SIMULATION OF THE RESTRICTED THREE-BODY PROBLEM USING THE TRICE-PB250 SYSTEM

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Section I. Summary

THIS REPORT describes a program for integrating the restricted three-body problem using the TRICE-PB250 system. Difficulties were overcome by the use of appropriate variables and programming techniques.

The accuracy of the computer program, together with the possibility of continuous visual output, resulted in a research tool of considerable flexibility and versatility.

Section II. Statement of the Problem

The restricted three-body problem describes the effect of two bodies of finite mass on the dynamic behavior of a body (particle) of infinitesimal mass. The two bodies of finite mass revolve around one another in circular orbits, and the particle of infinitesimal mass moves in their gravitational field. This situation is approximately realized, for example, in the motion of an artificial satellite in the Earth-Moon system. Figure 1 illustrates the geometry of the problem (the particle is assumed to move in the Earth-Moon plane).

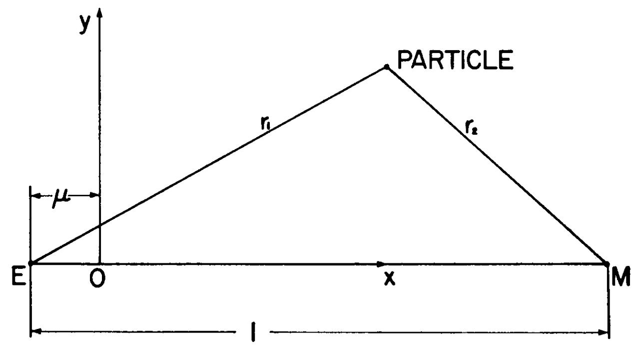


Figure 1. Geometric representation of the restricted three-body problem in a rotating coordinate system.

Section III. Equations of Motion

In a rotating coordinate system, the motion of the particle is described by the following equations ($\mu =$ the relative mass of the body of smaller mass):

$$\ddot{x} = x + 2\dot{y} - (1 - \mu)\frac{x + \mu}{r_1^3} - \mu\frac{x + \mu - 1}{r_2^3}$$

$$\ddot{y} = y - 2\dot{x} - (1 - \mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}$$

But major difficulties would be encountered if computation were attempted using the equations for the rotating coordinate system:

1. The integration step size would have to be changed frequently.
2. Study of trajectories that pass through a mass center would be impossible because these equations contain terms with r^3 in the denominator.
3. The position and velocity terms would vary greatly in range.
4. Scaling would be extremely difficult.

Therefore, at the suggestion of Dr. Arenstorf, staff mathematician of the Computation Division, a Thiele transformation was used to overcome many of the computational difficulties and also to provide a more flexible form of the equations. Use of the Thiele transformation provided a further advantage, in that broader studies of the problem were possible.

The Thiele transformation is accomplished by letting

$$x = {}^1/_2 - \mu + {}^1/_2 \cos u \cosh v$$

$$y = -{}^1/_2 \sin u \sinh v$$

The new independent variable is then defined by

$$s = \int_0^t \frac{dt}{r_1 r_2}$$

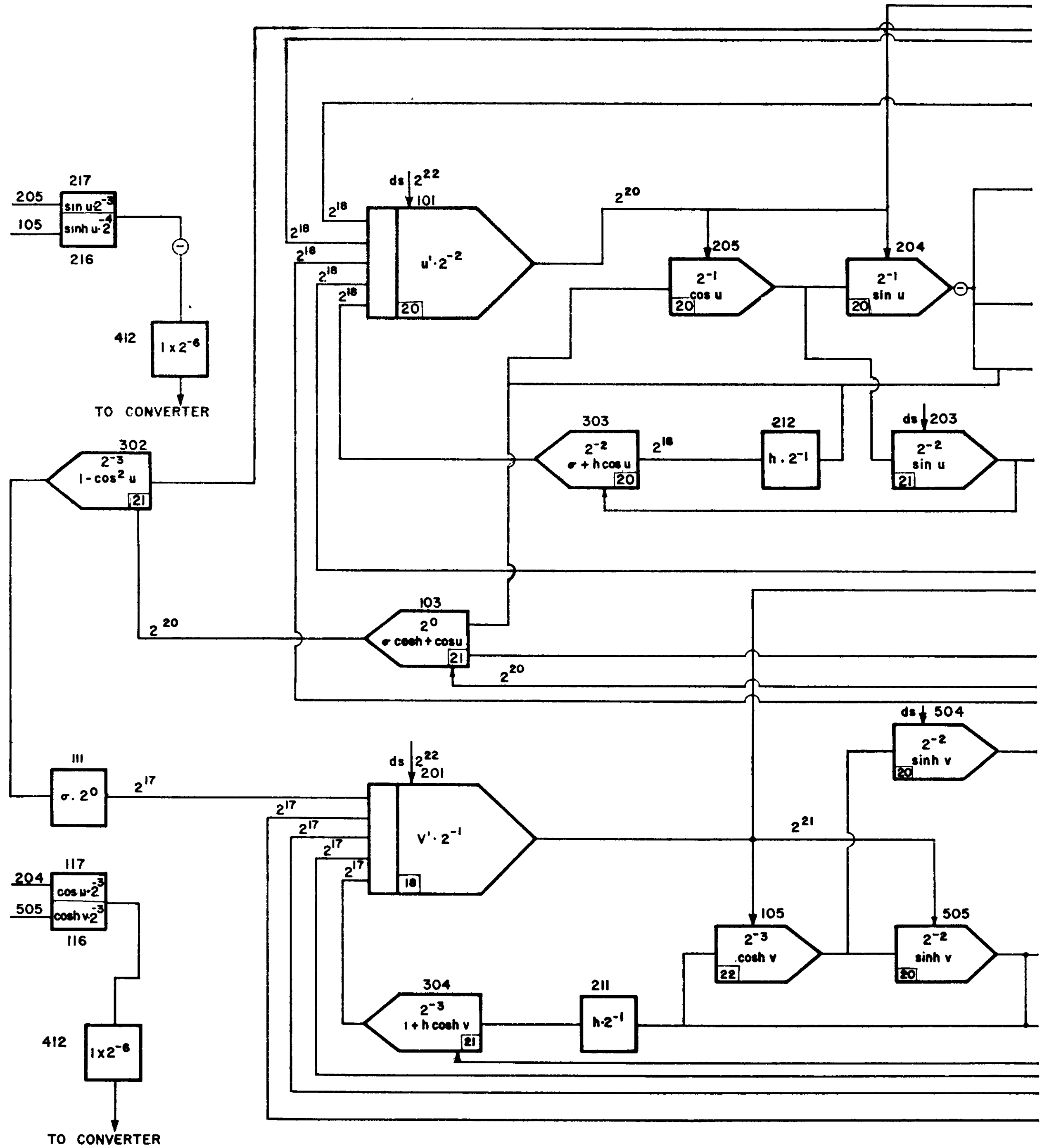


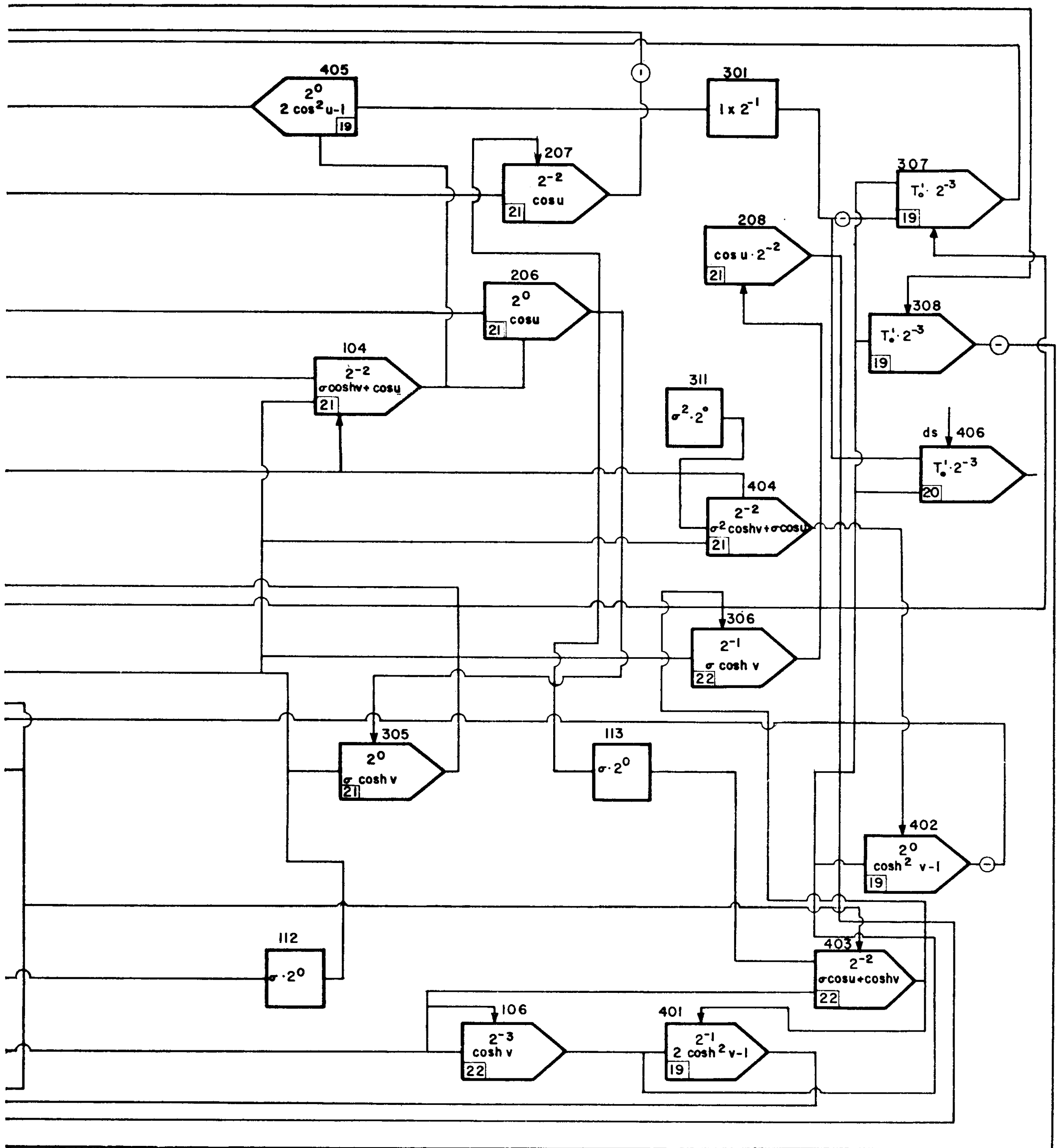
FIG. 2. TRICE PROGRAM for the restricted

Making these substitutions into the rotating form results in the following:*

$$u'' = {}^1/_2\,(\cosh^2 v - \cos^2 u)\, v' + \sin u \left[\frac{\sigma}{2} + \left(\frac{h}{2} + \frac{\sigma^2}{16}\right)\cos u - \frac{\sigma}{16}\cosh v\,(\cosh^2 v - 3\cos^2 u) + \frac{1}{16}\cos u \cos 2u\right]$$

$$v'' = -{}^1/_2\,(\cosh^2 v - \cos^2 u)\, u' + \sinh v \left[{}^1/_2\left(\frac{h}{2} + \frac{\sigma^2}{16}\right)\cosh v - \frac{\sigma}{16}\cos u\,(\cos^2 u - 3\cosh^2 v) + \frac{1}{16}\cosh v \cosh 2v\right]$$

*A prime (′) after a quantity denotes the first derivative of the quantity with respect to s; a double prime (″) denotes the second derivative with respect to s.



three-body problem in a rotating coordinate system.

where $\sigma = 1 - 2\mu$, and h (Jacobi's integral) is defined by the following equation:

$$(u')^2 + (v')^2 + \sigma \cos u - \cosh v - \frac{h}{2}(\cosh v - \cos u) - \frac{1}{16}\sin^2 \mu\,(\cos u + \cosh v)^2 - \frac{1}{16}\sinh^2 v\,(\sigma \cos u + \cosh v)^2 = 0$$

Via a third transformation, the motion of the particle was also described in a star-fixed coordinate system. The equation

$$t = \frac{1}{4}\int_0^s (\cosh^2 v - \cos^2 u)\, ds$$

was used to generate a new independent variable, and the following new equations were introduced:

$$z_1 = x \cos t - y \sin t \qquad z_2 = y \cos t + x \sin t$$

Section IV. The Computer Program

The configuration of TRICE elements that mechanize the Thiele form of the equations of motion is shown on Figure 2. The initial conditions for each trajectory were entered into the PB250 in terms of the rotating coordinate system variables μ, x_o, y_o, $\dot{x}_o$ and $\dot{y}_o$

The PB250, under control of an interpretive program language, computed a set of initial conditions that satisfied the Thiele form of the equations.

The following calculations were made:

1. $r_1 = \sqrt{(x_0 + \mu)^2 + y_0^2}$

2. $r_2 = \sqrt{(x_0 + \mu - 1)^2 + y_0^2}$

3. $\cosh v_0 = r_1 + r_2$

4. $\cos u_0 = r_1 - r_2$

5. $(\cosh v_0)^2 = (r_1 + r_2)^2$

6. $(\cos u_0)^2 = (r_1 - r_2)^2$

7. $T_0 = {}^1/_4 (\cosh^2 v_0 - \cos^2 \mu_0)$

8. $\sinh v_0 = \sqrt{\cosh v_0 - 1}$

9. $\sin u_0 = \sqrt{1 - \cos^2 u_0}$

10. $\sigma = 1 - 2\mu$

11. $\sigma^2 = (1 - 2\mu)^2$

12. $u_0' = {}^1/_2 (-\dot{y}_0 \cos u_0 \sinh v_0 + x_0 \sin u_0 \cosh v_0)$

13. $v_0' = {}^1/_2 (-\dot{y}_0 \sin u_0 \cosh v_0 + \dot{x}_0 \cos u_0 \sinh v_0)$

14. $$h = \frac{(u_0')^2 + (v_0')^2 + \sigma \cos u_0 - \cosh v_0}{{}^1/_2 (\cosh^2 v_0 - \cos^2 u_0)} - \frac{\frac{1}{16} (\sigma \cos u + \cosh v)^2 \sinh^2 v}{{}^1/_2 (\cosh^2 v_0 - \cos^2 u_0)} - \frac{\frac{1}{16} (\cos u_0 + \sigma \cosh v_0)^2 . \sin^2 u_0}{{}^1/_2 (\cosh^2 v_0 - \cos^2 u_0)}$$

15. $\sigma \cosh v_0$

16. $\sigma \cosh v_0 + \cos u_0$

17. $\sigma \cos u_0 + \cosh v_0$

18. $\sigma^2 \cos u_0 + \sigma \cosh v_0$

19. $2 \cos^2 u_0 - 1$

The results of these calculations were stored in the registers of the TRICE modules.

The major function of the TRICE program was to integrate the Thiele equations. Analog plots of the solution were made in the rotating coordinate system and in the star-fixed coordinate system.

The Thiele variables were transformed into the rotating coordinate system by TRICE modules generating the following functions:

$$x = {}^1/_2 - \mu + {}^1/_2 \cos u \cosh v$$

$$y = - {}^1/_2 \sin u \sinh v$$

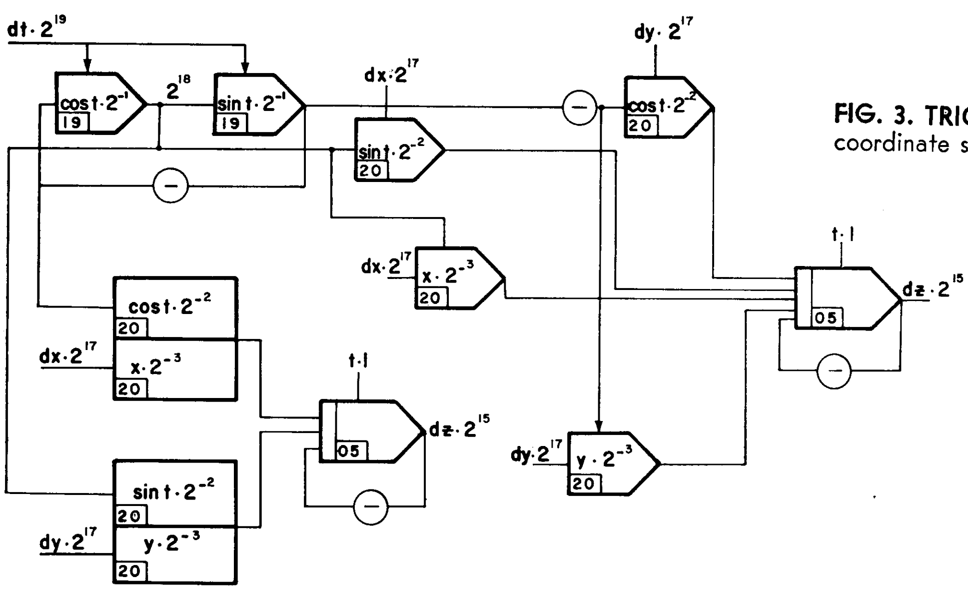


FIG. 3. TRICE PROGRAM for the star-fixed coordinate system.

The TRICE program for the star fixed transformation is shown in Figure 3.

Section V. Conclusions

A. ADVANTAGES OF THE THIELE FORM

Certain advantages of the Thiele form of the equations of motion are worth noting:

1. The choice of the independent variable

$$s\left(s = \int_0^t \frac{dt}{r_1\, r_2}\right)$$

eliminated the need to vary the integration step size as the trajectory passed through different regions of space. An equivalent effect was accomplished continuously during the solution as a result of the mathematical nature of the Thiele form. This effect can be clearly observed when trajectories approach either mass. Motion of the particle becomes very slow during the approach and increases in speed during departure from the region of a mass center.

2. As was noted earlier, because of terms containing $1/r^3$, the study of trajectories that pass through a mass center would be impossible using the equations for the rotating coordinate system. These terms do not appear in the Thiele form.

3. The terms in the Thiele equations are better behaved than the position and velocity terms in the rotating set. The sinusoidal terms are bounded between +1 and −1 for all real arguments. The hyperbolic functions, in contrast to the position and velocity terms in the rotating set, do not vary greatly in range.

4. The behavior of the new Thiele variables eliminated many scaling problems.

B. ADVANTAGES OF THE TRICE-PB250 SYSTEM

Use of the TRICE-PB250 system, which combines advantages of both digital and analog computers, contributed to an effective and efficient study of the three-body problem. Some of the advantages of the system include:

1. Parallel integration without the problem of drift
2. Generation of hyperbolic functions without stability problems
3. Continuous visual output provided by the converter and plotter
4. High accuracy and repeatability (6 decimal digits)
5. Automation provided by the PB250.

The orbits shown on Figures 4 through 6 were obtained using the TRICE-PB250 system. The study required that the initial conditions be adjusted until the trajectory became periodic. Then, using the visual output provided by the plotter, a human operator determined new initial conditions on the basis of symmetry and orthogonality (a periodic orbit is symmetrical about the horizontal axis and makes two orthogonal crossings). An automatic iteration process for finding proper initial conditions appeared too complex for expression in a useable mathematical form. Because of the relatively low cost and the ready availability of the system, the operator was able to

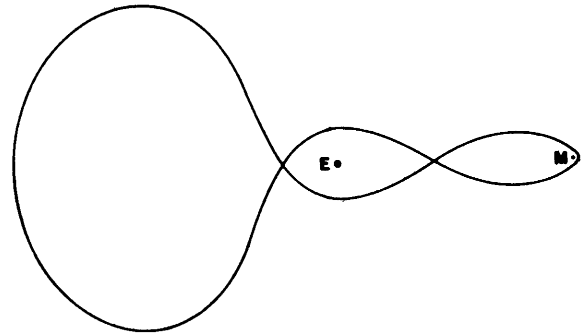


FIG. 4. TWO-LOOP periodic orbit computed using the TRICE-PB250, with initial conditions: $h = -1.197$; $x_0 = 0.994$; $y_0 = -2.114$; $\mu = 0.1227747$; $\frac{m}{k} = \frac{1}{2}$; $T = 5.44$.

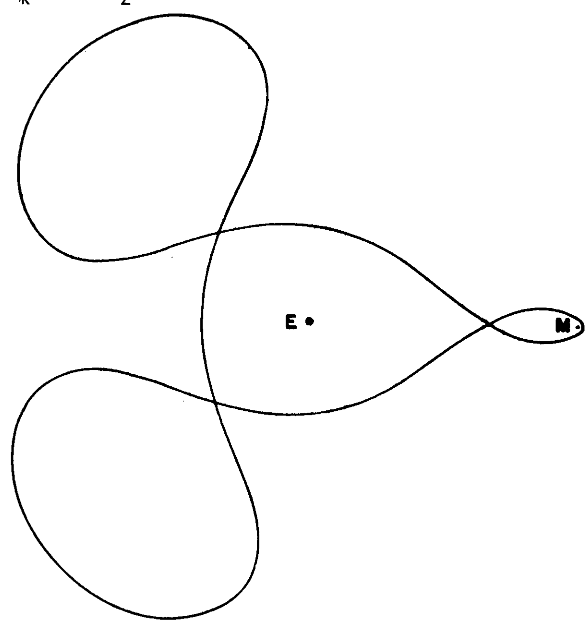


FIG. 5. THREE-LOOP periodic orbit computed using the TRICE-PB250, with initial conditions: $h = -1.367$; $x_0 = 0.994$; $y_0 = -2.0318$; $\mu = 0.12277471$; $m/k = 2/3$; $T = 11.12$.

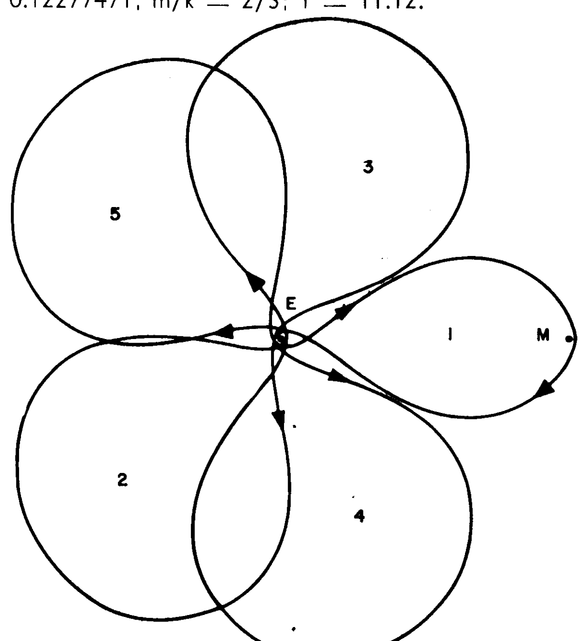


FIG. 6. FIVE-LOOP periodic orbit computed using the TRICE-PB250, with initial condition: $h = -0.45076$; $x_0 = 0.9927$; $y_0 = -2.64215$; $\mu = 0.122775$; $m/k = 3/7$ retrograde.

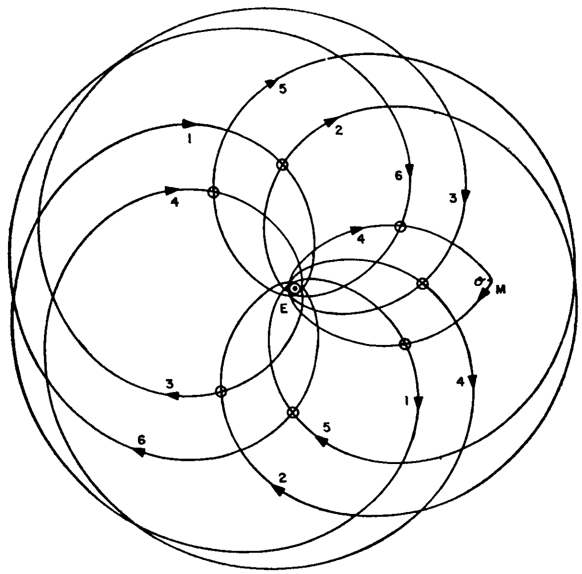


FIG. 7. SEVEN-LOOP periodic orbit using TRICE-PB250 with initial conditions: $x_0 = 1.01$; $\dot{x}_0 = 0$; $y_0 = 0$; $\dot{y}_0 = -1.369$; $\mu = 1/82$.

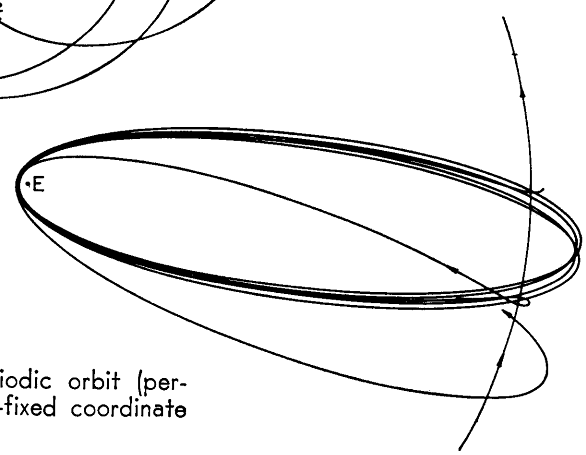


FIG. 8. FIVE-LOOP periodic orbit (perturbed ellipses) in a star-fixed coordinate system.

make the detailed studies required for this type of orbit.

The orbit shown on Figure 7 is a solution plotted in the rotating coordinate system. Figure 8 shows the trajectory as it appears in the star-fixed coordinate system.

Running times for a periodic solution depended on the initial conditions. Running times as long as 30 minutes were required for very-low-energy orbits, and as short as one minute for very-high-energy orbits.

Results to date have been of significant value as guidelines for further theoretical studies.

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