UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE M 2 - 118

TITLE	Automatic Inversion of a Symmetric Matrix
TYPE	Entire program
ACCURACY	Depends on the condition of the matrix to be inverted
DURATION	See text

METHOD OF USE

The code is read into the memory in the usual way. After a short time, Illiac will stop. Read in the elements of the matrix and parameters. Replace original tape in reader and restart computer. Since A is symmetric, its inverse is symmetric, and it will be necessary to print only half the off-diagonal elements. The first number that appears is the row number. This is followed by the diagonal elements and the upper off-diagonal elements. The inverse of the matrix should be scaled by the same factor by which the original has been scaled. A sample of the final result follows:

```
3
      7495.85849
-    14175.08696
     18455.76247
-    15562.46767
```

Since the original matrix was scaled down by ten, the above elements must be scaled down by ten.

PUNCHING OF DATA

The elements should be scaled such that the sum of the squares of the matrix elements is less than one fourth. Since A is a symmetric matrix, it is necessary to store only half the off-diagonal elements. The lower off-diagonal elements and diagonal elements are punched, row by row, as a sign followed by up to 12 decimal digits. The final fraction in the sequence is terminated by a J. This is followed by fF where f determines the number of places of the fraction desired to be printed. Finally, nN appears on the tape where n is the order of the matrix. The data tape should appear as follows:

$$a_{00}$$

$$a_{10}\ , a_{11}$$

$$a_{n-1,\ 0}\ , \ldots, a_{n-1,\ n-1} \qquad \text{JfFnN}$$

DESCRIPTION

The symmetric matrix A is diagonalized by a series of orthogonal transformations. The final matrix product can be written

$$D = B^T AB.$$

Then

$$A = B^{-T} DB^{-1} \quad (BB^T = I)$$

$$= BDB^T$$

and

$$A^{-1} = B^{-T} D^{-1} B^{-1}$$

$$= B D^{-1} B^T .$$

Since D is a diagonal matrix, the inverse has for elements the reciprocals of the diagonal elements of D. After Routine 42 finds the matrices B and D, this routine inverts D, performs the necessary matrix multiplication, and then prints out the result. This routine is essentially a combination of routines X 1, 21, 42, P 2, P 4, N 2. It is also possible to invert an asymmetric matrix using this routine. A matrix C can be made symmetric by multiplying it by its transpose. Thus

$$CC^T = A \text{ , where A is symmetric.}$$

Then

$$A^{-1} = C^{-T} C^{-1} \tag{1}$$

and

$$C^{-1} = C^T A^{-1} . \tag{2}$$

Operations (1) and (2) can be performed by use of Routine M 1 - 91.

LENGTH OF TIME

The length of time of the program, the accuracy, and the maximum matrix allowed to be inverted are essentially dependent on Routine 42. The maximum matrix that can be handled is of order 23. Routine 42 takes approximately $5n^3 \times 10^{-3}$ seconds to perform one iteration. The number of iterations varies directly with the order of the matrix, from about four iterations for a third order matrix to 7 for a twenty-third order matrix. To perform its various operations and print out the results, this routine requires approximately $4n (n + 1) (3n + 63 + 5f)$ milliseconds where n is the order of the matrix and f is the number of places the fraction is printed. It was found that slightly more than 14 minutes were necessary to invert a matrix of order 23 and print out the result.

CONDITION OF THE MATRIX

The accuracy of the result will naturally depend on the conditioning. The conditioning of the matrix is defined here as the ratio between the smallest and largest eigenvalues. As this ratio becomes very small, the accuracy becomes very poor. This ratio is printed as the last number and is followed by an N. The number of figures of accuracy will be no greater than 11 minus the number of zeros that appear in the rounded ratio.

DATE	November 11, 1953 rt. 6/16/55
CODED BY	Gene H. Golub
APPROVED BY	J.P. Nash

LOCATION	ORDER		NOTES
Decimal Order Input - X 1 - 18			
	00 30K		
Input a Sequence of Decimal Fractions - N 2 - 88			
	00 10K		
0	40 192F		
	50 L		
1	24 30F		Store fractions
	41 F		
2	81 4F		Bring in parameters and convert
	L0 11L		
3	32 6L		
	L4 11L		
4	50 F		
	74 11L		
5	S5 F		
	40 F		
6	26 2L		
	L5 F		
7	40 3F		
	L5 7L		
8	L4 5L		
	46 7L		
9	L1 L		
	40 L		
10	34 999F		
	22 1L		
11	00 F		
	00 10F		
	26 10N		
	00 10K		
0	00 F		
	00 192F		
1	80 F		
	00 F		
2	L5 3F		
	40 65F		

LOCATION	ORDER	NOTES
3	L5 4F	
	40 66F	
4	50 4F	
	74 4F	
5	S5 F	
	10 1F	
6	L4 L	
	40 7F	Form 192 + n(n+1) / 2
7	40 67F	
	42 10L	
8	L5 4F	
	L4 4F	
9	40 5F	
	41 F	
10	L5 1L	Generate unit matrix
	40 ( )F	
11	F5 F	
	40 F	
12	L0 4F	
	36 999F	
13	L5 10L	
	F4 4F	
14	40 10L	
	00 30F	Read around delay
15	26 10L	
	26 2L	
	26 1N	
	00 13K	
0	00 63F	
	26 190F	
	00 52K	

Square Root Routine - 21

LOCATION	ORDER	NOTES
	00 22K	
22	00 1F	
	00 1F	
23	00 S4	
	00 S4	
24	00 192F	
	00 192F	
25	80 S5	
	00 S5	
26	20 F	
	00 F	
27	00 F 00 0100	
	0000 0010 J	
28	80 F	
	00 F	
29	00 F	
	00 F	
30	J0 S7	
	74 S7	
	00 76K	

Eigenvalue Routine -- 42

Change word 102 to read

	36 999F	
	40 1F	

Change word 31 to read

	36 63F	
	L5 6F	

Change word 44 to read

	26 69F	
	50 120F	
	00 69K	

LOCATION	ORDER	NOTES
69	L5 10F	
	L0 75F	
70	32 71F	
	LJ 10F	
71	22 120F	
	L5 75F	
72	40 10F	These orders are necessary to avoid a division hang up
	L5 74F	
73	40 2F	
	26 122F	
74	7L 4095F	
	LL 4095F	
75	3L 4095F	
	LL 4095F	
	00 63K	
63	L3 6F	
	32 64F	
64	26 111F	
	27 113F	
	26 76N	
	00 82K	

Integer Print -- Routine P 4 - 55

Change word 22 to read

	82 4F	Decimal point will now be printed out
	92 643F	
	00 110K	

Fraction Print -- Routine P 11 - 148

Interlude II

	00 10K	
0	L5 65F	
	40 3F	

LOCATION	ORDER		NOTES
1	L5 17L		
	L4 66F		Bring out parameters
2	40 4F		
	L5 67F		
3	40 5F		
	F4 20L		
4	40 20L		
	L7 (194)F		
5	L2 (192)F		Find smallest eigenvalue
	32 7L		
6	L5 4L		
	00 20F		
7	46 5L		
	L5 20L		
8	F4 4L		
	42 4L		
9	L5 20L		
	L0 4F		
10	32 3L		
	19 38F		
11	40 6F		
	F5 6F		
12	40 6F		
	L5 5L		
13	46 15L		
	50 18L		
14	7J 19L		
	40 18L		
15	L2 ( )F	By 13	Find power of ten smaller than smallest eigenvalue
	32 11L		
16	26 999F		
	0F F		
17	80 F		
	00 F		

LOCATION	ORDER		NOTES
18	00 F 00 1000		
	0000 0000 J		
19	00 F 00 1000		
	0000 0000 J		
20	00 F		
	00 1F		
	26 10N		
	00 10K		
0	41 F		
	50 F		
1	L5 (192)F	By 4	
	40 (192)F		Restore diagonal elements λ_i consecutively
2	F5 F		
	40 F		
3	K4 F		
	00 20F		
4	F4 1L		
	40 1L		
5	L5 F		
	L0 4F		
6	36 1L		
	L5 69L		
7	40 F		
	L7 (193)F	By 11, from 13	Find smallest λ_i
8	L2 (192)F	By 10	
	32 10L		
9	L5 7L		
	00 20F		
10	46 8L		
	F5 7L	From 8	
11	42 7L		
	F5 F		
12	40 F		
	L0 4F		

LOCATION	ORDER		NOTES
13	32 7L		
	41 F		
14	L5 8L		
	46 18L		
15	L7 70L	From 18	
	10 1F		
16	40 70L		
	F5 45L		
17	42 45L		
	L7 70L		Find 2^{-n} smaller than smallest λ_i
18	L2 ( )F	By 14	
	36 15L		
19	L7 70L	From 24	
	66 (192)F	By 21	$2^{-n} / \lambda_i = d_i$
20	S5 130F		
	40 (192)F	By 22	
21	F5 19L		
	42 19L		
22	42 20L		
	F5 F		
23	40 F		
	L0 4F		
24	36 19L		
	41 6F		
25	41 9F	From 67	
	L5 6F		Print row number
26	J0 2F		
	50 26L		
27	26 82F		
	92 131F		
28	50 (192)F	By 30, from 34	
	7J (S5)	By 30	
29	40 (130)F	By 32, by 63	Evaluate $d_i\ a_{ki}$ $(i = 1, \ldots, n-1)$
	19 18F		
30	F4 28L		
	40 28L		

LOCATION	ORDER	NOTES	
31	19 18F		
	L4 29L		
32	40 29L		
	F5 9F		
33	40 9F		
	L0 4F		
34	36 28L		
	L5 6F		
35	40 7F		
	41 F	From 60	
36	41 1F		
	41 8F		
37	50 (130)F	By 42, by 58, from 44	
	7J (S5)	By 42, by 37	
38	50 1F		Perform matrix multiplication
	74 69L		
39	L4 F		
	40 F		
40	S5 192F		
	40 1F		
41	19 18F		
	F4 37L		
42	40 37L		
	F5 8F		
43	40 8F		
	L0 4F		Count
44	36 37L		
	L5 F		
45	50 1F		
	00 ( )F	By 17	Scale up
46	40 10F		
	S5 F		
47	40 11F		
	L5 10F		
48	36 53L		
	92 706F		Print negative sign

LOCATION	ORDER		NOTES
49	L3 11F		
	36 53L		
50	F5 10F		
	40 10F		
51	L5 11F		
	L4 68L		
52	40 11F		
	22 53L		
53	92 963F	From 48	
	L5 10F	From 49	Print integer
54	J0 S6		
	50 54L		
55	26 82F		
	L5 11F		Print fraction
56	J0 S3		
	50 56L		
57	26 110F		
	92 131F		
58	L5 20L		
	46 37L		
59	F5 7F		
	40 7F		
60	L0 4F		
	32 35L		Count
61	92 139F		
	L5 40L		
62	46 28L		
	L5 20L		
63	46 29L		
	L5 5F		
64	L4 4F		
	40 5F		
65	42 37L		
	F5 6F		
66	40 6F		
	L0 4F		

LOCATION	ORDER	NOTES	
67	36 25L		Count
	26 160F		
68	80 F		
	00 F		
69	00 F		
	00 1F		
70	80 F		
	00 F		
	00 160K		
0	L5 79F		
	40 F		
1	L7 (192)F	By 5, from 7	
	L2 (193)F	By 3	
2	36 4L		Test for smallest d_i ($d_i = k/\lambda_i$)
	F5 1L		
3	42 1L		
	26 6L		
4	L5 1L		
	00 20F		
5	46 1L		
	22 2L		
6	F5 F	From 3	
	40 F		
7	L0 4F		Count
	36 1L		
8	L5 18F		Bring out address of largest d_i
	10 20F		
9	42 14L		
	42 11L		
10	L5 1L		
	46 12L		
11	46 14L		
	L7 ( )F	By 9	
12	L2 ( )F	By 10	
	40 F		

LOCATION	ORDER		NOTES
13	L3 F		Test if $\lambda_S = \lambda_L$
	36 17L		
14	L5 ( )F	By 11	
	66 ( )F	By 9	
15	S5 11F		
	50 15L		Print out ratio
16	26 110F		
	22 17L		
17	92 66F	From 13	
	92 770F	From 16	Print out N
18	92 979F		Spaces
	OF F		
	26 10N		