UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

AUXILIARY
LIBRARY ROUTINE E 10 - 310

TITLE: Evaluation of Exponentially Weighted Semi-Infinite Integrals by Quadrature (Laguerre Quadrature)

TYPE: Closed subroutine, with one program parameter

NUMBER OF WORDS: 18 + 2N (see below)

DURATION: N(1.8 + T) milliseconds, where T is the duration in milliseconds of the auxiliary subroutine.

TEMPORARY STORAGE: Location 0 (may be used by auxiliary subroutine)

ENTRY: When this routine is located at y, entry is made by the orders:

p	-- aF
	50 pF
p + 1	26 yF

,

where a is the location of the auxiliary subroutine which computes the values of the function to be integrated. When control is returned to the right side of p + 1, the computed integral will be in the accumulator register and location y + 14.

DESCRIPTION: To evaluate the integral

$$\int_0^{\infty} e^{-x} f(x)\, dx,$$

this routine uses a form of Gaussian Quadrature appropriate to the interval $(0,\infty)$ and the weighting function e^{-x}:

$$\int_0^{\infty} e^{-x} f(x)\, dx \approx \frac{1}{2^p} \sum_{k=1}^{N} A_k f(x_k). \qquad (1)$$

The values A_k and x_k are chosen in a manner such as to give no truncation error when $f(x)$ is a polynomial of degree 2N -1 or less. In the case where the factor e^{-x} does not occur explicitly in the integrand,

$$\int_0^{\infty} g(x)\,dx = \int_0^{\infty} e^{-x}[e^{x}\,g(r)]\,dx \approx \frac{1}{2^Q}\sum_{k=1}^{N} A_k\, e^{x_k}\, g(x_k)$$

$$= \frac{1}{2^Q}\sum_{k=1}^{N} B_k\, g(x_k). \qquad (2)$$

It is assumed that the function $e^x\,g(x)$ may be closely approximated by a polynomial function.

Because the actual values of the points X_k and the weights A_k and B_k may exceed 1, they have been scaled down by powers of two. P and Q are defined in equations (1) and (2), and R is defined below.

N	R	P(for A_k)	Q(for B_k)
1	1	1	2
2	2	0	3
3	3	0	3
4	4	0	3
5	4	0	3
6	4	0	3
7	5	0	4
8	5	0	4
9	5	-	4
10	5	-	4
11	6	-	4
12	6	-	4
13	6	-	4
14	6	-	4
15	6	-	4

The auxiliary subroutine which computes $f(x_k)$ must take the scaling of these values of x_k into account. The function values computed by the auxiliary are assumed to lie in the range $-1 \leq f(x_k) < 1$.

The closed auxiliary subroutine is entered from the main routine with x_k^* in the accumulator and link in Q; control is returned to the main routine with $f(x_k)$ in the accumulator.

USE: To use this routine, the programmer copies the integration routine first on his program tape, and immediately after it the parameters, points x_k, and weights A_k or B_k appropriate to his needs. These latter numbers appear on the tail of the library tape, labeled by the number N of points at which the function is to be evaluated, and the type of weights (A_k or B_k) to be used.

SCALING: The scaling of the values of x_k is such that the auxiliary subroutine is presented with x_k^*, where

$$x_k^* = 2^{-R} x_k, \; 1 \leq R \leq 6,$$

and the largest x_k^* satisfies $1/2 \leq (x_k^*) \max < 1$ for all N. The computed integral is scaled down by 2^P (or 2^Q).

The A_k for N = 9 to 15 are not included because they decrease rapidly with increasing k and N, and become too small to be held in a single Illiac register.

For the convenience of the programmer, the above scale factors are contained in the subroutine parameter at location y+16, in the following form:

(y+16) = OR (y+18) OP (y+18+N),	for A weights
= OR (y+18) OQ (y+18+N)	for B weights.

ACCURACY: The truncation error due to omission of powers of x higher than 2N is

$$\frac{(n!)^2 f^{(2n)}(z)}{(2n)!}, \quad \text{where z is some point in } (0, \infty).$$

There will also be round-off errors which may be significant; for a discussion of these see the write-up of library routine E 5 - 195.

REFERENCES: H. E. Salzer, in Bulletin Am. Math. Soc., vol. 55, no. 10, p. 1004 (October, 1949) (also in Natl. Bur. of Standards AMS 37)

G. Szegö, Orthogonal Polynomials.

F. G. Tricomi, Vorlesungen über Orthogonalreihen

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LOCATION	ORDER	NOTES PAGE 1 E 10
	00K	
0	L5 16L	Preset point and weight addresses
	46 5L	
1	42 7L	
	K5 F	
2	46 6L	Plant subroutine entry and link
	42 13L	
3	41 14L	clear sum box
	2S 4L	
4	S5 F	Save L.S.P. of sum
	40 15L	
5	L5 ( )F	get x_k; link to Q
	50 5L	
6	26 ( )F	jump to auxiliary
	40 F	
7	L5 15L	L.S.P. to A, weight in Q
	50 ( )F	
8	74 F	x $f(x_k)$, accumulate M.S.P.
	L4 14L	
9	40 14L	
	L5 8L	
10	L4 5L	step x_k address
	46 5L	
11	F5 7L	step A_k address
	42 7L	
12	L0 17L	
	36 4L	
13	L5 14L	exit via link
	22 ( )F	
14	00 F	temporary store for M.S.P.
	00 F	
15	00 F	" " " L.S.P.
	00 F	
16	0R 18L	
	0P [18+N]L	Ceofficient addresses and scale constants

LOCATION	ORDER	NOTES PAGE 2 E 10
17	75 15L	end constant
	50 [18 + 2N]L	
18	x_1	
.	.	
.	.	
.	.	
17+N	x_N	
18+N	A_1 or B_1	
.	.	
.	.	
.	.	
17+2N	A_N or B_N	