubes themselves. Also the stages adjust themselves more readily. The plate currents should be those given under characteristics in the tube manual. In designing these care should be taken in regard to the expected voltage ranges.



The right-hand side of the first tube receives its signal as follows: When a signal is put on the input grid, say a positive voltage, the unby-passed cathode resistor will be subjected to an increase in current so that the cathode tends to follow the signal grid. But the right-hand grid is fixed so that we have an increase in voltage drop from right-hand cathode to grid. Thus the cathode is in general, except for the bias voltage, midway between the two grids and an increase in current on one side is approximately balanced by a decrease on the other side of the tube. In the use suggested, a transformer may be used to bring the signal back to ground level or in the final stages a push-pull output transformer used in the usual manner.

We have obtained $\frac{\partial\mu}{\partial x_j}$. With proper consideration of phasing, we can obtain an alternating voltage with amplitude

$$1 + \frac{\partial\mu}{\partial x_j}.$$

If we detect this linearly, we can easily obtain the size of $\frac{\partial\mu}{\partial x_j}$. This could be visually indicated by means of electron ray tubes to the operator.

Such an arrangement would permit a number of operators to work simultaneously, each minimizing μ by means of a certain set of variables. This arrangement would be useful if a device were constructed for a large system.

This is compromise in which the operator does not have to differentiate, but still performs the feedback.

If the matrix $((a_{ij}))$ is positive definite, we can use the feedback $\frac{dx_j}{dt} = -\varepsilon_j$. Since

$$\frac{\partial\mu}{\partial x_j} = \Sigma_{i=1}^{n} \varepsilon_i a_{ij}$$

we see that

$$\frac{d\mu}{dt} = \Sigma_{j=1}^{n} \frac{\partial\mu}{\partial x_j}\frac{dx_j}{dt} = -\Sigma_{i=1,\ j=1}^{n} a_{ij}\varepsilon_i\varepsilon_j$$

Since $((a_{i,j}))$ is positive definite, $\frac{d\mu}{dt}$ is always negative and, in fact, the same type of exponential decay for μ can be established as in the above.

7. The discussion of the feedback circuit in the previous section permits one to compare any other feedback circuit with this one. For to operate effectively, we must have $\frac{d\mu}{dt}$ negative in every case and we can observe in each case the manner in which μ decreases.

One very frequent feedback that is used can be described in the following manner. Let S (ε) be defined as the function, such that if $\varepsilon < -\delta$, S $(\varepsilon) = -1$, if $-\delta \leq \varepsilon \leq \delta$, we have S $(\varepsilon) = 0$, and if $\varepsilon > 0$, S $(\varepsilon) = 1$. Thus if δ is small, S (ε) is essentially sign (ε).

Now our feedback is precisely described if we specify $\frac{dx_j}{dt}$. In the case considered, an effort is made to obtain the feedback.

$$\frac{dx_j}{dt} = -k\,S(\varepsilon_j),\ \text{where}\ f_j = \varepsilon_j.$$

(k is positive and does not depend on j in general.) The reason for this is of course that it is very easy to mechanize this feedback. On each ε_j shaft, we have say a pair of contacts or a pair of sets of contacts. If ε_j departs in one direction from the zero position, one set of contacts are closed. These control a motor on the x_j shaft causing it to rotate in one direction. On the other hand, if the ε_j shaft goes in the other direction, another pair of contacts are made and x_j motor turns in the opposite direction.

Of course a complete analysis of the operation of such a device would have to study the equations of motion precisely. These would involve $\frac{d^2x_j}{dt^2}$ in general. Nevertheless, one can frequently assume that the equations

$$\frac{dx_j}{dt} = -k \text{ sign } (\varepsilon_j)$$

describe the situation with sufficient accuracy.

In order to insure that $\frac{d\mu}{dt}$ is negative, we must have sign ε_j = sign $(\frac{\partial\mu}{\partial x_j})$ in general, i.e., this condition is sufficient and this is the premise generally used. This may seem like a very heavy assumption but it should be realized that given a system of equations

$$f_i (x_1, \ldots, x_n) = 0, \; i=1, \ldots, n$$

we may solve any equivalent system. For instance, we may transform it by any matrix of rank n, i.e., consider the equivalent system

$$\Sigma_j a_{i,j} f_j = 0, \; i=1, \ldots, n$$

for which the determinant of the $a_{i,j}$'s is not zero. In special cases, this may permit one to obtain the desired result even when it is not true of the original system. For instance, if the jacobean matrix at the desired zero is known approximately, we may use its transpose as the set of $a_{i,j}$'s.

Now if the jacobean of the system, i.e., the matrix $((f_{i,x_j}))$ is bounded away from zero for the permissible range of values of x, then we can show that there is a constant C such that

$$\Sigma_j |\Sigma_i \varepsilon_i f_{i,x_j}| \geq C \sqrt{\mu}.$$

(We assume that the f_{i,x_j} are continuous on a closed compact set including the permissible values.) For if we consider the quantities

$$y_j = \Sigma_i \varepsilon_i f_{i,x_j}$$

we see that under our assumptions, we can solve for the ε_i's as a linear combination of the y_j's with uniformly bounded coefficients. Thus there is a constant D such that

$$D (\Sigma_j |y_j|) \geq |\varepsilon_i|$$

This yields

$$\sqrt{n} \; D (\Sigma_j |y_j|) \geq \sqrt{\mu}$$

when we square both sides, add over i and extract the square root. This leads immediately to the desired result when we divide by $D\sqrt{n}$.

If we substitute

$$\frac{dx_j}{dt} = -k \text{ sign } (\frac{\partial\mu}{\partial x_j})$$

in the expression

$$\frac{d\mu}{dt} = \Sigma^n_{j=1} \frac{\partial\mu}{\partial x_j}\frac{dx_j}{dt} = -k \; \Sigma^n_{j=1} |\frac{\partial\mu}{\partial x_j}|$$

$$\leq -k \; C \sqrt{\mu}$$

or

$$\frac{d}{dt}(2\sqrt{\mu}) = \frac{\frac{d\mu}{dt}}{\sqrt{\mu}} \leq -k \; C$$

Thus

$$\sqrt{\mu} - \sqrt{\mu_0} \leq -\tfrac{1}{2} k \; C \; (t - t_0)$$

or

$$\sqrt{\mu} \leq \sqrt{\mu_0} - \tfrac{1}{2} k \; C \; (t - t_0)$$

Thus μ must approach zero in a finite time. In general, however, the x_j's are moving with a finite velocity at the point where $\mu = 0$ and there is a tendency to overshoot and the machine oscillates or "hunts." There are, however, devices to slow the x_j in the region around $\varepsilon_j = 0$.

8. In view of the fact that many questions of analysis can be approximately answered by finite approximations, we discuss in the present section the possibility of using a general feedback device for problems involving functions.

For definiteness, let us suppose that we have a second order differential equation

$$F \left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0$$

to be solved in an interval $a \leq x \leq b$, with boundary conditions

$$G \left(y[a], \frac{dy}{dx}\Big]_{x=a}\right) = 0$$

$$H \left(y[b], \frac{dy}{dx}\Big]_{x=b}\right) = 0.$$

We then let

$$\mu = \int_a^b F^2 \, dx + G^2 + H^2$$

(In the interesting special case in which $F (x,y\frac{dy}{dx}, \frac{d^2y}{dx^2})$ is the Euler equation for minimizing an integral

$$\int_a^b E \left(x, y, \frac{dy}{dx}\right) dx$$

we may substitute this integral for $\int_a^b F^2 \, dx$.)

At this point, we introduce for the unknown function $y = f(x)$ a finite linear combination $\sigma(x)$, which is supposed to approximate f and for which $\sigma'(x)$ and $\sigma''(x)$ are supposed to approximate $f'(x)$ and $f''(x)$ respectively. We have seen in Part II, Chapter V, Sec. 1, how such a σ could be constructed for the interval $-\pi \leq x \leq \pi$. σ may be expressed in the form

$$\sigma(x) = a_0 + d_1 f_1 + d_2 f_2 + \Sigma^N_{n=1} (a_n \sin nx + b_n \cos nx)$$

We have two alternative procedures at this point. One of these is to realize σ as a function of the time as indicated in Part II, Chapter V, Sec. 1, and then obtain μ by applying the necessary operations. The other possibility at this point is to express μ as a function of the coefficients in the above expression for σ and realize this function.

The net effect, however, in each case is to produce μ as a function of the coefficients. We may then try to minimize μ as in Sec. 5 or 6 above. If we have a feedback device, we must have the partial derivatives of μ relative to the coefficients as outputs of a part of the device.

In connection with the feedback, the following operational considerations are worth noting. We can consider our problem as concerning an operator from an infinite dimensional vector space consisting of triples of functions in the form $[y, y', y'']$ to a function space with elements $[F(x), A, B]$ consisting of the functions $F(x)$ on the interval and with two extra dimensions. The norm is obtained from the usual L_2 norm in an obvious manner.

Let us consider

$$\mu(y) = \int_a^b F^2(x, y, y', y'')\, dx + G^2(y, y')]_{x=a} + H^2(y, y')]_{x=b}.$$

This is clearly the norm squared of the transformation

$$T[y, y', y''] = [F(x, y, y', y''), G(y, y'), H(y, y')]$$

between the spaces mentioned.

Let us try to find the vectorial increment $(\delta y, \delta y', \delta y'')$, which will minimize μ. In the usual notation of the calculus of variation, we have

$$\tfrac{1}{2}\delta\mu = \int_a^b (F F_y\, \delta y + F F_{y'}\, \delta y' + F F_{y''}\, \delta y'')\, dx$$
$$+ G(G_y\, \delta y + G_{y'}\, \delta y')]_{x=a} + H(H_y\, \delta y + H_{y'}\, \delta y')]_{x=b}.$$

Let us introduce $K(x) = \int_a^x F F_y\, dx$ and integrate the first term under the integral sign by parts.

$$\tfrac{1}{2}\delta\mu = \int_a^b (F F_y - K[x])\, \delta y' + F F_{y''}\, \delta y'')\, dx + G(G_y\, \delta y + G_{y'}\, \delta y']_{x=a}$$
$$+ ([H H_y - K]\, \delta y + H_y H_{y'}\, \delta y')]_{x=b}.$$

Now let $C_1 = (H H_y - K)]_{x=b}$. We insert this constant in the coefficient of $\delta y'$ under the integral sign. The net result is the following:

$$\tfrac{1}{2}\delta\mu = \int_a^b [(F F_y - K(x) + C_1)\delta y' + F F_{y''}\, \delta y'']\, dx$$
$$+ ([G G_y + C_1]\, \delta y + G G_{y'}\, \delta y')]_{x=a} + H_y H_{y'}\, \delta y']_{x=b}.$$

The purpose of this manoeuvre was to eliminate the δy at $x = b$ terms.

Next let

$$L(x) = \int_a^x (F F_y - K(x) + C_1)\, dx$$

and again integrate the first term under the integral sign by parts. The result is

$$\tfrac{1}{2}\delta\mu = \int_a^b (F F_{y''} - L[x])\, \delta y''\, dx + ([G G_y + C_1]\, \delta y + G G_{y'}\delta y')]_{x=a}$$
$$+ (H H_y + L[x])\, \delta y']_{x=b}$$

We then let $C_2 = (H H_y + L)]_{x=b}$ and insert this constant under the integral sign. We then have

$$\tfrac{1}{2}\delta\mu = \int_a^b (F F_{y''} - L[x] + C_2)\, \delta y''\, dx + (G G_y + C_1)\, \delta y]_{x=a}$$
$$+ (G G_{y'} + C_2)\, \delta y']_{x=a}$$

Let $k_1 = G G_y + C_1]_{x=a}$, $k_2 = G G_{y'} + C_2]_{x=a}$. It is now evident that if we let δy be any negative multiple of

$$\nabla\mu = k_1 + k_2\,(x-a) + \int_a^x \int_a^{x'} (F F_{y''} - L + C_2)\, dx''\, dx'$$

$\delta\mu$ will be negative. Under suitable continuity restrictions, this will imply that the actual variation in μ will be negative provided the variation in δy is small.

The formulas thus obtained could be utilized to plan a feedback, no matter what method is used to represent the functions. In the case of a linear F, the expression for $\nabla\mu$ is the well-known expression T^*Ty.

9. The basic ideas of the preceding section generalize with little difficulty to problems in two or more variables. To illustrate this let us briefly consider a problem of the following sort. Suppose we have a region S whose boundary consists of a rectifiable curve which has a continuously turning tangent except for at most a finite number of points. For simplicity in the formulas, we will suppose that S can be described in both of the following ways: 1. There are two functions $\varphi_1(x)$, $\varphi_2(x)$ defined on the interval $a \le x \le b$ such that S is the set of points (x, y) with $a < x < b$ and $\varphi_1(x) < y < \varphi_2(x)$. 2. There are two functions $\psi_1(y)$, $\psi_2(y)$ defined on an interval $c \le y \le d$ such that S consists of points (x, y) for which $c < y < d$ and $\psi_1(y) < x < \psi_2(y)$. These restrictions are by no means essential. Their sole purpose is to simplify the formulas and for the same reason we will also suppose that boundary does not contain line segments parallel to an axis, which is again a nonessential restriction.

Let us suppose that our problem is to obtain a solution of the equation

$$F(x, y, z, p, q, r, s, t) = 0$$

on the region S, where

$$p = \frac{\partial z}{\partial x},\quad q = \frac{\partial z}{\partial y},\quad r = \frac{\partial^2 z}{\partial x^2},\quad s = \frac{\partial^2 z}{\partial x \partial y},\quad t = \frac{\partial^2 z}{\partial y^2},$$

subject to the boundary condition

$$G(x, y, z, p, q) = 0.$$

F is to have continuous partial derivatives relative to the various apparent variables.

We consider the equivalent of the discussion of the previous section. Incidently, this discussion should be of interest in connection with existence theorems.

We let

$$\mu = \iint_S F^2\, dS + \int_B G^2\, ds$$

We then have in the usual notation

$$\tfrac{1}{2}\delta\mu = \iint_S F(F_z\delta z + F_p\delta p + F_q\delta q + F_r\delta r + F_s\delta s + F_t\delta t)\, dS$$
$$+ \int_B G(G_z\delta z + G_p\delta p + G_q\delta q)\, ds.$$

Now if we integrate the last term of the spatial integral by parts, we obtain

$$\iint_S F F_t \delta t\, dy\, dx = \int_B F F_t \delta q\, dx - \iint_S \frac{\partial}{\partial y}(F F_t)\, \delta q\, dS$$

Let $K_1(x,y) = \int_{\psi_1(y)}^x \frac{\partial}{\partial y}(F F_t)\, dx$.

Then

$$\iint_S F\,F_t\,\delta t\,dS = \int_B \delta q\,(F\,F_t\,dx - K_1\,dy) + \iint_S K_1\,(x,y)\,\delta s\,dS.$$

Similarly, if $K_2 = \int_{\varphi_1(x)}^{y} \frac{\partial}{\partial x}(F\,F_r)\,dy$, we have

$$\iint_S F\,F_r\,\delta r\,dS = \int_B \delta p\,(F\,F_r\,dy - K_2\,dx) + \iint_S K_2\,(x,y)\,\delta s\,dS.$$

Thus

$$\tfrac{1}{2}\delta\mu = \iint_S (F\,F_z\,\delta z + F\,F_p\,\delta p + F\,F_q\,\delta q + [F\,F_s + K_1 + K_2]\,\delta s)\,dS$$

$$+ \int_B G\,G_z\,\delta z\,ds + \delta p\,(G\,G_p\,ds + F\,F_r\,dy - K_2\,dx)$$

$$+ \delta q\,(G\,G_q + F\,F_t\,dx - K_1\,dy).$$

We now integrate the first term under the integral sign by parts, letting $K_3 = \int_{\psi_1(y)}^{x} F\,F_z\,dx$. We obtain

$$\tfrac{1}{2}\delta\mu = \iint_S [(F\,F_p - K_3)\,\delta p + F\,F_q\,\delta q + (F\,F_s + K_1 + K_2)\,\delta s]\,dS$$

$$+ \int_B \delta z\,(G\,G_z\,ds + K_3\,dy) + \delta p\,(G\,G_p\,ds + F\,F_r\,dy - K_2\,dx)$$

$$+ \int_B \delta q\,(G\,G_q\,ds + F\,F_t\,dx - K_1\,dy).$$

Now consider

$$\int_B (G\,G_z\,ds + K_3\,dy)\,\delta z.$$

Let us choose a fixed point P_0 on the boundary and let δz have the value k_0 at P_0. Then on the boundary we have

$$\delta z = k_0 + \int_{P_0}^{P} (\delta p\,dx + \delta q\,dy) = k_0 + \int_{P_0}^{P} \frac{d}{ds}\left(\frac{d\delta z}{ds}\right) ds.$$

Let $K_4(s) = \int_{P_0}^{P} (G\,G_z\,ds + K_3\,dy)$. This integration, of course, extends along a branch of the boundary whose arc length is s. Let $K_{4,0}$ denote

$$\int_B (G\,G_z\,ds + K_3\,dy).$$

Then

$$\int_B \left(G\,G_z + K_3\,\frac{dy}{ds}\right) \delta z\,ds$$

$$= \int_B \left(G\,G_z + K_2\,\frac{dy}{ds}\right)\left(k_0 + \int_{P_0}^{P} \frac{d}{ds}[\delta z]\,ds\right) ds$$

$$= k_0\,K_{4,0} + \int_B \frac{dK_4}{ds} \int_{P_0}^{P} \frac{d}{ds}(\delta z)\,ds$$

$$= k_0\,K_{4,0} + \left(\int_B \frac{d}{ds}[\delta z]\,ds\right) K_{4,0} - \int_B K_4\,\frac{d}{ds}(\delta z)\,ds.$$

Now

$$\int_B \frac{d}{ds}(\delta z)\,ds = \delta z\Big]_{P_0}^{P_0} = 0 \text{ and } \frac{d}{ds}(\delta z)\,ds = \delta p\,dx + \delta q\,dy.$$

Hence,

$$\int_B \left(G\,G_z + K_3\,\frac{dy}{ds}\right) \delta z\,ds = k_0\,K_{4,0} - \int_B K_4\,(\delta p\,dx + \delta q\,dy).$$

Thus

$$\tfrac{1}{2}\delta\mu = \iint_S [F\,F_p - K_3)\,\delta p + F\,F_q\,\delta q + (F\,F_s + K_1 + K_2)\,\delta s]\,dS$$

$$+ \int_B \delta p\,(G\,G_p\,ds + F\,F_r\,dy - K_2\,dx - K_4\,dx)$$

$$+ \int_B \delta q\,(G\,G_q\,ds + F\,F_t\,dx - K_1\,dy - K_4\,dy)$$

$$+ k_0\,K_{4,0}.$$

We now let $K_5 = \int_{\varphi_1(x)}^{y} (F\,F_p - K_3)\,dy$ and let us integrate the first term by parts. The result is

$$\tfrac{1}{2}\delta\mu = \iint_S [F\,F_q\,\delta q + (F\,F_s + K_1 + K_2 - K_5)\,\delta s]\,dS$$

$$+ \int_B \delta p\,[G\,G_p\,ds + F\,F_r\,dy + (K_5 - K_2 - K_4)\,dx]$$

$$+ \int_B \delta q\,[G\,G_q\,ds + F\,F_t\,dx - (K_1 + K_4)\,dy]$$

$$+ k_0\,K_{4,0}.$$

Similarly, we let $K_6 = \int_{\psi_1(y)}^{x} (F\,F_q)\,dx$ and obtain

$$\tfrac{1}{2}\delta\mu = \iint_S [(F\,F_s + K_1 + K_2 - K_5 - K_6)\,\delta s]\,dS$$

$$+ \int_B \delta p\,[G\,G_p\,ds + F\,F_r\,dy + (K_5 - K_2 - K_4)\,dx]$$

$$+ \int_B \delta q\,[G\,G_q\,ds + F\,F_t\,dx + (K_6 - K_1 - K_4)\,dx]$$

$$+ k_0\,K_{4,0}.$$

Now, for $i = 1, 2$, let

$$H_i(x) = [G\,G_p\,\frac{ds}{dx} + F\,F_r\,\frac{dy}{dx} + (K_5 - K_2 - K_4)]_{y=\varphi_i(x)}.$$

Then

$$\int_B \delta p\,[G\,G_p\,ds + F\,F_r\,dy + (K_5 - K_2 - K_4)\,dx]$$

$$= \int_a^b \delta p]_{y=\varphi_1} H_1\,dx - \int_a^b \delta p]_{y=\varphi_2} H_2\,dx$$

$$= \int_a^b \delta p]_{y=\varphi_1} (H_1 - H_2)\,dx - \int_a^b H_2\,(\delta p]_{y=\varphi_1}^{\varphi_2})\,dx$$

$$= \int_a^b \delta p]_{y=\varphi_1} (H_1 - H_2)\,dx - \int_a^b H_2(x) \int_{\varphi_1}^{\varphi_2} \delta s\,dy\,dx$$

$$= \int_a^b \delta p]_{y=\varphi_1} (H_1 - H_2)\,dx - \iint_S H_2\,(x)\,\delta s\,dS.$$

Similarly, if we put for $i = 1, 2$

$$H_{i+2}(y) = G\,G_q\,\frac{ds}{dy} + F\,F_t\,\frac{dx}{dy} + (K_6 - K_1 - K_4)]_{x=\psi_i(y)}$$

we obtain

$$\int_B \delta q\,[G\,G_q\,ds + F\,F_t\,dx - (K_1 + K_4)\,dy]$$

$$= \int_c^d \delta q]_{x=\psi_1} (H_4 - H_3)\,dy + \iint_S H_4(y)\,\delta s\,dS.$$

Substituting in the expression for $\delta\mu$, we obtain

$$\tfrac{1}{2}\delta\mu = \iint_S (F\,F_s + K_1 + K_2 - K_5 - K_6 + H_4 - H_2)\,\delta s\,dS$$

$$+ \int_a^b \delta p]_{y=\varphi_1} (H_1 - H_2)\,dx + \int_c^d \delta q]_{x=\psi_1} (H_4 - H_3)\,dy$$

$$+ k_0\,K_{4,0}.$$

We have chosen a point P_0 with coordinates (x_0, y_0) Let

$$L_1(x) = \int_{x_0}^{x} (H_1 - H_2)\,dx$$

$$L_2(y) = \int_{y_0}^{y} (H_4 - H_3)\, dy$$

$$H_5(x,y) = F F_s + K_1 + K_2 - K_5 - K_6 + H_4 - H_2$$

and let $S(x,y)$ denote the set of points (η, ζ) in S with $\eta \leq x$, $\zeta \leq y$. Let

$$\nabla\mu = K_{4,0} + L_1(x) + L_2(y) + \iint_{S(x,y)} H_5\, dS$$

Now one readily shows that $\nabla\mu(x_0, y_0) = K_{4,0}$ $\frac{\partial\nabla\mu}{\partial x} = H_1 - H_2$ when $y = \varphi_1$, $\frac{\partial\nabla\mu}{\partial y} = H_4 - H_3$ when $x = \varphi_1$ and $\frac{\partial^2\nabla\mu}{\partial x\partial y} = H_5$. Thus if our increment δz is proportional to $-\nabla\mu$, we see that $\delta\mu$ is negative.

10. We wish also to point out that a device whose output is a quadratic form can be easily modified so as to permit one to obtain the characteristic values and vectors of a Hermitean matrix.

Suppose $(a_{i,j})_{i,j=1,\ldots,n}$ is the matrix and

$$\mu = \Sigma_i \Sigma_j a_{i,j} x_i x_j$$

is the corresponding quadratic form. Now a quadratic form can be expressed as the difference of two sums of squares of linear combinations of the x_i's. (Cf. Dickson, *Modern Algebraic Theories*, pp. 68-74.)

Thus, in general, we have

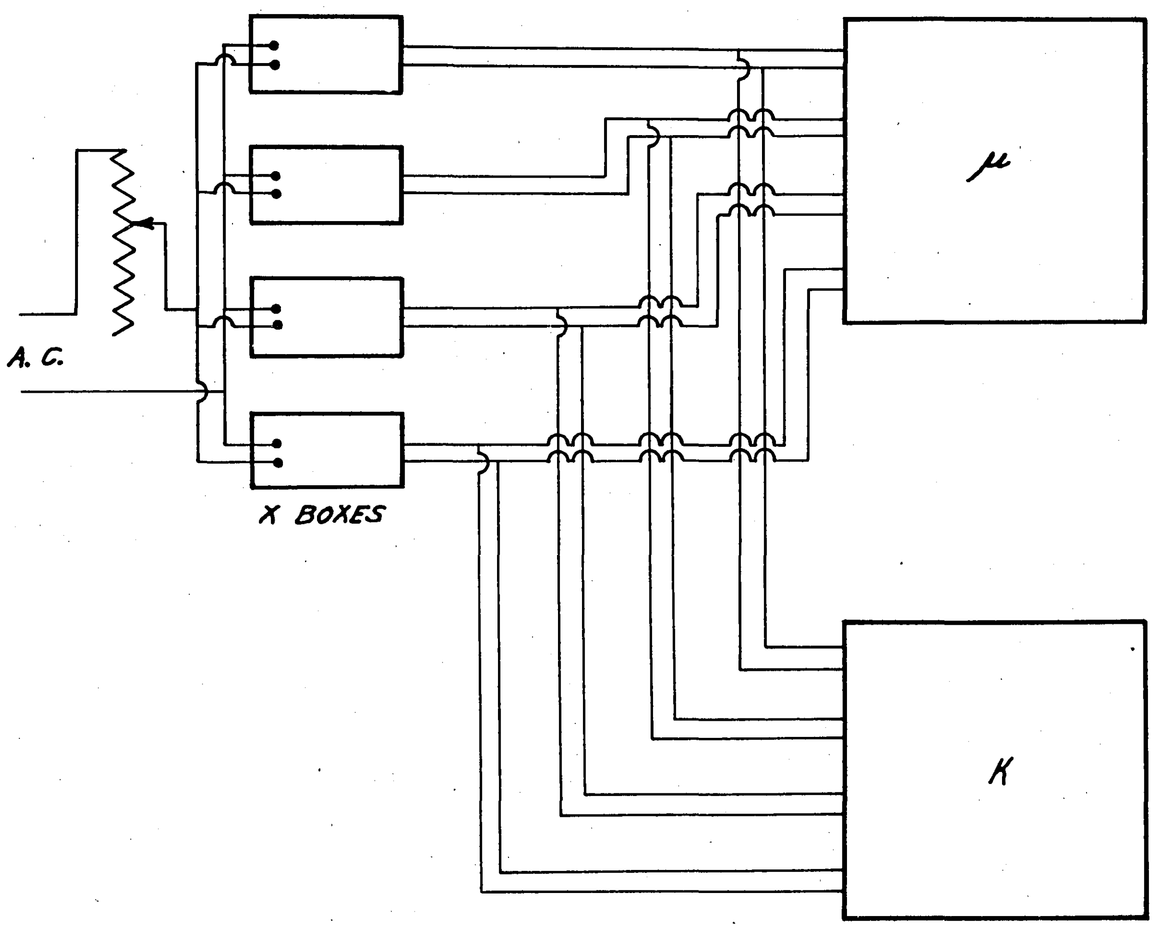
$$\mu = \Sigma_{i=1}^{q} (\Sigma_{j=1}^{n} b_{i,j} x_j)^2 - \Sigma_{i=q+1}^{n} (\Sigma_{j=1}^{n} b_{i,j} x_j)^2$$

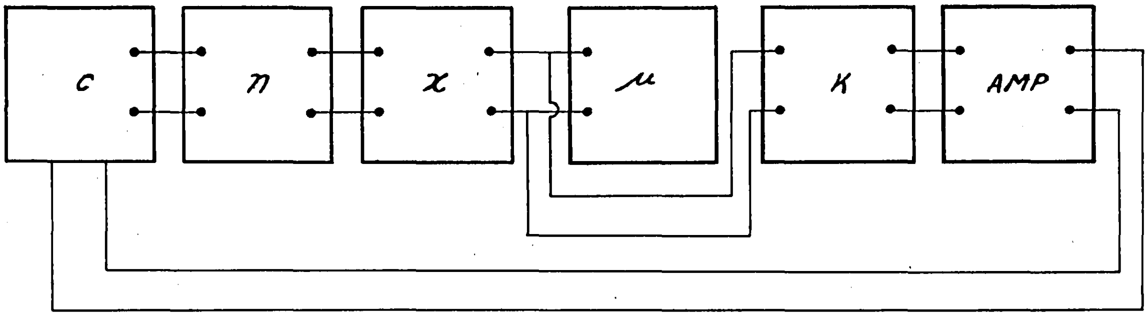
although in certain cases, the total number of squares may be less than n.

Consequently the device described in Sec. 5 can be readily modified to produce such a μ. For instance, one may produce both sums of squares and obtain minus the second sum by a one-stage direct current amplifier with gain one and then one averages.

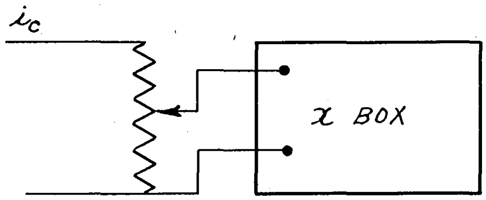
There is a characteristic vector $(x_1, \ldots, x_n)$ which maximizes μ subject to the condition $\Sigma x_i^2 = 1$. Thus, we must vary the x vector subject to this condition so as to maximize μ. This could be done by hand if one had an auxiliary device which produces $\Sigma_{i=1}^{n} x_i^2 = 1$. For instance, if we had a power control on the line before the x-boxes, we could use the x dials simply to indicate the direction numbers for the vectors x and by varying the power input insure that $k = \Sigma_{j=1}^{n} x_i^2 = 1$.

Since, however, this means that two controls must be manipulated simultaneously, it would probably be desirable to have an automatic feedback to control the length of the vector x. This could readily be





arranged in the case in which the x boxes are power amplifying circuits. The input signal for each box can be arranged with a current from C, whose value will be determined so as to maintain K constant, across a potentiometer whose contact position will correspond to the direction number



η_i for the corresponding x_i. (The sign, of course, would be determined by a d.p.d.t. switch which is not shown.) We now determine C by an integrating circuit arranged so that

$$\frac{dc}{dt} = - h(K-1) = - h(C^2[\Sigma\ \eta_i^2] - 1$$

where h is a large positive constant. For brevity let $\theta^2 = 1/\Sigma\ \eta_i^2$. (We can regard θ as a constant since the changes which occur in it as the η_i's are manipulated are supposed to be relatively slow.) Then this differential equation becomes

$$\frac{\frac{dC}{dt}}{C^2 - \theta^2} = - h\ \theta^2$$

which integrates into

$$\log_e \left(\left|\frac{C - \theta}{C + \theta}\right|\right) = - 2\ h\ \theta^3\ t + k$$

or

$$\frac{C - \theta}{C + \theta} = A\ e^{-2h\theta^3 t}$$

Solving for C this becomes

$$C = \left(\frac{1 + A\ e^{-2\theta^3 ht}}{1 - A\ e^{-2\theta^3 ht}}\right)\ \theta.$$

Now if C is positive at $t = 0$, then A will have absolute value < 1 and hence C will approach θ in a manner which is similar to an exponential approach.

V. Electronic Digital Machines

1. The present chapter is devoted to the two types of electronic digital devices which have been developed. The simpler of these consists of electronic counters which are used in experimental physics and industrial production controls. The other type is the high speed electronic computer.

The input to these counters are electrical pulses. The simplest type is a binary counter whose result is translated by the operator into a decimal value. However, there are now well developed methods of using an essentially binary setup to obtain a decimal result. There are also counters which are essentially decimal.

The high speed electronic computers are expensive and complicated devices suitable for performing very intricate calculations. They involve devices for the usual arithmetrical operations, for "memory" and the automatic control of these operations. In view of the general similarity of purposes, we will also treat the electromechanical computers in this chapter. Their general organization is similar although their performance is slower.

Although interesting refinements have been introduced, the general principles of the arithmetrical organs are similar to those of the digital devices previously studied. In order to obtain high speed and automatic functioning, these principles have been translated into electronic circuits. For the arithmetrical organs, we will indicate this translation, assuming that the reader is familiar with the principles given before.

However, the advance represented by these machines consists of the use of large memories, high speed computation and automatic controls. These present very interesting mathematical questions and the successful use of these machines requires a preliminary mathematical analysis of the problems which must satisfy certain criterions which we will discuss. We shall see also that their iterative abilities permit them to proceed on a more abstract level than ordinary computing.

In the present chapter we will first discuss the representation of numbers and, in particular, digits in these devices. We then indicate the nature of the arithmetrical organs developed, the memories and the control systems. Finally we will discuss the specific problems that arise in the use of these machines.

2. In these machines, the various digits are represented either by states of certain circuits or by pulses, coded relative to a fundamental time cycle. In a time coded machine, a number is represented by a timed sequence of signals. There is a signal for each radix place and each signal has a pulse character which indicates the digit in the corresponding place. In these the binary system is employed and a digit is represented by either a pulse or its absence. A storage element involves a closed circuit with a time delay in which the signal sequence corresponding to a number circulates.

In a position coded machine, each digit is normally represented by a state of a circuit and a number is represented by means of a set of such circuits, one for each radix place. When a number is operated on, each digit is transformed into a pulse coded signal but the signals travel in parallel paths, one for each decimal place.

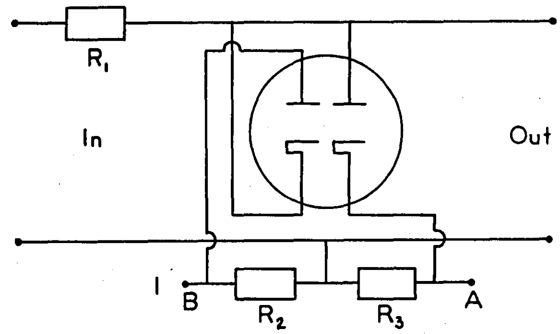
In the Eniac and in the Harvard calculator the base ten is used, in a position coded system. An active signal in the Eniac, is a set of pulses equal in number to the digit represented. In the Harvard calculator the subinterval of the fundamental cycle in which a signal occurs indicates the digit. In the newest I.B.M. sequence calculator, the decimal system is used but each digit is expressed as a binary. This is called the binary decimal system.

In the newer machines, the binary system is used. An active signal for a position coded device is just a pulse or its absence.

In the course of a long computation, the various numbers which appear may differ considerably in size. This may be taken care of by using registers of adequate length or by writing the number in the form $\alpha_0, \alpha_1, \alpha_2 \ldots x\, r^n$ where r is the radix used and recording both $\alpha_0, \alpha_1, \alpha_2 \ldots$ and n. This last method is referred to as the "floating decimal point." Its use does tend to complicate the arithmetical units but the simpler system requires much more mathematical analysis to insure the proper functioning of the machine and to prevent unnecessary loss of accuracy.

In both types of devices, it is necessary to introduce the electronic equivalent of a switch. These are referred to as "gates." A gate has an input channel, an output channel and a control channel. A signal on the input channel appears in the output only when the control is activated. For example, a double grid tube or "mixer" can be used as gate. The first control grid is the input, the plate is the output. Normally the second control grid is below the cut-off but a control voltage can bring it into the active region in which case a signal on the first grid will appear on the plate. The screen grid of a pentode or the suppressor grid can also be used for cut-off purposes.

An alternative gate can be obtained by means of a double diode, which is to function as a variable resistance shunted across the main channel.



Normally no current flows in resistors R_2 and R_3 and the output is essentially shorted through the diodes and R_2 and R_3. If a current flows through R_2 and R_3 from A to B, then each diode plate is below its cathode and the diode is effectively a large resistance and the output voltage is a considerable fraction of the input voltage.

A third gate possibility is connected with grid bias. The control voltage determines the grid bias. Normally this is far below the cut-off but when a signal is to be passed the voltage is brought just to the bias point and the tube will pass a positive pulse.

3. The fundamental arithmetrical unit for a high speed digital machine of the position coded type is a device to correspond to a digit. For each place in a register, we must have a unit which is capable of receiving the digit it is to represent, storing the digit indefinitely, indicating the digit when signalled and it must be capable of clearing.

These four properties are the minimum requirements for a place in a register. They are adequate for registers which correspond to memory or temporary storage. For an accumulator or adder, the addition operation must also be provided for. In current practice, this requirement is satisfied by making this type of digit unit a cyclic counter controlled by pulses and with a cycle equal to the radix. This counter can either count unit pulses or it may be subject to initiating and terminating pulses. When a cycle is completed a pulse is emitted which is to activate a carry arrangement.

A trigger circuit is the obvious unit for a binary counter. A pulsed output can be obtained from the plate of either tube. Let us refer to the tubes which are conducting in the 0 and 1 state as the 0 tube and 1 tube respectively. An output connected to the 0 tube will give a positive pulse when the circuit changes from 0 to 1, a negative pulse when it changes from 1 to 0. The polarities are reversed for the 1 tube.

A set of such circuits can be readily combined into a binary counter. For instance, if the trigger circuits respond to negative pulses, we connect a sequence of these, so that the input trigger circuit corresponds to the 2^0 place. The 0-tube output of this circuit is connected to the input of the 2^1 trigger circuit and so forth. This corresponds to a counter. For an accumulator provision must be made for a carry system. Notice that with the binary system there is no advantage in using a simultaneous feed and successive carries since the latter would take as long as a successive feeding of the original addition. Two systems have been proposed. One of these involves two carry registers. When the first addition occurs the first carry register receives the carries. It then adds into the main register and any second carry appears in the second register. The second register is now used as an addend and the process is repeated until both registers are cleared.

The other is, of course, the binary equivalent of the "standing on nine" carry. Consider the place corresponding to 2^n. If after the main addition, the 2^n place stands on one, its carry is empty. Consequently the 2^{n+1} place can be connected directly to the entrance of the 2^n place so that if the 2^n place receives a pulse, the 2^{n+1} place receives one also. This channel is to be open only in the carry time following the main addition. After the carry time, the carry registers are cleared. Shifting one place can be accomplished by performing the main feed for $1\cdot2^N + 1\cdot2^{N-1}+\cdot+1.2^0$, thus causing the original number to appear in the carry register. The main register is then cleared and the carry performed.

•For each decimal place, the Eniac uses a ring counter consisting of ten trigger circuits, with a mutual cathode bias which favors the state in which only one trigger circuit is on. As one trigger circuit is pulsed off, its successor is turned on. The Harvard Calculator uses a rotating brush set up for each decimal place with a solenoidal clutch feed for addition. It also uses the "standing on nine" carry.

In a time coded device, the addition of two quantities is accomplished by a circuit involving gates. The two terms are fed into the circuit, as a sequence of signals in the order $\alpha_0 2^0 + \alpha_1 2^1 + \ldots + \alpha_n 2^n$ and the sum appears as the output. Actually it is necessary to have a gate arrangement in which there are three inputs at a time and two outputs. The carry from the addition in the previous place must also be treated as an input along with digits of the terms and output must also indicate the carry which enters a time delay network from which it emerges at the input in time for the addition in the next place.

It is not difficult to set up a gate arrangement to accomplish this. A pulse can be used as a control voltage either to open a gate or to close it. Thus two gates with the output of the first going in to the second will have three inputs and a pulse will appear on the output only if each input has a prescribed character. Thus we can arrange it so that a pulse will appear in a prescribed output if, for instance, a pulse appears on the second and third input but none on the first.

There are eight possible states for the three inputs. One of these (1,1,1) should cause pulses to appear in both outputs, one should cause no pulses and can be ignored and for each of the others, a pulse should appear in one but not the other outlet.

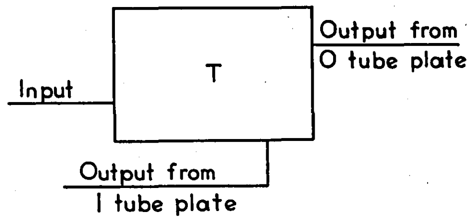
Apparently this type of addition will take a long time but this disadvantage can be compensated for in many problems by performing many additions at once.

4. In many physical experiments, in particular, those involving Geiger counters, it is desired to count high speed pulses. For these electronic pulse counters were developed, originally in a binary form as a simple sequence of trigger circuits, but eventually with a decimal indication to the operator. High speed counters with a decimal indication were also developed for production controls. These are used, for instance, in packaging in conjunction with a photocell to count the objects passing on a conveyor belt. In a high speed electronic computer which uses the binary system, it is necessary to convert decimal inputs into binary form and also perform the reverse operation.

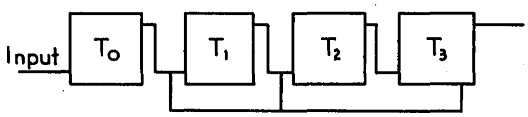
Four trigger circuits in a sequence would normally constitute a counter with modulus of 16. However, it is possible to introduce auxiliary circuits which will reset the device to zero at the tenth pulse. In these counters, it is necessary to provide a method for sending a signal to one grid of every pair so that the circuit can be reset to the (0,0,0,0) state. This reset arrangement can also be used to make the cycle ten pulses in length. Starting with (0,0,0,0) one finds that the first time at which both the first and the fourth trigger circuit are in the one state is for (1,0,0,1) or 9. Suppose we have a gate which is controlled by these two circuits so that when they are both in the 1 state, a path is opened to the reset arrangement. Thus the tenth pulse will reset the circuit to (0,0,0,0). It is also possible to set up an arrangement in which the states 6-16 are used.

The usual output of a trigger circuit is the negative pulse which occurs at the 0 tube plate when the circuit passes from the 1 to the 0 state and in a binary counter, this is fed to the stage for the next higher power of two. However, it is also possible to obtain a negative pulse output from the plate of the 1 tube when the circuit passes from the 0 to the 1 state. Thus we can schematize the usual trigger circuit as shown in the following diagram.

The second output can be used in a number of

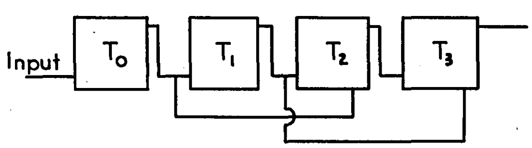


ways to cause the four stage binary counter to skip 6 states and thus become a decade counter. These are discussed in a paper by I. E. Grosdorf, *R.C.A. Reviews*, Vol. VII, No. 3 (Sept. 1946), 438. We may have, for instance,



The stages T_0, T_1, T_2 constitute a scale of 8 counter which will pass through the states corresponding to 0, 1, ..., 7 without any output to T_3. When the first three stages reach the (1,1,1) state corresponding to $7 = 1\cdot2^0 + 1\cdot2 + 1\cdot2^2$ and a pulse is received, these three stages clear and a pulse is sent to T_3 . T_3 then changes to 1 and sends a pulse from its second output to T_1 and T_2. These also go into a 1 state so that the circuit passes from the (1,1,1,0) state to the (0,1,1,1) state, which corresponds to $1\cdot2 + 1\cdot2^2 + 1\cdot2^3 = 14$. The next pulse changes the circuit to (1,1,1,1) and then the next pulse clears in the usual fashion. Thus the circuit passes through the binary stages 0,1,2, ..., 7,14,15,0 and has a cycle of ten.

There is still another possibility involving two feedbacks. This combination passes from (0,0,0,0)



to (1,1,0,0) in response to the first three pulses. As above the effect of the next pulse is to send it to the (0,1,1,0) stage, i.e., the counter passes from 3 to 6 skipping two states. The next state is (1,1,1,0) or 7. The next pulse, at first, clears the system to (0,0,0,1). However, there is also a feedback from the last stage so the circuit goes to (0,0,1,1) or 12. [One would also expect that when T_2 turns to 1 from 0 in this process, that another pulse is returned to T_1. However, T_2 is in the 0 state so briefly that the capacitative connections have not adjusted themselves to the 0 state, so the feedback signal from T_2 to T_1 is not adequate to change the state of T_1. The circuit continues then through 13, 14, 15 to 0. Thus the circuit goes through the binary states equivalent to 0,1,2,3,6,7,12,13,14,15 and back to 0.

The state of this circuit is indicated by neon lamps. A neon lamp will glow only if the total voltage across it exceeds a certain amount. In the R.C.A. counter described by Grosdorf, a neon lamp is used for each state. For each state two points are chosen so that only when the circuit is in this state is the full voltage available. The lamp is connected between these points.

For odd numbers, the negative point is taken to be the plate of the 1 tube in the T_0 stage and for even numbers the plate of the 0 tube in the T_0 stage. The positive point is obtained as follows: For each of the pairs (0,1), (2,3), (4,5), (6,7), (8,9), we find a pair of tubes which are both off for this pair of states but no other. Let the plate current for these two tubes go through a common resistor from B+ before it goes into the individual load resistor. Let the lower end of this resistor be the positive point for both lamps of the pair. If neither tube is conducting this point is at B+ voltage, otherwise lower. Thus only when both tubes are off, will the positive end of these lamps be high enough to permit glow. For the pairs (0,1), the tube pair is $T_3$1 and $T_1$1; for (2,3), $T_1$0 and $T_2$1; for (4,5), $T_2$0 and $T_3$1; for (6,7), $T_1$1 and $T_3$0, and for (8,9), it is $T_1$0 and $T_3$0.

Another interesting type of counter is the pentode pair ring counter developed by Regener; *Rev. of Sci. Instruments*, XVII (1946), pp. 180-89. (Two papers). For each decimal place, this consists of a ring of ten pentodes. These ten pentodes can be considered as equally spaced around a circle and connected so that diametrically opposed tubes constitute a trigger pair, i.e., the screen grid of each tube is directly connected to the plate of its opposite number. In addition, resistors are connected between adjacent pairs of plates, so that the voltage on a plate is partly determined by the plate voltage of its neighbor and this tends to keep adjacent pairs in the same state. Consequently as we go around the circle the most stable configuration is five adjacent pentodes on and five adjacent ones off.

The above connections are intended primarily to maintain a state for the ring. The method of changing state is the following: The control grids of the pentodes are connected together. A positive pulse received by an off tube will bring the plate voltage down but only slightly affect the on tube. Now each tube plate is connected to the suppressor grid of the next clockwise tube. When the pulse is received each tube which is the successor of an off tube receives a negative pulse on its suppressor which would turn it off if it were on. But the only tube of these which is on, is the first clockwise on tube and this is turned off and its successor turned on.

A cathode ray tube can be used to indicate the state of the decade.

5. An accumulator is a register designed for addition. In most automatic calculators, negative numbers are represented by their nines' complement or ones' complement, and subtraction is referred to addition. The manipulation of signs in the elementary operations, for such a system can be best understood from an example. We will use the decimal system in the example since it will be easy to pass from the decimal to binary system, while the reverse process may be a little obscure.

Suppose then we wish to represent the numbers between -.499 and +.499 to three decimal places. We have three decade counters one for each decimal place and a binary place for the sign. This binary place receives a carry from the decimal place besides the decimal point and sends a carry to the last decimal place. A positive number +.246 then is represented by 0.246 while a negative number is represented by its nines' complement with a 1, i.e., -.324 is represented by 1.675 where the first one is to stand for both the negative sign and the extra one that must be added in last place in the complementation process. We suppose also that there is some alarm system which operates when the numbers are off scale, i.e., when numbers appear between .500 and 1.500 in the register. Alternately, we can suppose that we have a "floating decimal point" and when a number is off scale an appropriate shift in the number and exponent occurs. In the "floating decimal point" case, addition or subtraction must be preceded by a shifting process which yields equal exponents.

Now one can readily show that the addition of algebraic quantities is obtained by straight addition in this accumulator. Since positive quantities are less than .5, the addition of two of them will not introduce any end carry. Consider next the addition of a positive and a negative quantity. If the positive quantity exceeds the negative one in absolute value, for instance, if we are adding .325 to 1.847 (i.e., -.152), then there will be a carry from the decimal place next to the decimal point to the binary place. Added to the one in the binary place, this causes this place to clear to zero and a one is sent to the last decimal place. This one compensates for the fact that the usual complement for -.152 would be .848 not .847. The result is .173 as it should be. If the negative number exceeds the positive in absolute value, we will have no carry from the decimal place next to the decimal point and the result appears in the correct code. If two negative numbers are added, one can verify again that all the carries operate and leave the result in the correct code.

Multiplication is most easily accomplished by taking absolute values and obtaining the sign separately.

The binary equivalent of the above discussion is obvious.

Since electronic accumulators and registers are available, one can readily construct an arithmetrical organ analogous to an ordinary desk calculator, provided one introduces a method of shifting. Multiplication is by repeated addition which, in the binary case, coincides with the split multiplication table method. As in the case of the desk calculator one needs a register for the multiplier, another for the multiplicand and an accumulator for the product.

The multiplication process begins by entering the multiplier and multiplicand into their respective registers. The next step, in the binary case, consists of sensing the digit in the last place of the multiplier and if it is 1, adding the multiplicand into the accumulator for the product. Then both product and multiplier are shifted one to the right. The accumulator has n places and in this shift the digit on the right-hand end of the accumulator is shifted to the left end of the multiplier register. The digit of the multiplier which has been used disappears and the digit for the next higher place in the multiplier appears at the right end of the multiplier register. Thus we can repeat the above process of sensing the right end digit of the multiplier, adding if necessary and shifting until all the digits of the multiplier have been used and the product appears with the digits for the higher places in the accumulator and for the lower places in the multiplier register.

In the time coded binary machines, multiplication is again by repeated additions. In the Harvard calculator, which uses the decimal system, the multiplication begins by storing the nine digital multiples of the multiplier and then a process analogous to the above is carried on, with the appropriate multiple of the multiplier being added to the accumulator at each step. A split multiplication table method of multiplication is used in the Eniac, involving a bank of tubes. Cf. A. W. Burks, *Proc. of I.R.E.*, XXXV (1947), pp. 756-67.

There is another method of representing a number which has the multiplicative advantages of the binary system. This uses the radix three but digits - 1,0,1 are used instead of the usual 0,1,2. Two carry systems are needed but sign procedures are simpler.

If a number of accumulator registers are available in a decimal machine, they can be used for multiplication. For instance, suppose we have three such accumulators. We associate these with the digits 1, 3 and 5. Each digit of the multiplier is expressed in terms of 5, 3 and ±1. Then at each place, a properly shifted version of the multiplicand is fed into the appropriate accumulators. For instance, if the multiplier digit is 4 = 5 - 1, the multiplicand is fed in to the five accumulator and subtracted from one. Shifting of the multiplicand occurs at every step and at the end the content of the five accumulator is added five times to the one accumulator and the content of the three accumulator is added three times.

Division can be carried out in the usual fashion by subtracting multiples of the quotient. In the Harvard machine, the various multiples are available in the multiplier and the appropriate one is selected by a selective comparison process. In the binary devices, division must begin with appropriate shifting. One can proceed in the usual manner or one can permit the sign of the remainder to vary. Another method for division involves finding a reciprocal and multiplying by it. Reciprocals can be found approximately by means of a table and then made precise by interpolation in the table or by an iterative process such as the following. Suppose N is a number whose reciprocal is desired and let x_0 be such that $|(1-Nx_0)| < 1$. Consider the sequence defined by the relation $x_{n+1} = x_n(2-Nx_n)$. One can verify that $1-Nx_{n+1} = (1-Nx_n)^2 = (1-Nx_0)^{2(n+1)}$ and hence approaches zero as $n \to \infty$.

There are a number of methods for extracting the square root. The usual method involves a trial process. If x is our approximation to $\sqrt{N}$ then we seek a Δx such that $2x\Delta x + (\Delta x)^2 < N - x^2$. Such a trial process can be carried out on these machines but other methods are preferable. For instance, we can first find an integer n such that $N = n^2 N_1$ with $\frac{1}{4} < N_1 < 1$. This reduces the problem to that of extracting the root of a quantity less than 1. For N_1 the following sequence can be used. Let $x_0 = N_1$, $x_{n+1} = x_n + \frac{1}{2}(N_1 - x_n^2)$. One can readily show that

$$N_1 - x_{n+1}^2 = (1 - x_n - \tfrac{1}{4}(N_1 - x_n^2))\ (N_1 - x_n^2)$$

Let us suppose for the moment that $x_n < \sqrt{N_1}$. Consider

$$\begin{aligned} \theta_n &= 1 - x_n - \tfrac{1}{4}\ (N_1 - x_n^2) \\ &= 1 - x_n - \tfrac{1}{4}\ (\sqrt{N_1} - x_n)\ (\sqrt{N_1} + x_n) \\ &\geq 1 - x_n - \tfrac{1}{2}\ (\sqrt{N_1} - x_n) \\ &= 1 - \tfrac{1}{2}\ (\sqrt{N_1} + x_n) > 0 \end{aligned}$$

This result and the above equation for $N_1 - x_{n+1}^2$ shows that $x_{n+1} < \sqrt{N_1}$. We also have $\theta_n < 1$ and thus $x_n < x_{n+1}$. Therefore, $\theta_n < 1 - x_0$ and $N_1 - x_{n+1}^2 < (1 - x_0)^{n+1}\ (N_1 - x^2)$. Hence the sequence x_n converges to $\sqrt{N_1}$.

Functions are taken care of by tables in which the values of the function are given at various intervals and the coefficients for interpolation in the Newtonian formula. In the I.B.M. sequence calculator, a table is constructed by the use of selective switching, utilizing the argument, and a number of tapes.

6. Punched cards are, of course, the most common form of memory for a computing device. The Harvard calculator uses this form of memory with other forms. However, a punched card memory is a sequential memory, i.e., to get to a specific place in this memory, it is necessary to go through a number of cards until one comes to the desired card. Card reading and card punching are slow relative to the speed of electronic devices and for this reason cards are used in the purely electronic devices only for record purposes.

A tape or a magnetic wire recorder is another type of sequential memory. These are used to store functions. However, as a rule the functions are not stored as a table but selected values and the interpolation coefficients are given so that, other values can be obtained by using Newton's Formula. This procedure is the desirable one for the machine since the sequential nature of the memory is minimized.

Notice that if a stack of cards is used as a memory it is desirable to punch on each card not only the number to be recalled but also another number which acts like an index and indicates the place in the memory. A memory is a function of one variable. The index or key is this variable and the value of the function is the number to be recalled.

For many types of problems, it is desirable to have a memory with an immediate response, i.e., one that is not sequential. There are in the Eniac and also in the Harvard calculator a large number of accumulators and these are used for this purpose. In addition, there are registers, in which each decade can be set by hand.

However, the new binary machines will be designed around an electronic memory in which the response to the key is essentially of the same order of magnitude in time as the other operations of the machine.

The time coded machines will have a memory which is essentially a delay line. The pulses and omissions travel along a mercury tube as shock pulses. When they reach the end of the tube, they are detected, amplified and impressed on the initial end of the tube. This memory has a slightly sequential character, since eight numbers, each of forty binary digits are present in the tube and one must wait until the desired one appears.

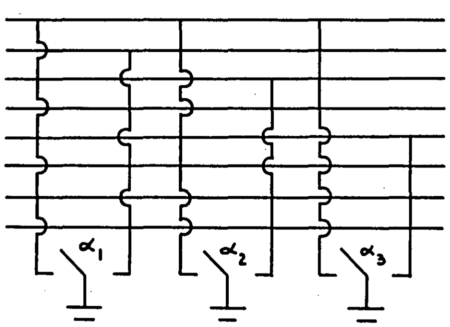
The position coded machine will have a special memory tube, which is equivalent to a large number of trigger circuits. However, the basis of the device is not a matter of external circuits but depends upon the secondary emission characteristics of certain substances entirely within the tube. Suppose we have a substance, which is such that when an electron with a certain velocity strikes it more than one electron is ejected but if the velocity is lower than a critical value, the electron is simply absorbed.

Now suppose we have a number of pieces of this material in the path of an electron beam. The speed v_0 corresponds to a certain voltage e_0. If a piece has a higher positive voltage than e_0, the electrons which strike the piece will have higher velocity than the critical value and more electrons will be emitted than received and hence the voltage of the piece will tend to become more positive. On the other hand, a piece at the same potential as the source will receive low velocity electrons and since no secondary emission will occur, it will tend to become more negative. Thus a steady stream of electrons will tend to maintain each piece in one of two states and thus each piece can be used as a stage in a binary register.

The emitted electrons are collected on a suppressor grid. The pieces of emitting material are placed on one side of a sheet of dielectric and the other side is metal coated. This arrangement constitutes a number of condensers with a common plate, which is connected to the suppressor by means of a resistor. Now if a slowly moving beam is concentrated on a piece which is positively charged then the corresponding condenser is discharged and the discharging current flows through the resistor. The latter is the desired external signal.

The keying of the memory may depend upon positioning the beam vertically and horizontally and in addition the suppressor grid can be split for selection purposes. The more keying parameters that are available the better, for if there are k parameters, each with n possible values, the total memory positions are n^k.

For memory selection and also for fulfilling orders, it is necessary to have a method by which a binary number signal will select one of a number of channels. The diode clamping gate can be used for selection purposes, since it can be used to ground a number of circuits at once. For instance, suppose we have eight channels, one of which is to be selected by a three-digit binary, α_1, α_2, α_3. Each α_i chooses between two clamping circuits so that the input of one of these is grounded, the other then is open to ground. This is equivalent to a single pole, double throw switch with center grounded and the situation then can be described by the diagram.



Notice that for each choice of positions for all the switches one and only one channel remains ungrounded. If the signals are all of given polarity, only one diode is necessary for grounding. A matrix selecting switch system can also be used in multipliers to obtain the two digits in the split multiplication table method and circuits of this type were originally developed by Rajchman and Crawford for this purpose. Multiple contact relays are used for selection also.

7. The new digital calculators are intended to carry through calculations which are primarily cyclic in nature. These cycles are in general complex with subcycles and sub-subcycles which may vary in character.

Each step of such a calculation is governed by an order which is stored in the machine either on tapes or in the memory. After each order has been executed, the next order appears. A cycle or subcycle of a calculation then corresponds to a sequence of orders.

In many mathematical procedures, a given cycle of computation is repeated until a desired result is obtained, for instance, until a given equation is satisfied. The sequence of orders corresponding to such a cycle will end then with a conditional order which causes the machine either to return to the first order of the sequence or to go to a new sequence of orders.

For a more detailed discussion of the various orders that are to be used in the new binary digital computers, the reader is referred to the reports by Burks, A. W., H. H. Goldstine, and John von Neumann, *Preliminary Discussion of the Logical Design of an Electronic Computing Instrument*, 2nd edition. Princeton, New Jersey: The Institute for Advanced Study, 1947.

In general, an order will appear as a binary number. This number will be broken into two parts, $\alpha_1, \ldots, \alpha_u, \beta_1, \ldots, \beta_v$. The digits $\alpha_1, \ldots, \alpha_u$ will be applied to an order switch circuit similar to that described in the previous section and a channel is selected. When a pulse is sent along this channel, the appropriate circuits are put into the correct states to execute the order. For instance, suppose a number is to be read out of a certain register A, and into another register B. For each position in the A register we have a gate to the main bus which is opened or closed depending on the digit in the place. The order opens a common input channel to these gates so that a pulse is sent out on the main bus in each place where there is a one in the A register. At the B register, gates are opened so that the pulses which appear on the main bus enter the register.

One type of order then refers to the transfer of a number from one register to another. Frequently it is also desirable to clear the receiving register. However, in the case of an accumulator, the number received is added to the content of the register, and one can have orders to multiply or divide or carry through a sequence of arithmetrical operations when this sequence is wired into the arithmetrical unit.

Another type of order selects a position in the memory and orders the contents of this position read into the memory register. The B's are used to determine the memory position and are zero otherwise.

The steps in the ordering and the mathematical processes are timed by a crystal controlled oscillator, called the "clock," which sends out micro-second pulses. The time to carry out an order varies and the control system must sense when the order has been carried out in order that a new one may be initiated.

A control system for a digital calculator then consists of a register or registers to indicate the memory position it is to use. Normally this acts like a counter which changes by one at each step in the order sequence but a particular order may reset it to the initiating position. There will be two switching matrices, one for orders, one for selecting a memory position. Finally, for each order there will be various circuits associated with other components in the computer.

The above refers in the main to the projected binary digital computers. The order sequence in the Harvard calculator is tape controlled and the gating by which the orders are executed is accomplished by relays. The control system for the Eniac is decentralized and each unit in the Eniac has a control associated with it. The control initiates the operation of the unit, including gating, when signaled and after its function has been performed it sends a signal to another unit. In the Eniac, there are a large number of accumulators some with special features, and a large multiplying unit. There is a set of control circuits associated with each unit. Instead of one bus, provision is made for connecting the various units by a number of multi-channelled connectors, which are set up in advance of the computation.

In the new I.B.M. Sequence Controlled Calculator, orders appear on tapes. There are a number of order tapes and each order contains a direction as to which tape contains the order which is desired next. This permits multiple subsequencing in a convenient way. The selective switching is based on relays.

A tape controlled device can also be used effectively for a long sequence of non-repetitious calculations, provided one intends to do this a number of times. The order sequence is recorded on tapes, independently of the numerical values involved and this tape used repeatedly. The magnetic wire memory of an electronic device can be used for the same purpose.

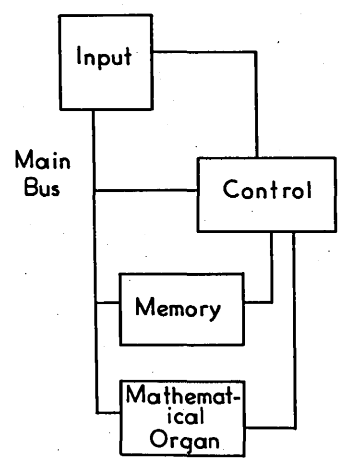
The input system for a binary calculator must involve a provision for transforming a decimal number to the binary system. Purely from the mathematical point of view, the easiest way to accomplish this transformation, is in two steps. Firstly, each digit is expressed as a binary, i.e., the number is changed to the "binary decimal" system. Thus, if the number is $\alpha_1\alpha_2\alpha_3$. in the decimal system we can express $\alpha_1 = \beta_{i,0} + \beta_{i,1}2 + \beta_{i,2}2^2 + \beta_{i,3}2^3$ where $\beta_{i,j}$ is 0 or 1. We then have

$$\begin{aligned}\alpha_1\alpha_2\alpha_3 &= \alpha_1 10^2 + \alpha_2 10 + \alpha_3 \\ &= (\beta_{1,0} + \beta_{1,1}2 + \beta_{1,2}2^2 + \beta_{1,3}2^3)\,10^2 \\ &+ (\beta_{2,0} + \beta_{2,1}2 + \beta_{2,2}2^2 + \beta_{2,3}2^3)\,10 \\ &+ (\beta_{3,0} + \beta_{3,1}2 + \beta_{3,2}2^2 + \beta_{3,3}2^3)\end{aligned}$$

If we express 10^2 and 10 in the binary system, it is clear that by a distributive process, the original number will appear in the binary system.

This can be used in a binary computer as follows. Suppose we begin with a twelve-digit number. We express this first in the binary decimal system. This is done in the input circuits and the result stored in the memory, using a memory place for each digit. Each digit is then multiplied by the appropriate power of ten expressed as a binary and the results added. This requires a total of 22 memory places and 12 multiplication times but this is much faster than any method of feeding the device, indeed, the conversion could occur digit by digit as the numbers enter the machine. The converse process is best carried out by reversing this process, i.e., the given binary is first divided by the highest power of ten used until the remainder is less than this power. The quotient is then a decimal digit expressed in the binary form and as such is sent to output for conversion. The remainder is then divided by the next highest power of ten and so on.

8. We are now in a position to give a schematic for the proposed digital computers.



As we have explained in the previous section, the machine will continue to carry out a given cycle of computation until a desired result is obtained. This cycle, which in turn may consist of subcycles, is executed by means of a sequence of orders stored in the memory. The control system ordinarily proceeds from one order to the next. It has registers which contain the parameters which specify the step in the cycle involved and consequently the memory position from which the next order is to be obtained. Ordinarily this register changes by a single step. However, a conditional order may reset this counter to its initial value if one desires to repeat the cycle.

In a tape controlled system, each order appears on a tape and each order contains a line which indicates from which tape the next order is to be obtained. It is obvious that various subcycles may be carried out by this process. A subcycle is initiated by referring to a specified tape which contains the subcycle orders. The orders continue to refer to this tape until the subcycle is completed or until a desired result is obtained. In the latter case, by means of a conditional order, the subcycle may be repeated a number of times.

In the purely electronic devices, there will be at least one order register and we will simplify our discussion by assuming that there are order registers for any cyclic parameter we need. However, the memory can be used in such a fashion that one can proceed quite conveniently with only one order register.

We will illustrate the behavior of these devices in general by indicating the way they would be used to solve a system of linear equations

$$\Sigma_{j=1}^{n} a_{i,j}\, x_j = b_i;\ i = 1, \ldots, n$$

by the Gauss Seidel method. As a practical matter, there are other methods more suitable for the solution of simultaneous linear equations in a device of this type. (Cf. the paper of Goldstine, H. H., and J. von Neumann, *Bulletin of the Amer. Math. Soc.*, LIII (1947), 1021-99.) However, its procedures do give a simple example of the type of situation to which a sequence calculator is well suited.

Let us suppose that k is the number of cycles which have been completed and that we are engaged in the k + 1st cycle. This cycle consists of n steps. Suppose we have completed t-1 steps. We now have as values for $x_1, \ldots, x_{t-1}$ the values $x_1^{(k+1)}, \ldots, x_{t-1}^{(k+1)}$ and for $x_t, x_{t+1}, \ldots, x_n$, the values $x_t^{(k)}, x_{t+1}^{(k)}, \ldots, x_n^{(k)}$. Mathematically we wish to calculate with these values of x

$$x_t^{(k+1)} = (b_t - \Sigma_{j=1, j\neq t}^{n} a_{t,j}\, x_j)/a_{t,t}$$

and substitute this value for x_t. We must also note where $x_t^{(k+1)}$ differs from $x_t^{(k)}$ and when no difference occurs during an entire cycle we stop the procedure.

It is clear that t is an order parameter. But in executing a step, we must carry out n substeps which we enumerate by means of the parameter j. Substep j consists of the orders:

A. Compare j with t. If j = t the step is completed and we change j to t + 1 and start over again. Otherwise

B. Locate $a_{t,j}$ in the memory and bring to the memory register (one order) and then transfer to multiplicand register of the arithmetrical organ. (These two orders would normally appear on one line.)

C. Locate x_j in the memory, transfer to the memory register and then to the multiplier register.

D. Multiply $a_{t,j}$ and x_j; locate $\Sigma_{\alpha=1, \alpha\neq t}^{j-1} a_{t,\alpha}\, x_\alpha$ in the memory and transfer to the memory register.

E. Add $\Sigma_{\alpha=1, \alpha\neq t}^{j-1} a_{t,\alpha}x_\alpha$ to $a_{t,j}x_j$ in the accumulator and compare j with n.

F. If $j \neq n$, transfer $\Sigma_{\alpha=1, \alpha\neq t}^{j} a_{t,\alpha}x_\alpha$ to proper memory position and set j to j + 1; and return to A. Otherwise

G. If j = n, locate b_t in the memory and form $b_t - \Sigma_{j=1, j\neq t}^{n} a_{t,j}x_j$.

H. Locate $a_{t,t}$ in the memory and transfer to divisor register.

I. Divide $b_t - \Sigma_{\alpha=1, \alpha\neq t}^{n} a_{t,\alpha}x_\alpha$ by $a_{t,t}$ and locate $x_t^{(k)}$ in the memory.

J. Subtract $x_t^{(k)}$ from $x_t^{(k+1)}$ and if result is zero proceed to (L) below. If result is not zero, add $x_t^{(k)}$ to $x_t^{(k+1)} - x_t^{(k)}$ and

K. Transfer $x_t^{(k+1)}$ to memory place for x_t and send a unit to the place in the memory where the condition number for the full cycle is stored. (We suppose that the memory is set up so that a position is cleared just before anything is written into it. Thus $x_t^{(k)}$ is erased in the x_t place and if we already had a unit in the condition number place, this unit is erased and then replaced.)

L. Compare t with n. If $t \neq n$, change t to t + 1 and repeat the full step. If t = n, consider

M. Recall the condition number for the full cycle. If this is zero stop the machine. If not reset t to zero, put a zero in the condition number place and repeat the process.

We may schematize this by

$$(\,[\,(A(?)BCDE(F?))_{j=1}^{n}\ \ GHIJ(K?)L]_{t=1}^{n}\ M)^{k}.$$

Notice that we can add to or subtract from a quantity in the accumulator immediately with an addend or subtrahend from any other register. The conditional orders can be set up by means of a switching circuit. One simple arrangement would involve a register to control this switching circuit. To execute a conditional order, we use a full line which consists of two orders. (We have indicated this in the above.) The first part of the first order causes the number to appear in the condition register. If this register now reads zero, the machine is referred to the memory position given by the $\beta_1\beta_2\ldots$ of this order for its next order. If this register does not read zero, the machine proceeds to take the second order on this line.

Of course, many variations on the design of such a computer are possible. For instance, since most operations involve two numbers, one may have two main busses and two memory registers. The arithmetrical organ will contain two registers and an accumulator at least. It may be convenient in certain circumstances to have one other register here.

9. There are many complex problems associated with the use of the new computers. One of the most important of these is checking. Certain checking procedure can be considered as maintenance for electronic devices. For instance, one can have routine inspections which will spot and replace any unit when it enters a marginal condition and before failure occurs.

In addition, however, other checks which are mathematical in character are usually considered necessary. When a mathematical procedure has been decided upon, check runs should be made in the first place to check the mathematics. This can be done on hand machines by operators and should be, in order to provide a record for study. When the procedure has been set up as a sequence of orders, further test runs should be made on the device itself to test the ordering set up.

Two proposals have been made to check the mathematical functioning of the machine. One of these would construct two machines to operate in parallel with cross connections between the accumulators in the arithmetrical organ and certain other registers. These cross connections would detect any discrepancy between analogous registers and stop the computation in order to permit an inspection to determine the error. The alternate proposal is to program the usual computational checks as part of the ordering. These are well known and, in particular, smoothness checks have been well developed and are relatively easy to apply. If any of these criteria are not satisfied, the machine is stopped.

10. It is clear that a preliminary mathematical analysis of the computation is necessary for the use of a digital computer. The validity of each step must be carefully examined so that a precise sequence of orders will be laid out. The length of a computation is important if it is possible for round off errors to have an accumulative effect or if the time needed is impractically large.

There is considerable room for mathematical investigations of the type represented by the paper of Goldstine and von Neumann referred to above which precisely specifies the error in the elimination method for solving linear equations. While the use of these devices will undoubtedly provide empirical information on the validity of various computational procedures, precise theoretical analyses are also essential.

VI. Accuracy, Noise and Stability

1. In this Chapter, we wish to briefly introduce the reader to the stability problem in the design of mechanisms. The stability question has been considered in a number of connections and reasonable methods for dealing with it have been developed in certain cases. In general for each problem, there is at least one method of obtaining a stable solution to a problem, i.e., essentially the "down the gradient" method of Sections 6-10 of Chapter IV above. However, this will be an "expensive" solution in most cases and the practical problem is one of obtaining a compromise which is still stable.

We will first briefly discuss the question of noise in calculating procedures in general and the related question of accuracy. We will then consider the problem of linear stability and then the non-linear problem. Our treatment will be introductory with references to the literature wherever practical.

2. In most calculating procedures, either by machine or by hand, we have definite inputs, say numbers, and specific outputs are desired. But the actual output will in general be the sum of three terms, one of these is the actual answer, the second is a definite error of the device, which may be the function of the inputs and, finally, there is a chance variable called the noise, which is a consequence of chance occurrences which happen during the computation.

The definite error of the machine, itself, can be analyzed in more detail. Certain errors are due to specific macroscopic causes and can be analyzed directly in the mechanism. Another type of error is due to a large number of relatively small causes and it may be impractical to consider such a sum causally. On the other hand, for design purposes this second type of error may be considered as a chance variable which is the sum of a large number of small chance variables. For instance, a certain error may be due to the variations of a relatively large number of resistors, and while each variation is small, the resultant may not be negligible.

At the design stage, the resistors are not available but the probability function for the variations may be known to a certain extent. The central limit theorem in probability states that the sum $\Sigma_{i=1}^{n} \varepsilon_i$ of a large number of chance variables, in general, is normally distributed with variance the square root of the sum of the squares of the variances of the ε_i to a good approximation. Thus, we can predict the probable errors for such a sum quickly. For instance, if each ε_i is such that the probability is .5 and that the ε_i will be at least e in size, then the probability is .5 that the sum will be at least $e\sqrt{n}$ in size. (We have assumed that the expected value of each ε_i is zero.) Procedures of this type are treated in probability textbooks. For instance, a proof of the central limit theorem is given in Kamke, E. *Wahrscheenlichtkeitstheorie*. Leipzig: S. Hirzel, 1932, pp. 148-60.

The noise term in the answer in general is due to sudden discontinuities which occur by chance during the computation. These can occur in a number of ways. A sudden application of a load may change the line voltage and affect an electronic calculator, or a computer may make an error. Smaller errors can occur, for instance, the rounding errors or deviations from a statistical equilibrium, such as the thermal noise in resistors or the "shot effect" in vacuum tubes.

Noise then is a chance variable which is to be treated by statistical methods. For the type of error represented by an error of a computer that occurs rarely but when it does has a large effect, the so-called Poisson Distribution is available. (Cf. Kamke, *loc. cit.*, pp. 129-30.) For the more common type of noise which results from a large number of small errors which occur at random, one has the noise theory developed for electronic circuits and given in the two articles by Rice, S. O., *The Bell System Technical Journal*, Vol. 23, (1944), 282-332, and Vol. 24 (1945), 46-156.

3. The relationship between noise and stability can be immediately recognized in the case where the system is governed by a linear system of differential equations with constant coefficients. If the motion is uniquely determined by these and the initial conditions, then we can proceed to solve the system by eliminating all but one unknown function. For this remaining unknown function $x(t)$ we will have a linear differential equation of the nth order

$$x^{(n)} + a_1 x^{(n-1)} + \ldots + a_n x = f(t)$$

with constant coefficients. The corresponding characteristic equation

$$p^n + a_1 p^{n-1} + \ldots + a_n = 0$$

will have n roots, $p_1, p_2, \ldots, p_n$.

A noise effect introduces a sudden discontinuity in the behavior of the function x at a time t_o. This means that after the time t_o, $x(t)$ is a somewhat different solution of the differential equation from that before the noise occurred. The difference between these two solutions is a consequence of the noise and must satisfy the homogeneous differential equation obtained by setting $f(t) = 0$ in the above. This difference function is therefore a linear combination in the form $\Sigma_{i=1}^{n} c_i \exp(p_i t)$. If all the p_i's have negative real parts, such a linear combination will approach zero with time. If any p_i has a positive real part, any noise effect can introduce a difference function of this sort which increases indefinitely with time. Thus the system will be stable under the effect of random noise, if and only if, all the p_i's have negative real parts.

Rational conditions on the coefficients of a polynomial which will insure that the roots are all on the left of the imaginary axis were originally obtained by Hurwitz (*Mathematische Annalen*, XLVI (1895), 273-84). The question of stability in electrical circuits has been carefully considered and practical methods for the design of stable circuits with prescribed characteristics have been developed. Cf., for instance, the discussion in Bode, H. W., *Network Analysis and Feedback Amplifier Design*. New York: D. Van Nostrand, 1945, pp. 103-69, or MacColl, L. A., *Servo Mechanisms*. New York: D. Van Nostrand Co., 1945, pp. 21-34.

It is customary to regard instability as associated with energy as in amplifiers, so that when a slight signal can effect the introduction of energy into the system, which may in turn by some devious route cause a larger version of the original signal to appear. Of course, in a discussion of stability all possible paths for the transfer of signals must be considered and the study of electrical circuits has shown the importance of "parasitic" paths for stability. In a mechanical device, besides the intentional connections, one may have other paths between units, for instance, through the frame of the device and mountings for the units.

However, instability can appear in a purely mathematical procedure, in which a chance deviation is repeatedly magnified during the course of a computation. Here the "feedback" which causes the instability is in mathematics itself. The stability of a long computation such as those which are the objectives in electronic digital computers must always be considered. Numerical methods in which errors are repeatedly reduced are desirable since an introduced error merely slows down rather than destroys the computation. "Down the gradient" procedures such as indicated in Chapter IV above are of this character. However, one may have a cyclic procedure, which at a certain point considers the errors and tends to rectify them in a stable fashion, but is not a complete "down the gradient" procedure. Indeed, gradient procedures in general indicate the direction of a step but not its size.

4. For the study of stability in the case in which one does not have linear equations with constant coefficients, the modern theories of nonlinear mechanics are available as given, for instance, in Minorsky, N., *Introduction to Non-*

Linear Mechanics, Parts I-IV. Washington, D. C.: David Taylor Model Basin, 1944-46.

In the general case it is necessary to study the solutions of differential equations in the large. It is clear that the definite stability necessary for a mathematical machine requires that every solution should be asymptotic to the correct solution, but in the present state of theory this requires practically an individual investigation in each case. The study of systems of differential equations in the large goes back to Poincaré and G. D. Birkhoff, but the possibilities are exceedingly complex. Cf. the paper of N. Levinson, *Annals of Math. (2)*, 45 (1944), pp. 723-37, for a number of earlier references.

There are really two stability problems which appear in the use of mathematical machines. One of these is the stability of the mathematical procedure involved, the other is the problem of the stability of the specific device used and its components. There is no way for component stability to compensate for a non-stable mathematical procedure. Since we must start from a situation in which the answer is unknown, perturbations analogous to noise are always present. It is for this reason that "down the gradient" procedures have been emphasized in this book. These are stable even though as we have pointed out, they may not be the most economical stable method in individual cases.

One presupposes, of course, the stability of the components. But one should also consider in this connection, the accuracy of the components since these can destroy the stability of the procedure. For instance, the Gauss Seidel iteration method for solving linear equations is stable if the matrix is positive definite. But if a positive definite matrix is inaccurately realized, this property may be lost.

PART FOUR: MATHEMATICAL INSTRUMENTS

I. Planimeters

1. The present part is devoted to a brief discussion of mathematical instruments. Many of these are relatively inexpensive time-saving devices whose mathematical theory is quite interesting.

There is little point in trying to draw a hard and fast distinction between mathematical instruments and continuous computing devices in general. Our reason for the distinction is purely one of convenience. In general, mathematical instruments operate on a graph. They are relatively simple in construction, are used for a single mathematical operation and are not intended for joint operation with other devices but are complete in themselves.

An instrument whose purpose is to measure the area enclosed by a curve is called a planimeter. For convenience we divide planimeters into two types, one of which consists of those planimeters which operate directly on areas. The second type consists of those which operate on the boundary of the area to be measured.

The purpose of an integrometer is to provide integrals such as $\int_a^b y^2\,dx$, $\int_a^b y^3\,dx$, etc. An integraph is an instrument to draw the graph of an indefinite integral of a given function. The harmonic analyzers are designed to yield the Fourier coefficients of a function. There are also instruments, curvometers, to yield the arc length of a given curve.

For this part of the course, our two main references will be: A Galle, *Mathematische Instrumente*, Leipzig: B. T. Teubner, 1912, and H. de Morin, *Les Appareils d'Integration*, Paris: Gauthier Villars, 1913.

2. The remainder of the present chapter will be devoted to considering those planimeters which operate directly on areas.

Perhaps the simplest such device is a piece of glass ruled into squares. One places the glass on the area to be measured and counts the number of squares which lie wholly within the area and estimates the remaining area around the boundary.

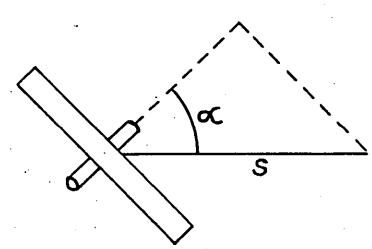
The "harp planimeter" is designed to assist one in forming a sum $\Sigma_{i=1}^{n} y_i\,\Delta x$. One has a large number of threads strung in parallel on a frame. To find the area under a given curve we lay the frame so that the threads are perpendicular to the x-axis. A compass is used to measure the ordinates of the points on the curve midway between two threads. The sum of the ordinates of these points can be obtained by a ratchet and counter arrangement on the compass or by laying the ordinates off along a straight line.

If in the latter case, the total length is excessive we may use a fixed length l and whenever this length is exceeded, we may shift back by l. Thus if the first k coordinates have a sum which exceeds l we take another compass or ruler and measure back l from the final point of the sum. We may then continue to lay off the ordinates using the point obtained by shifting back as the starting point. The final answer must allow for the shifting.

Instead of the threads and compass, a glass slide may be fitted in the frame and used to measure the ordinates. Marks along the side of the slide parallel to the x axis can be used to show where the ordinates are to be taken. The slide moves parallel to the y axis until the proper mark is on the curve. The displacement of the slide measures the ordinate.

Of course, an ordinary adding machine can be used in conjunction with a formula for numerical integration for obtaining areas. This is particularly valuable in the case in which the function is given in the form of a table rather than a graph. For a discussion of numerical integration, the reader is referred to: C. Jordan, *Calculus of Finite Differences*, Budapest: Eggenberger, 1939, pp. 512-27.

3. Another device based on the formula $\Sigma\, y_i\,\Delta x$ is also described by Galle, *op. cit.*, pp. 67-68. The ordinates are measured by a roller wheel principle which is of great importance in the theory of instruments. The principle is the following. Suppose we have a wheel of radius r, resting on paper with its plane perpendicular to the plane of the paper. Suppose, then, that we shove the wheel across the paper an amount s in a direction which

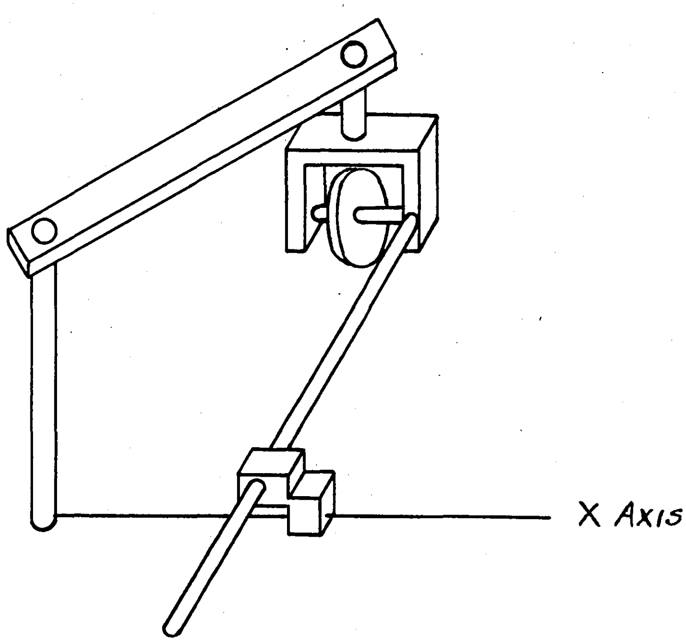


makes an angle α with the axle of the wheel. During this displacement the axle remains parallel to its original position. The component of the displacement parallel to the axle will induce no rotation, while if the wheel turns freely, there will be no slipping perpendicular to the axle. Consequently, the wheel will turn through an angle φ such that $r\varphi = s \sin \alpha$.

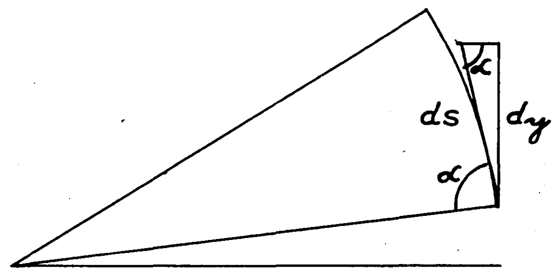
This formula generalizes readily to the case in which the wheel is displaced so that the point of contact moves along an arc C in such a fashion that α, the angle between the tangent and the axle, is a Riemann integrable function of the arc length. For instance, if α is continuous except possibly at a finite number of points where it has a right and left limit, it satisfies this condition. Since α is Riemann integrable, one can show that $\sin \alpha$ is also a Riemann integrable function of s. Thus if we consider the motion as a limit of polygonal motions on sets of chords, we obtain that the wheel will turn through an angle φ such that

$$r\ \varphi = \int_C \sin \alpha \ ds$$

Now let us suppose that we have a wheel which is initially in contact with the x axis and with its axle parallel to the x axis. Suppose the wheel is constrained to move in a circle with center on the x axis and in such a way that the axle is always parallel to the x axis. For instance, one might have a bar pivoted at the circle center. The other end of this bar has a pivot in which there is a u-shaped yoke, which holds the axle of the wheel. The yoke has a rigidly attached rod extending perpendicular to the axle of the wheel. In turn this rod slips freely through a collar which is rigidly attached to another collar which slides along a rod parallel to the x-axis. Since the two collars are mutually perpendicular, the rod extension of the yoke is always perpendicular to the x axis and the wheel axle is always parallel to the x axis.



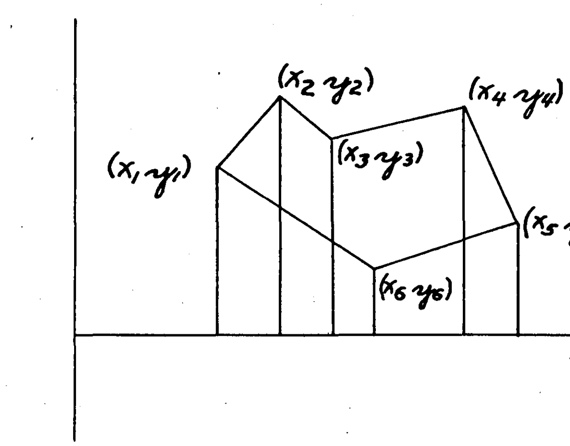
Now then if the wheel is moved from its original position of contact on the x axis to a point (x,y), the wheel will turn through an angle φ such that $r\ \varphi = y$, where r is the radius of the wheel. This is evident from the above integral formula since $dy = \sin \alpha \ ds$. Thus if we attach a counter to measure the revolution of such a wheel, we can measure the ordinate of a point.



The actual planimeter described by Galle is constructed so that while an indicator moves from a point (a,o) to the point (x,y), the actual wheel moves in a motion which is a mirror image of this. There is a worm-gear positioned for each quarter turn so that the wheel apparatus is displaced parallel to the x axis equal amounts between the measurements of the ordinates. It is also necessary that when the wheel is returned to a position at which the measurement of an ordinate begins, the counter does not register.

4. Galle also describes the conversion planimeter which obtains the area of a polygon by converting it into a triangle with the same area. We give a discussion of this based on analytic geometry.

We first obtain a formula for the area of a polygon. It is easily seen, for instance, that the



accompanying polygon has area

$$\tfrac{1}{2}(y_1+y_2)(x_2-x_1) + \tfrac{1}{2}(y_2+y_3)(x_3-x_2) + \tfrac{1}{2}(y_3+y_4)(x_4-x_3)$$
$$+\tfrac{1}{2}(y_4+y_5)(x_5-x_4) + \tfrac{1}{2}(y_5+y_6)(x_6-x_5) + \tfrac{1}{2}(y_1+y_6)(x_1-x_6)$$
$$= \tfrac{1}{2}[y_1x_2-x_1y_2 + y_2x_3-x_2y_3 + y_3x_4-x_3y_4 + y_4x_5-x_4y_5$$
$$+ y_5x_6-x_5y_6 + y_6x_1-x_6y_1].$$

In general, we see that the formula

$$A = \Sigma_{j=1}^{n-1}\ (y_jx_{j+1} - x_jy_{j+1}) + y_nx_1 - x_ny_1$$

holds for a polygon with vertices (x_1,y_1), (x_2,y_2), ..., (x_n,y_n).

Now we can construct a triangle with this area as follows: Suppose we have a slide which moves parallel to the x axis. On this slide, we have a pivot and on the pivot we have a ruler. We set the ruler perpendicular to the x axis and slide until (x_1,y_1) is on the edge. Next we turn the ruler until it passes through (x_2,y_2). The ruler now has slope $m_1 = y_2/(x_2-x_1)$.

We then slide the ruler parallel to itself until it passes through (x_1,y_1). The equation for the edge is then

$$y - y_1 = [y_2/(x_2 - x_1)]\ (x - x_1)$$

and the x- intercept a_1 is such that

$$a_1\, y_2 = x_1\, y_2 - y_1\, x_2 + x_1\, y_1 = \text{(say) } A_1.$$

We then turn the ruler until it passes through (x_3,y_3). The slope is then

$$m_2 = y_2\, y_3/(x_3\, y_2 - A_1)$$

We then slide the ruler parallel to itself until it passes through (x_2,y_2). The equation of the edge is then

$$y - y_2 = [y_2\, y_3/(x_3\, y_2 - A_1)]\ (x - x_2)$$

and the x intercept a_2 of this line is such that

$$y_3 a_2 = y_3 x_2 - x_3 y_2 + A_1 = \Sigma_{j=1}^{2}(x_j y_{j+1} - y_j x_{j+1}) + x_1 y_1 = A_2.$$

The procedure in general is the following: Suppose the ruler passes through (x_k,y_k) at the end of the k'th step. We suppose that the k'th intercept a_k is such that

$$a_k y_{k+1} = \Sigma_{j=1}^{k}\ (x_j y_{j+1} - y_j x_{j+1}) + x_1 y_1 = A_k.$$

We then turn the ruler until the edge is on $(x_{k+2},\ y_{k+2})$. The slope of the edge is then

$$m_{k+1} = y_{k+2}\ y_{k+1}/(x_{k+2}\ y_{k+1} - A_k).$$

We slide the rule until the edge passes through $(x_{k+1},\ y_{k+1})$. The equation for the edge is then

$$y - y_{k+1} = [y_{k+2}\ y_{k+1}/(y_{k+1}\ x_{k+2} - A_k)]\ (x - x_{k+1}).$$

The new intercept a_{k+1} is such that

$$y_{k+2}\ a_{k+1} = x_{k+1}\ y_{k+2} - y_{k+1}\ x_{k+2} + A_k = A_{k+1}.$$

Thus

$$A_{k+1} = \Sigma_{j=1}^{k+1}\ (y_j\ x_{j+1} - x_j\ y_{j+1}) + x_1\ y_1.$$

Hence if we carry this process through to the n'th step, we get

$$a_n\, y_{n+1} = A_n = \Sigma_{j=1}^{n}\ (y_j x_{j+1} - x_j y_{j+1}) + x_1\, y_1$$

This yields, since $(x_{n+1},\ y_{n+1}) = (x_1,\ y_1)$

$$y_1\ (x_1 - a_n) = \Sigma_{j=1}^{n-1}\ (x_j\ y_{j+1} - x_{j+1}\ y_j) + y_n\, x_1 - x_n y_1 = 2A.$$

Or $\quad \frac{1}{2}\ y_1\ (x_1 - a_n) = A.$

Thus the right triangle, one leg of which is the ordinate y_1 and the other is $x_1 - a_n$ has the same area as the given polygon.

The apparatus described by Galle consists of a bar along which the slide moves. On this slide, we have the pivoted ruler. For the above result, of course, the x axis can be taken in any direction. The pivoted ruler is first set up perpendicular to the bar. The bar is then located so that y_1 has a fixed value and then fixed with two pins. Consequently the quantity $x_1 - a_n$ is always proportional to the area and a directly reading scale is located on the bar.

5. One old method for the evaluation of an area is to cut out a replica from some material of uniform density and thickness and to weigh the result. A modern method is to make a mask from which the desired area is cut out and then to measure the total illumination which passes through such a mask by means of a photo tube. With less delicate photo sensitive methods, the entire light may be concentrated by means of lens onto the tube itself but, in general, it is preferable to disperse the light in a cavity and to determine the general level of illumination. Of course, the entire light should be concentrated first in the latter case so that it will enter the cavity through a small aperture.

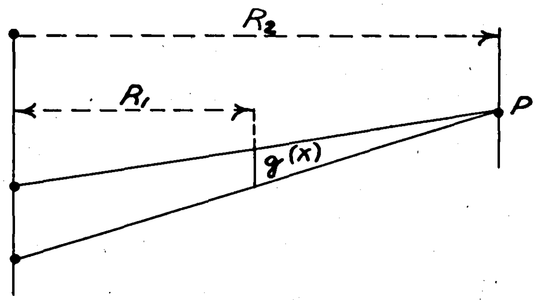
Since the output of a photo tube is not a linear function of the illumination falling on it, the answer is obtained in general by a bridge method. One has besides the above a duplicate arrangement associated with another photo tube with however an adjustable shutter instead of the mask. One adjusts the shutter until the output of the two tubes are identical. Then the opening of the shutter will have the same area as the hole in the mask.

This method of integration is used in the "cinema integraph" described in the paper of Hazen and Brown, *Jour. Franklin Institute*, Vol. 230 (1940) pp. 19-44 and 183-205, which also gives an historical account of the development of the instrument. The cinema integraph is designed to evaluate quantities in the form

$$\int_a^b f(x \pm y)\ g(x)\ dx$$

and similar quantities.

The following ingenious method is used to obtain the product. Let us consider the case of two positive functions f(x) and g(x). Masks are cut out for each function but with different x scales. The source of light is linear and the masks for g(x) and f(x) are wrapped around concentric cylinders whose common axis is the linear light source; g(x) will be on the inner cylinder. The ordinates on the mask run along the elements of the cylinder while the x axis is perpendicular to the elements. Now consider a value of x. Corresponding to x, we have a plane containing the linear light source. Now let us consider a point P on the outer cylinder in this plane. The mask on the inner cylinder is such that there is an opening of height g(x), through which the linear light source is visible from this point. The actual length of the light source which is visible is $\frac{R_2}{R_1}g(x)$. Consequently, if we ignore the slight variation in intensity due to distance, we see that the illumination on the outer cyclinder at any point on the x plane is the same and proportional to g(x). The f(x) mask, of course, permits a fraction of this proportional to f(x) to pass.



For the general case in which f(x) and g(x) vary in sign, allowance is made for the four possibilities in signs. Let $g_+ = \max\,[g(x), 0]$, $g_- = \max\,[-g(x), 0]$, f_+ and f_- are defined similarly. In the device, there are two photo tubes whose balance indicates the result. The integrals $\int_a^b g_+f_+\,dx$, $\int_a^b f_-g_-\,dx$, $\int_a^b f_+g_-\,dx$ and $\int_a^b f_-g_+\,dx$ are obtained as indicated above but the illumination from the first two goes to one photo tube, that of the other two to the second tube. A biasing light also enters the first tube so that it is always possible to balance the arrangement by positive illumination on the second tube. The balancing shutter is controlled by a servo motor.

Of course, one may readily rotate the outer cylinder to obtain the integrals

$$\int_a^b f(x+y)\ g(x)\ dx.$$

For the reader who is interested in actually using photo tubes, the brief pamphlet: *R.C.A. Phototubes*, R.C.A. Manufacturing Co., will be useful. It describes the various types of tubes available and gives circuits for various purposes.

II. Planimeters

1. In the present chapter, we will discuss planimeters, i.e., devices for obtaining areas. The earliest type of planimeter was based on the variable speed drive but these have been superseded by the mechanically simpler fixed length planimeters.

A variable speed drive can be utilized to evaluate the integral $\int_a^b y\,dx$ from a graph in an obvious fashion. One has a pointer which traces the curve and this is attached to the variable speed drive in such a way that the ordinate of the point traced is the linear or rate input of the variable speed drive and the abscissa is the rotatory or disk input.

This can be done in a number of ways. We may have a carriage on broad rimmed wheels which rolls across the paper parallel to the x axis. The rotation of the wheels then yields the abscissa. The ordinate is entered by means of an extension of variable length which remains parallel to the y axis.

A large variety of such devices can be found in the references, Galle and Morin cited in the beginning of the previous chapter. Historically these devices are important since they led to the development of the variable speed drive itself.

2. The most common type of modern planimeter is concerned with the area swept out by a line segment of fixed length. This line segment is generally represented in planimeters by a bar, one end of which carries a pointer which traces the curve C. There is also an integrating wheel, i.e., a wheel with a counter on it to measure the amount of rotation. The axle of this wheel is parallel to the bar and if the point of contact of this wheel traces a curve C', then the wheel will register

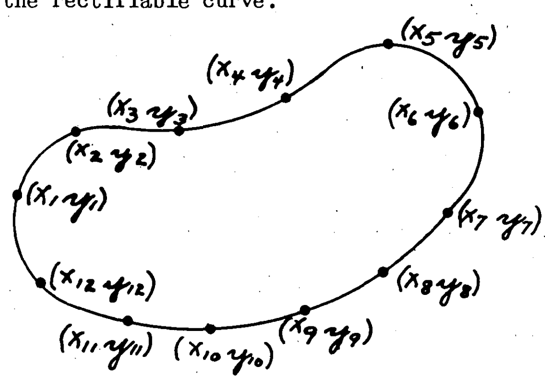
$$\int_{C'} \sin\alpha\ ds$$

where α is the angle between the axle of the wheel and ds (cf. Section 3 of the preceding chapter). In general, α is also the angle between the bar and ds.

It will simplify matters if we go through certain mathematical preliminaries before discussing the devices themselves. We do this in the present section.

Although it is not customary in mathematical discussions, we will suppose that in traversing a simple closed curve, the arc length is increasing when the point moves in a clockwise direction.

Now let us consider a simple closed rectifiable curve and let us take n points on it. These can be chosen so that they are the successive vertices of a polygon whose area approximates the area enclosed by the rectifiable curve.



The area of this polygon has been proven in Sec. 4 of the preceding chapter to be

$$A = \tfrac{1}{2}\Sigma_{j=1}^{n-1}\ (y_j\ x_{j+1} - x_j\ y_{j+1}) + \tfrac{1}{2}\ (y_n\ x_1 - x_n\ y_1)$$

If we let $x_{j+1} = x_j + \Delta x_j$, $y_{j+1} = y_j + \Delta y_j$, $x_1 = x_n + \Delta x_n$, $y_1 = y_n + \Delta y_n$, where these appear in this formula, we obtain

$$A = \tfrac{1}{2}\ \Sigma_{j=1}^{n}(y_j\ \Delta x_j - x_j\ \Delta y_j).$$

If we pass to the limit, we obtain

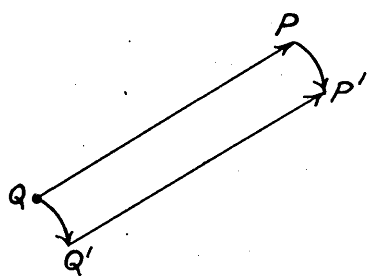
$$A = \tfrac{1}{2}\int_C (y\,dx - x\,dy)$$

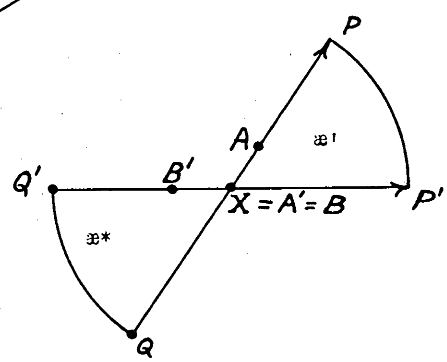
a formula which, of course, is well known but which we derive in order to establish the sign.

Consider the area swept out by a line segment of length l moving in a plane. This is an essential notion in our present discussion and our immediate objective is to define it. We suppose that the motion is continuous and smooth. Thus we suppose that each point describes an arc with a continuously turning tangent. (It would be sufficient for this if just two distinct points on the line segment move in this manner.)

We first discuss the question of the definition of this area and in particular the matter of sign. Suppose we have a directed line segment QP which moves to a new position Q'P'. We suppose that this motion is small which is justifiable in view of the fact that any motion of the postulated sort can be considered as the consequence of a number of small motions.

There are essentially two possibilities. In one





of these, the new position Q'P' does not intersect QP, in the second case, it does. In the first case the area is readily defined. The point P describes an arc $\breve{PP'}$ and the point Q an arc $\breve{QQ'}$ and these together with the initial and final position of the line segment enclose an area which is given by the formula

$$\Delta A = \left[\int_{QP} + \int_{\breve{PP'}} - \int_{Q'P'} - \int_{\breve{QQ'}}\right] (y\,dx - x\,dy).$$

This formula clearly specifies the sign of the area. An area such as that shown will be positive, while if the line segment had moved upwards, the area would be negative.

In the second case the situation is not immediately clear since in general points not in the two triangles shown may be covered in the motion. Let X be the point on the plane where the initial and final position of the line segment intersect. Suppose X is the image of a point A on the initial segment. We will call X, B, when it is considered to be on the initial segment, i.e., X = A' = B. Now A is either on BP or QB of the initial segment and no essential generality will be lost if we assume the former. Then B' the image of B is on either Q'A' or A'P'. In the first case the line segment BA in moving to its new position would turn through a total angle equal to PXP' while in the second case, it would turn through an angle equal to PXQ'. Now the motion is supposed to be small and hence only the smaller rotation is possible. Thus if A is on XP, we may suppose B' is on XQ'. Hence the line segment AP moves onto XP' = A'P' and these have no common points and QX = QB moves onto Q'B' and these two also have no common points. Now we suppose that the motion is so slight that the area is not greatly altered by using the line segments AA' or BB' as boundaries instead of the actual paths. But then it is clear that we can regard the area swept out by AP and QB as in the first case.

Since only a small motion is considered, the actual displacement A'A = XA = BA is small and hence the area swept out can be considered as negligible. Let us then arbitrarily assign to it the value

$$\left(\int_{BA} + \int_{\breve{AA'}} - \int_{B'A} - \int_{\breve{BB'}}\right) (y\,dx - x\,dy).$$

For the area swept out by AP and QB, we have, of course, the previous formulas. If we add these three expressions for the area, we find that we have exactly the same formula for ΔA. In this case, it is clear that the area is broken up into two parts, æ' and æ*. The former is positive, the latter is negative.

Let us take now a motion which is not restricted in size and, as we have suggested, consider it as a number of small motions to each of which the above formula for ΔA is applicable. We then find that if C_1 is the path of P and C_2 that of Q then

$$A = \left[\int_{QP} - \int_{O'P'} + \int_{C_1} - \int_{C_2}\right] (y\,dx - x\,dy).$$

If the large motion is such that the line segment returns to its original position then

$$A = \int_{C_1} (y\,dx - x\,dy) - \int_{C_2} (y\,dx - x\,dy).$$

Notice that our argument really depended very little on the nature of the integrand. We could have used any $F(x,y,\frac{dy}{dx})\,ds$, provided that F was continuous in the three variables.

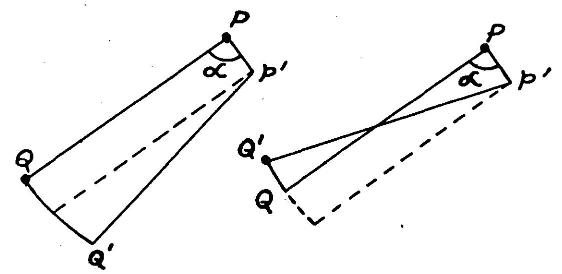
In the area case in particular, it is worth noting that if the points P and Q circumscribe an area A_1 and A_2 respectively, i.e., C_1 and C_2 are the boundaries of these curves, then $A = A_1 - A_2$.

3. We have described the integrating wheel on the fixed length planimeters and the fact that it registers

$$\int \sin\alpha\,ds.$$

This expression is also associated with the area swept out by a fixed length. Consider a differential dA of the area. Let φ denote the clockwise rotation between the original position and the present position. It is readily seen that

$$dA = l \sin \alpha \, ds + \tfrac{1}{2} l^2 \, d\varphi.$$



Thus if the fixed length moves from one position to another and C is the path of P we have

$$A = l \int_C \sin \alpha \, ds + \tfrac{1}{2} l^2 \varphi + k.$$

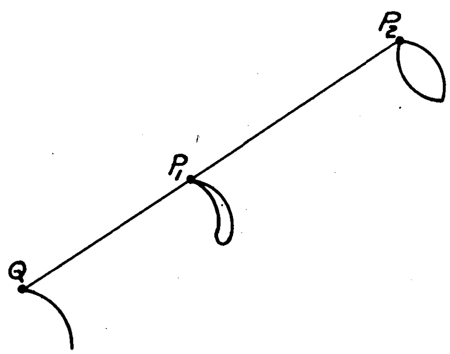
But it is evident that the constant of integration is zero and thus

$$A = l \int \sin \alpha \, ds + \tfrac{1}{2} l^2 \varphi.$$

If the fixed length returns to its original position without making a complete revolution we have

$$A = l \int_C \sin \alpha \, ds.$$

Let us now consider three points, Q, P_1, P_2, on the fixed length. Let us suppose that the fixed length moves and returns to its original position. P_1 traces out a path C_1. Q is however constrained to move on a path either a line segment or an arc of a circle in such a way that when it has returned to its original position, no area has been enclosed. We will suppose that P_1 lies between Q and P_2. The other cases are treated in an entirely analogous manner so we will not consider them further.



Let l_1 denote the length Q P_1, l_2 the length $P_1 P_2$. Let A_1 denote the area enclosed by C_1. (If P_1 circumscribes the area in the usual sense a number of times in traversing C_1, then A_1 is a multiple of the area as usually understood.)

We first notice that

$$l_2 \int_{C_1} \sin \alpha \, ds = \text{Area swept out by } P_1 P_2 = l_2 \int_{C_2} \sin \alpha \, ds.$$

Thus the integral $\int_C \sin \alpha \, ds$ is independent of the position of the point P on the line.

On the other hand, if $l = Q P_2$ we have

$$l \int_C \sin \alpha \, ds = \text{Area swept out by } Q P_2 = A_2.$$

4. The last two formulas of the previous section are the basis of many planimeters. The operator moves a pointer at P_1 so that it circumscribes the desired area. The integrating wheel is at some other point F along the fixed length. The planimeters are classified as linear if the point Q moves along a straight line or polar if Q moves along the arc of a circle.

If one mounts the fixed length on a carriage with broad wheels or on a track in such a way that the fixed length is pivoted at Q, then Q will move in a straight line. If one connects the fixed length at Q by means of a hinge to an arm which itself is pivoted at a fixed point, we obtain a polar planimeter.

It is desirable that the curve C and the permissible path of Q be such that if we take any circle of radius $l = P_2 Q$ with center on a point of C, then this circle will intersect the path of Q at only one point. This will insure that when the operator returns the point P to its original position after traversing the curve, Q will return to its original position.

For a more detailed description of the various instruments the reader is referred to the references of Chapter I, Sec. 1, i.e., Galle or de Morin.

5. If we refer again to the formula

$$A = l \int_C \sin \alpha \, ds + \tfrac{1}{2} l^2 \varphi$$

of Sec. 3 for the area swept out by a moving line, we see that there is one other interesting possibility. Let us suppose that C refers to the path of Q. Let us suppose that at Q we have a knife edge parallel to the fixed length and resting on the paper. Now if we move the P end of the fixed length, then Q will move along a path which is tangent to the fixed length at every instant. Consequently $\sin \alpha \equiv 0$ on C. Thus if we enclose the area A_p in our motion of the point P and φ_0 is the total change in the angle in this process then

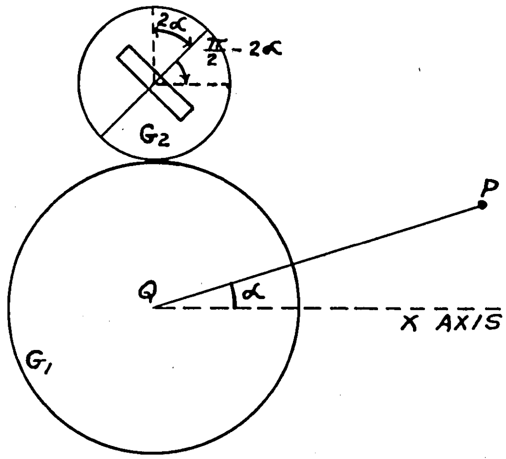
$$\tfrac{1}{2} l^2 \varphi_0 = \text{Area swept out by a fixed length} = A_p - A_Q.$$

In general, A_Q will be small or we can measure it by repeating the process. This is the principle of the Prytz or "hatchet" planimeter.

III. Integrometers

1. Integrometers are very interesting developments of linear planimeters designed to evaluate integrals in the form $\int_C y^2 \, dx$, $\int_C y^3 \, dx$ and so forth. The principle upon which these are based can be readily understood from a discussion of the part of such a device concerned with the evaluation of $\int y^2 \, dx$ and we will confine our attention to this for the present.

The frame of the device is constrained to move parallel to the x-axis. The fixed length will be denoted QP. P will trace out the curve and Q is pivoted on the frame so that Q moves on the x-axis. We have a gear G_1 which rotates with PQ. We have another gear G_2 meshing with G_1 whose radius is one half that of G_1. Thus when PQ rotates through an angle α from a position coinciding with x-axis, G_2 will rotate through an angle 2α.



Mounted on G_2 there is an integrating wheel, whose point of contact with the paper is on the axis for G_2. The integrating wheel is arranged so that when $\alpha = 0$, its angle is perpendicular to the x-axis. Consequently when PQ rotates through an angle α, the axle will turn through an angle 2α and make angle $\frac{\pi}{2} - 2\alpha$ with the x-axis.

As P moves around a closed curve C, Q will also move along the x-axis. We can call its path on the x-axis D. D is closed and of course folded on itself. It is clear that the integrating wheel measures

$$\int_D \sin\left(\frac{\pi}{2} - 2\alpha\right) dx = \int_D \cos 2\alpha \, dx.$$

I believe it is evident that by using gears G_3, G_4, etc., instead of G_2, with radii, $\frac{1}{3}$, $\frac{1}{4}$, etc., of the radius of G_1 and by orienting the original position of the axle correctly we can obtain

$$\int_D \sin 3\alpha \, dx, \int_D \cos 4\alpha \, dx, \text{ etc.}$$

2. Let us now consider the integral $\int_C y^2 \, dx$ around a closed curve C which is traced by the point P. Let α denote the angle PQ makes with the x-axis. Then $y = l \sin \alpha$ and

$$\int_C y^2 \, dx = l^2 \int_C \sin^2 \alpha \, dx = (l^2/2) \int_C (1 - \cos 2\alpha) \, dx$$

$$= (-l^2/2) \int_C \cos 2\alpha \, dx$$

since the integral $\int_C dx$ around a closed curve is zero.

Since the integrating wheel registers $\int \cos 2\alpha \, dx$ many of the books seem to think that this settles the matter. But the wheel registers

$$\int_D \cos 2\alpha \, dx$$

while the quantity desired is $\int_C \cos 2\alpha \, dx$. Of course, the angle α is the same but the dx is clearly different in general. For instance, Q may even remain fixed while P traverses an arc of the circle, thus dx on D is zero but not on C.

Thus it is necessary to show that

$$\int_D \cos 2\alpha \, dx = \int_C \cos 2\alpha \, dx$$

around any closed curve C. We now prove this.

Denote the coordinates of P by (x,y) and those of Q by $(\eta,0)$. We can consider x, y and η as functions of s the arc length variable on the curve C. Indeed

$$\eta = x - \sqrt{l^2 - y^2}$$

$$d\eta = dx + \frac{y}{\sqrt{l^2 - y^2}} dy.$$

We also recall that $\cos 2\alpha = 1 - 2\sin^2 \alpha = (1 - 2\frac{y^2}{l^2})$. Hence

$$\int_D \cos 2\alpha \, dx = \int_C \cos 2\alpha \, d\eta = \int_C \cos 2\alpha \, dx$$

$$+ \int_C \left(1 - 2\frac{y^2}{l^2}\right) \frac{y}{\sqrt{l^2 - y^2}} dy.$$

Since any integral in the form $\int_C F(y) \, dy$, where C is a closed curve, is zero we have

$$\int_D \cos 2\alpha \, dx = \int_C \cos 2\alpha \, dx.$$

3. It is readily seen how the above can be generalized to evaluate integrals $\int_C y^3 \, dx$, $\int_C y^4 \, dx$, etc. Thus

$$y^3 = l^3 \sin^3 \alpha = l^3 \sin \alpha \, \tfrac{1}{2} (1 - \cos 2\alpha)$$
$$= \tfrac{1}{2} l^3 (\sin \alpha - \sin \alpha \cos 2\alpha) = \tfrac{1}{4} l^3 (3 \sin \alpha - \sin 3\alpha)$$

and

$$y^4 = l^4 \tfrac{1}{4} (1 - 2\cos 2\alpha + \cos^2 2\alpha)$$
$$= \frac{l^4}{8} (3 - 4\cos 2\alpha + \cos 4\alpha).$$

As mentioned in Sec. 1 above, by the use of gears G_3 and G_4 with different gear ratios relative to G_1, we can obtain the integrals

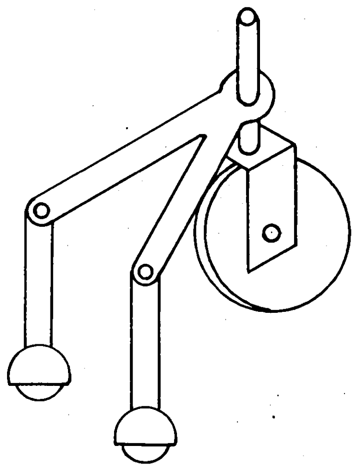
$$\int_D \sin 3\alpha \, dx, \int_D \cos 4\alpha \, dx, \text{ etc.}$$

and the proof that the D integral is equal to the C integral is quite analogous to that of Sec. 2. The only difference is that in the F(y) mentioned at the end of Sec. 2, the factor $(1 - 2\,y^2/l^2)$ is replaced by other functions of y: $3\,y/l - 4\,y^3/l^3$, $8\,(y^4/l^4 - y^2/l^2) + 1$, etc.

IV. Integraph

In the present chapter we wish to describe briefly the integraph. An integraph is an instrument used to draw the graph of a function for which the derivative is given. In certain modern developments, this has become an instrument for solving differential equations.

The principle of the integraph is essentially that of the steering wheel on a tricycle. To describe the situation precisely let us introduce a theoretical device. This is similar to a tricycle except that the rear wheels have been replaced by a pair of spherical ball bearings in sockets.



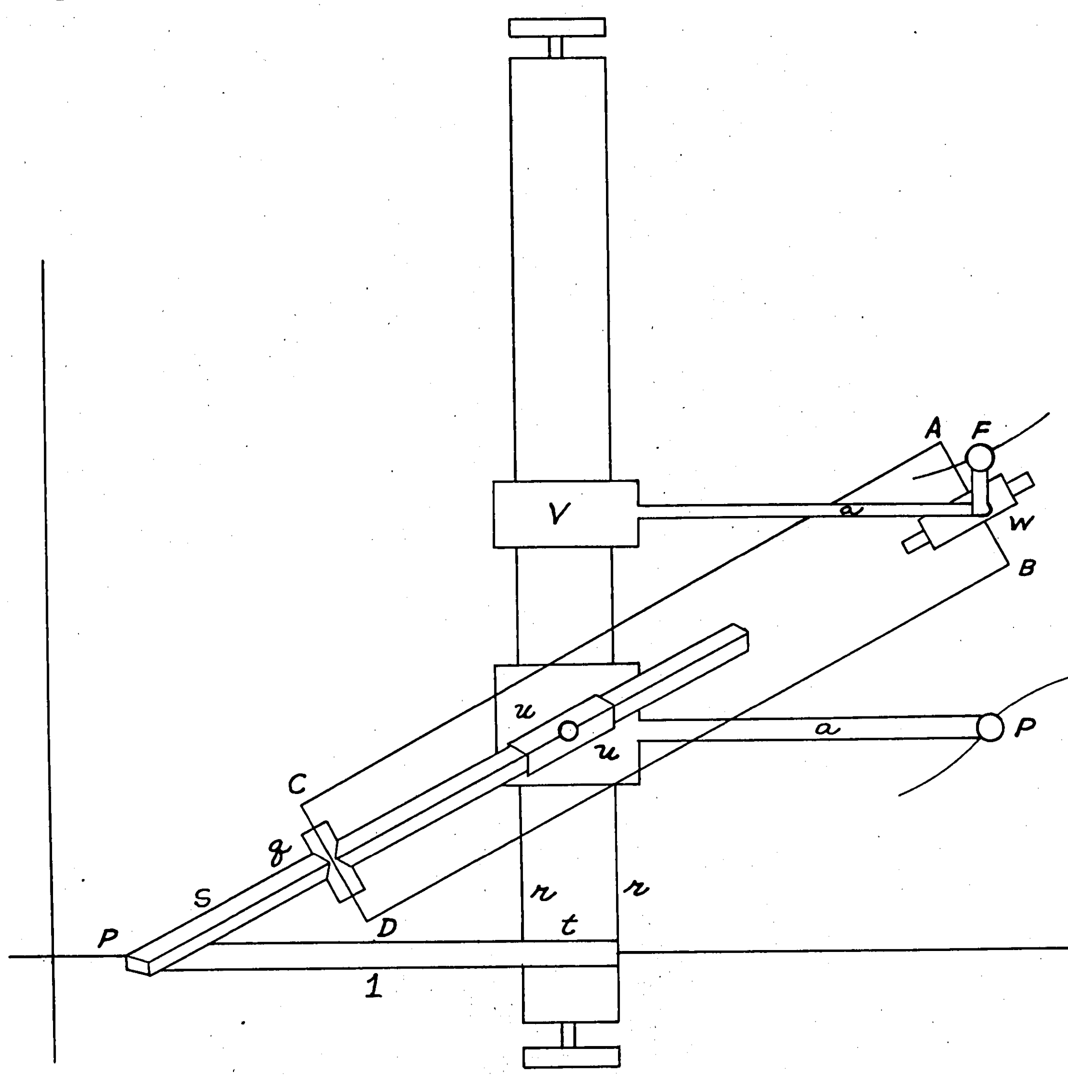
Now if a force F which is not too great is applied to this tricycle in a direction not perpendicular to the front wheel, then as in the case of the integrating wheel described in Section 3 of Chapter I above, the tricycle will move in a path to which the front wheel is tangent. The component of F which is parallel to the axle of the wheel is counteracted by a friction force at the point of contact of the wheel and the plane on which it moves while the component of F perpendicular to the axle will cause the tricycle to move. We suppose that the turning moments are counteracted in some other manner.

To set up an integrating device then we must steer this tricycle so that the line of the steering wheel always makes an angle α with the x-axis such that $\tan \alpha = f(x)$ where $f(x)$ is the function whose integral is desired.

2. The above principle is applied in a number of ways which can be roughly classified under two headings. One type is represented by the Conradi instrument, described by Galle, *loc. cit.*, pp. 158-59. We present a diagram of it below, looking at it from above.

The frame of the device consists of two parallel rails, r, which are mounted on wheels so that the frame moves parallel to the x-axis. There are two carriages, u and v, which move along r.

One of these carriages, u, has an extension on which is a pointer P which follows the given curve.



On this carriage we have a pivoted collar through which rides a bar s. The other end of this bar is pivoted at the point p which is on a fixed extension of frame. This arrangement of collar and pivot insures that the slope of the bar s equals f(x).

On s we have another slide q (it actually is a carriage but we show it as a collar) which contains a line CD which remains perpendicular to s. CD is part of a parallelogram ABCD, whose other side AB determines the direction of the wheel W. The carriage v has a pivot upon which the mounting for W turns. From this arrangement the wheel W always has the same slope as s and the motion of W is also the motion of the carriage v. Since the direction of W always has slope f(x), the point of contact of the wheel with the basic plane moves on a curve which is an integral of f(x). The pencil F traces this motion essentially, since the upward displacement merely changes the constant of integration.

3. Notice that the arrangement of the bar s and the slide q is such that the slope of the line CD is determined but not its position. This is essential since the only permitted restraint on the wheel W is the determination of its slope relative to the x-axis.

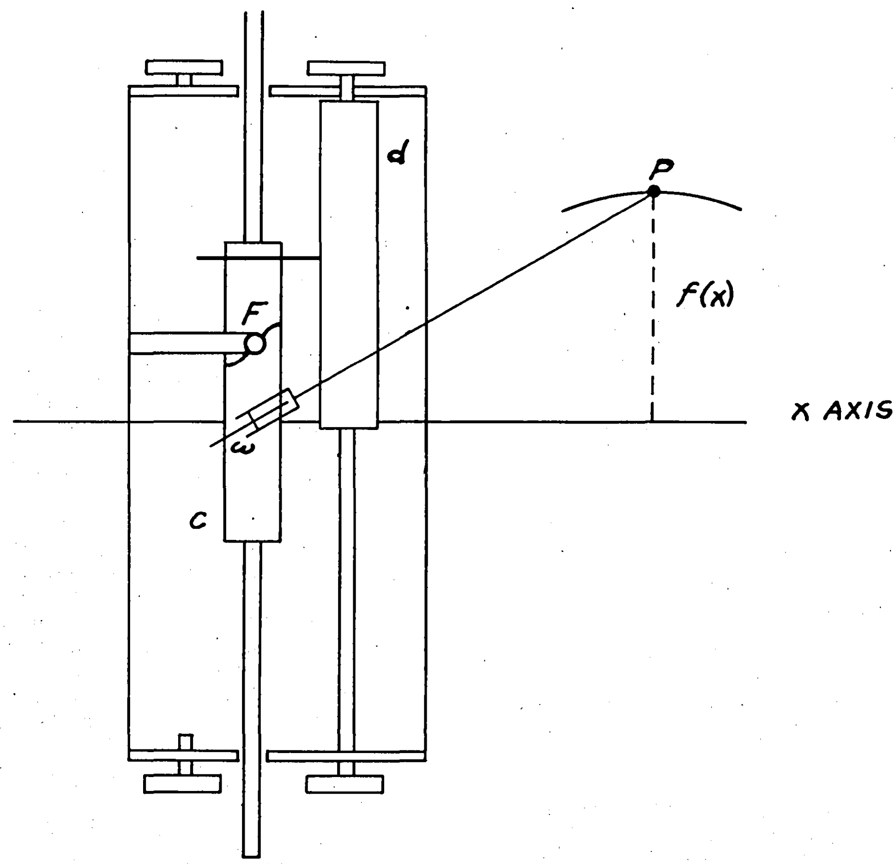
There is one other way in which the desired connection between the given curve and the slope of the wheel can be obtained which is due to Abdank Abankanowicz. Instead of having the steering wheel on the original plane, we permit it to ride on a cylinder which is free to displace itself parallel to the y-axis. This cylinder turns at the same rate as the tracing point covers the x-axis. However, the wheel is fixed on the frame and so the desired relative motion of wheel and cylinder is obtained by the displacement of the cylinder. (Cf. H. de Morin, *loc. cit.*, pp. 136-41.)

In the accompanying diagram, the point P traces the given derivative curve f(x). This determines the slope of the wheel w. The front wheels of the carriage turn the cylinder d (which can be toothed like a gear), whose rotation is communicated to the cylinder C. Owing to the slope of the wheel w, C rides up or down in the carriage as it turns. A pencil F fixed on the carriage and pressing against C will record the motion of C. Of course, the integral curve can wind around C a number of times.

4. A number of steering wheel integrators can be combined into a device for solving differential equations. A modern example of this is given by the device described in: Myers, D. M., *Jour. Sci. Instruments*, Vol. XVI (1939), pp. 209-22. This involves two integrating wheels and is suitable for solving differential equations

$$a\frac{d^2z}{dx^2} + b\frac{dz}{dx} + cz = d$$

where b and c are constants but a and d may be functions of any one of the variable x, z or $\frac{dz}{dx}$.



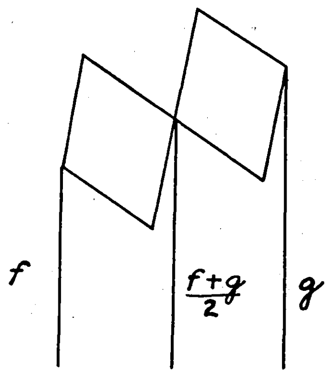
The two integraph wheels are connected to realize the pair of equations

$$\frac{dy}{dx} = \frac{1}{a}(d - b\,y - cz)$$

$$\frac{dz}{dx} = y$$

The connection to the integraph wheels is essentially the parallelogram arrangement of the Conradi integraph. Consequently z and $y = \frac{dz}{dx}$ are present as linear displacements in the device. This permits one operator to enter a as a function of either one of these variables or of x from a graph. Similarly d can be entered.

The multiplications involved are based on similar triangle principles. This permits one to readily multiply by 1/a. Addition is accomplished by means of a "lazy tongs" form of a linear differential.



V. Harmonic Analyzers

1. An harmonic analyzer is a device for evaluating the Fourier coefficients of a function f(x) on the interval $0 \leq x \leq 2\pi$.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

(If the interval given is not this, we can easily change scale.)

In general, they are direct calculating devices utilizing elements, many of which we are familiar with.

Naturally any of the numerical methods for computing integrals can be used. In this connection the reader is referred to the following: H. Lipson and C. A. Beevers, *Proc. Phys. Soc.*, Vol. 48 (1936), pp. 772-80.

There is a rather well-known mathematical treatment upon which both calculational procedures and many harmonic analyzers and synthesizers have been based and this we now give.

Let us consider now the simplest numerical approximation to an integral. Let $x_p = p\pi/n$ for $p = 1, \ldots, 2n$, $y_p = f(x_p)$. The points x_p divide the interval $0 \leq x \leq 2\pi$ into 2n equal subintervals. From the obvious approximation for the integral we obtain

$$a_0' = (1/2n)\,\Sigma_{p=1}^{2n} y_p$$

$$a_q' = (1/n)\,\Sigma_{p=1}^{2n} y_p \cos q\,x_p \qquad q = 1, \ldots, n$$

$$b_q' = (1/n)\,\Sigma_{p=1}^{2n} y_p \sin q\,x_p .$$

(Note that $b_n' = 0$ for $n\,x_p = p\pi$ and $\sin p\pi = 0$.)

Now form the expression

$$a_0' + \Sigma_{q=1}^{n-1}(a_q' \cos q\,x_r + b_q' \sin q\,x_r)$$

$$+ \tfrac{1}{2}(a_n' \cos n\,x_r + b_n' \sin n\,x_r)$$

$$= \frac{1}{2n}\Sigma_{p=1}^{2n} y_p\,(1 + 2\,\Sigma_{q=1}^{n-1}(\cos q\,x_p \cos q\,x_r$$

$$+ \sin q\,x_p \sin q\,x_r) + \cos n\,x_p \cos n\,x_r$$

$$+ \sin n\,x_p \sin n\,x_r) =$$

$$\frac{1}{2n}(\Sigma_{p=1}^{2n} y_p\,[1 + 2\,\Sigma_{q=1}^{n-1}\cos q\,(x_p - x_r) + \cos n\,(x_p - x_r)]).$$

Now let $\theta = x_p - x_r$. Since for $\theta \neq 0$, we have $\cos q\,\theta = \frac{\sin(q+\frac{1}{2})\theta - \sin(q-\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$, we have for $x_p - x_r \neq 0$,

$$1 + 2\,\Sigma_{q=1}^{n-1}\cos q\,(x_p - x_r) + \cos n\,(x_p - x_r) =$$

$$1 + \frac{\sin(n-\frac{1}{2})(x_p - x_r) - \sin\frac{1}{2}(x_p - x_r)}{\sin\frac{1}{2}(x_p - x_r)} + \cos n\,(x_p - x_r).$$

Now if $x_p - x_r \neq 0$, we have $x_p - x_r = k\,\pi/n$. Thus $n\,(x_p - x_r) = k\pi$ and $\sin n\,(x_p - x_r) = 0$. Hence $\sin(n-\frac{1}{2})(x_p - x_r) = -\sin\frac{1}{2}(x_p - x_r)\cos n\,(x_p - x_r)$. Substituting in the above expression yields that if $x_p - x_r \neq 0$ we have $1 + 2\,\Sigma_{q=1}^{n-1}\cos q\,(x_p - x_r) + \cos n\,(x_p - x_r) = 0$. On the other hand if $x_p - x_r = 0$, it is clear that $1 + 2\,\Sigma_{q=1}^{n-1}\cos q\,(x_p - x_r) + \cos n\,(x_p - x_r) = 2n$. Consequently when we go back to the first expression and substitute in the formula previously obtained, we have

$$a_0' + \Sigma_{q=1}^{n-1} a_q' \cos q\,x_r + \Sigma_{q=1}^{n-1} b_q' \sin q\,x_r$$

$$+ \tfrac{1}{2}(a_n' \cos n\,x_r + b_n' \sin n\,x_r) = y_r .$$

In other words, if we calculate the coefficients a_p' and b_p' by the above approximation formulas, we get a trigonometric polynomial which takes on the values $y_1, \ldots, y_{2n}$ (For convergence reasons this is in general a desirable procedure only for the situation in which y is continuous and $y(2\pi) = y(0)$.)

Now the above formal calculations essentially shows that if we consider the system of equations

$$y_p = a_0 + \Sigma_{q=1}^{n-1} a_q \cos q\,x_p + \Sigma_{q=1}^{n-1} b_q \sin q\,x_p$$

$$+ \tfrac{1}{2} a_n \cos n\,x_p$$

as a system of 2n equations on the 2n quantities

$a_0, a_1, \ldots, a_n, b_1, \ldots, b_{n-1}$, then the system

$$a_0 = \frac{1}{2n} \Sigma_{p=1}^{2n} y_p$$

$$a_q = \frac{1}{n} \Sigma_{p=1}^{2n} y_p \cos q\, x_p, \quad q = 1, \ldots, n$$

$$b_q = \frac{1}{n} \Sigma_{p=1}^{2n} y_p \sin q\, x_p, \quad q = 1, \ldots, n-1$$

has a matrix which is inverse to the matrix of the first set. Consequently each of these systems is non-singular.

This has the very important consequence that if a device is constructed to yield the various linear combinations

$$l_p = \Sigma_{q=1}^{n} k_q \cos p\, x_q$$

$$r_p = \Sigma_{q=1}^{n} s_q \sin p\, x_q$$

then it can be used to either obtain the Fourier coefficients of a given function or given the Fourier coefficients, we can obtain the value of the function at the specified points. (The formulas do vary slightly but this can be readily taken care of.)

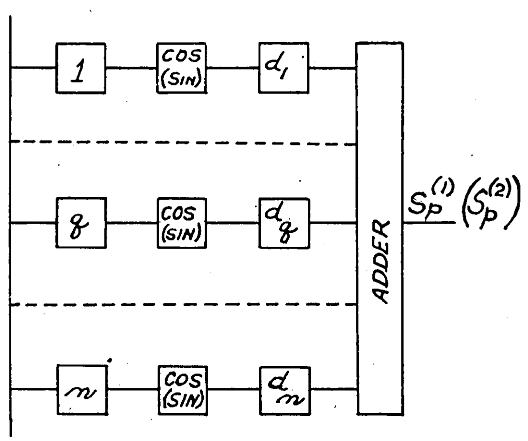
Naturally these formulas are readily adapted to calculations based on ordinary arithmetical machines or punched card machines.

2. It is clear from the discussion of the preceding section that what is desired is a device to produce linear combinations

$$s_p^{(1)} = \Sigma_{q=1}^{n} d_q \cos (p\, q\, \pi/n)$$

$$s_p^{(2)} = \Sigma_{q=1}^{n} e_q \sin (p\, q\, \pi/n)$$

As an example let us consider the harmonic analyzer of Michelson and Stratton. For this we have the schematic diagram



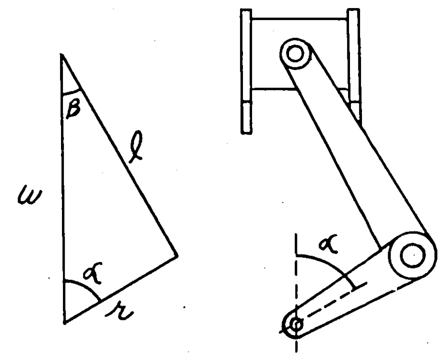
This arrangement is such that if we replace the input $p\,\pi/n$ by a continuous variable x, we get the sums

$$s_p' = \Sigma_{q=1}^{n} d_q \cos q\, x$$

$$s_p^{(2)} = \Sigma_{q=1}^{n} e_q \sin q\, x.$$

By adding these to a_0, we evaluate the function whose Fourier coefficients are the d_q's and e_q's. This fact is used to draw a complete graph of the function.

In the Michelson Stratton instrument, the input $p\,\pi/n$ or x is an angle. The multiplication by q is obtained by a gear ratio. The cosine is obtained by an eccentric.



(Notice that in the accompanying diagram

$$w = r \cos \alpha + l \cos \beta = r \cos \alpha + l \sqrt{1 - \sin^2 \beta}$$

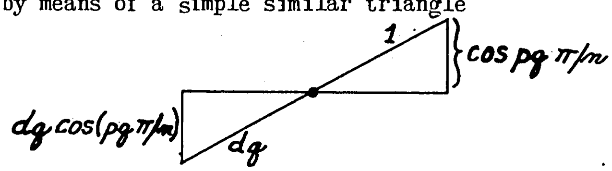
$$= r \cos \alpha + l \sqrt{1 - \sin^2 \alpha \frac{r^2}{l^2}} = l + r \cos \alpha +$$

$$+ l \left(\sqrt{1 - \sin^2 \alpha \frac{r^2}{l^2}} - 1 \right)$$

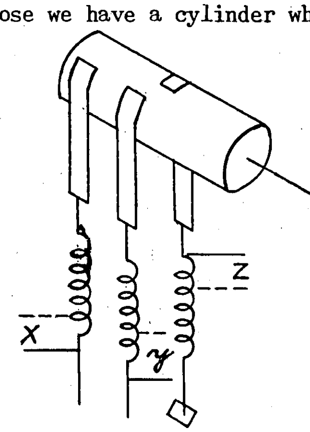
$$= l + r \cos \alpha - r \sin^2 \alpha \left(\frac{r}{l + \sqrt{l^2 - r^2 \sin^2 \alpha}} \right)$$

Thus the percentage error which results when we consider the other end of the eccentric to have a harmonic motion is about 100 r/2l.)

The multiplication by the constant d_q is obtained by means of a simple similar triangle



The method of addition that is used is very interesting since it is a mechanical counterpart to the voltage averaging which we have previously discussed. Let us describe it in the case of two addends since this case contains the essential ideas. Suppose we have a cylinder which can rotate



around its axis. On one side of the cylinder, we have two bands. One end of these bands is fastened to the cylinder, the other end to a spring. The band is partly wrapped around the cylinder and extends down vertically to the corresponding spring. Each band and spring corresponds to an input. On the opposite side of the cylinder is a similar arrangement of band and spring for the output.

We suppose that the two input springs are similar. We suppose that these are normally extended an amount l_1. By Hooke's Law, the force exerted by each of these springs is $k_1\ l_1$ where k_1 is the force necessary to extend the spring a unit length. Let k_2 and l_2 be the corresponding quantities for the output spring. Since the cylinder is in equilibrium, a consideration of moments shows that $2\ k_1\ l_1 = k_2\ l_2$.

Now suppose we move the other end of the input springs down amounts x and y respectively. The cylinder will rotate and the output spring whose lower end is fixed will be extended an amount z. The input springs will be extended amounts x - z and y - z respectively. The moment equation will still be

$$k_1\ (l_1 + x-z) + k_1\ (l_1 + y-z) = k_2\ (l_2 + z)$$

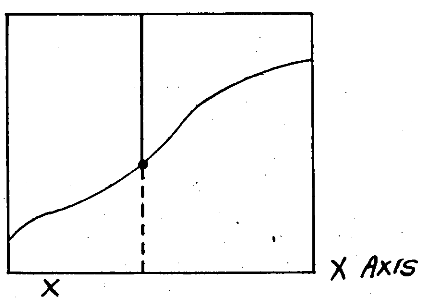
or

$$k_1\ (x+y) = (2\ k_1+k_2)\ z$$

It should be clear how any number of inputs can be introduced into such a device.

The output is then a linear displacement.

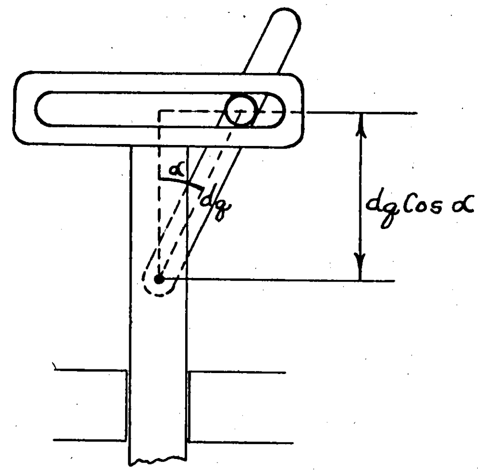
When the machine is used as a synthesizer, i.e., to graph a function y whose Fourier coefficients are given, the output appears as the displacement of a pencil above a horizontal line which corresponds to the x-axis. This pencil presses against



a piece of paper on a vertical drawing board. As we mentioned above, in this case we have a continuous input x and the drawing board is continuously displaced to the left with this input. Thus while the pencil remains in the same vertical plane, the graph of y appears on the drawing board.

3. A modern development of the above is given in S.L., Brown, *Jour. Franklin Institute*, Vol. 228 (1939), pp. 675-94. The schematic diagram is the same, the $p\pi/n$ input is again an angle, the multiplication by q is again by a gear box, so that $pq\pi/n$ appears as an angle.

The cosines and sines of this angle are obtained by taking projections on the x-axis of a line segment which makes an angle α with x-axis. This is readily accomplished by a sliding arrangement.



The multiplication by d_q is obtained by varying the length of the segment. Addition is by means of an endless chain.

Another modern analyzer and synthesizer is described in the reference: F. W. Kranz, *Jour. Franklin Inst.*, Vol. 204, pp. 245-62.

In these two references the following point is made. For the synthesizer, we need essentially the linear combinations

$$\Sigma_{q=1}^{n}\ a_q \cos\ (qp\pi/n)$$

and

$$\Sigma_{q=1}^{n}\ b_q \sin\ (pq\pi/n)$$

so that we have only n terms. However, the corresponding analyzer formulas are

$$a_q = \Sigma_{p=1}^{2n}\ y_p \cos\ (qp\pi/n)$$

$$b_q = \Sigma_{p=1}^{2n}\ y_p \sin\ (qp\pi/n)$$

which include 2 n terms. Nevertheless, it is relatively simple as these authors point out to obtain

$$\Sigma_{p=1}^{2n}\ y_p \cos\ (pq\pi/n)$$

from a device which yields

$$\Sigma_{p=1}^{n}\ d_p \cos\ (pq\pi/n).$$

For

$$a_q = \Sigma_{p=1}^{2n}\ y_p \cos\ (pq\pi/n) = \Sigma_{p=1}^{n}\ y_p \cos pq\pi/n$$

$$+ \Sigma_{p=1}^{n}\ y_{n+p} \cos\ (q\pi + pq\pi/n)$$

$$= \Sigma_{p=1}^{n}\ (y_p + [-1]^q\ y_{n+p}) \cos\ (pq\pi/n)$$

Similarly

$$\Sigma_{p=1}^{2n}\ y_p \sin\ (pq\pi/n) = \Sigma_{p=1}^{n}\ (y_p + [-1]^q\ y_{n+p}) \sin\ (pq\pi/n)$$

This reasoning generalizes readily. Suppose we wish to obtain

$$A_q = \Sigma_{r=1}^{2kn} y_r \cos (rq\pi/kn)$$

and

$$B_q = \Sigma_{r=1}^{2kn} y_r \sin (rq\pi/kn)$$

from a device which can produce

$$\Sigma_{p=1}^{n} d_p \cos (tp) \text{ and } \Sigma_{p=1}^{n} e_p \sin (tp)$$

where t is an input which can assume any value.

Now let $r = sn + p$ where $p \leq n$. Then if $y_r = y_{s,p}$

$$A_q = \Sigma_{r=1}^{2kn} y_r \cos (rq\pi/kn)$$
$$= \Sigma_{p=1}^{n} \Sigma_{s=0}^{2k-1} y_{s,p} \cos (sq\pi/k + pq\pi/kn)$$
$$= \Sigma_{p=1}^{n} (\Sigma_{s=0}^{2k-1} y_{s,p} \cos sq\pi/k) \cos (pq\pi/kn)$$
$$- \Sigma_{p=1}^{n} (\Sigma_{s=0}^{2k-1} y_{s,p} \sin sq\pi/k) \sin (pq\pi/kn).$$

Now let

$$d_{p,q} = \Sigma_{s=0}^{2k-1} y_{s,p} \cos qs\pi/k$$
$$e_{p,q} = \Sigma_{s=0}^{2k-1} y_{s,p} \sin qs\pi/k.$$

Then

$$A_q = \Sigma_{p=1}^{n} d_{p,q} \cos (pq\pi/kn)$$
$$- \Sigma_{p=1}^{n} e_{p,q} \sin (pq\pi/kn).$$

Thus A_q is the sum of two expressions, which can be obtained from the device by letting $t = q\pi/kn$ in the expressions given above for the output. If we let $t = q\pi/k$, we see that we also have

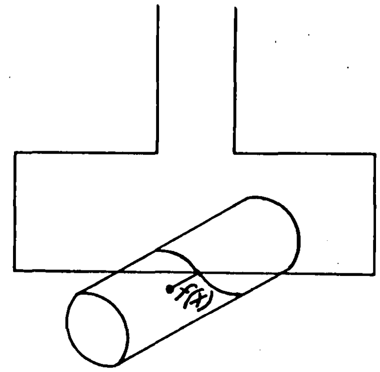
$$d_{p,q} = \Sigma_{s=0}^{2k-1} y_{s,p} \cos st$$
$$e_{p,q} = \Sigma_{s=0}^{2k-1} y_{s,p} \sin st.$$

Thus if 2k-1 is $\leq$ n, we may obtain these from the device also. (We may have to set certain coefficients zero.) Theoretically the expressions for $d_{p,q}$ and $e_{p,q}$ could also be compounded in the way A_q was if $2k-1 > n$. Hence, theoretically the analyzer could be used to calculate any number of coefficients. The limitation on this process is the accuracy with which $t = q\pi/kn$ could be entered in the device. An error of ε in radians yields in general an error of $(kn/\pi)\ \varepsilon$ 100 per cent in the coefficients.

4. Naturally we can use an ordinary integrator to evaluate the integrals

$$\int_0^{2\pi} f(x) \cos kx\, dx;\ \int_0^{2\pi} f(x) \sin kx\, dx.$$

Thus Galle describes an instrument (Sommerfeld-Wiechert, *loc. cit.*, pp. 145-48) in which the integrand is obtained by projecting a line segment of length f(x). The graph of f(x) is wrapped around a horizontal cylinder C. As x varies, the element upon which f(x) is represented appears uppermost. At the same time the cylinder itself rotates about a vertical axis so that the axis of the cylinder makes an angle kx with a horizontal line l. An arrangement with a wire perpendicular to l can move parallel to l. The operator keeps the wire on the uppermost point of curve. The linear displacement of the arrangement is then f(x) cos kx.



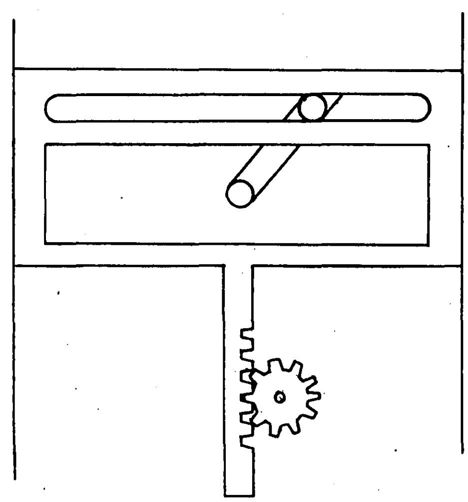
This linear displacement is the linear input of a simple disk integrator. This consists of a disk which rotates an amount x and an integrating wheel on it which is displaced from the center an amount equal to the linear input.

An alternate to the arrangement above is to apply f(x) to the linear displacement and let the rotatory input $\alpha = (1/k) \sin kx$. Since the output is

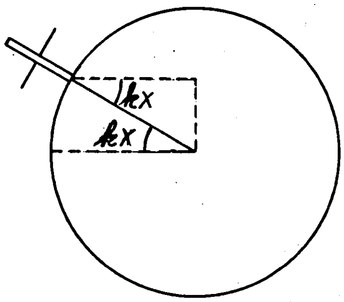
$$\int_0^{2\pi} f(x)\, d\alpha = \int_0^{2\pi} f(x) \cos kx\, dx$$

this yields the desired result.

This can be done in a number of ways. For instance α may be produced as a linear displacement by means of projecting arrangement and then changed to a rotation by means of a rack and pinion.



An alternative method of getting the desired result is to use a spherical integrator. This is the basis of the Henrici-Conradi analyzer. The basic calculating device consists of a sphere, whose rotation is the function f(x). The integrating wheel is at a spherical distance $k\alpha$ from the plane perpendicular to the axis of rotation. It is clear that if the axle of the integrating wheel is in



the same plane as the axis of rotation, the instantaneous change in the output is

$$d\beta = h \cos k x \, d\, f(x)$$

where h is the ratio of the radius of the wheel to that of the sphere. Thus if x goes from 0 to 2π we have

$$\beta = \int_0^{2\pi} h \cos hxdf(x) = h \cos kxf(x)]_0^{2\pi}$$

$$+ kh \int_0^{2\pi} \sin kxf(x)\, dx$$

$$= h\, [f(2\pi) - f(0)] + kh \int_0^{2\pi} \sin kx\, f(x)\, dx.$$

The cos kx integral is obtained when $\pi/2 - kx$ is substituted for kx.

A modern version of the Henrici integrator is described in an article by D. C. Miller, *Jour. Frankl n Inst.*, Vol. 182, pp. 285-322.

6. There are a number of relatively simple devices which permit one to use a planimeter to evaluate the Fourier coefficients of a function. An auxiliary of this sort is accredited by Galle to Yule (*loc. cit.*, pp. 134-35). The interval $0 \leq x \leq 2\pi$ is divided into 2n parts. For each subinterval a rectangle is formed. If y_k is the value of f(x) at the midpoint of the subinterval, the sides of the rectangle have length y_k and $\sin([k+1]\, q\pi/n) - \sin(kq\pi/n) = 2 \cos([k+\frac{1}{2}]q\pi/n) \sin(q\pi/2n) \simeq (q\pi/n) \cos([k+\frac{1}{2}]q\pi/n)$. A planimeter is used to obtain the sum of the areas of these rectangles. Thus the reading is

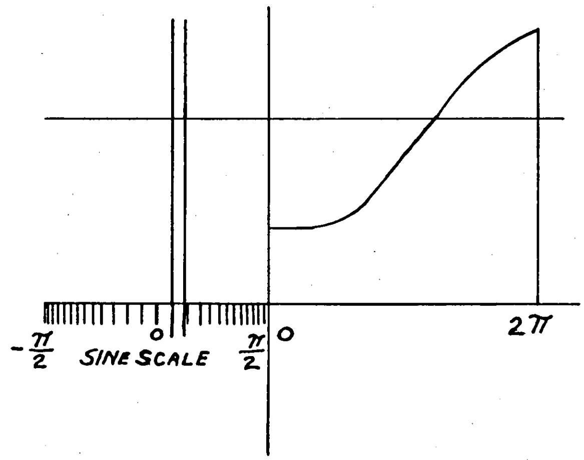
$$2\, \Sigma_{k=1}^{2n} f([k+\tfrac{1}{2}]\pi/n) \cos([k+\tfrac{1}{2}]q\pi/n) \sin(q\pi/2n)$$

$$\simeq \frac{q\pi}{n} (\Sigma_{k=1}^{2n} f[(k+\tfrac{1}{2})\pi/n] \cos[(k+\tfrac{1}{2})q\pi/n])$$

$$\simeq q \int_0^{2\pi} f(x) \cos qx\, dx = q\pi a_q.$$

There are a number of ways in which the desired rectangles can be constructed. Consider the drawing board upon which the graph of the function is drawn. The graph covers the x-interval from 0 to 2π. We have a wire which moves parallel to the x-axis and this we can set on the ordinate of the curve at the desired point. This wire and the x-axis will constitute opposite sides of the rectangle.

The other sides of the rectangle are positioned by means of a sine scale along the left-hand side of the x-axis. We have two slides with wires paral-



lel to the y-axis. To set up a rectangle with one pair of sides having the length $\sin\beta - \sin\alpha$, we set the wire of one slide on the value α of the sine scale and the other on the value β. The distance between the wires is then $\sin\beta - \sin\alpha$. The x-axis, the first slide wire and these last two now enclose a rectangle. (This is not the device described by Galle, *loc. cit.*, but the idea is the same.)

Notice that if we take our interval end points in the form $(k+\frac{1}{2})\pi/n$, so that the midpoints are in the form $k\pi/n$, the individual rectangles have the area

$$f(k\pi/n)\, (\sin[(k+\tfrac{1}{2})q\pi/n] - \sin[(k-\tfrac{1}{2})q\pi/n])$$

$$= 2\, f(k\pi/n) \cos(q\pi k/n) \sin(q\pi/2n)$$

The sum of these rectangles

$$\frac{q}{2} \frac{\sin(q\pi/2n)}{q\pi/2n} (\pi/n)\, \Sigma_{k=1}^{2n} f(k\pi/n) \cos(q\pi k/n)$$

differs only by the factor $\frac{\sin(q\pi/2n)}{q\pi/2n}\,(q\pi/2)$ from the expression

$$\frac{1}{n} \Sigma_{k=1}^{2n} f(k\pi/n) \cos(q\pi k/n)$$

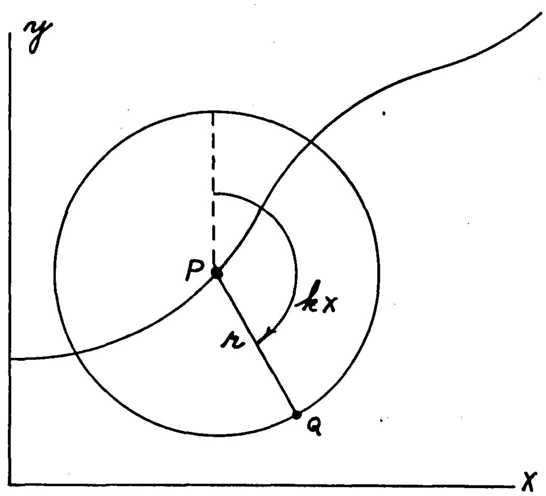
which we have considered in the preceding sections of this chapter. The situation relative to the sine expression is similar.

Another type of device for the evaluation of the Fourier coefficients by means of a planimeter is based on the following geometrical construction. Suppose that as the point P traces the curve $y = f(x)$, it carries a wheel with it. The wheel revolves in a plane parallel to the plane of the graph around an axis through P. The rotation is kx. Let us consider a point Q on the circumference of the wheel, which is uppermost when $x = 0$.

Suppose Q has the coordinates (η, ζ), then

$$\eta = x + r \sin kx$$

$$\zeta = y + r \cos kx$$



Consider now the integral which corresponds to the area under the curve traced by Q. We have

$$\int_Q \zeta d\eta = \int_0^{2\pi} (y + r \cos kx)\,(1 + kr \cos kx)\,dx$$

$$= \int_0^{2\pi} y\,dx + kr \int_0^{2\pi} y \cos k\,x\,dx$$

$$+ \int_0^{2\pi} r \cos kx\,dx + kr^2 \int_0^{2\pi} \cos^2 kx\,dx$$

$$= \int_0^{2\pi} y\,dx + kr \int_0^{2\pi} y \cos kx\,dx + k\pi r^2$$

since k is integral. Thus if we have a planimeter to find the area under the given curve and to trace the Q curve and find the area under it, then we know both $\int_0^{2\pi} y\,dx$ and $\int_Q \zeta d\eta$ and from these the desired integral

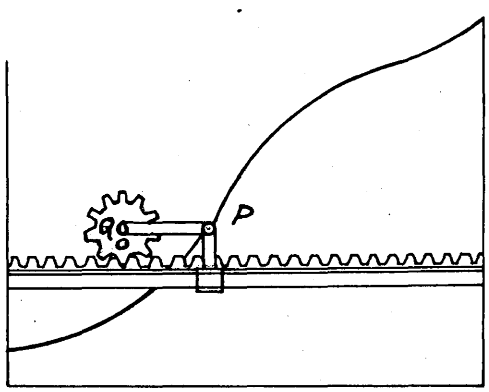
$$\int_0^{2\pi} y \cos kx\,dx$$

can be readily obtained. To obtain the integral

$$\int_0^{2\pi} y \sin kx\,dx$$

we start with a point Q whose radius is initially horizontal.

There are a number of ways in which the desired result can be obtained. One of these, described by Galle as due to Yule consists of a pinion and a rack parallel to the x-axis. The rack can slide up and down in grooves, but remains parallel to its original position. Let P denote the pointer that is used to follow the curve. We have a connection from P to the rack such that the pitch



line of the rack has an ordinate which differs from that of P by a fixed amount. Another connection to the pinion insures that the center of the wheel has the same ordinate as P and the abscissae differ by a constant. The rate of rotation of the pinion is determined by the choice of the radius. There is a hole in the pinion in which the follower of a planimeter is inserted to trace the Q curve.

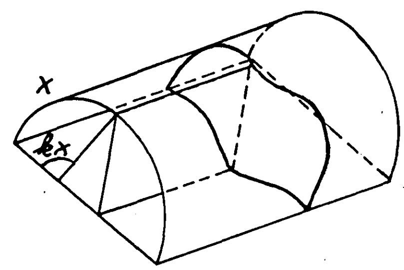
A number of other ways of doing this are also discussed by Galle.

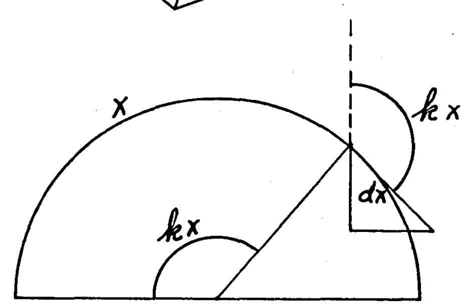
8. It is clear that the devices which produce integrals in the form

$$\int_a^b f(x)\,g(x)\,dx$$

can be used to obtain the Fourier coefficients of a function. For instance, there is the "cinema integraph" described above. Another device for this purpose is that described by J. A. Van der Akker, *Journ. Opt. Soc. of Amer.*, Vol. 29 (1939), pp. 364-69 and 501. This consists apparently of two input boards, a similar triangle multiplier and a disk integrator.

There is one other geometrical construction which has been utilized in harmonic analyzers. Consider a half-cylinder of diameter $2/k$. The circumference of the semicircle is π/k. Let us measure off a distance x along the semicircle from one end and wrap the graph of $f(x)$ on the interval $0 \le x \le \pi/k$ around the half-cylinder. Now consider the projection on the diametrical half plane of the area under the curve and with abscissa less than x. This





is a function of x, A(x). The central angle subtended by the arc of length x is kx and it is readily seen that

$$d\,A = f(x) \sin kx\,dx.$$

Thus the total area of the projection is

$$\int_0^{\pi/k} f(x) \sin kx\,dx.$$

(This assumes f(x) is positive, a result which can always be attained by adding a constant, an operation which does not affect any of the Fourier coefficients except the first.)

A similar argument shows that if we use the portion of the graph from π/k to $2\pi/k$, we obtain

$$-\int_{\pi/k}^{2\pi/k} f(x) \sin kx \, dx$$

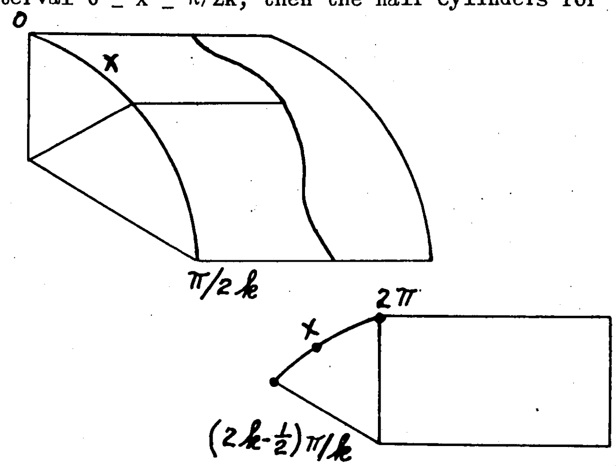
and if we use the portion $2\pi/k \le x \le 3\pi/k$, we get

$$\int_{2\pi/k}^{3\pi/k} f(x) \sin kx \, dx.$$

It is clear from these, we can build up

$$\int_0^{2\pi} f(x) \sin kx \, dx.$$

If we begin with a quarter cylinder using the interval $0 \le x \le \pi/2k$, then the half cylinders for



intervals $(1 + \frac{1}{2})\,\pi/k \le x \le (1 + 3/2)\,\pi/k$ and finally using the quarter cylinder for the interval $(2k - \frac{1}{2})\,\pi/k \le x \le 2\pi$, we get the integrals

$$\int_0^{\pi/2k} f(x) \cos kx \, dx, \; -\int_{\frac{1}{2}\pi/k}^{3\pi/2k} f(x) \cos kx \, dx, \text{ etc.}$$

$$\int_{(2k-\frac{1}{2})\pi/k}^{2\pi} f(x) \cos k \, dx$$

and from these we can construct

$$\int_0^{2\pi} f(x) \cos kx \, dx.$$

The projections involved can be done photographically. Thus, if the cylinders are transparent and the graph of function is cut out as a mask, then if we have light rays perpendicular to the diametrical plane of the semicylinder, we can obtain a shadow corresponding to the area projection. The area projection can be measured by a planimeter or a photocell bridge.

Presumably, in the latter case, it would be more convenient to have a flat slide, which controls the illumination along an ordinate, so that it is proportional to sin kx. This could be done by having varying opaqueness, differently spaced lines or dots of different denseness. Two masks would be needed for each coefficient to take care of the sign of sin kx.

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A complete bibliography for mathematical machines would require a search of the technical literature of many fields. Much of this would be repetitious from our point of view, so the following material is intended merely to give leads to the literature rather than to cover it completely. Since in our discussion, we have emphasized the mathematical principles at the expense of technical detail, the emphasis in the bibliography is in the other direction.

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INDEX

Note: The Roman number refers to the part. The page or range of pages follow; "a" denotes the left-hand column on a page, "b" the right-hand column.

THEORY OF FUNCTIONS

J. F. RITT

DAVIES PROFESSOR OF MATHEMATICS, COLUMBIA UNIVERSITY

Here is the basic material of a course in the theory of functions which the author has given at Columbia University for some years. Published a year ago in mimeographed form, it met with such success that it is now presented in more substantial format, entirely revised and edited by the author and by Dr. Milton Sobel. The emphasis is on the complex variable. From this standpoint, about two-fifths of the work is on the real variable with the remaining time given to the complex. Halfway through, the student will find himself equipped for higher courses on the real variable or for a course on existence theorems for differential equations. At the end he can study almost any topic of mathematical analysis; above all, he has learned what it means to understand mathematics and can deal with any mathematical book.

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