

33-214

PRIMER ON ANALOG COMPUTATION

Introduction

In engineering, the analog computer can be an effective tool in either of two categories: model building — an inductive process, and model analysis — a deductive process. In model building, an analytical relation between variables is hypothesized to describe the physical system of interest. Forcing functions identical to those in the physical system then can be applied to the hypothetical model so that a comparison of the response of the model with that of the actual physical system will indicate changes that should be made in the model to better fit it to the system and thus establish the validity of the model. Because of the ease with which parameter variations and model changes can be accomplished, the analog computer is useful in conducting many "trial and error" experiments on the model to obtain the best fit to the physical system. It is also possible to have the computer mechanized so that it will seek the best fit automatically once a desired optimum is quantitatively defined.

On the other hand, in model analysis, it is necessary to have a mathematical statement available which describes the physical system to be studied. These mathematical statements (equations) are often supplemented by graphical information as well as logic statements, and a definite range for the parameter values are assigned for the study. Experiments are then performed, varying the inputs and the parameters describing the system, to obtain, finally, an optimum response for the system, or to develop a better understanding of the intrinsic nature of the system by studying input-output relations. Here, again, because of its speed, flexibility, and similarity to the physical system, the analog computer is an effective engineering tool.

Although the analog computer has been used for both model building and model analysis, we will develop its principles of operation and programming from the point of view of model analysis. This approach is preferable since computer use in analysis is straight-forward and more readily defined than in the case of model building.

Concept of the Analog Computer

An analog computer is a collection of operational devices which are capable of performing basic mathematical operations. By interconnecting components which can integrate, add, multiply by -1, multiply by a constant, and multiply variables, we can analyze dynamic systems which are described by ordinary differential equations.

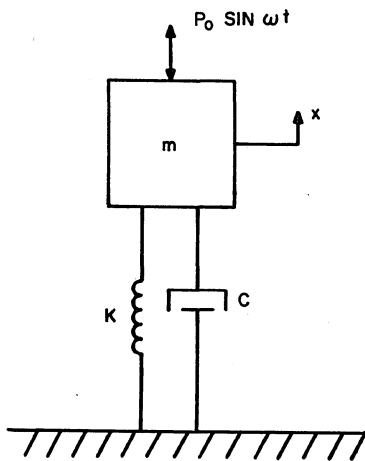


Figure 1. Schematic Diagram of a Simple Linear Mechanical System

The use of these mathematical building blocks can be illustrated by analyzing the mechanical system

shown in Figure 1. Equation 1 defines dynamic equilibrium for this simple spring-mass-dash pot combination:

$$m \frac{d^2 x}{dt^2} + c \frac{dx}{dt} + kx = P_0 \sin \omega t \quad (1)$$

Here, we are interested in investigating the motion of the mass, m , subjected to an external exciting force, $P_0 \sin \omega t$. If x represents the motion of the mass, and we look upon the problem as a mathematical exercise, then we must find the dependent variable, x , as a function of the independent variable time, t [i.e. $x = f(t)$].

Let us re-write equation 1 so that we have $m(d^2x/dt^2)$, the inertial force, on one side of the equation:

$$m \frac{d^2 x}{dt^2} = -c \frac{dx}{dt} - kx + P_0 \sin \omega t \quad (2)$$

If we assume that $m(d^2x/dt^2)$ is known, then double integration and multiplication by $1/m$ are required to produce the dependent variable, x . With dx/dt available as the first integral of d^2x/dt^2 , x available as the integral of dx/dt , and $P_0 \sin \omega t$, a known function of time, we can produce

the three terms on the R.H.S. of Equation 2 by simple mathematical operations, and then sum them as shown in the mathematical block diagram of Figure 2. To force equality between $m (d^2x/dt^2)$ and the R.H.S. of Equation 2, we connect Point a to Point b on the diagram, as shown by the dotted line.

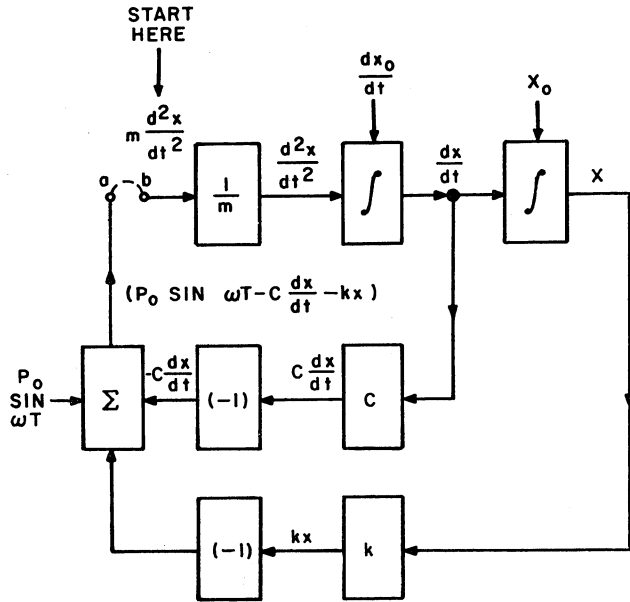


Figure 2. Mathematical Block Diagram for the Solution of Equation 2

In the physical system, the mass could have an initial velocity dx_0/dt and an initial deflection x_0 . This is introduced by allowing for initial condition inputs to the integrating devices. With initial conditions applied, the output of the integrator is now the definite integral of the input between the limits $t = 0$ and some instant of time $t = t_1$.

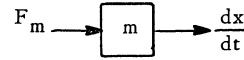
Dynamic Model of Physical System

While analysis of the system in Figure 1 was taken from a mathematical point of view, it can be shown that the mechanical elements have transfer functions (input/output relationships) which can be used to describe the system in a manner similar to the mathematical block diagram approach of Figure 2.

Table 1 describes the relations between force and motion (transfer function) for simple mechanical elements. If we consider that mass motion (velocity) is due to an applied force, and that spring or dashpot force is the result of a relative motion across the element, then we can picture a mass as an integrator (of forces) and a multiplier (by the constant $1/m$), and a spring as an integrator (of motion) and a multiplier (by k). The dashpot is simply multiplication (of velocity) by a constant c .

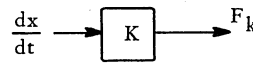
TABLE I. RELATIONS BETWEEN FORCE AND MOTION FOR MECHANICAL ELEMENTS

MASS



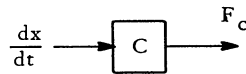
$$\text{Eq 3a } F_m = m \frac{d}{dt} \left(\frac{dx}{dt} \right) \quad \text{Eq 3b } \frac{dx}{dt} = \frac{1}{m} \int F_m dt$$

SPRING



$$\text{Eq 3c } F_k = k \int \frac{dx}{dt} dt = kx$$

DASHPOT



$$\text{Eq 3d } F_c = C \frac{dx}{dt}$$

From Newton's second law, ($F = ma$), we know that the forces (F_k , F_c , $P_0 \sin \omega t$) exerted on the mass, m , produce the inertial force, $m(d^2x/dt^2)$ (acting in the plus direction, positive upward). Figure 3

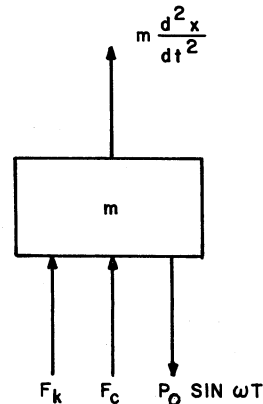


Figure 3. Free Body Diagram for Mass of Figure 1

shows the forces acting on m as a free body. The resulting motion of m is

$$\frac{dx_m}{dt} = \frac{1}{m} \int F_m dt \quad (3)$$

Since the spring and dashpot each have one end attached to the mass, while the other end is fixed to the earth (a rigid reference), the motion across these elements is also dx_m/dt . For positive motion of the mass, there is a negative force exerted by the spring and dashpot on the mass. For the dashpot, this force is $-c dx_m/dt$, and for the spring, $-kx_m$, where x_m is the instantaneous deflection of the spring. For dynamic equilibrium we must connect a to b .

Initial kinetic energy of the system is defined by the initial velocity of the mass, dx_0/dt . Initial potential energy is introduced by an initial deflection, x_0 , of the spring. Figure 4 shows a transfer function diagram of the system. Comparison of this with Figure 2 reveals that the interconnection of mathematical computing components produces a diagram similar to the one obtained by considering the dynamic equilibrium for the physical system.

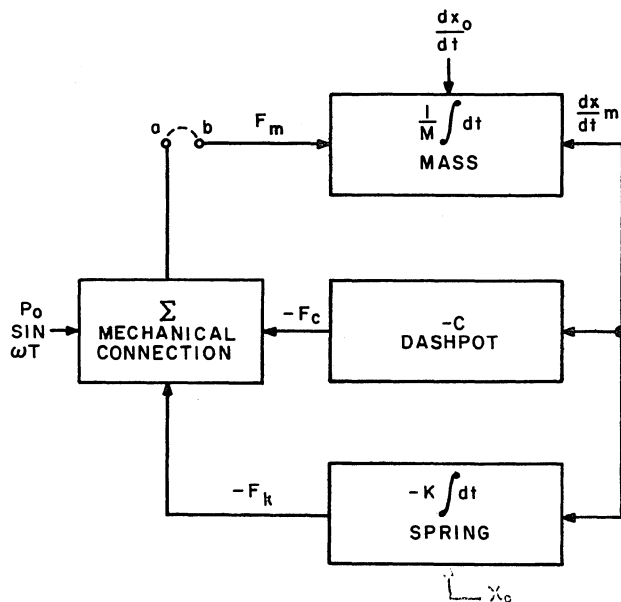


Figure 4. Transfer Function Diagram for Figure 1

Computational Elements

We have next to consider the appropriate physical medium in which the mathematical operations outlined above should be performed. Although analog computers can and have used mechanical, hydrau-

lic, pneumatic and other elements, an electrical medium is most desirable for several reasons:

1. wide dynamic range,
2. ease of interconnection,
3. availability of recording devices activated by electrical signals,
4. availability of precision components,
5. availability of highly developed analytical techniques.

We will, therefore, devote our attention to the electronic general purpose analog computer, or electronic differential analyzer, in which the problem variables and their derivatives appear as voltages.

Attenuator: The attenuator is the simplest computing element. Its output is a voltage which is the product of the input voltage times a constant, say ρ ($0 \leq \rho \leq 1.0$). The programmer's symbol and input-output relation for an attenuator are shown in Figure 5. The attenuators are usually precision potentiometers with calibrated dials.

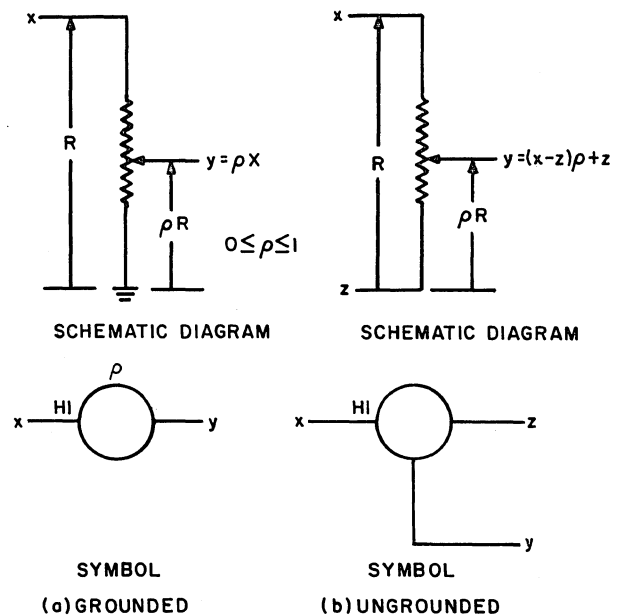
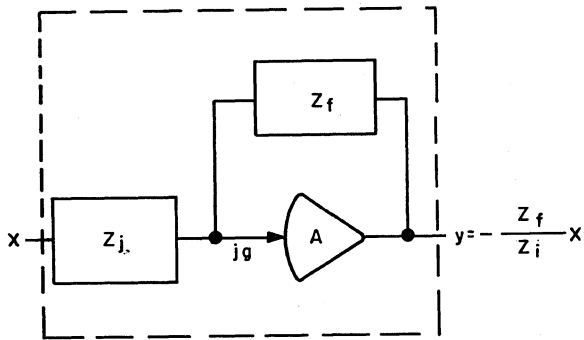


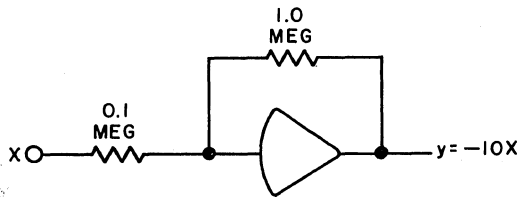
Figure 5. Attenuator

Operational Amplifier: The operational amplifier, shown schematically and symbolically in Figure 6, is the basic component of the modern electronic analog computer. The area shown enclosed by the dashed line in Figure 6a is a block diagram of the operational amplifier. The triangular symbol

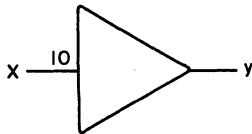
represents a high-gain, direct-coupled amplifier having a wide frequency range usually extending from zero to beyond 25 kilocycles, and effectively zero grid current. Z_f and Z_i are passive electrical



a. BLOCK DIAGRAM FOR GENERAL OPERATIONAL AMPLIFIER



SCHEMATIC DIAGRAM



SYMBOL

b. MULTIPLICATION BY -10 WITH AN OPERATIONAL AMPLIFIER

Figure 6. Operational Amplifier

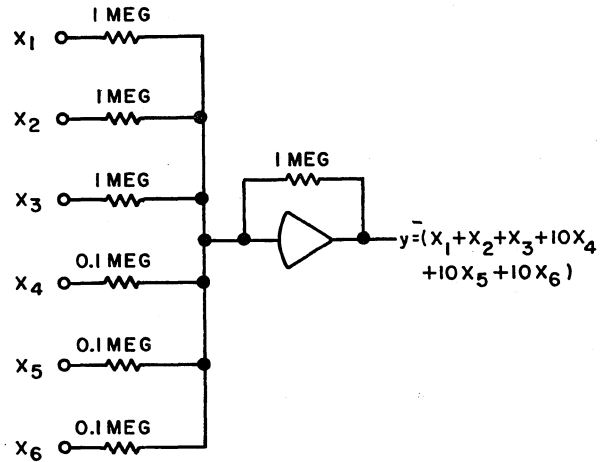
components with impedance (voltage-to-current ratio) that can be matched to better than 0.01 percent, an accuracy more than adequate for the majority of engineering calculations. The general relationship between the output, y , and the input, x , for the operational amplifier is

$$y = (-Z_f/Z_i) x \quad (4)$$

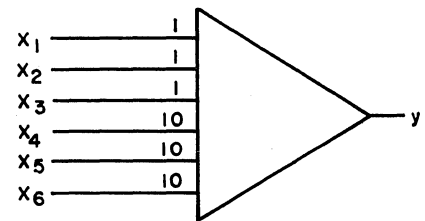
The amplifier inherently performs the operation of mathematical inversion (multiplication by minus 1), while Z_f and Z_i establish the specific mathematical operation of the device. For example, if Z_f were 1 megohm (10^6 ohms) and Z_i were 0.1 megohm (10^5 ohms), the input-output relation, as shown in Figure 6b, would be

$$y = -(10^6/10^5)x = -10x \quad (5)$$

By introducing additional input paths to the DC amplifier, as seen in Figure 7, we can perform summation with the same device.



SCHEMATIC DIAGRAM

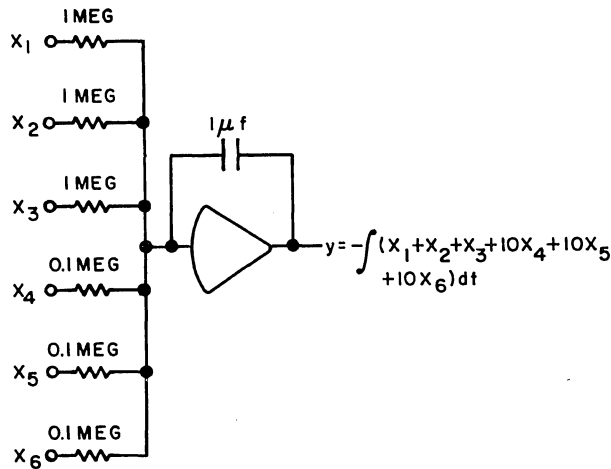


SYMBOL

Figure 7. Summer Amplifier

In addition to inversion, multiplication by a constant, and summation, the operation of integration also can be performed using a feedback capacitor with the DC amplifier. Figure 8a is a schematic diagram for an integrator (integration is with respect to time). High quality operational amplifiers are chopper-stabilized to reduce drift, i.e. amplifier output with zero input, which is characteristic of DC amplifiers.

Multipliers: In linear problems with time-varying coefficients, as well as in nonlinear problems, in general, it is necessary to produce the product of one variable times another. This occurs in such problems as the calculation of the mass change of a missile during flight due to fuel consumption, the measurement of energy dissipation where damping coefficients are a function of position as well as velocity, the analysis of flexible elements having cubic elasticity, etc. Although there are several methods for multiplying two variables, the most common techniques are the servo multiplier and the quarter-square multiplier.



SCHMATIC DIAGRAM

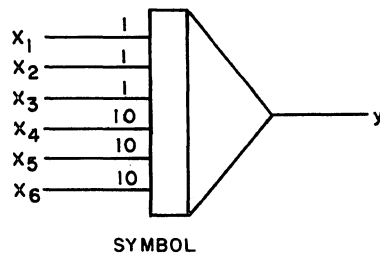
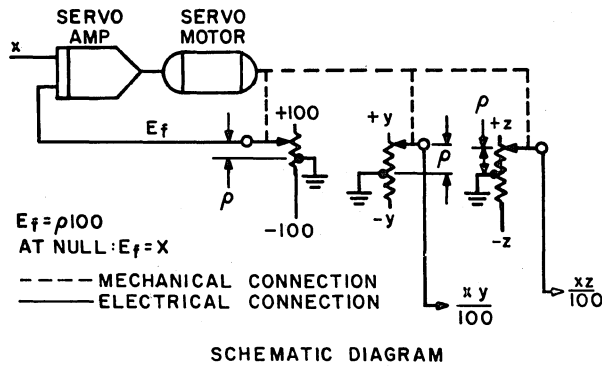


Figure 8. Integrator Amplifier

In the servo mechanical multiplier, as shown in Figure 9, a potentiometer wiper is positioned as a function of one variable, say x , by means of a position feedback control loop. The voltage applied to the potentiometer is a function of a second



SCHMATIC DIAGRAM

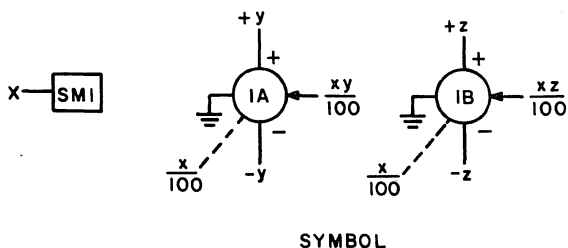


Figure 9. Servo Multiplier

variable, say y , and thus the output of the wiper will be proportional to the product of the two variables. By ganging several potentiometers, it is possible to obtain, simultaneously, products of x times the several variables applied to the additional potentiometers (XZ , XW , etc.).

The quarter square multiplier operation is based on the relation

$$= \frac{1}{4} \left[(x + y)^2 - (x - y)^2 \right] \quad (6)$$

which reduces multiplication to the operations of summation and squaring. Squaring is done by means of solid state electronic switching, in conjunction with a DC amplifier which produces a straight line segment approximation of the relation $k(x + y)^2$. The programmer's symbol for a quarter-square multiplier is shown in Figure 10.

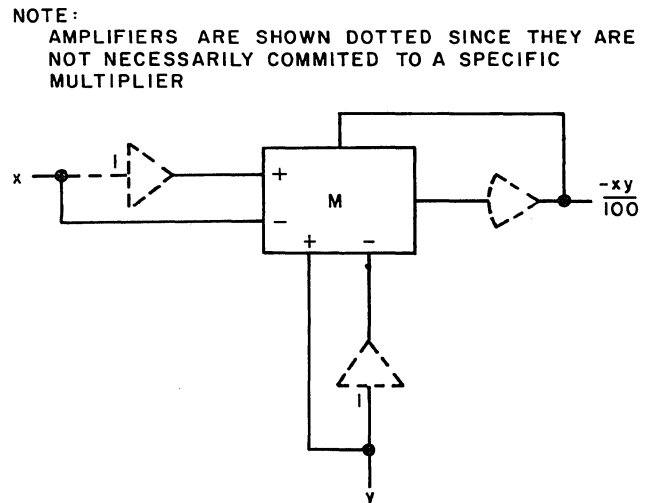


Figure 10. 1/4 Square Multiplier

This multiplier, when used as the feedback element of a DC amplifier, can perform division and square rooting.

Function Generators: In certain problems, two variables may be related but have no analytical description existing for the relationship. This usually occurs where data is obtained experimentally. When these relationships cannot be approximated by an analytical expression, it is necessary to use a function generator to approximate the function. The general technique for function generation is first to divide the curve into a number of straight line segments, as shown in Figure 11. These segments then can be reproduced by means of a function generator on which either the segment slopes, or breakpoints and slopes, or both can be set to provide the best fit to the curve.

Two common analog computer function generators are the tapped servo potentiometer and the diode

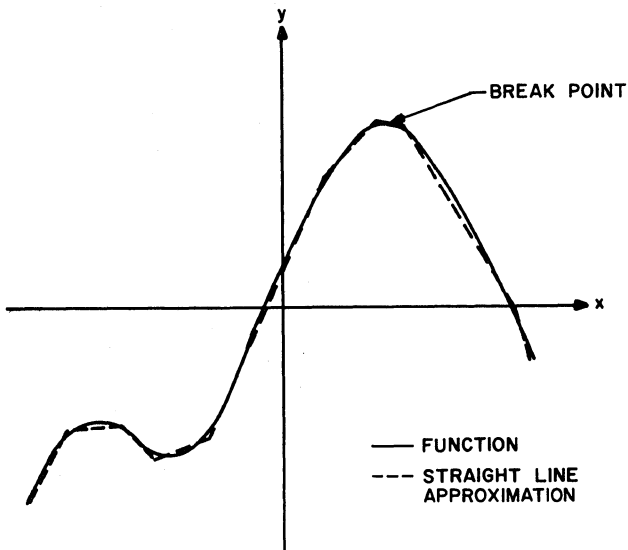
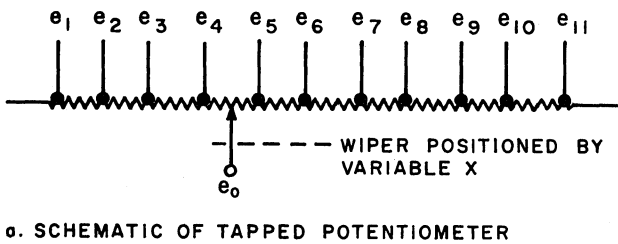
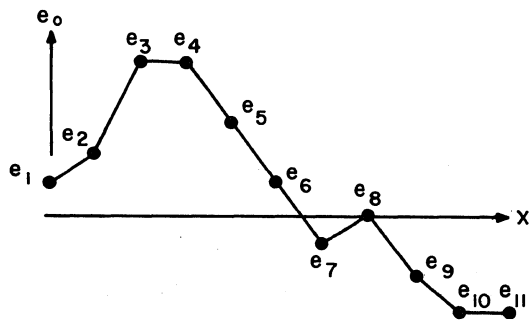


Figure 11. Straight Line Approximation of a Non-Analytic Function

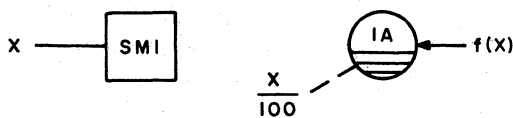
function generator (DFG), both of which produce single-valued functions.



a. SCHEMATIC OF TAPPED POTENTIOMETER



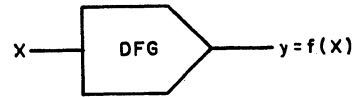
b. OUTPUT OF TAPPED POTENTIOMETER AS A FUNCTION OF X.



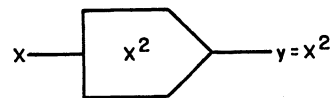
c. SYMBOL

Figure 12. Tapped Servo Potentiometer Function Generator

Figure 12a is a schematic diagram of a tapped multiplying potentiometer of a standard servomultiplier, with the wiper positioned by the variable x . Voltages e_1 to e_n are established at fixed intervals along the potentiometer by means of an external source to produce an output voltage similar to the one shown in Figure 12b. The symbol for a tapped servo potentiometer is shown in Figure 12c.



a. VARIABLE DIODE FUNCTION GENERATOR



b. FIXED D.F.G.: x^2

Figure 13. Diode Function Generators

The DFG (Figure 13) is a series of diodes and resistor networks used in conjunction with two DC amplifiers. By means of potentiometers, both breakpoints and slopes of the straight line segments can be adjusted. Ten (10) to twenty (20) straight line segments are normally available on this unit.

Fixed function generators are also available to produce a given function of a variable such as x^2 , x^4 , $\log x$, etc. Their operation is essentially the same as the variable DFG.

The trigonometric functions, $\sin \theta$ and $\cos \theta$, are produced by using special potentiometers on servo multipliers, shown symbolically in Figure 14, or by using shaping networks in conjunction with DC amplifiers.

With $\sin \theta$ and $\cos \theta$ available, transformation from rectangular-to-polar coordinates and polar-to-rectangular coordinates is possible.

Special Devices: In certain problems, it is necessary to include discontinuous phenomena such as limits, backlash, and dead zone. These functions can be simulated by means of biased diodes used in conjunction with high gain amplifiers (Figure 15).

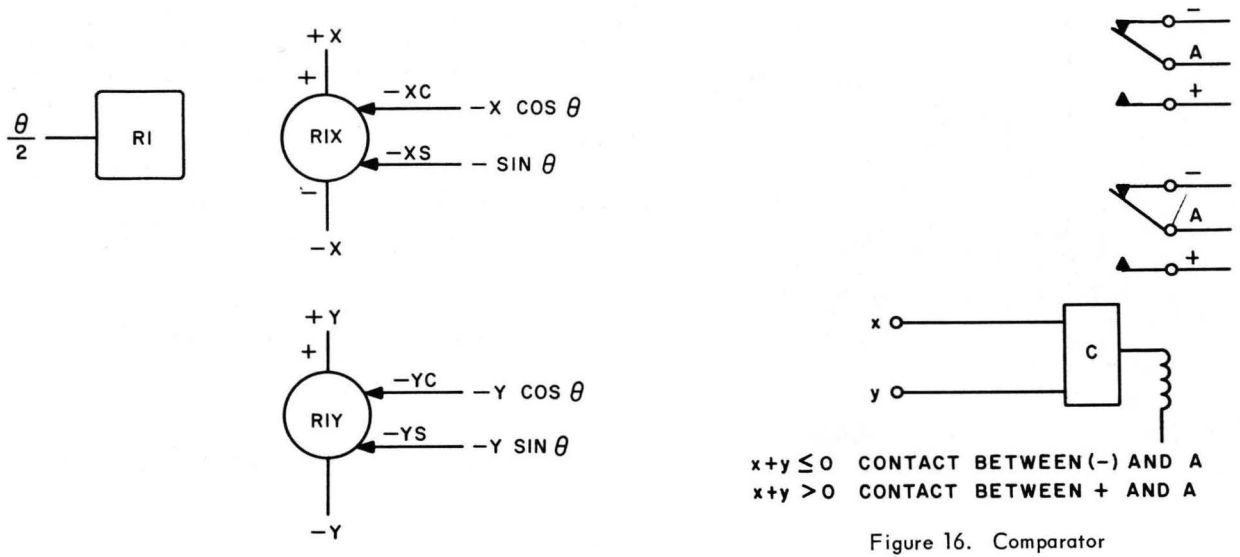


Figure 14. Servo Resolver

means to interconnect the output of one unit to the input of one or more other units. For some computers, this is accomplished by connecting directly from the output of one device into the input of another by means of a cord or wire, as shown in Figure 17. By terminating all the input

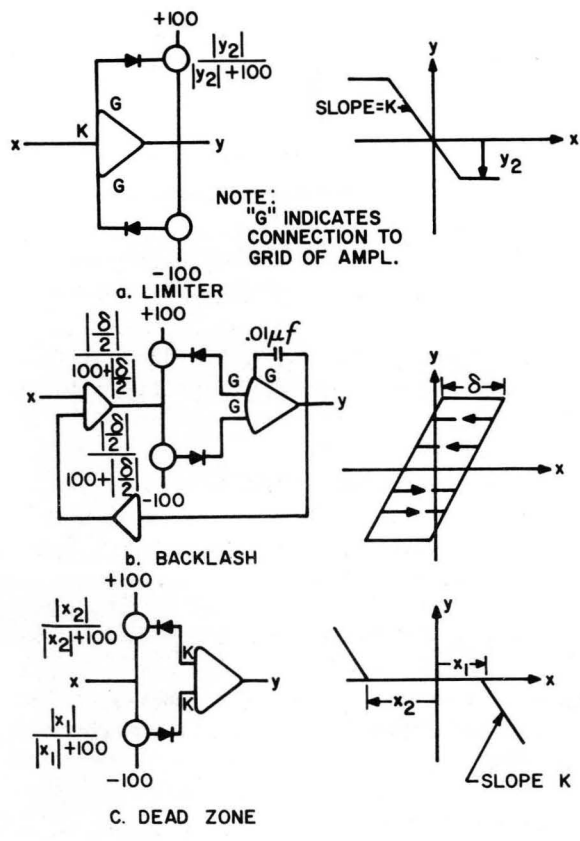


Figure 15. Simple Diode Circuits to Represent Discontinuous Functions

In addition, high-speed relays—comparators—(Figure 16), and electronic switches are available to perform additional logic operations.

Connecting Components

Once an assemblage of computer building blocks is available, it is then necessary to provide a

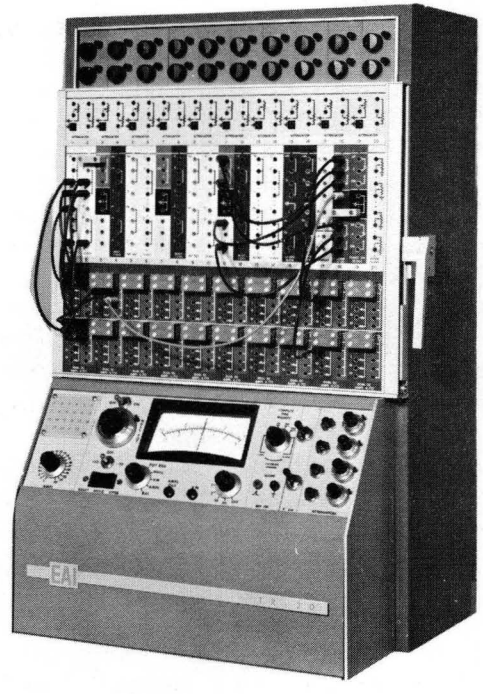


Figure 17. TR 20 Patch Area

and output points for the building blocks at a single area on the computer, however, it is possible to make connections by means of a removable patch panel, as shown in Figure 18. When the patch

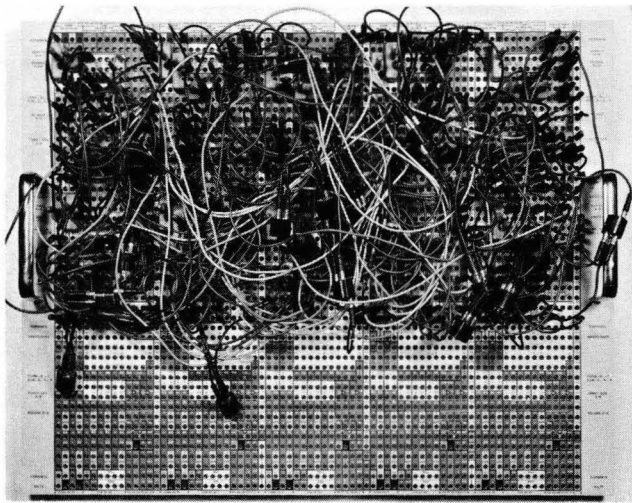


Figure 18. 231R Patch Panel

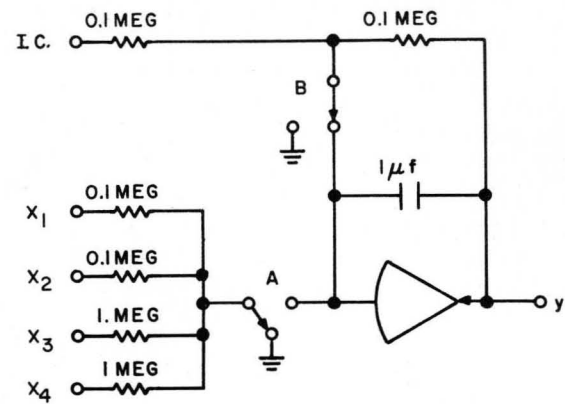
panel is put on the computer, the component interconnections are made by means of the patch cords on the patch panel. Thus, one problem can be run on the computer while connections are being made away from the machine on separate patch panels for other problems.

Computer Control

Analog computers have two basic operating modes—reset (or initial condition) and operate—which are analogous to the physical systems which they model. In reset, on the physical system, initial conditions are introduced and parameter adjustments or system changes made. To do this, the system is usually de-energized, permitting changes in valve settings, liquid level in tanks, relative position of elements, deflection of springs, replacement of weights, etc. In operate, power is applied to the system and the dynamic response (or operation) under imposed initial conditions or forcing functions is allowed to take place.

On the computer, mode control is accomplished by means of switching associated with the integrators. Figure 19 shows an integrator network and amplifier and the two relays used to produce the RESET (initial condition) and OPERATE modes. The relays for all integrators are thrown simultaneously and are controlled by master Operate and Initial Condition switches located on the control panel of the computer.

It is worthwhile to note that by opening all the Operate switches (input to the integrators) during a problem run—leaving the reset switch open—it is possible to “freeze” the variables and their



MODE	SWITCH A	SWITCH B	OUTPUT y
RESET	OPEN	CLOSED	-I.C.
HOLD	OPEN	OPEN	VOLTAGE AT y WHEN A IS OPENED
OPERATE	CLOSED	OPEN	$-\int(10X_1 + 10X_2 + 10X_3 + 10X_4)dt - I.C.$

Figure 19. Integrator Network and Control Circuit

derivatives at a particular instant in the problem solution. This is the HOLD mode of the computer.

When the computer is returned to the OPERATE mode, it will have, in effect, a new set of initial conditions, i.e. the conditions existing when the computer was placed in the HOLD mode (or modified during HOLD by the programmer).

Computer Readout

There are several methods available for displaying the results obtained in a computer solution.

A Digital Voltmeter-Printer Combination allows monitoring and subsequent printout of all voltages in a problem at a particular instant, or when the voltage reaches a fixed value, as shown in Figure 20. Readout is available to 0.01%.

In addition to the digital voltmeter readout, a continuous record can be made of problem variables as functions of time. Strip chart recorders, as shown in Figure 21, with either single or multi-channel capabilities, are available for this task.

If it is desired to plot one variable versus another, as in the case of a phase-plane plot, an X-Y

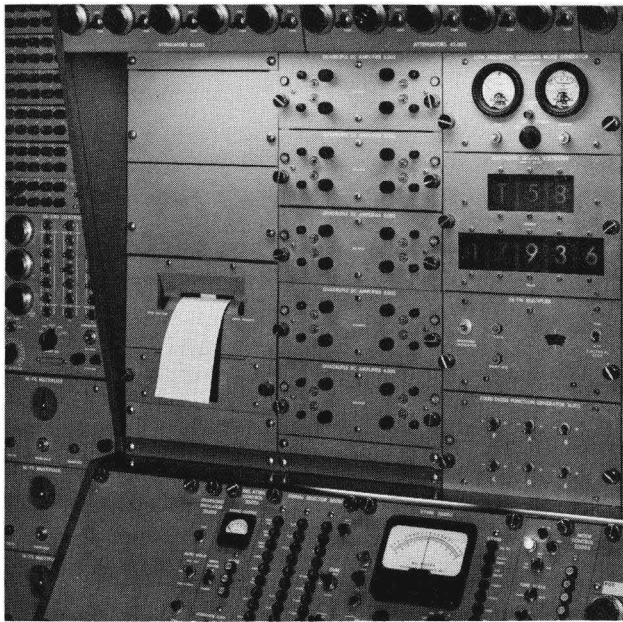


Figure 20. Digital Voltmeter Printer



Figure 22. X-Y Plotter



Figure 21. Strip Chart Recorder

plotter, as shown in Figure 22, can be used. By means of these devices, dynamic response data for the problem can be obtained.

Programming

Using the symbols we have developed, let us illustrate computer programming by preparing a computer diagram to solve Equation (1). To do this, we shall use the following routine as a programming procedure:

1. Separate the highest order derivative(s) (nth order) to the left hand side of the equation.
2. Assuming that the highest derivative(s) is available, integrate n times to produce the dependent variable, as shown in Figure 23.

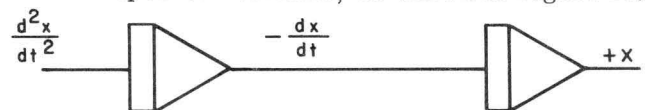


Figure 23. Generating the Dependent Variable from the Highest Derivative

3. Using the variable(s) and the (n-1)th order derivatives produced by integration, generate the terms on the right hand side of the equation(s), as shown in Figure 24.
4. Sum the terms produced by step 3, invert, and introduce them as inputs to the first integrator, as shown in Figure 24. (If d^2x/dt^2 is not required explicitly, the voltages representing the terms on the R.H.S. could be summed in integrator 1, saving amplifiers 4 and 5.
5. Introduce required initial condition inputs to integrators. The outputs of integrators

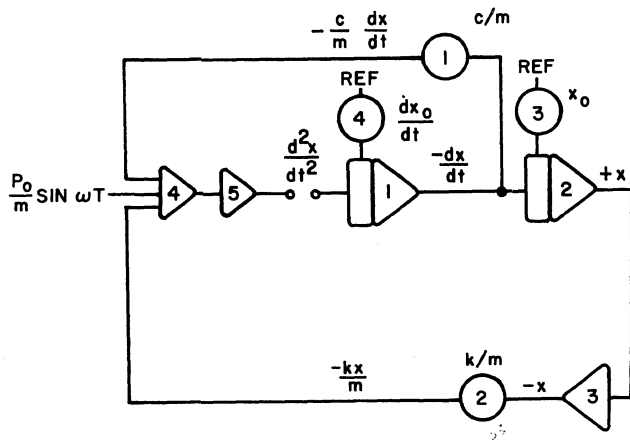


Figure 24. Computer Program for Solution of Equation 1

1 and 2, as shown in Figure 24, will be voltages varying in a manner analogous to the problem variables dx/dt and x , respectively. Problem parameters are adjusted by potentiometers 1 and 2. By changing the I.C. potentiometers on the integrators, we can change problem initial conditions.

Scaling

In physical systems, we have such variables as pressure, displacement, velocity, time, etc. On the analog computer, voltage is the dependent variable while the independent variable is time. To fit a problem to the computer, it is necessary to provide definite relations between the problem variables and the computer voltage (voltage scaling), and also between the problem independent variable and time on the computer (time scaling). Let us first consider voltage scaling.

Voltage Scaling: On the basis of amplifier design, there is a voltage range in which the computer can operate to produce the mathematical functions described previously. On modern general purpose analog computers, this voltage will usually be either ± 10 volts or ± 100 volts. We shall use ± 100 volts as the basis for further discussions of scaling.

In a physical system, the maximum value of the dependent variable can be quite small, ≤ 1.0 (e.g. motions of intricate mechanisms). On the other hand, acceleration forces due to shock blast result in a very large variable ($\geq 10^3$ pounds force). In order to transform the problem variable(s) into a computer voltage, and assure that the maximum allowable voltage on the computer will not be exceeded when the maximum value of the variable

in the problem is reached, we must develop a set of scaled relations between problem variables and computer voltages.

In relating problem variables to voltage, we need scale factors of the form *volts per inch*, *volts per degree*, *volts per foot per second*, and so forth. To obtain these scale factors, we must first estimate the maximum values of the variables which will occur in the problem. Estimates of maximum values of the variables and their derivatives are guided by the experience of the engineer, coupled with information obtained from the analysis of simplified forms of the equations describing the physical system. However, errors in estimating maximum values are not serious since, once the problem is put on the computer, these errors are quickly indicated by overloads of computing elements, and then one can rescale as necessary. (Most analog computers have both an audible as well as a visual alarm to indicate overload of computing elements.) In addition to not exceeding the voltage limitation of the computing elements, proper scaling also requires that the voltages be greater than some minimum voltage level at some time during a problem. It is desirable, therefore, to check the voltage levels at the outputs of computing components to be certain that they exceed some reasonable minimum value.

In order to develop a system of scale factors for a problem that will permit simple conversion from problem variable to voltage and vice versa, and to avoid introducing a new set of symbols, the following voltage scaling procedure is suggested:

1. Estimate maximum values for all the variables and their derivatives.
2. Round off estimated maximum values (in the upward direction) and express them as powers of 10 times such integers as 1, 2, 4, 5, and 8.
3. Obtain the scale factors for the problem variable and their derivatives by the ratio:

$$\frac{\text{Maximum Computer Voltage}}{\text{Maximum Value of Variable from Step (2)}}$$

Scale factors can be identified by the symbol S with an appropriate subscript for each variable. For example, the scale factor for x could be designated S_x , for \dot{x} , $S_{\dot{x}}$, etc.

4. Set up a table similar to Table 11, listing problem variables and their derivatives (show units), scale factors (show units),

and scaled variables which are now voltages which we will produce on the computer.

TABLE II. DEVELOPING SCALE FACTOR FOR PROBLEM VARIABLES

PROBLEM VARIABLE	ESTIMATED MAXIMUM VALUE	SCALE FACTOR = $\frac{100}{\text{ROUND OFF OF MAX.}}$	COMPUTER VARIABLE
X_1 in.	32	$\frac{100}{50} = 2$ volts/in.	$[2 X_1]$
$(X_1 - X_2)$ in.	190	$\frac{100}{200} = 0.5$ volts/in.	$[0.5(X_1 - X_2)]$
$10^6 e^{-kx}$ lbs.	10^6	$\frac{100}{10^6} = 10^{-4}$ volts/in.	$[10^2 e^{-kx}]$
$\sin \theta$	1	$\frac{100}{1} = 100$	$[100 \sin \theta]$

To prepare the computer patching diagram, use scaled voltage equations which can be obtained from the original problem equations, and scaled voltage variables, obtained by the following procedure:

1. Use the problem equation(s) in the form where the highest derivative(s) of the dependent variable(s) is alone on the left hand side of the equation.
2. Multiply through the equation by the scale factor for the highest derivative (or the scale factor for the $(n-1)^{\text{th}}$ derivative, if the highest derivative is not required explicitly as an output voltage of a computing element). This converts each of the terms in the equation from units of the problem variable to a voltage.
3. On the right hand side of the equation, replace the problem variables by the scaled variables. In order to maintain equality in the equation, it is then necessary to multiply each term by the reciprocal of the scale factor. Each term now consists of two parts: a voltage and a coefficient.
4. Factor the coefficient into a potentiometer setting (a number less than one for all values of the parameters) and an amplifier gain. (The magnitude of the amplifier gains developed in this step are an indication of the possible need for time scaling the problem. This will be discussed below.)

Time Scaling: Whether or not time scaling is required in a problem is determined by several factors:

1. Limitation of dynamic response of computing elements.
2. Limitation of dynamic response of recording elements.
3. Length of time required for problem solution.

Time scaling can be reduced to a straight-forward procedure which will not affect the voltage scale factors previously established for the dependent variables of the problem.

To increase or decrease the time required for a phenomenon to occur, it is necessary simply to increase or decrease the rate at which the phenomenon takes place. Since the inputs to all integrators are rates, to increase or decrease the rate of problem solution on the computer we have only to increase or decrease the gain of the integrators.

Let us relate t , the time required for solution of the physical problem to τ , the time required on the machine, by $\tau = \beta t$. If a physical system takes ten seconds to reach steady state, a β equal to ten would result in the solution requiring 100 seconds on the computer. The relation between τ and t can be mechanized by having integrator inputs modified by the factor $1/\beta$. Thus, for β equal to 10, $1/\beta$ will be a number less than one, the rate to the integrators will be decreased, and the solution time on the computer will be increased. For β less than 1, $1/\beta$ will be a number greater than 1, and the solution time on the computer will be reduced accordingly, due to the increase in integrator gain (rate).

In the preliminary analysis of a problem, it is often possible to estimate frequencies (or time constants) that will occur in the system. Time scaling may be necessary if these frequencies are either less than 0.01 cycles per second or greater than 5 cycles per second. During programming and voltage scaling, the resulting amplifier gains are also an excellent indication as to whether or not time scaling is required. If large integrator gains are required, then it may be desirable to slow the problem solution by introducing large values for β . On the other hand, if small pot settings, say less than 0.01, are required throughout a problem, time scaling is usually appropriate, using a β less than 1.0 to increase problem solution speed, and to obtain suitable pot settings.

If time scaling is accomplished by changing the gain of integrators, it is not necessary to rescale the voltage scale factors. One simply has to note

on the computer diagram that β is other than one and to take this into consideration when evaluating response data as a function of the independent variable.

Problem Checking

Once the computer program has been completed, it is of utmost importance to ascertain that the final computer program is actually the model of the original problem equations. This can be accomplished as follows:

1. Select an arbitrary set of initial conditions for all the derivatives and variables on the right hand side of the equation. (Initial conditions selected should not result in zero terms.)
2. With the initial conditions chosen in Step (1) and appropriate values selected for the parameters, calculate the voltages that should exist at each point in the computer diagram (directly on the diagram) as well as the inputs to all integrators.
3. Substitute the same initial condition and parameter into the original problem equations and solve for the highest derivatives of the unknown. This should be related to the calculated voltage from Step (2) representing the highest derivative (input to integrators) by the following expression:

$$\frac{d^n x}{dt^n} \text{ (from equation)} = \left(\frac{\beta}{\text{scale factor for } \frac{d^n x}{dt^n}} \right) \left(\begin{array}{l} \text{voltage calculated} \\ \text{from computer diagram} \\ \text{for } \frac{d^n x}{dt^n} \end{array} \right) \quad (7)$$

This verifies that the computer diagram is the correct mechanization of the original equation.

When the problem is actually set up on the computer, the problem check initial condition and parameters are programmed as a static check run. The voltage existing at the input to the integrator (whose input is analogous to $d^n x/dt^n$) should equal the voltage calculation in Step (2) above. This final check verifies computer patching, pot settings, amplifier gains, and functioning of the nonlinear computing elements.

Partial Differential Equations

In general, the analog computer can handle differential equations with only one independent vari-

able. However, partial differential equation can be solved if they can be replaced by a set of simultaneous ordinary differential equations. This can be illustrated by considering Equation (8), describing the transverse vibration of a beam:

$$\rho \frac{\partial^2 y}{\partial t^2} = - \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) \quad (8)$$

The right hand side of the equation can be written as four first order derivatives with respect to x using the following relations:

$$S \text{ (shear force)} = \frac{\partial}{\partial x} \left(-EI \frac{\partial^2 y}{\partial x^2} \right) = \frac{\partial M}{\partial x}$$

$$M \text{ (bending moment)} = -EI \frac{\partial^2 y}{\partial x^2} = -EI \frac{\partial \theta}{\partial x}$$

$$\theta \text{ (slope)} = \frac{\partial y}{\partial x}$$

to give us

$$\rho \frac{\partial^2 y}{\partial t^2} = \frac{\partial S}{\partial x} \quad (9)$$

$$S = \frac{\partial M}{\partial x} \quad (10)$$

$$M = -EI \frac{\partial \theta}{\partial x} \quad (11)$$

$$\theta = \frac{\partial y}{\partial x} \quad (12)$$

Replacing the first order derivatives by a finite difference approximation will result in a set of N simultaneous ordinary differential equations of the form:

$$\rho \frac{d^2 y_i}{dt^2} = \frac{S_{i+1/2} - S_{i-1/2}}{\Delta x} \quad (13)$$

where

$$S_{i+1/2} = \frac{M_{i+1} - M_i}{\Delta x} \quad (14)$$

$$M_i = E_i I_i \left(\frac{\theta_{i+1/2} - \theta_{i-1/2}}{\Delta x} \right) \quad (15)$$

$$\theta_{i+1/2} = \frac{y_{i+1} - y_i}{\Delta x} \quad (16)$$

These equations can now be handled in the same manner as ordinary differential equations.

Illustrative Example

To illustrate the application of the analog computer to a specific problem let us analyze the motion of a setback leaf system which is used as a safety mechanism in a projectile fuse*.

Setback Leaf Analysis:

With optimum parameters, the setback leaf system shown in Figure 25 will always arm a projectile

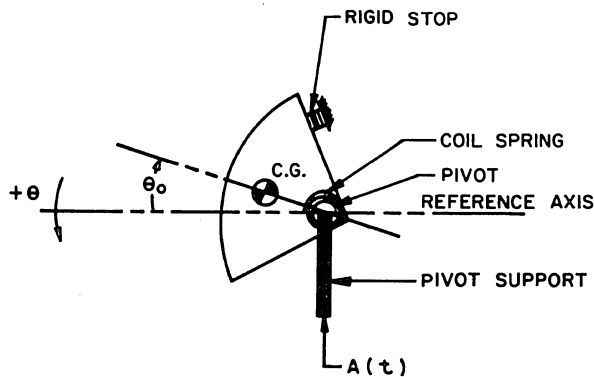


Figure 25. Setback Leaf Geometry

fuse when subjected to an acceleration of sufficient magnitude and duration. It will keep the fuse safe when subjected to short term accelerations that can occur when the fuse is accidentally or parachute dropped. In the system considered, three, interlocked, spring-restrained cams are rotated by the force due to acceleration. If Hooke's Law is assumed for the springs, and the coefficient of friction is assumed to have the form $k_1 e^{-C_5 \theta}$, then the general differential equation of motion for each cam becomes:

$$\ddot{\theta} = C_1 A(t) \left[\cos \theta - C_2 e^{-C_5 \theta} \right] - (C_3 + C_4 \theta) - C_5 e^{-C_5 \theta} \dot{\theta} \quad (17)$$

where

$$\begin{aligned} A(t) &= \text{driving force (Figure 28)} \\ &0 < A(t) < 1000 \\ C_1 A(t) \cos \theta &= \text{torque due to acceleration} \end{aligned}$$

* The data and information on this problem was given through the courtesy of the Diamond Ordnance Fuze Laboratory, National Bureau of Standards. The original analysis was performed by Mr. Arthur Hausner of D.O.F.L.

$$C_1 = \frac{ma}{I}$$

$$C_3 + C_4 \theta = \text{torque due to torsion spring}$$

$$C_3 = \frac{k_1}{I}$$

$$C_4 = \frac{k_2}{I}$$

$$C_5 e^{-C_5 \theta} \dot{\theta} = \text{torque due to leaf on leaf friction (anal. approx. to exp. data)}$$

$$C_2 e^{-C_5 \theta} \dot{\theta} = \text{bearing friction for spring vs. shaft and leaf}$$

Tabulation of Parameter Values

C_1	=	900
C_2	=	0.4875
C_3	=	100,000
C_4	=	40,000
C_5	=	25,000
C_6	=	0.0023

Tabulation of Initial Conditions and Variable Maxima

θ_0	=	-20°
θ_r	=	$+40^\circ$
$\dot{\theta}$	<	2000 rad/sec
$\ddot{\theta}$	<	10^6 rad/sec

Initial Condition

$$t = 0, \theta = \theta_0, \dot{\theta} = 0$$

Because of the interlock design, only one cam moves initially. When $\theta = \theta_r$, the next cam releases. The value of $A(t)$ at that time is an initial condition for the next cam. If $\ddot{\theta} < 0$ when $A(t)$ starts, the first leaf does not move; i.e. $\theta = \theta_0$ until $\ddot{\theta} > 0$.

Cams are analyzed separately until all three are analyzed or until one does not reach θ_r . In this case, the system does not operate.

Function Generation (Exponential Function): The function $e^{-C_5 \theta}$ can be generated analytically.

$$\text{Let } y = e^{-C_5 \theta}$$

Differentiating y with respect to t we obtain

$$\frac{dy}{dt} = -C_6 e^{-C_6 \dot{\theta}} \frac{d\dot{\theta}}{dt} \quad (18)$$

The value of y can be obtained by integrating $\frac{dy}{dt}$, which is obtained from the circuit outlined in Figure 26. Since the maximum of y occurs at $\dot{\theta} = 0$, we can scale to generate $100 e^{-C_6 \dot{\theta}}$.

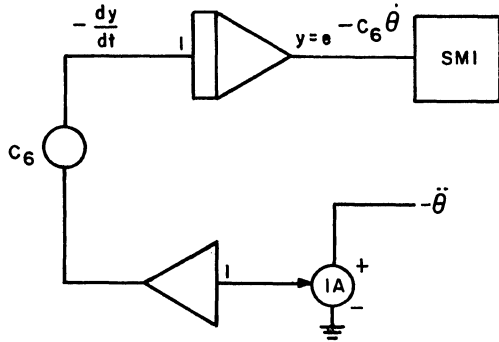


Figure 26. Generating the Function $e^{-C_6 \dot{\theta}}$

Acceleration Pulse: Rather than utilize a function generator, the acceleration pulse can be closely approximated by summing three exponential decay functions:

$$A(t) \approx \sum_{i=1}^3 A_i e^{-k_i t} \quad (19)$$

These functions are the solution of first order differential equations and can be mechanized as shown in Figure 27.

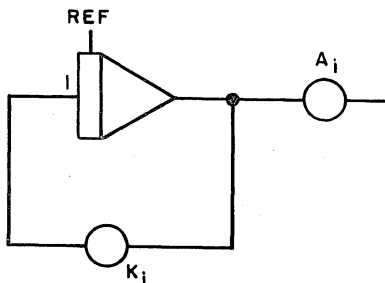


Figure 27. Generating AC + 7

Figure 28 compares the shape of the actual acceleration pulse with the computer generated function. The parameters α_i and K_i are obtained experimentally by means of the computer to provide the best fit to the function $A(t)$.

Cosine: The $\cos \theta$ can be produced by means of a servo resolver which will produce the cosine and sine of the input variable. The input to the resolver must be scaled so as not to exceed 90 volts for the maximum value of the input. An appropriate scale factor is $\left[\frac{\theta}{2}\right]$, allowing for angles up to 180° .

Scaling: On the basis of previous experience, estimates of the problem variables are available as shown in Table III.

TABLE III - SAMPLE PROBLEM - VOLTAGE SCALE FACTORS

VARIABLE	ESTIMATED MAXIMUM	SCALE FACTOR	COMPUTER VARIABLE
θ	1 rad	$\frac{100}{1} = 100 \text{ volts/rad}$	$[100 \theta]$
$\dot{\theta}$	2000 rad/sec	$\frac{100}{2000} = .05 \text{ volts/rad/sec}$	$[.05 \dot{\theta}]$
$\ddot{\theta}$	40,000 rad/sec ²	$\frac{100}{10^6} = 10^{-4} \text{ volts/rad/sec}^2$	$[10^{-4} \ddot{\theta}]$
$e^{-C_6 \dot{\theta}}$	1	$\frac{100}{1}$	$[100 e^{-C_6 \dot{\theta}}]$
$A(t)$	900	$\frac{100}{1000}$	$\left[\frac{A(t)}{10}\right]$
$\cos \dot{\theta}$	1	$\frac{100}{1}$	$[100 \theta]$

Since the duration of the acceleration pulse is of the order of 20-30 milliseconds, it is obvious that the problem will have to be time scaled. A factor of $\beta = 1000$ would be reasonable.

Scaled Voltage Equation: The scaled voltage equation was obtained using the procedures suggested above:

1. Multiply equation (14) by the scale factor for $\ddot{\theta}$

$$\begin{aligned} [10^{-4} \ddot{\theta}] &= 10^{-4} C_1 A(t) \left[\cos \theta - C_2 e^{-C_6 \dot{\theta}} \right] \\ &- 10^{-4} C_3 - 10^{-4} C_4 \theta - 10^{-4} C_5 e^{-C_6 \dot{\theta}} \end{aligned} \quad (20)$$

2. Replace remaining problem variables by computer voltages. Combine exponential functions to reduce the number of separate products. Factor of (100) is scale factor of multiplier.

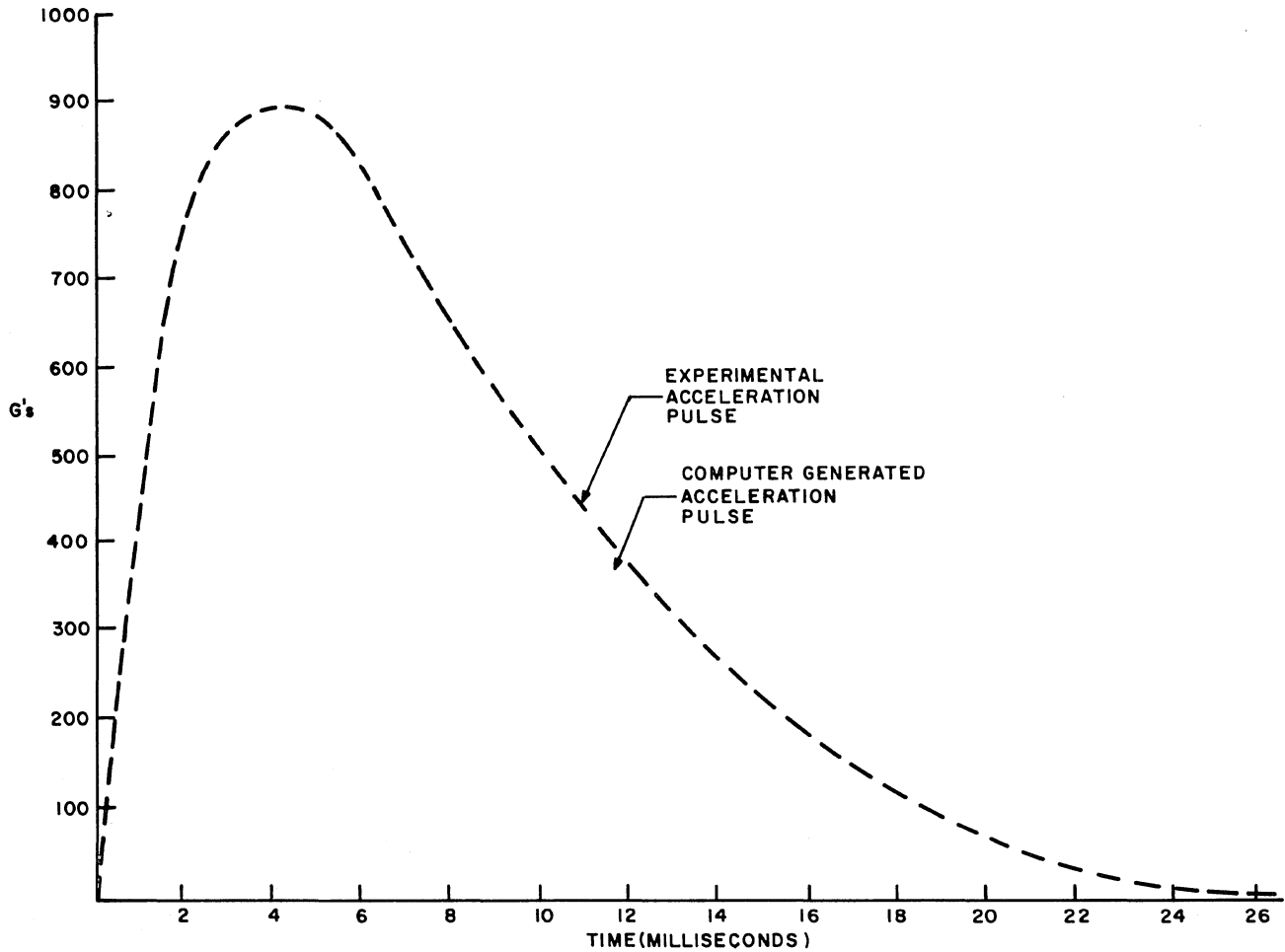


Figure 28. Acceleration Function

$$[10^{-4} \theta] = \frac{C_1}{10^3} \left[\frac{A(t)}{10} \right] [\cos \theta] - \frac{[100 e^{-C_6 \dot{\theta}}]}{(100)}$$

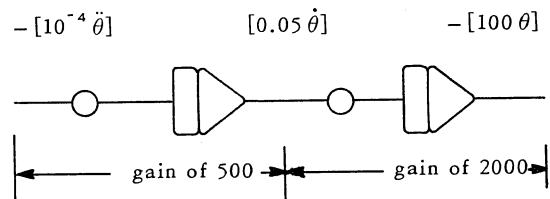
$$\left\{ \frac{C_5}{10^4} + \frac{C_1 C_2}{10^3} \left[\frac{A(t)}{10} \right] \right\} - \frac{C_3}{10^4} - \frac{C_4}{10^6} [100 \theta] \quad (21)$$

3. Factor the coefficients on the R.H.S. of the equation into a potentiometer setting and an amplifier gain (or reference voltage in the case of constants)

$$[10^{-4} \ddot{\theta}] = \frac{C_1}{10^3} \left[\frac{A(t)}{10} \right] [\cos \theta] - \frac{[100 e^{-C_6 \dot{\theta}}]}{(100)}$$

$$\left\{ 100 \frac{C_5}{10^6} + \frac{C_1 C_2}{10^3} \left[\frac{A(t)}{10} \right] \right\} - 100 \frac{C_3}{10^6} - \frac{C_4}{10^6} [100 \theta] \quad (22)$$

4. The integrator gains are established from the ratios of scaled values of θ , $\dot{\theta}$, and $\ddot{\theta}$



From this integrator gain it is also indicated that a time scaling of $\beta = 1000$ is required.

Static Check Calculation: Using the following values for the initial conditions and forcing functions, together with parameters assigned, we can calculate θ from our original equation:

$$A(t) = 39.80 \text{ g's}$$

$$\theta = -0.3500 \text{ radians}$$

$$\dot{\theta} = 2000 \text{ rad/sec}$$

$$e^{-C_6 \dot{\theta}} = 0.5000$$

$$\begin{aligned} \ddot{\theta} &= 900(-398)[+0.9400 - (0.4875)(0.5000)] - 100,000 \\ &\quad - (40,000)(-0.3500) - 25,000(0.5000) \\ &= -249,427 - 100,000 + 14,000 - 12,500 \\ &= -347,927 \text{ rad/sec}^2 \end{aligned}$$

Input to integrator whose output is $(0.1\dot{\theta})$ should be

$$\begin{aligned} -[10^{-4} \ddot{\theta}] \times \frac{500}{\beta} &= -10^{-4} (-347,927) \times \frac{500}{1000} \\ &= +17.39 \text{ volts} \end{aligned}$$

Equipment Assignment Sheets: Figure 29 is the final scaled computer diagram. Equipment assignment sheets are presented in Figures 30 and 31 showing potentiometer settings and voltages that should exist to verify computer patching.

*(See Amplifier Assignment Sheet—Amplifier 00)

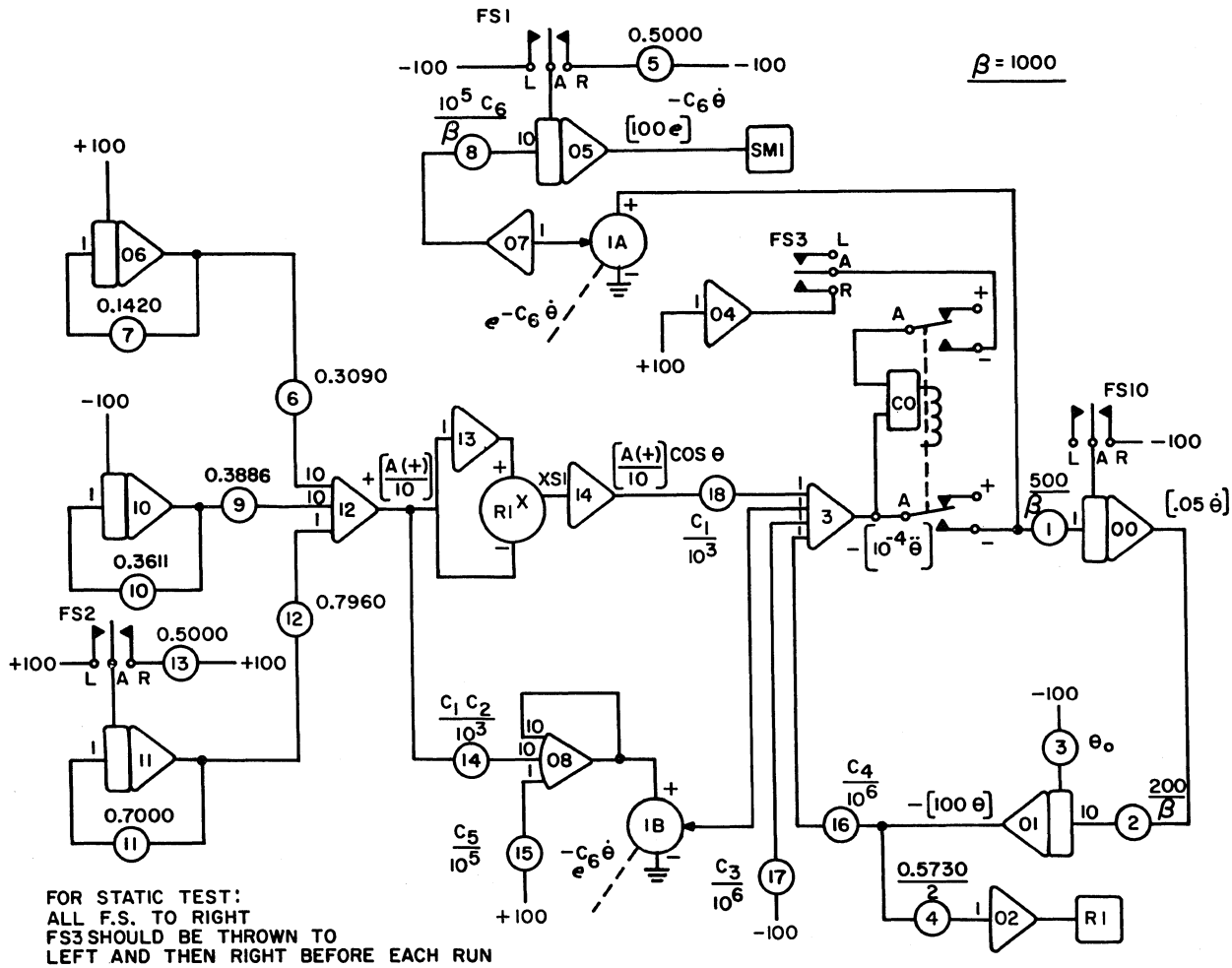


Figure 29. Computer Mechanization of Set Back Leaf System

AMPLIFIER ASSIGNMENT SHEET

PROBLEM Set Back Leaf Analysis
DATE 9-29-61

$\beta = 1000$

AMP NO.	FB	FUNCTION, AND/OR VARIABLE	STATIC CHECK		DYNAMIC CHECK		NOTES
			CHECK PT.	OUTPUT	CHECK PT.	OUTPUT	
0 0	I	$+ [.10 \dot{\theta}]$	+ 17.39	+100.00			
1	I	$- [100 \theta]$	+200.00	+ 35.00			
2	Inv	$[57.3 \frac{\theta}{2}]$		- 10.00			
3							
4	Inv			-100.00			
5	I	$+ [100 e^{-C_6 \dot{\theta}}]$	- 40.00	+ 50.00			
6	I	$- 100 e^{-\frac{142.0 t}{\beta}}$	- 14.20	-100.00			
7	Inv	$(10^{-4} \ddot{\theta}) (e^{-C_6 \dot{\theta}})$		- 17.38			
8							
9							
1 0	I	$+100 e^{-\frac{361.1 t}{\beta}}$	+ 36.11	+100.00			
1	I	$-100 e^{-\frac{700 t}{\beta}}$	- 35.00	- 50.00			
2	S	$+ \left[\frac{A(t)}{10} \right]$		- 39.80			
3	Inv	$- \left[\frac{A(t)}{10} \right]$		+ 39.80			
4	Inv	$\left(\frac{A(t)}{10} \right) (\cos \theta)$		- 37.40			
5							
6							
7	H.G.	$- e^{-C_6 \dot{\theta}} \left(\frac{C_1 C_2}{10^4} A(t) + \frac{C_5}{10^4} \right)$		+ 14.92			
8	S	$- [10^{-4} \theta]$		+ 34.76			
9							

Figure 30. Amplifier Assignment Sheet

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POTENTIOMETER ASSIGNMENT SHEET

PROBLEM Set Back Leaf Analysis
DATE 9-29-61

$\beta = 1000$

POT NO.	PARAMETER DESCRIPTION	SETTING STATIC CHECK	SETTING RUN NO.	SETTING RUN NO.	SETTING RUN NO.	NOTES	POT NO.
1	$500/\beta$	0.5000					
2	$200/\beta$	0.2000					
3	θ_0	0.3500					
4	$0.573/2$	0.2865					
5	Static Test I.C.	0.5000				Set with FSI to Right	
6		0.3090					
7	$142.0/\beta$	0.1420					
8	$10^5 C_6/S$	0.2300					
9	$388.6/\beta$	0.3886					
10	$361.1/\beta$	0.3611					
11	$700.0/\beta$	0.7000					
12		0.7960					
13	Static Test I.C.	0.5000				Set with FS2 to Right	
14	$C_1 C_2/10^3$	0.4388					
15	$C_5/10^5$	0.2500					
16	$C_4/10^6$	0.0400					
17	$C_3/10^6$	0.1000					
18	$C_1/10^3$	0.9000					
19							
20							
21							
22							
23							
24							

Figure 31. Potentiometer Assignment Sheet

Results

The results for θ vs. time are plotted in Figure 32. The magnitude of $A(t)$ at the instant of release ($\theta = 40^\circ$) would be used as the input to the second leaf to analyze its motion.

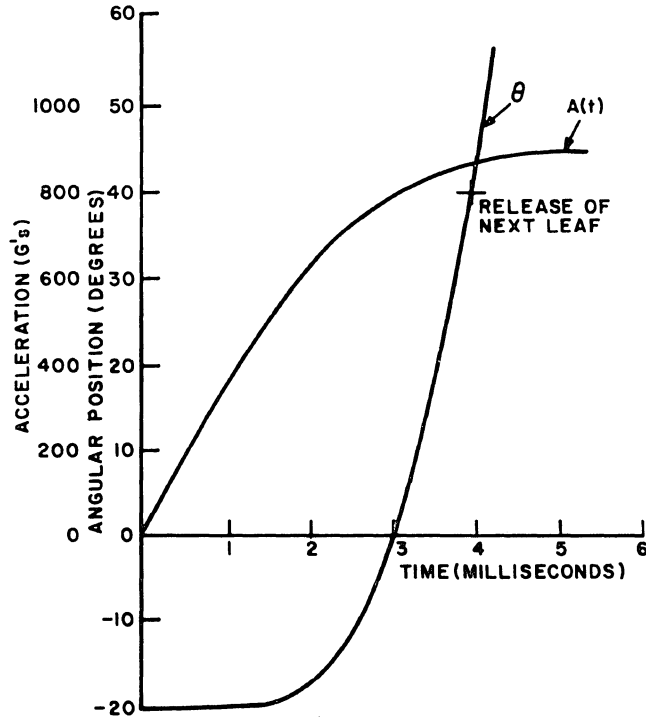


Figure 32.

It should be noted that this paper—touching only lightly on the high spots of analog computation—is, at best, but a brief survey of the field and one intended more to introduce than to edify. Several

good reference works are available, as indicated in the following list, which cover in detail both the operation and application of analog computers in engineering analysis and design work.

- (1) Rogers, A.E. and Connolly, T. W., Analog Computation in Engineering Design, McGraw-Hill Book Co., New York, NY, 1960
- (2) Lauher, V. A. and Williams, T. J., Demonstration Problems—Analog Computer Applications Course, Monsanto Chemical Co., St. Louis, Mo., 1959
- (3) Handbook of Analog Computation, in preparation by the Education and Training Group of Electronic Associates Inc., Princeton Computation Center, Princeton, New Jersey
- (4) Truitt, T. D. and Rogers, A. E., Basics of Analog Computers, J. F. Rider, New York, NY, 1960



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