Signals, Systems, and Society

By:
Carlos E. Davila

Signals, Systems, and Society

By:

Carlos E. Davila

C O N N E X I O N S

Rice University, Houston, Texas

Table of Contents

Chapter 1

Introduction to Continuous-Time Signals

1.1 Continuous-Time Signals: Introduction[1]

The Merrian-Webster dictionary defines a signal as:

> A detectable physical quantity or impulse (as a voltage, current, or magnetic field strength) by
> which messages or information can be transmitted.

These are the types of signals which will be of interest in this book. Indeed, signals are not only the means by which we perceive the world around us, they also enable individuals to communicate with one another on a massive scale. So while our primary emphasis in this book will be on the theoretical foundations of signal processing, we will also try to give examples of the tremendous impact that signals and systems have on society. We will focus on two broad classes of signals, *discrete-time* and *continuous-time*. We will consider discrete-time signals later on in this book. For now, we will focus our attention on continuous-time signals. Fortunately, continuous-time signals have a very convenient mathematical representation. We represent a continuous-time signal as a function $x(t)$ of the real variable t. Here, t represents *continuous time* and we can assign to t any unit of time we deem appropriate (seconds, hours, years, etc.). We do not have to make any particular assumptions about $x(t)$ such as *boundedness* (a signal is bounded if it has a finite value). Some of the signals we will work with are in fact, not bounded (i.e. they take on an infinite value). However most of the continuous-time signals we will deal with in the real world are bounded.

---

[1]This content is available online at <http://cnx.org/content/m32862/1.2/>.

1
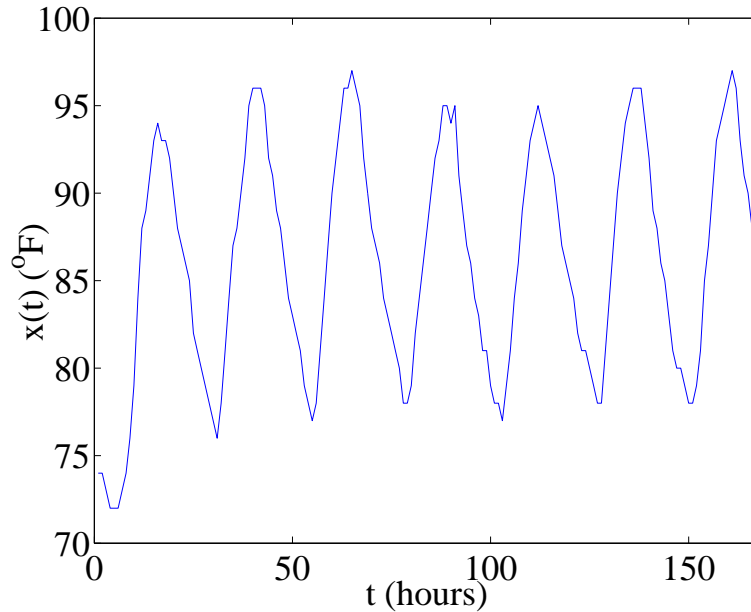
Figure 1.1: Temperature signal recorded in Dallas, Texas from Aug. 16 to Aug. 22, 2002.

We actually encounter signals every day. Suppose we sketch a graph of the temperature outside the Jerry Junkins Electrical Engineering Building on the SMU campus as a function of time. The graph might look something like in Figure 1.1. This is an example of a signal which represents the physical quantity temperature as it changes with time during the course of a week. Figure 1.2 shows another common signal, the *speech* signal. Human speech signals are often measured by converting sound (pressure) waves into an electrical potential using a microphone. The speech signal therefore corresponds to the air pressure measured at the point in space where the microphone was located when the speech was recorded. The large deviations which the speech signal undergoes corresponds to vowel sounds such as "ahhh" or "eeeeh" (voiced sounds) while the smaller portions correspond to sounds such as "th" or "sh" (unvoiced sounds). In Figure 1.3, we see yet another signal called an electrocardiogram (EKG). The EKG is a voltage which is generated by the heart and measured by subtracting the voltage recorded from two points on the human body as seen in Figure 1.4. Since the heart generates very low-level voltages, the difference signal must be amplified by a high-gain amplifier.
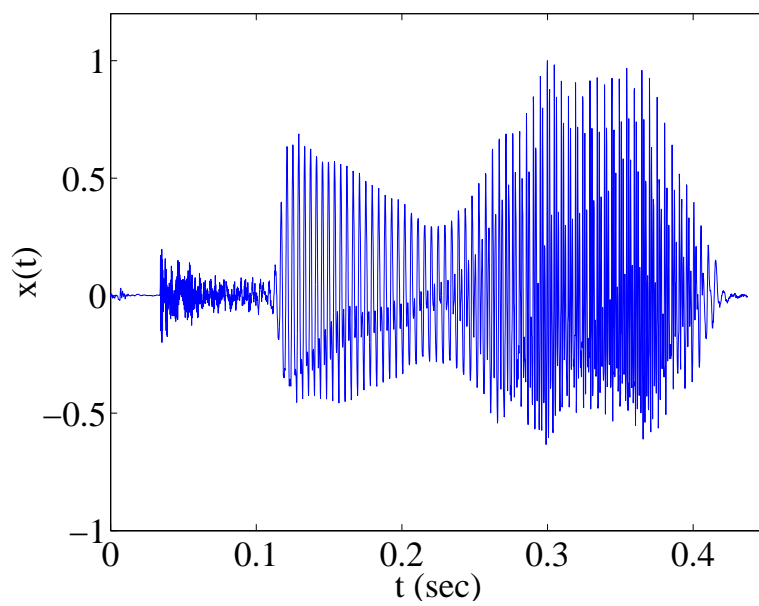
Figure 1.2: Speech signal.
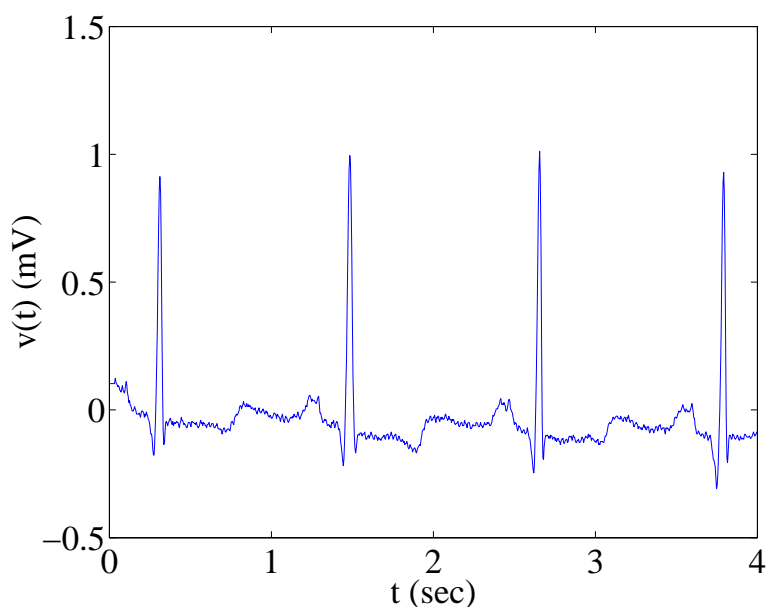
Figure 1.3: Human electrocardiogram (EKG) signal.
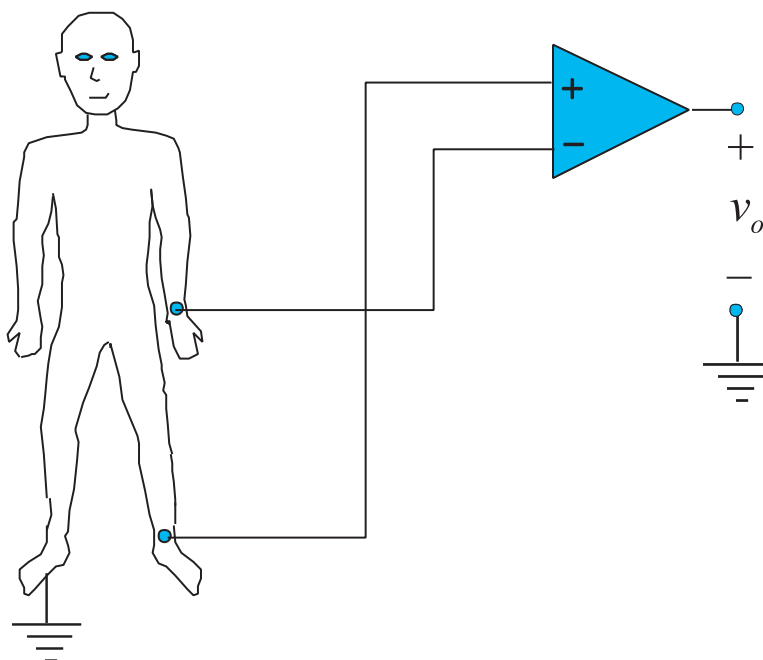
Figure 1.4: Measurement of the electrocardiogram (EKG).

1.2 Signal Power, Energy, and Frequency[2]

Signals can be characterized in several different ways. Audio signals (music, speech, and really, any kind of sound we can hear) are particularly useful because we can use our existing notion of "loudness" and "pitch" which we normally associate with an audio signal to develop ways of characterizing any kind of signal. In terms of audio signals, we use "power" to characterize the loudness of a sound. Audio signals which have greater power sound "louder" than signals which have lower power (assuming the pitch of the sounds are within the range of human hearing). Of course, power is related to the amplitude, or size of the signal. We can develop a more precise definition of power. The signal power is defined as:

$$p_x = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2\left(t\right) dt \tag{1.1}$$

The energy of this signal is similarly defined

$$e_x = \int_{-\infty}^{\infty} x^2\left(t\right) dt \tag{1.2}$$

We can see that power has units of energy per unit time. Strictly speaking, the units for energy depend on the units assigned to the signal. If $x\left(t\right)$ is a voltage, than the units for e_x would be volts^2-seconds. Notice also that some signals may not have finite energy. As we will see shortly, periodic signals do not have finite

[2]This content is available online at $<$http://cnx.org/content/m32864/1.3/$>$.

energy. Signals having a finite energy are sometimes called *energy signals*. Some signals that have infinite energy however can have finite power. Such signals are sometimes called *power signals*.

We use the concept of "frequency" to characterize the pitch of audio signals. The frequency of a signal is closely related to the variation of the signal with time. Signals which change rapidly with time have higher frequencies than signals which are changing slowly with time as seen Figure 1.5. As we shall see, signals can also be represented in terms of their frequencies, $X(j\Omega)$, where Ω is a frequency variable. Devices which enable us to view the frequency content of a signal in real-time are called *spectrum analyzers*.
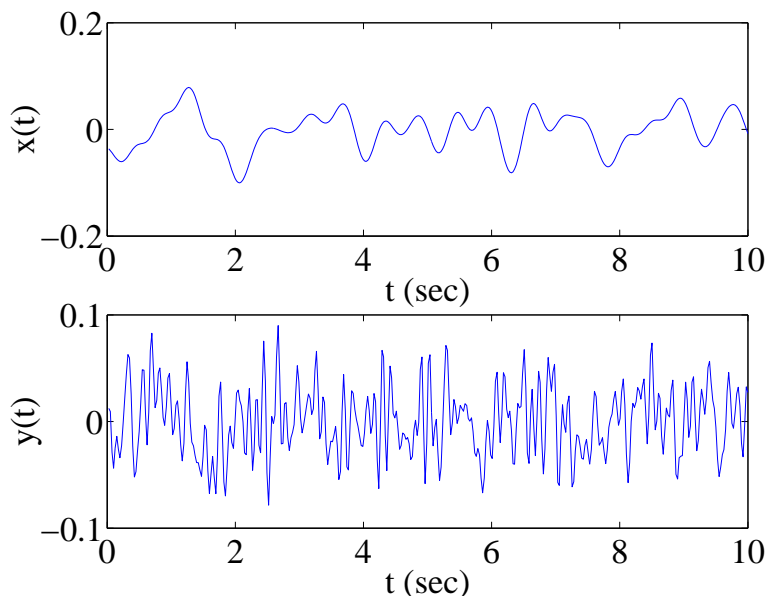
Figure 1.5: The signal $y(t)$ contains a greater amount of high frequencies than $x(t)$.

Something to keep in mind is that the signals shown in Figures Figure 1.1, Figure 1.2, and Figure 1.3 each have different units (degrees Fahrenheit, pressure, and voltage, respectively). So while we can compare relative frequencies between these signals, it doesn't make much sense to compare their power since each signal has different units. We will take a more formal look at the frequency of signals starting in Chapter 2.

1.3 Basic Signal Operations[3]

We will be considering the following basic operations on signals:

- Time shifting:
$$y(t) = x(t - \tau) \tag{1.3}$$

    The effect that a time shift has on the appearance of a signal is seen in Figure 1.6. If τ is a positive number, the time shifted signal, $x(t - \tau)$ gets shifted to the right, otherwise it gets shifted left.
- Time reversal:
$$y(t) = x(-t) \tag{1.4}$$

---

[3]This content is available online at $<$http://cnx.org/content/m32866/1.2/$>$.

Time reversal flips the signal about $t = 0$ as seen in Figure 1.6.
- Addition: any two signals can be added to form a third signal,

$$z(t) = x(t) + y(t) \tag{1.5}$$

- Time scaling:

$$y(t) = x(\Omega t) \tag{1.6}$$

Time scaling "compresses" the signal if $\Omega > 1$ or "stretches" it if $\Omega < 1$ (see Figure 1.7).
- Multiplication by a constant, α:

$$y(t) = \alpha x(t) \tag{1.7}$$

- Multiplication of two signals, their product is also a signal.

$$z(t) = x(t) y(t) \tag{1.8}$$

Multiplication of signals has many useful applications in wireless communications.
- Differentiation:

$$y(t) = \frac{dx(t)}{dt} \tag{1.9}$$

- Integration:

$$y(t) = \int x(t)\, dt \tag{1.10}$$

There is another very important signal operation called *convolution* which we will look at in detail in Chapter 3. As we shall see, convolution is a combination of several of the above operations.
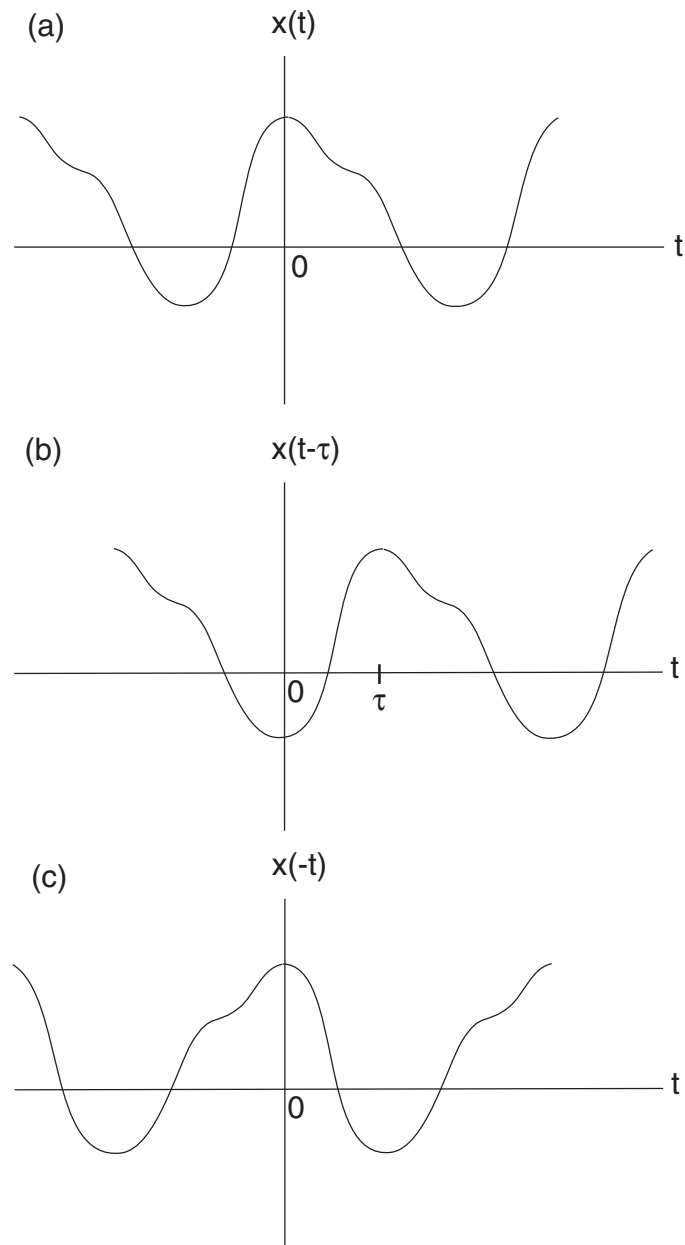
Figure 1.6: (a) original signal, (b) time-shift, (c) time-reversal.

(a)   x(t)

(b)   x(Ωt)

(c)   x(Ωt)
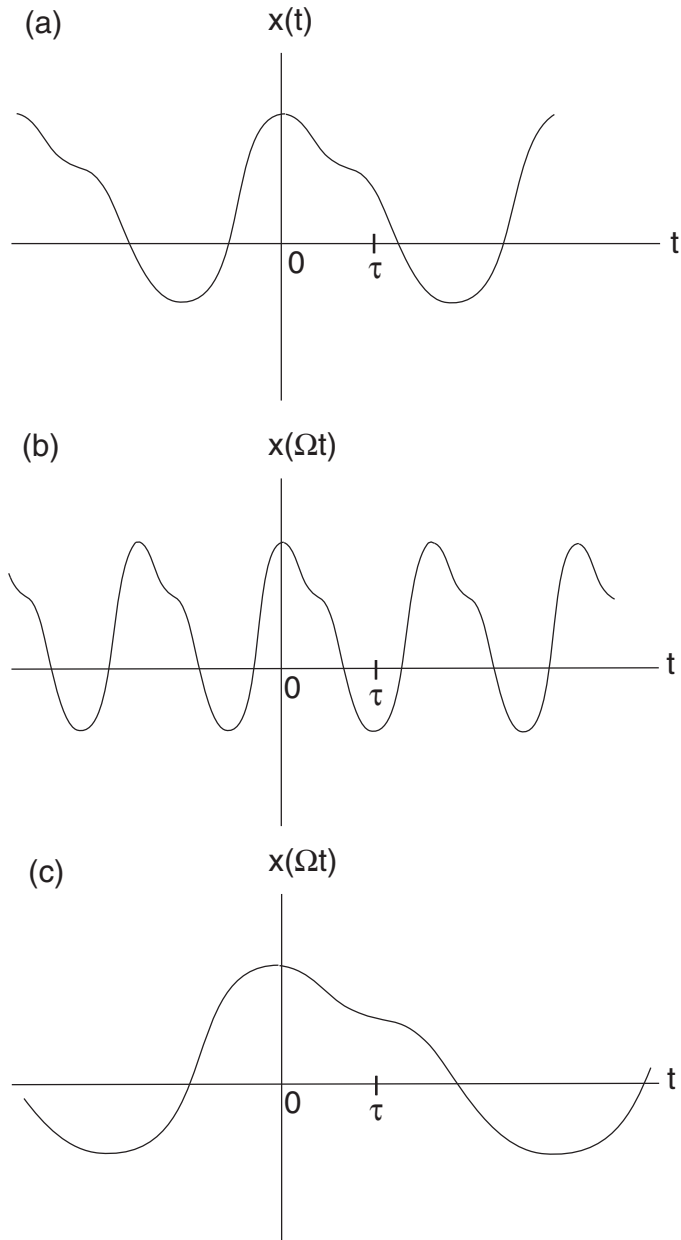
Figure 1.7: (a) original signal, (b) $\Omega > 1$, (c) $\Omega < 1$.

1.4 Complex Numbers and Complex Arithmetic[4]

Before we begin studying signals, we need to review some basic aspects of complex numbers and complex arithmetic. The *rectangular coordinate* representation of a complex number z is z has the form:

$$z = a + jb \tag{1.11}$$

where a and b are real numbers and $j = \sqrt{-1}$. The *real part* of z is the number a, while the *imaginary part* of z is the number b. We also note that $jb\,(jb) = -b^2$ (a real number) since $j\,(j) = -1$. Any number having the form

$$z = jb \tag{1.12}$$

where b is a real number is an *imaginary number*. A complex number can also be represented in *polar coordinates*

$$z = re^{j\theta} \tag{1.13}$$

where

$$r = \sqrt{a^2 + b^2} \tag{1.14}$$

is the *magnitude* and

$$\theta = arctan\left(\frac{b}{a}\right) \tag{1.15}$$

is the *phase* of the complex number z. The notation for the magnitude and phase of a complex number is given by $|z|$ and $\angle z$, respectively. Using Euler's Identity:

$$e^{\pm j\theta} = cos\,(\theta) \pm jsin\,(\theta) \tag{1.16}$$

it follows that $a = rcos\,(\theta)$ and $b = rsin\,(\theta)$. Figure 1.8 illustrates how polar coordinates and rectangular coordinates are related.
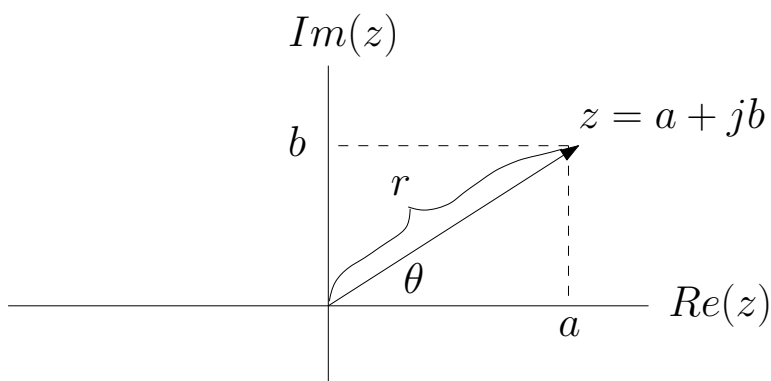
Figure 1.8: Relationship between rectangular and polar coordinates.

Rectangular coordinates and polar coordinates are each useful depending on the type of mathematical operation performed on the complex numbers. Often, complex numbers are easier to add in rectangular coordinates, but multiplication and division is easier in polar coordinates. If $z = a + jb$ is a complex number then its *complex conjugate* is defined by

$$z^* = a - jb \tag{1.17}$$

in polar coordinates we have

$$z^* = re^{-j\theta} \tag{1.18}$$

note that $zz^* = |z|^2 = r^2$ and $z + z^* = 2a$. Also, if $z_1, z_2, ..., z_N$ are complex numbers it can be easily shown that

$$(z_1 + z_2 + \cdots + z_N)^* = z_1^* + z_2^* + \cdots + z_N^* \tag{1.19}$$

and

$$(z_1 z_2 \cdots z_N)^* = z_1^* z_2^* \cdots z_N^* \tag{1.20}$$

Table 1.1 indicates how two complex numbers combine in terms of addition, multiplication, and division when expressed in rectangular and in polar coordinates.

operation	rectangular	polar
$z_1 + z_2$	$(a_1 + a_2) + j(b_1 + b_2)$	
$z_1 z_2$	$a_1 a_2 - b_1 b_2 + j(a_1 b_2 + a_2 b_1)$	$r_1 r_2 e^{j(\theta_1 + \theta_2)}$
z_1 / z_2	$\frac{(a_1 a_2 + b_1 b_2) + j(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$	$\frac{r_1}{r_2} e^{j(\theta_1 - \theta_2)}$

Table 1.1: Operations on two complex numbers, $z_1 = a_1 + jb_1 = r_1 e^{j\theta_1}$ and $z_2 = a_2 + jb_2 = r_2 e^{j\theta_2}$. The sum of two complex numbers is cumbersome to express in polar coordinates, and is not shown.

1.5 Periodic Signals[5]

Periodic signals have the following property:

$$x(t) = x(t + kT) \tag{1.21}$$

where k is an integer and T is called the *fundamental period*. Periodic have the property that they "repeat" every T seconds. For periodic signals, the power can be defined as

$$p_x = \frac{1}{T} \int_{t_0}^{t_0 + T} x^2(t) \, dt \tag{1.22}$$

Figure 1.9 shows an example of a periodic signal. We will study the frequency content of periodic signals in some detail in Chapter 2.

---

[5]This content is available online at $<$http://cnx.org/content/m32869/1.2/$>$.
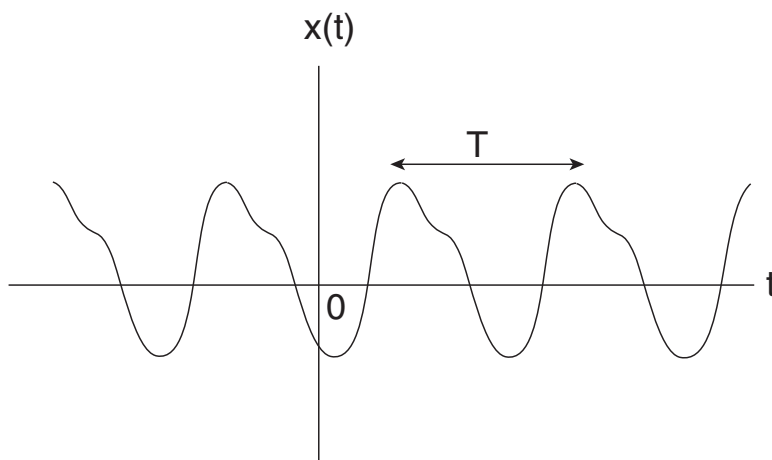
Figure 1.9: General periodic signal.

1.6 Sinusoidal Signals[6]

Sinusoidal signals are perhaps the most important type of signal that we will encounter in signal processing. There are two basic types of signals, the *cosine*:

$$x\left(t\right) = cos\left(\Omega t\right) \tag{1.23}$$

and the *sine*:

$$x\left(t\right) = sin\left(\Omega t\right) \tag{1.24}$$

Plots of the sine and cosine signals are shown in Figure 1.10. Sinusoidal signals are periodic signals. The period of the cosine and sine signals shown above is given by $T = 2\pi/\Omega$. The frequency of the signals is $\Omega = 2\pi/T$ which has units of *rad/sec* . Equivalently, the frequency can be expressed as $1/T$, which has units of sec^{-1}, *cycles/sec*, or *Hz*. The quantity Ωt has units of radians and is often called the *phase* of the sinusoid. Recalling the effect of a time shift on the appearance of a signal, we can observe from Figure 1.10 that the sine signal is obtained by shifting the cosine signal by $T/4$ seconds, i.e.

---

[6]This content is available online at $<$http://cnx.org/content/m32870/1.4/$>$.
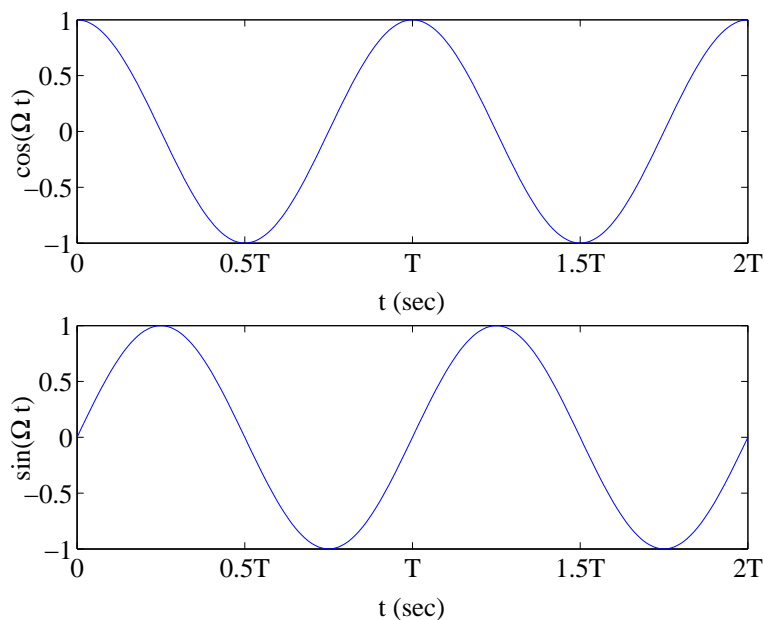
Figure 1.10: Cosine and sine signals. Each signal is periodic with period $T = 2\pi/\Omega$.

$$sin\left(\Omega t\right) = cos\left(\Omega\left(t - T/4\right)\right) \tag{1.25}$$

and since $T = 2\pi/\Omega$, we have

$$sin\left(\Omega t\right) = cos\left(\Omega t - \pi/2\right) \tag{1.26}$$

Similarly, we have

$$cos\left(\Omega t\right) = sin\left(\Omega t + \pi/2\right) \tag{1.27}$$

Using Euler's Identity, we can also write:

$$cos\left(\Omega t\right) = \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right) \tag{1.28}$$

and

$$sin\left(\Omega t\right) = \frac{1}{2j}\left(e^{j\Omega t} - e^{-j\Omega t}\right) \tag{1.29}$$

The quantity $e^{j\Omega t}$ is called a *complex sinusoid* and can be expressed as

$$e^{\pm j\Omega t} = cos\left(j\Omega t\right) \pm jsin\left(j\Omega t\right) \tag{1.30}$$

There are a number of trigonometric identities which are sometimes useful. These are shown in Table 1.2. Table 1.3 shows some basic calculus operations on sine and cosine signals.

$sin\left(\theta\right) = cos\left(\theta - \pi/2\right)$
$cos\left(\theta\right) = sin\left(\theta + \pi/2\right)$
$sin\left(\theta_1\right)sin\left(\theta_2\right) = \frac{1}{2}\left[cos\left(\theta_1 - \theta_2\right) - cos\left(\theta_1 + \theta_2\right)\right]$
$sin\left(\theta_1\right)cos\left(\theta_2\right) = \frac{1}{2}\left[sin\left(\theta_1 - \theta_2\right) - sin\left(\theta_1 + \theta_2\right)\right]$
$cos\left(\theta_1\right)cos\left(\theta_2\right) = \frac{1}{2}\left[cos\left(\theta_1 - \theta_2\right) + cos\left(\theta_1 + \theta_2\right)\right]$
$acos\left(\theta\right) + bsin\left(\theta\right) = \sqrt{a^2 + b^2}cos\left(\theta - tan^{-1}\left(\frac{b}{a}\right)\right)$
$cos\left(\theta_1 \pm \theta_2\right) = cos\left(\theta_1\right)cos\left(\theta_2\right) \mp sin\left(\theta_1\right)sin\left(\theta_2\right)$
$sin\left(\theta_1 \pm \theta_2\right) = sin\left(\theta_1\right)cos\left(\theta_2\right) \pm sin\left(\theta_1\right)cos\left(\theta_2\right)$

Table 1.2: Useful trigonometric identities.

$\frac{d}{dt}cos\left(\Omega t\right) = -\Omega sin\left(\Omega t\right)$
$\frac{d}{dt}sin\left(\Omega t\right) = \Omega cos\left(\Omega t\right)$
$\int cos\left(\Omega t\right)dt = \frac{1}{\Omega}sin\left(\Omega t\right)$
$\int sin\left(\Omega t\right)dt = -\frac{1}{\Omega}cos\left(\Omega t\right)$
$\int_0^T sin\left(k\Omega_o t\right)cos\left(n\Omega_o t\right)dt = 0$
$\int_0^T sin\left(k\Omega_o t\right)sin\left(n\Omega_o t\right)dt = 0, k \neq n$
$\int_0^T cos\left(k\Omega_o t\right)cos\left(n\Omega_o t\right)dt = 0, k \neq n$
$\int_0^T sin^2\left(n\Omega_o t\right)dt = T/2$
$\int_0^T cos^2\left(n\Omega_o t\right)dt = T/2$

Table 1.3: Derivatives and integrals of sinusoidal signals.

Now suppose that we have a sum of two sinusoids, say

$$x\left(t\right) = cos\left(\Omega_1 t\right) + cos\left(\Omega_2 t\right) \tag{1.31}$$

It is of interest to know what the period T of the sum of 2 sinusoids is. We must have

$$\begin{aligned} x\left(t - T\right) &= cos\left(\Omega_1\left(t - T\right)\right) + cos\left(\Omega_2\left(t - T\right)\right) \\ &= cos\left(\Omega_1 t - \Omega_1 T\right) + cos\left(\Omega_2 t - \Omega_2 T\right) \end{aligned} \tag{1.32}$$

It follows that $\Omega_1 T = 2\pi k$ and $\Omega_2 T = 2\pi l$, where k and l are integers. Solving these two equations for T gives $T = 2\pi k/\Omega_1 = 2\pi l/\Omega_2$. We wish to select the shortest possible period, since any integer multiple of the period is also a period. To do this we note that since $2\pi k/\Omega_1 = 2\pi l/\Omega_2$, we can write

$$\frac{\Omega_1}{\Omega_2} = \frac{k}{l} \tag{1.33}$$

so we seek the smallest integers k and l that satisfy (1.33). This can be done by finding the greatest common divisor between k and l. For example if $\Omega_1 = 10\pi$ and $\Omega_2 = 15\pi$, we have $k = 2$ and $l = 3$, after dividing out 5, the greatest common divisor between 10 and 15. So the period is $T = 2\pi k/\Omega_1 = 0.4$ sec. On the other hand, if $\Omega_1 = 10\pi$ and $\Omega_2 = 10.1\pi$, we find that $k = 100$ and $l = 101$ and the period increases to $T = 2\pi k/\Omega_1 = 20$ sec. Notice also that if the ratio of Ω_1 and Ω_2 is not a rational number, then $x\left(t\right)$ is not periodic!

If there are more than two sinusoids, it is probably easiest to find the period of one pair of sinusoids at a time, using the two lowest frequencies (which will have a longer period). Once the frequency of the first two sinusoids has been found, replace them with a single sinusoid at the composite frequency corresponding to the first two sinusoids and compare it with the third sinusoid, and so on.

1.7 Introduction to Continuous-Time Signals: Exercises[7]

1. Consider the signals shown in Figure 1.11. Sketch the following signals[8]:

    a. $x_1(t) + 2x_2(t)$.
    b. $x_1(-t) - x_2(t-1)$.
    c. $x_1(-t+1)$.
    d. $x_2(t-1)$.
    e. $x_1(2t)$.
    f. $x_1(t/2)$.
    g. $x_1(t)x_3(t+1)$.
    h. $x_3(2t-4)$.
    i. $x_3(-2t-4)$.

2. For each of the signals in Figure 1.12:

    a. What is the period?
    b. Sketch $x(t-0.25)$, and $x(t+1)$.
    c. Find the power for each signal.

3. Suppose that $z_1 = 3 + j2$, $z_2 = 4 + j5$. Find:

    - $|z_1|, \angle z_1, |z_2|, \angle z_2$
    - $z_1 + z_2$ in rectangular coordinates.
    - $z_1 z_2$ in rectangular and polar coordinates.
    - z_1/z_2 in polar coordinates.

4. What is the period, frequency, and power of the sinusoidal signal $x(t) = 2cos(5t)$.
5. Can you find a general formula for the power of the sinusoidal signal $x(t) = Acos(\Omega t)$?
6. Express $2cos(10t) + 3sin(10t)$ as a single sinusoidal signal.
7. Sketch $x(t) = sin(t - \theta)$ for $\theta = \pi/4, \pi/2$, and $3\pi/4$.
8. Sketch $x(t) = sin(2t - \theta)$ for $\theta = \pi/4, \pi/2$, and $3\pi/4$.
9. Consider the motion of the second hand of a clock. Assume the length of the second hand is 1 meter. (a) what is the angular frequency of the second hand. (b) find an expression for the horizontal and vertical displacements of the tip of the second hand, assuming the origin is at the clock center and $t = 0$ when the second hand is over the 3.

---

[7]This content is available online at <http://cnx.org/content/m32871/1.3/>.
[8]Assume that for step discontinuities, the signal takes on the greater of the two values.
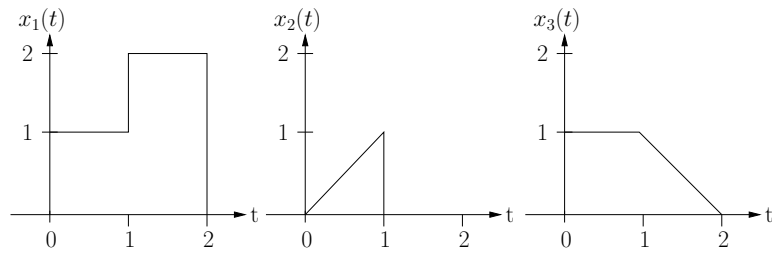
Figure 1.11: Signals for problem 1.
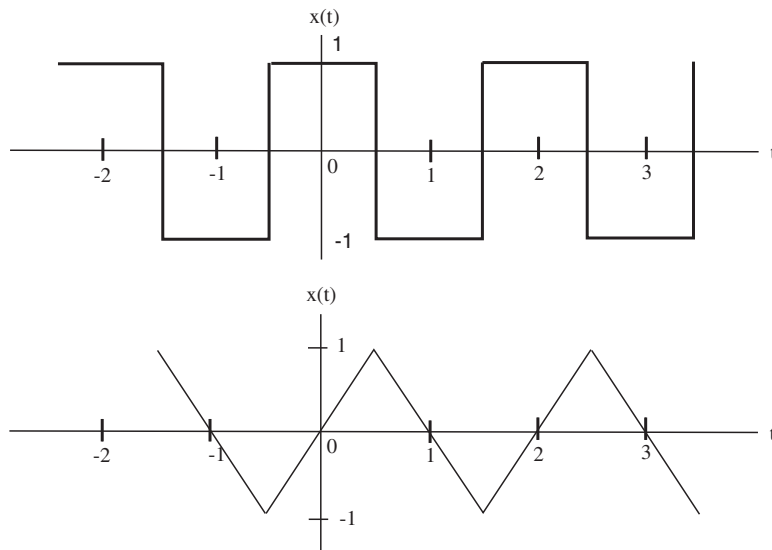
Figure 1.12: Signals for problem 2.

$$(1.34)$$

Chapter 2

Fourier Series of Periodic Signals

2.1 Symmetry Properties of Periodic Signals[1]

A signal has *even symmetry* of it satisfies:

$$x(t) = x(-t) \tag{2.1}$$

and *odd symmetry* if it satisfies

$$x(t) = -x(-t) \tag{2.2}$$

Figure 2.1 shows pictures of periodic even and odd symmetric signals. If $x(t)$ is an odd symmetric periodic signal, then we must have:

$$\int_{t_0}^{t_0+T} x(t)\,dt = 0 \tag{2.3}$$

This is easy to see if we choose $t_0 = -T/2$.

---

[1]This content is available online at $<$http://cnx.org/content/m32875/1.2/$>$.
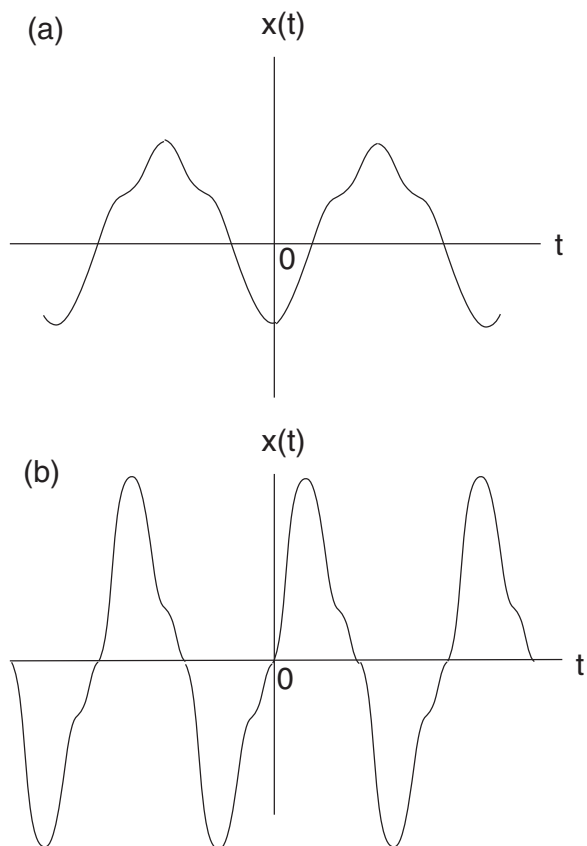
Figure 2.1: (a) Even-symmetric, and (b) odd-symmetric periodic signals. Note that the integral over any period of an odd-symmetric periodic signal is zero.

We also note that the product of two even signals is also even while the product of an even signal and an odd signal must be odd. Finally, the product of two odd signals must be even. For example, suppose $x_o(t)$ has odd symmetry and $x_e(t)$ has even symmetry. Their product has odd symmetry because if $y(t) = x_o(t) x_e(t)$, then $y(-t) = x_o(-t) x_e(-t) = -y(t)$.

2.2 Trigonometric Form of the Fourier Series[2]

A major goal of this book is to develop tools which will enable us to study the frequency content of signals. An important first step is the *Fourier Series*. The Fourier Series enables us to completely characterize the frequency content of a periodic signal[3]. A periodic signal $x(t)$ can be expressed in terms of the Fourier

---

[2]This content is available online at <http://cnx.org/content/m32879/1.4/>.

[3]There are periodic signals for which a Fourier series doesn't exist, conditions for existence of the Fourier series are given below.

Series, which is given by:

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty} a_n cos\left(n\Omega_0 t\right) + \sum_{n=1}^{\infty} b_n sin\left(n\Omega_0 t\right) \tag{2.4}$$

where

$$\Omega_0 = \frac{2\pi}{T} \tag{2.5}$$

is the *fundamental frequency* of the periodic signal. Examination of (2.4) suggests that periodic signals can be represented as a sum of suitably scaled cosine and sine waveforms at frequencies of $\Omega_0, 2\Omega_0, 3\Omega_0, ...$ The cosine and sine terms at frequency $n\Omega_0$ are called the nth *harmonics*. Evidently, periodic signals contain only the fundamental frequency and its harmonics. A periodic signal cannot contain a frequency that is not an integer multiple of its fundamental frequency.

In order to find the Fourier Series, we must compute the *Fourier Series coefficients*. These are given by

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x\left(t\right) dt \tag{2.6}$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x\left(t\right) cos\left(n\Omega_0 t\right) dt, \quad n = 1, 2, ... \tag{2.7}$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x\left(t\right) sin\left(n\Omega_0 t\right) dt, \quad n = 1, 2, ... \tag{2.8}$$

From our discussion of even and odd symmetric signals, it is clear that if $x\left(t\right)$ is even, then $x\left(t\right) sin\left(n\Omega_0 t\right)$ must be odd and so $b_n = 0$. Also if, $x\left(t\right)$ has odd symmetry, then $x\left(t\right) cos\left(n\Omega_0 t\right)$ also has odd symmetry and hence $a_n = 0$ (see exercise). Moreover, if a signal is even, since $x\left(t\right) cos\left(n\Omega_0 t\right)$ is also even, if we use the fact that for any even symmetric periodic signal $v\left(t\right)$,

$$\int_{-T/2}^{T/2} v\left(t\right) dt = 2 \int_{0}^{T/2} v\left(t\right) dt \tag{2.9}$$

then setting $t_0 = -T/2$ in (2.7) gives,

$$a_n = \frac{4}{T} \int_{0}^{T/2} x\left(t\right) cos\left(n\Omega_0 t\right) dt, \quad n = 1, 2, ... \tag{2.10}$$

This can sometimes lead to a savings in the number of integrals that must be computed. Similarly, if $x\left(t\right)$ has odd symmetry, we have

$$b_n = \frac{4}{T} \int_{0}^{T/2} x\left(t\right) sin\left(n\Omega_0 t\right) dt, \quad n = 1, 2, ... \tag{2.11}$$

Example 2.1 Consider the signal in Figure 2.2. This signal has even symmetry, hence all of the $b_n = 0$. We compute a_0 using,

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x\left(t\right) dt \tag{2.12}$$

which we recognize as the area of one period, divided by the period. Hence, $a_0 = \tau/T$. Next, using (2.7) we get

$$a_n = \frac{2}{T} \int_{-\tau/2}^{\tau/2} cos\left(n\Omega_0 t\right) dt \tag{2.13}$$

Note how the limits of integration only go from $-\tau/2$ to $\tau/2$ since $x(t)$ is zero everywhere else. Evaluating this integral leads to

$$a_n = \frac{2\tau}{T} \frac{sin(n\Omega_0\tau/2)}{n\Omega_0\tau/2}, \quad n = 1, 2, ... \tag{2.14}$$

Figure 2.3 shows the first few Fourier Series coefficients for $\tau = 1/2$ and $T = 1$. If we attempt to reconstruct $x(t)$ based on only a limited number (say, N) of Fourier Series coefficients, we have

$$\hat{x}(t) = a_0 + \sum_{n=0}^{N} a_n cos(n\Omega_0 t) \tag{2.15}$$

Figures Figure 2.4 and Figure 2.5 show $\hat{x}(t)$ for $N = 10$, and $N = 50$, respectively. The ringing characteristic is known as *Gibb's phenomenon* and disappears only as N approaches ∞.

The following example looks at the Fourier series of an odd-symmetric signal, a *sawtooth* signal.

Example 2.2 Now let's compute the Fourier series for the signal in Figure 2.6. The signal is odd-symmetric, so all of the a_n are zero. The period is $T = 3/2$, hence $\Omega_0 = 4\pi/3$. Using (2.8), the b_n coefficients are found by computing the following integral,

$$b_n = \frac{8}{3} \int_{-1/2}^{1/2} tsin(4\pi nt/3)\, dt \tag{2.16}$$

After integrating by parts, we get

$$b_n = 3\frac{sin(2\pi n/3)}{(\pi n)^2} - 2\frac{cos(2\pi n/3)}{\pi n}, \quad n = 1, 2, ... \tag{2.17}$$

These are plotted in Figure 2.7 and approximations of $x(t)$ using $N = 10$ and $N = 50$ coefficients are shown in Figures Figure 2.8 and Figure 2.9, respectively.
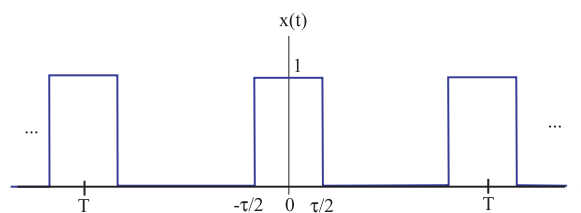
Figure 2.2: Example "Trigonometric Form of the Fourier Series". This signal is sometimes called a pulse train.

Figure 2.3: Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".

Figure 2.4: Approximation to $x(t)$ based on the first 10 Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".

Figure 2.5: Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".

Figure 2.6: Example "Trigonometric Form of the Fourier Series". Sawtooth signal.
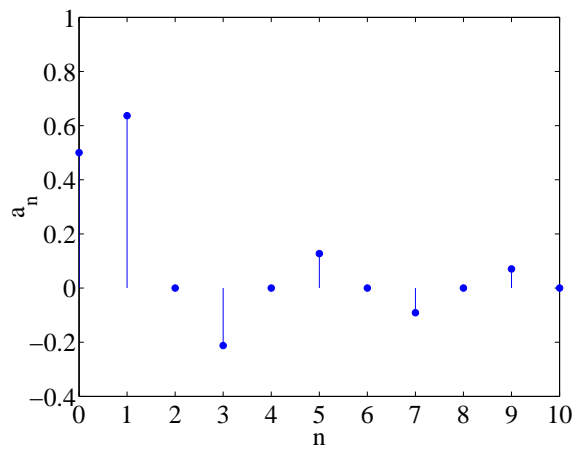
Figure 2.7: Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".
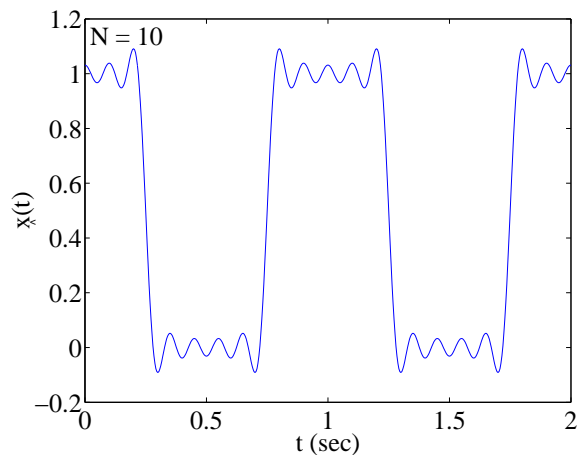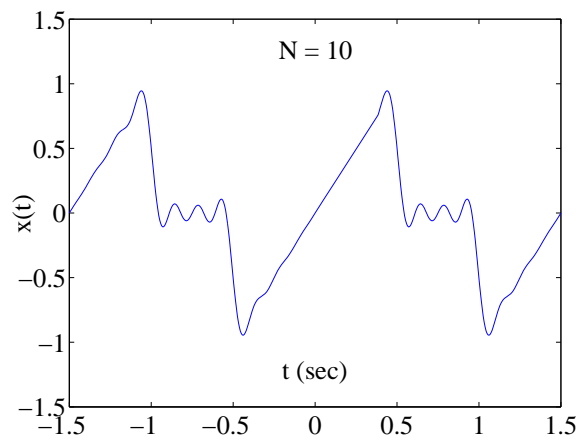

Figure 2.8: Approximation to $x(t)$ based on the first 10 Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".
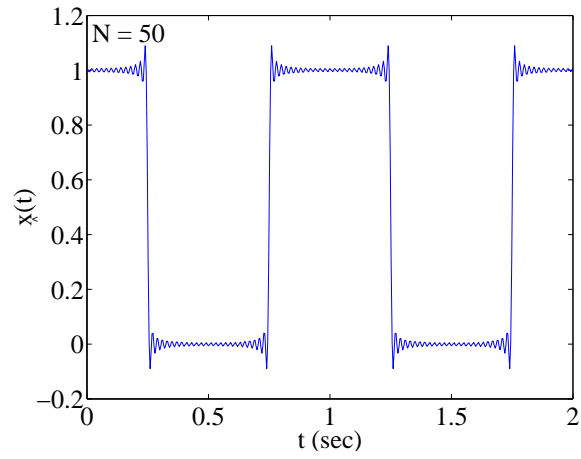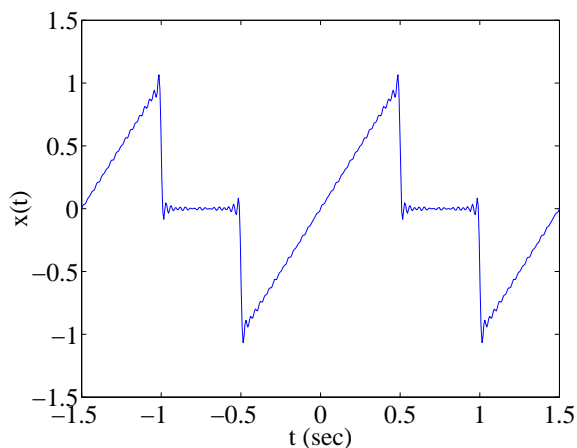
Figure 2.9: Fourier Series coefficients for Example "Trigonometric Form of the Fourier Series".
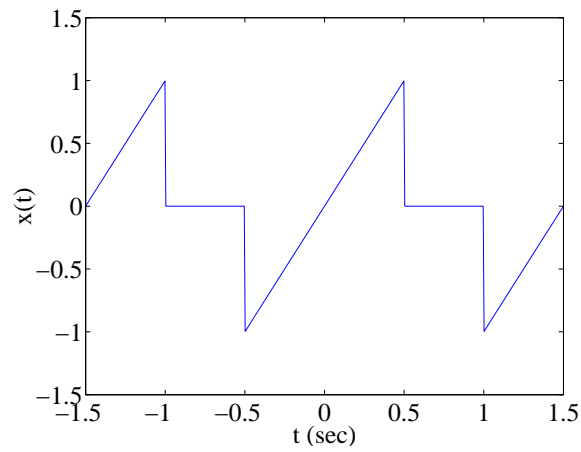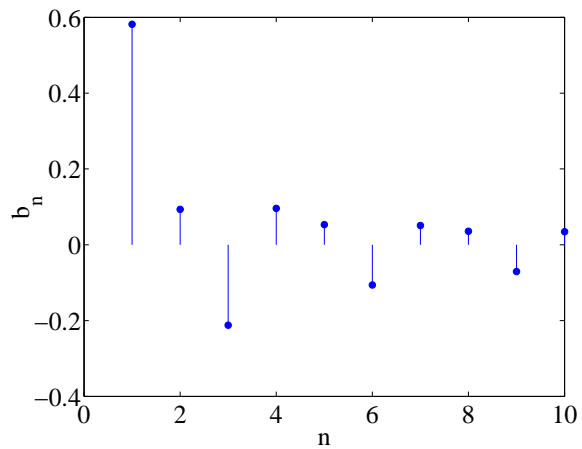
References (Chapter 5)

2.3 Half-Wave Symmetry[4]

2.3.1 Half-Wave Symmetry

Periodic signals having half-wave symmetry have the property

$$
\begin{aligned}
x\left(t\right) &= -x\left(t - T/2\right) \\
x\left(t\right) &= -x\left(t + T/2\right)
\end{aligned}
\tag{2.18}
$$

It turns out that signals with this type of symmetry only have odd-numbered harmonics, the even harmonics are zero. To see this, lets look at the formula for the coefficients a_n:

$$
\begin{aligned}
a_n &= \tfrac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)\cos\left(n\Omega_0 t\right)dt \\
&= \tfrac{2}{T}\left[\int_{t_0}^{t_0+T/2} x\left(t\right)\cos\left(n\Omega_0 t\right)dt + \int_{t_0+T/2}^{t_0+T} x\left(t\right)\cos\left(n\Omega_0 t\right)dt\right] \\
&= \tfrac{2}{T}\left[I_1 + I_2\right]
\end{aligned}
\tag{2.19}
$$

Making the substitution $\tau = t - T/2$ in I_2 gives

$$
\begin{aligned}
I_2 &= \int_{t_0}^{t_0+T/2} x\left(\tau + T/2\right)\cos\left(n\Omega_0\left(\tau + T/2\right)\right)d\tau \\
&= -\int_{t_0}^{t_0+T/2} x\left(\tau\right)\cos\left(n\Omega_0\left(\tau + T/2\right)\right)d\tau
\end{aligned}
\tag{2.20}
$$

The quantity $cos\left(n\Omega_0\left(\tau + T/2\right)\right) = cos\left(n\Omega\tau + n\pi\right)$ can be simplified using the trigonometric identity

$$
cos\left(u \pm v\right) = cos\left(u\right)cos\left(v\right) \mp sin\left(u\right)sin\left(v\right)
\tag{2.21}
$$

---

[4]This content is available online at $<$http://cnx.org/content/m32877/1.4/$>$.

We have

$$cos\left(n\Omega\tau + n\pi\right) \quad = cos\left(n\Omega\tau\right)cos\left(n\pi\right) - sin\left(n\Omega\tau\right)sin\left(n\pi\right)$$
$$= (-1)^n cos\left(n\Omega\tau\right) - 0$$

(2.22)

Therefore

$$I_2 = -(-1)^n \int_{t_0}^{t_0+T/2} x\left(\tau\right)cos\left(n\Omega_0\tau\right)d\tau$$

(2.23)

and we can write:

$$a_n = \frac{2}{T}\left(1 - (-1)^n\right)\int_{t_0}^{t_0+T/2} x\left(t\right)cos\left(n\Omega_0 t\right)dt$$

(2.24)

From this expression we find that $a_n = 0$ whenever n is even. In fact, we have

$$a_n = \{ \begin{array}{ll} \frac{4}{T}\int_{t_0}^{t_0+T/2} x\left(t\right)cos\left(n\Omega_0 t\right)dt, & n, \text{ odd} \\ 0, & n, \text{ even} \end{array}$$

(2.25)

A similar derivation leads to

$$b_n = \{ \begin{array}{ll} \frac{4}{T}\int_{t_0}^{t_0+T/2} x\left(t\right)sin\left(n\Omega_0 t\right)dt, & n, \text{ odd} \\ 0, & n, \text{ even} \end{array}$$

(2.26)

A good choice of t_0 can lead to a considerable savings in time when calculating the Fourier Series of half-wave symmetric signals. Note that half-wave symmetric signals need not have odd or even symmetry for the above formulae to apply. If a signal has half-wave symmetry and in addition has odd or even symmetry, then some additional simplification is possible. Consider the case when a half-wave symmetric signal also has even symmetry. Then clearly $b_n = 0$, and (2.25) applies. However since the integrand in (2.25) is the product of two even signals, $x\left(t\right)$ and $cos\left(n\Omega_0 t\right)$, it too has even symmetry. Therefore, instead of integrating from, say, $-T/4$ to $T/4$, we need only integrate from 0 to $T/4$ and multiply the result by 2. Therefore the formula for a_n for an even, half-wave symmetric signal becomes:

$$a_n = \{ \begin{array}{ll} \frac{8}{T}\int_0^{T/4} x\left(t\right)cos\left(n\Omega_0 t\right)dt, & n, \text{ odd} \\ 0, & n, \text{ even} \end{array}$$

(2.27)

$$b_n = 0$$

(2.28)

For an odd half-wave symmetric signals, a similar argument leads to

$$a_n = 0$$

(2.29)

$$b_n = \{ \begin{array}{ll} \frac{8}{T}\int_0^{T/4} x\left(t\right)sin\left(n\Omega_0 t\right)dt, & n, \text{ odd} \\ 0, & n, \text{ even} \end{array}$$

(2.30)

2.4 Convergence of the Fourier Series[5]

Consider the trigonometric form of the Fourier series

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n cos(n\Omega_0 t) + \sum_{n=1}^{\infty} b_n sin(n\Omega_0 t) \tag{2.31}$$

It is important to state under what conditions this series (the right-hand side of (2.31)) will actually converge to $x(t)$. The nature of the convergence also needs to be specified. There are several ways of defining the convergence of a series.

1. *Uniform convergence:* define the finite sum:

$$x_N(t) = a_0 + \sum_{n=1}^{N} a_n cos(n\Omega_0 t) + \sum_{n=1}^{N} b_n sin(n\Omega_0 t) \tag{2.32}$$

   where N is finite. Then the series converges uniformly if the absolute value of $x(t) - x_N(t)$ satisfies

$$|x(t) - x_N(t)| < \varepsilon \tag{2.33}$$

   for all values of t and some small positive constant ε.

2. *Point-wise convergence:* as with uniform convergence, we require that

$$|x(t) - x_{N(t)}(t)| < \varepsilon \tag{2.34}$$

   for all t. The main difference between uniform and point-wise convergence is that for the latter, the number of terms in the summation $N(t)$ needed to get the error below ε may vary for different values of t.

3. *Mean-squared convergence:* here, the series converges if for all t:

$$\lim_{N \to \infty} \int_{t_0}^{t_0+T} |x(t) - x_{N(t)}(t)|^2 dt = 0 \tag{2.35}$$

   Gibb's phenomenon, mentioned in some of the examples above, is an example of mean-squared convergence of the series. The overshoot in Gibb's phenomenon occurs only at abrupt discontinuities. Moreover, the height of the overshoot stays the same independently of the number of terms in the series, N. The overshoot merely becomes less noticeable because it becomes more and more narrow as N increases.

Dirichlet has given a series of conditions which are necessary for a periodic signal to have a Fourier series. If these conditions are met, then

- the Fourier series has point-wise convergence for all t at which $x(t)$ is continuous.
- where $x(t)$ has a discontinuity, then the series converges to the midpoint between the two values on either side of the discontinuity.

The *Dirichlet Conditions* are:

1. $x(t)$ has to be *absolutely integrable* on any period:

$$\int_{t_0}^{t_0+T} |x(t)| dt < \infty \tag{2.36}$$

2. $x(t)$ can have only a finite number of discontinuities on any period.
3. $x(t)$ can have only a finite number of extrema on any period.

Most periodic signals of practical interest satisfy these conditions.
  References (Chapter 5)

---
[5]This content is available online at <http://cnx.org/content/m32880/1.4/>.

2.5 Complex Form of the Fourier Series[6]

The trigonometric form of the Fourier Series, shown in (2.4) can be converted into a more convenient form by doing the following substitutions:

$$cos\left(n\Omega_0 t\right) = \frac{e^{jn\Omega_0 t} + e^{-jn\Omega_0 t}}{2} \tag{2.37}$$

$$sin\left(n\Omega_0 t\right) = \frac{e^{jn\Omega_0 t} - e^{-jn\Omega_0 t}}{j2} \tag{2.38}$$

After some straight-forward rearranging, we obtain

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\Omega_0 t} + \sum_{n=1}^{\infty}\left[\frac{a_n + jb_n}{2}\right]e^{-jn\Omega_0 t} \tag{2.39}$$

Keeping in mind that a_n and b_n are only defined for positive values of n, lets sum over the negative integers in the second summation:

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\Omega_0 t} + \sum_{n=-1}^{-\infty}\left[\frac{a_{-n} + jb_{-n}}{2}\right]e^{jn\Omega_0 t} \tag{2.40}$$

Next, let's assume that a_n and b_n are defined for both positive and negative n. In this case, we find that $a_n = a_{-n}$ and $b_n = -b_{-n}$, since

$$\begin{aligned}
a_{-n} &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)cos\left(-n\Omega_0 t\right)dt \\
&= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)cos\left(n\Omega_0 t\right)dt \\
&= a_n
\end{aligned} \tag{2.41}$$

and

$$\begin{aligned}
b_{-n} &= \frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)sin\left(-n\Omega_0 t\right)dt \\
&= -\frac{2}{T}\int_{t_0}^{t_0+T} x\left(t\right)sin\left(n\Omega_0 t\right)dt \\
&= -b_n
\end{aligned} \tag{2.42}$$

Using this fact, we can rewrite (2.40) as

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\Omega_0 t} + \sum_{n=-1}^{-\infty}\left[\frac{a_n - jb_n}{2}\right]e^{jn\Omega_0 t} \tag{2.43}$$

If we define[7] $c_0 \equiv a_0$, and

$$c_n \equiv \frac{a_n - jb_n}{2} \tag{2.44}$$

then we can rewrite (2.43) as

$$x\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \tag{2.45}$$

---

[6]This content is available online at <http://cnx.org/content/m32887/1.4/>.
[7]The notation "≡" is often used instead of the "=" sign when defining a new variable.

which is called the *complex form* of the Fourier Series. Note that since $a_{-n} = a_n$ and $b_{-n} = -b_n$, we have

$$c_{-n} = c_n^* \tag{2.46}$$

This means that

$$|c_{-n}| = |c_n| \tag{2.47}$$

and

$$\angle c_{-n} = -\angle c_n \tag{2.48}$$

Next, we must find formulas for finding the c_n given $x(t)$. We first look at a property of complex exponentials:

$$\int_{t_0}^{t_0+T} e^{jk\Omega_0 t} dt = \begin{cases} T, & k = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.49}$$

To see this, we note that

$$\int_{t_0}^{t_0+T} e^{jk\Omega_0 k t} dt = \int_{t_0}^{t_0+T} \cos(k\Omega_0 t) + j \int_{t_0}^{t_0+T} \sin(k\Omega_0 t) \, dt \tag{2.50}$$

It's easy to see that $k\Omega_0$ also has period T, hence the integral is over k periods of $\cos(k\Omega_0 t)$ and $\sin(k\Omega_0 t)$. Therefore, if $k \neq 0$, then

$$\int_{t_0}^{t_0+T} e^{jk\Omega_0 t} dt = 0 \tag{2.51}$$

otherwise

$$\int_{t_0}^{t_0+T} e^{jk\Omega_0 t} dt = \int_{t_0}^{t_0+T} dt = T \tag{2.52}$$

We use (2.49) to derive an equation for c_n as follows. Consider the integral

$$\int_{t_0}^{t_0+T} x(t) e^{-jn\Omega_0 t} dt \tag{2.53}$$

Substituting the complex form of the Fourier Series of $x(t)$ in (2.53), (using k as the index of summation) we obtain

$$\int_{t_0}^{t_0+T} \left[\sum_{k=-\infty}^{\infty} c_n e^{jk\Omega_0 t} \right] e^{-jn\Omega_0 t} dt \tag{2.54}$$

Rearranging the order of integration and summation, combining the exponents, and using (2.49) gives

$$\sum_{k=-\infty}^{\infty} c_n \int_{t_0}^{t_0+T} e^{j(k-n)\Omega_0 t} dt = T c_n \tag{2.55}$$

Using this result, we find that

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\Omega_0 t} dt \tag{2.56}$$

Example 2.1 Let's now find the complex form of the Fourier Series for the signal in Example p. 20. The integral to be evaluated is

$$c_n = \frac{2}{3} \int_{-0.5}^{0.5} 2t e^{-j\frac{4\pi}{3}nt} dt \tag{2.57}$$

Integrating by parts yields

$$c_n = \frac{j}{n\pi} cos\,(2\pi n/3) - \frac{j3}{(n\pi)^2} sin\,(2\pi n/3) \tag{2.58}$$

Figure 2.10 shows the magnitude of the coefficients, $|c_n|$. Note that the complex Fourier Series coefficients have even symmetry as was mentioned earlier.
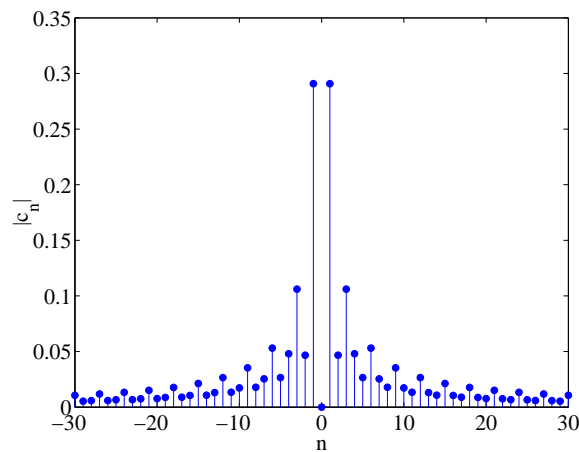

Figure 2.10: Fourier Series coefficients for Example "Complex Form of the Fourier Series".

Can the basic formula for computing the c_n in (2.56) be simplified when $x(t)$ has either even, odd, or half-wave symmetry? The answer is yes. We simply use the fact that

$$c_n = \frac{a_n - jb_n}{2} \tag{2.59}$$

and solve for a_n and b_n using the formulae given above for even, odd, or half-wave symmetric signals. This avoids having to integrate complex quantities. This can also be seen by noting that (setting $t_0 = T/2$ in (2.56)):

$$
\begin{aligned}
c_n &= & \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t} dt \\
&= & \frac{1}{T} \int_{-T/2}^{T/2} x(t) cos\,(n\Omega_0 t)\, dt - \frac{j}{T} \int_{-T/2}^{T/2} x(t) sin\,(n\Omega_0 t)\, dt \\
&= & \frac{1}{2} a_n - \frac{j}{2} b_n
\end{aligned}
\tag{2.60}
$$

Alternately, if $x(t)$ has half-wave symmetry, we can use (2.60), (2.24), and (2.25) to get

$$c_n = \{ \begin{array}{ll} \frac{2}{T} \int_{-T/4}^{T/4} x(t)\, e^{-jn\Omega_0 t} dt, & n \text{ odd} \\ 0, & n \text{ even} \end{array} \tag{2.61}$$

Unlike the trigonometric form, we cannot simplify this further if $x(t)$ is even or odd symmetric since $e^{-jn\Omega_0 t}$ has neither even nor odd symmetry.

Example 2.2 In this example we will look at the effect of adjusting the period of a pulse train signal. Consider the signal depicted in Figure 2.11.
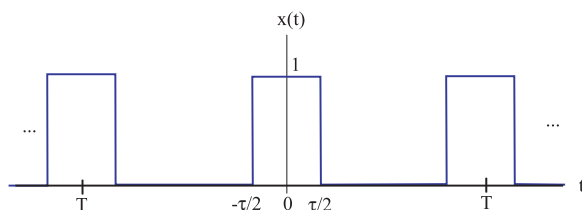

Figure 2.11: Pulse train having period T used in Example "Complex Form of the Fourier Series".

The Fourier Series coefficients for this signal are given by

$$
\begin{aligned}
c_n &= \frac{1}{T} \int_{-\tau/2}^{\tau/2} e^{-jn\Omega_0 t} dt \\
&= \frac{-1}{jn\Omega_0 T} \left(e^{-jn\Omega_0 \tau/2} - e^{jn\Omega_0 \tau/2} \right) \\
&= \frac{\tau}{T} \frac{\sin(n\Omega_0 \tau/2)}{n\Omega_0 \tau/2} \\
&\equiv \frac{\tau}{T} \text{sinc}\left(n\Omega_0 \tau/2 \right)
\end{aligned}
\tag{2.62}
$$

Figure 2.12 shows the magnitude of $|c_n|$, the *amplitude spectrum*, for $T = 1$ and $\tau = 1/2$ as well as the Fourier Series for the signal based on the first 30 coefficients

$$
\hat{x}(t) = \sum_{n=-30}^{30} c_n e^{n\Omega_0 t}
\tag{2.63}
$$

Similar plots are shown in Figures Figure 2.13, and Figure 2.14, for $T = 4$, and $T = 8$, respectively.
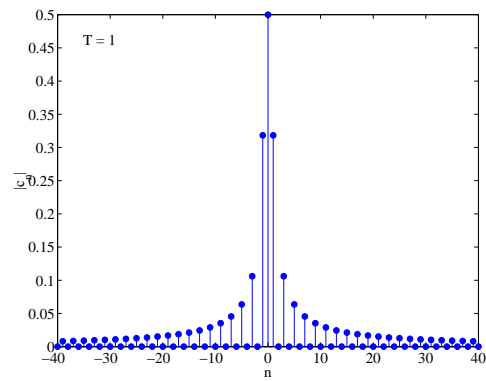
(a)
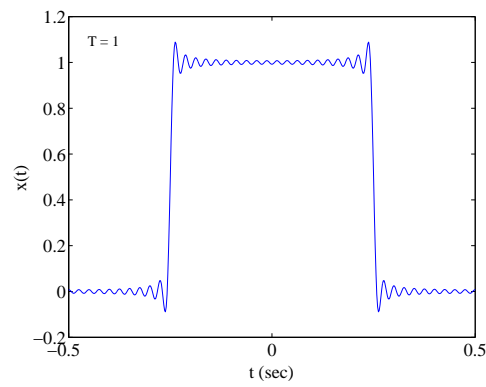
(b)

Figure 2.12: Example "Complex Form of the Fourier Series", $T = 1, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$.
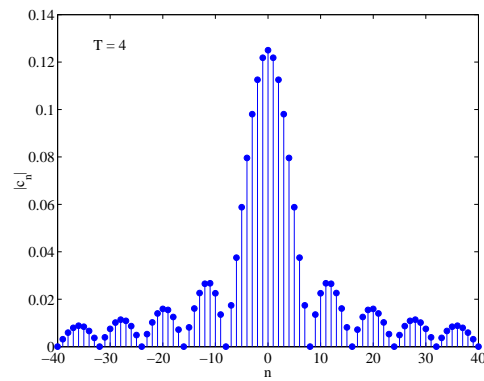
(a)
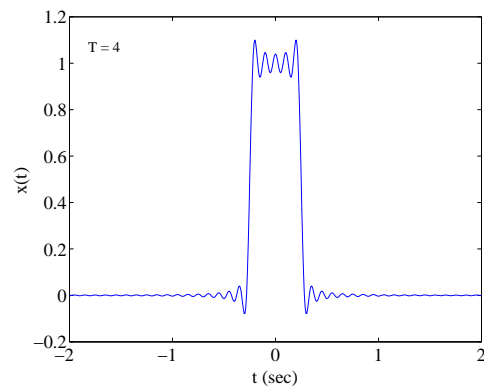
(b)

Figure 2.13: Example "Complex Form of the Fourier Series", $T = 4, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$.
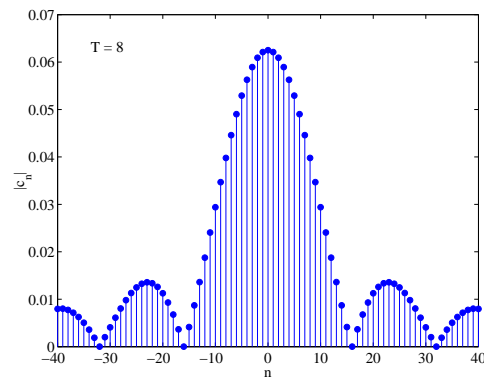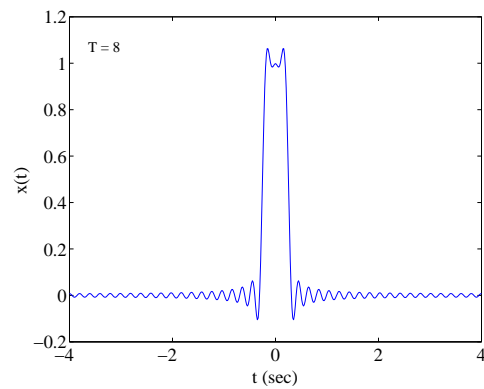
Figure 2.14: Example "Complex Form of the Fourier Series", $T = 8, \tau = 1/2$: (top) Fourier Series coefficient magnitudes, (b) $\hat{x}(t)$.

This example illustrates several important points about the Fourier Series: As the period T increases, Ω_0 decreases in magnitude (this is obvious since $\Omega_0 = 2\pi/T$). Therefore, as the period increases, successive Fourier Series coefficients represent more closely spaced frequencies. The frequencies corresponding to each n are given by the following table:

n	Ω
0	0
± 1	$\pm\Omega_0$
± 2	$\pm 2\Omega_0$
$\vdots$	$\vdots$
$\pm n$	$\pm n\Omega_0$

Table 2.1

This table establishes a relation between n and the frequency variable Ω. In particular, if $T = 1$, we have $\Omega_0 = 2\pi$ and

n	Ω
0	0
± 1	$\pm 2\pi$
± 2	$\pm 4\pi$
$\vdots$	$\vdots$
$\pm n$	$\pm 2n\pi$

Table 2.2

If $T = T$, then $\Omega_0 = \pi/2$ and

n	Ω
0	0
± 1	$\pm \pi/2$
± 2	$\pm \pi$
$\vdots$	$\vdots$
$\pm n$	$\pm n\pi/2$

Table 2.3

and if $T = 8$, we have $\Omega_0 = \pi/4$ and

n	Ω
0	0
± 1	$\pm \pi/4$
± 2	$\pm \pi/2$
$\vdots$	$\vdots$
$\pm n$	$\pm n\pi/4$

Table 2.4

Note that in all three cases, the first zero coefficient corresponds to the value of n for which $\Omega = 4\pi$. Also, as T gets bigger, the c_n appear to resemble more closely spaced samples of a continuous function of frequency (since the $n\Omega$ are more closely spaced). Can you determine what this function is?

As we shall see, by letting the period T get large (infinitely large), we will derive the *Fourier Transform* in the next chapter.

References (Chapter 5)

2.6 Parseval's Theorem for the Fourier Series[8]

Recall that in Chapter 1, we defined the power of a periodic signal as

$$p_x = \frac{1}{T} \int_{t_0}^{t_0+T} x^2(t)\, dt \tag{2.64}$$

where T is the period. Using the complex form of the Fourier series, we can write

$$x(t)^2 = \left(\sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \right) \left(\sum_{m=-\infty}^{\infty} c_m e^{jm\Omega_0 t} \right)^* \tag{2.65}$$

where we have used the fact that $x(t)^2 = x(t)\, x(t)^*$, i.e. since $x(t)$ is real $x(t) = x(t)^*$. Applying (1.19) and (1.20) gives

$$
\begin{aligned}
x(t)^2 &= \left(\sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \right) \left(\sum_{m=-\infty}^{\infty} c_m^* e^{-jm\Omega_0 t} \right) \\
&= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_n c_m^* e^{j(n-m)\Omega_0 t} \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2 + \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t}
\end{aligned} \tag{2.66}
$$

Substituting this quantity into (2.64) gives

$$
\begin{aligned}
p_x &= \frac{1}{T} \int_{t_0}^{t_0+T} \left[\sum_{n=-\infty}^{\infty} |c_n|^2 + \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} \right] dt \\
&= \sum_{n=-\infty}^{\infty} |c_n|^2 + \frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} dt
\end{aligned} \tag{2.67}
$$

It is straight-forward to show that

$$\frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} dt = 0 \tag{2.68}$$

This leads to *Parseval's Theorem* for the Fourier series:

$$p_x = \sum_{n=-\infty}^{\infty} |c_n|^2 \tag{2.69}$$

which states that the power of a periodic signal is the sum of the magnitude of the complex Fourier series coefficients.

2.7 The Fourier Series: Exercises[9]

1. Show that an even-symmetric periodic signal has Fourier Series coefficients $b_n = 0$ while an odd-symmetric signal has $a_n = 0$.
2. Find the trigonometric form of the Fourier Series of the periodic signal shown in Figure 2.15.
3. Find the trigonometric form of the Fourier Series for the periodic signal shown in Figure 2.16.
4. Find the trigonometric form of the Fourier Series for the periodic signal shown in Figure 2.17 for $\tau = 1$, $T = 10$.
5. Suppose that $x(t) = 5 + 3\cos(5t) - 2\sin(3t) + \cos(45t)$.

    a. Find the period of this periodic signal.
    b. Find the trigonometric form of the Fourier Series.

6. Find the complex form of the Fourier Series of the periodic signal shown in Figure 2.15.

7. Find the complex form of the Fourier Series of the periodic signal shown in Figure 2.16.

8. Find the complex form of the Fourier Series for the signal in Figure 2.17 using:

   a. $\tau = 1$, $T = 10$.
   b. $\tau = 1$, $T = 100$.

For each case plot the magnitude of the Fourier Series coefficients. You may use Matlab or some other programming language to do this.

9. Show that

$$\frac{1}{T} \int_{t_0}^{t_0+T} \sum_{n \neq m} c_n c_m^* e^{j(n-m)\Omega_0 t} dt = 0 \qquad (2.70)$$
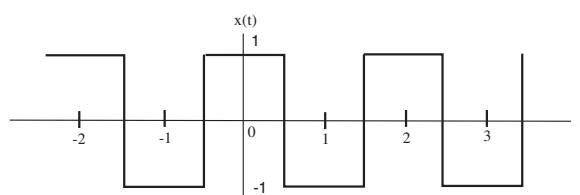

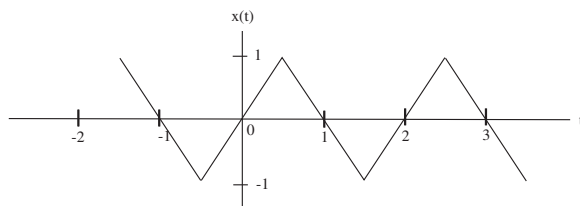**Figure 2.15:** Signal for problems list, p. 35 and list, p. 35.

Figure 2.16: Signal for problem list, p. 35 and list, p. 35.
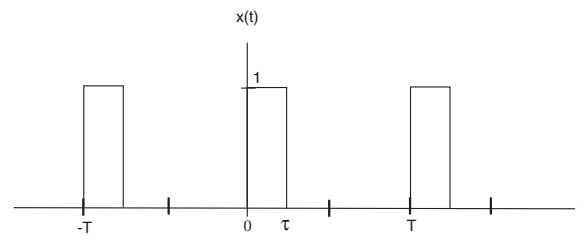
Figure 2.17: Pulse train signal for problems list, p. 35 and list, p. 35.

Chapter 3

The Fourier Transform

3.1 Derivation of the Fourier Transform[1]

Let's begin by writing down the formula for the complex form of the Fourier Series:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t} \tag{3.1}$$

as well as the corresponding Fourier Series coefficients:

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\Omega_0 t} dt \tag{3.2}$$

As was mentioned in Chapter 2, as the period T gets large, the Fourier Series coefficients represent more closely spaced frequencies. Lets take the limit as the period T goes to infinity. We first note that the fundamental frequency approaches a differential

$$\Omega_0 = \frac{2\pi}{T} \rightarrow d\Omega \tag{3.3}$$

consequently

$$\frac{1}{T} = \frac{\Omega_0}{2\pi} \rightarrow \frac{d\Omega}{2\pi} \tag{3.4}$$

The nth harmonic, $n\Omega_0$, in the limit approaches the frequency variable Ω

$$n\Omega_0 \rightarrow \Omega \tag{3.5}$$

From equation (3.2), we have

$$c_n T \rightarrow \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \tag{3.6}$$

The right hand side of (3.6) is called the *Fourier Transform* of $x(t)$:

$$X(j\Omega) \equiv \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \tag{3.7}$$

---

[1]This content is available online at <http://cnx.org/content/m32889/1.3/>.

Now, using (3.6), (3.4), and (3.5) in equation (3.1) gives

$$x\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\Omega\right)e^{j\Omega t}d\Omega \tag{3.8}$$

which corresponds to the *inverse Fourier Transform*. Equations (3.7) and (3.8) represent what is known as a *transform pair*. The following notation is used to denote a Fourier Transform pair

$$x\left(t\right) \leftrightarrow X\left(j\Omega\right) \tag{3.9}$$

We say that $x\left(t\right)$ is a *time domain* signal while $X\left(j\Omega\right)$ is a *frequency domain* signal. Some additional notation which is sometimes used is

$$X\left(j\Omega\right) = F\{x\left(t\right)\} \tag{3.10}$$

and

$$x\left(t\right) = F^{-1}\{X\left(j\Omega\right)\} \tag{3.11}$$

References (Chapter 5)

3.2 Properties of the Fourier Transform[2]

The Fourier Transform (FT) has several important properties which will be useful:

1. *Linearity:*
$$\alpha x_1\left(t\right) + \beta x_2\left(t\right) \leftrightarrow \alpha X_1\left(j\Omega\right) + \beta X_2\left(j\Omega\right) \tag{3.12}$$

    where α and β are constants. This property is easy to verify by plugging the left side of (3.12) into the definition of the FT.
2. *Time shift:*
$$x\left(t - \tau\right) \leftrightarrow e^{-j\Omega\tau}X\left(j\Omega\right) \tag{3.13}$$

    To derive this property we simply take the FT of $x\left(t - \tau\right)$

$$\int_{-\infty}^{\infty} x\left(t - \tau\right)e^{-j\Omega t}dt \tag{3.14}$$

    using the variable substitution $\gamma = t - \tau$ leads to

$$t = \gamma + \tau \tag{3.15}$$

    and

$$d\gamma = dt \tag{3.16}$$

    We also note that if $t = \pm\infty$ then $\tau = \pm\infty$. Substituting (3.15), (3.16), and the limits of integration into (3.14) gives

$$\begin{aligned} \int_{-\infty}^{\infty} x\left(\gamma\right)e^{-j\Omega(\gamma+\tau)}d\gamma &= e^{-j\Omega\tau}\int_{-\infty}^{\infty} x\left(\gamma\right)e^{-j\Omega\gamma}d\gamma \\ &= e^{-j\Omega\tau}X\left(j\Omega\right) \end{aligned} \tag{3.17}$$

    which is the desired result.

---

[2]This content is available online at $<$http://cnx.org/content/m32892/1.3/$>$.

3. *Frequency shift:*

$$x\left(t\right)e^{j\Omega_0 t} \leftrightarrow X\left(j\left(\Omega - \Omega_0\right)\right) \tag{3.18}$$

Deriving the frequency shift property is a bit easier than the time shift property. Again, using the definition of FT we get:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x\left(t\right)e^{j\Omega_0 t}e^{-j\Omega t}dt &= \int_{-\infty}^{\infty} x\left(t\right)e^{-j(\Omega-\Omega_0)t}dt \\
&= X\left(j\left(\Omega-\Omega_0\right)\right)
\end{aligned} \tag{3.19}
$$

4. *Time reversal:*

$$x\left(-t\right) \leftrightarrow X\left(-j\Omega\right) \tag{3.20}$$

To derive this property, we again begin with the definition of FT:

$$\int_{-\infty}^{\infty} x\left(-t\right)e^{-j\Omega t}dt \tag{3.21}$$

and make the substitution $\gamma = -t$. We observe that $dt = -d\gamma$ and that if the limits of integration for t are $\pm\infty$, then the limits of integration for γ are $\mp\gamma$. Making these substitutions into (3.21) gives

$$
\begin{aligned}
-\int_{\infty}^{-\infty} x\left(\gamma\right)e^{j\Omega\gamma}d\gamma &= \int_{-\infty}^{\infty} x\left(\gamma\right)e^{j\Omega\gamma}d\gamma \\
&= X\left(-j\Omega\right)
\end{aligned} \tag{3.22}
$$

Note that if $x\left(t\right)$ is real, then $X\left(-j\Omega\right) = X(j\Omega)^{*}$.

5. *Convolution:* The convolution integral is given by

$$y\left(t\right) = \int_{-\infty}^{\infty} x\left(\tau\right)h\left(t-\tau\right)d\tau \tag{3.23}$$

The convolution property is given by

$$Y\left(j\Omega\right) \leftrightarrow X\left(j\Omega\right)H\left(j\Omega\right) \tag{3.24}$$

To derive this important property, we again use the FT definition:

$$
\begin{aligned}
Y\left(j\Omega\right) &= \int_{-\infty}^{\infty} y\left(t\right)e^{-j\Omega t}dt \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\left(\tau\right)h\left(t-\tau\right)e^{-j\Omega t}d\tau dt \\
&= \int_{-\infty}^{\infty} x\left(\tau\right)\left[\int_{-\infty}^{\infty} h\left(t-\tau\right)e^{-j\Omega t}dt\right]d\tau
\end{aligned} \tag{3.25}
$$

Using the time shift property, the quantity in the brackets is $e^{-j\Omega\tau}H\left(j\Omega\right)$, giving

$$
\begin{aligned}
Y\left(j\Omega\right) &= \int_{-\infty}^{\infty} x\left(\tau\right)e^{-j\Omega\tau}H\left(j\Omega\right)d\tau \\
&= H\left(j\Omega\right)\int_{-\infty}^{\infty} x\left(\tau\right)e^{-j\Omega\tau}d\tau \\
&= H\left(j\Omega\right)X\left(j\Omega\right)
\end{aligned} \tag{3.26}
$$

Therefore, convolution in the time domain corresponds to multiplication in the frequency domain.

6. *Multiplication (Modulation):*

$$w\left(t\right) = x\left(t\right)y\left(t\right) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(j\left(\Omega-\Theta\right)\right)Y\left(j\Theta\right)d\Theta \tag{3.27}$$

Notice that multiplication in the time domain corresponds to convolution in the frequency domain. This property can be understood by applying the inverse Fourier Transform (3.8) to the right side of (3.27)

$$
\begin{aligned}
w\left(t\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(j\left(\Omega-\Theta\right)\right)Y\left(j\Theta\right)e^{j\Omega t}d\Theta d\Omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}Y\left(j\Theta\right)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(j\left(\Omega-\Theta\right)\right)e^{j\Omega t}d\Omega\right]d\Theta
\end{aligned}
\tag{3.28}
$$

The quantity inside the brackets is the inverse Fourier Transform of a frequency shifted Fourier Transform,

$$
\begin{aligned}
w\left(t\right) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}Y\left(j\Theta\right)\left[x\left(t\right)e^{j\Theta t}\right]d\Theta \\
&= x\left(t\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}Y\left(j\Theta\right)e^{j\Theta t}d\Theta \\
&= x\left(t\right)y\left(t\right)
\end{aligned}
\tag{3.29}
$$

The properties associated with the Fourier Transform are summarized in Table 3.1.

Property	$y\left(t\right)$	$Y\left(j\Omega\right)$
Linearity	$\alpha x_1\left(t\right)+\beta x_2\left(t\right)$	$\alpha X_1\left(j\Omega\right)+\beta X_2\left(j\Omega\right)$
Time Shift	$x\left(t-\tau\right)$	$X\left(j\Omega\right)e^{-j\Omega\tau}$
Frequency Shift	$x\left(t\right)e^{j\Omega_0 t}$	$X\left(j\left(\Omega-\Omega_0\right)\right)$
Time Reversal	$x\left(-t\right)$	$X\left(-j\Omega\right)$
Convolution	$x\left(t\right)*h\left(t\right)$	$X\left(j\Omega\right)H\left(j\Omega\right)$
Modulation	$x\left(t\right)w\left(t\right)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}X\left(j\left(\Omega-\Theta\right)\right)W\left(j\Theta\right)d\Theta$

Table 3.1: Fourier Transform properties.

3.3 Symmetry Properties of the Fourier Transform[3]

When $x\left(t\right)$ is real, the Fourier transform has *conjugate symmetry*, $X\left(-j\Omega\right)=X(j\Omega)^*$. It is not hard to see this:

$$
\begin{aligned}
X(j\Omega)^* &= \left[\int_{-\infty}^{\infty}x\left(t\right)e^{-j\Omega t}dt\right]^* \\
&= \int_{-\infty}^{\infty}\left[x\left(t\right)e^{-j\Omega t}\right]^*dt \\
&= \int_{-\infty}^{\infty}x\left(t\right)e^{j\Omega t}dt \\
&= X\left(-j\Omega\right)
\end{aligned}
\tag{3.30}
$$

where the second equality uses the definition of a Riemann integral as the limiting case of a summation, and the fact that the complex conjugate of a sum is equal to the sum of the complex conjugates. The third equality used the fact that the complex conjugate of a product is equal to the product of complex conjugates.

Letting $X\left(j\Omega\right)=a\left(j\Omega\right)+jb\left(j\Omega\right)$, it follows that

$$
X\left(-j\Omega\right)=a\left(-j\Omega\right)+jb\left(-j\Omega\right)
\tag{3.31}
$$

and

$$
X(j\Omega)^*=a\left(j\Omega\right)-jb\left(j\Omega\right)
\tag{3.32}
$$

---

[3]This content is available online at $<$http://cnx.org/content/m33894/1.1/$>$.

Equating (3.31) and (3.32) gives $a\left(j\Omega\right) = a\left(-j\Omega\right)$ and $b\left(-j\Omega\right) = -b\left(j\Omega\right)$, which implies that the real and imaginary parts of $X\left(j\Omega\right)$ have even and odd symmetry, respectively. A consequence of this is that $\left|X\left(j\Omega\right)\right| = \left|X(j\Omega)^*\right| = \left|X\left(-j\Omega\right)\right|$, that is, the magnitude of the Fourier transform has even symmetry. It can similarly be shown that the phase of the Fourier transform has odd symmetry.

3.4 The Unit Impulse Function[4]

3.4.1 The Unit Impulse Function

The unit impulse is very useful in the analysis of signals, *linear systems*, and *sampling*. Consider the plot of a rectangular pulse in Figure 3.1. Note the height of the pulse is $1/\tau$ and the width of the pulse is τ. So we can write

$$\int_{-\infty}^{\infty} x_p\left(t\right) dt = 1 \qquad (3.33)$$

As we let τ get small, then the width of the pulse gets successively narrower and its height gets progressively higher. In the limit as τ approaches zero, we have a pulse which has infinite height, and zero width, yet its area is still one. We define the *unit impulse function* as

$$\delta\left(t\right) \equiv \lim_{\tau \to 0} x_p\left(t\right) \qquad (3.34)$$
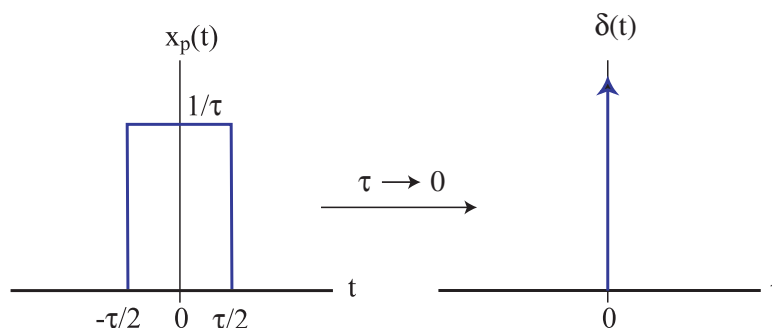

Figure 3.1: Rectangular pulse, $x_p\left(t\right)$ approaches the unit impulse function, $\delta\left(t\right)$, as τ approaches zero.

The area under $\delta\left(t\right)$ is one, and so we can write

$$\int_{-\infty}^{\infty} \delta\left(t - \tau\right) dt = 1 \qquad (3.35)$$

If we multiply the unit impulse by a constant, K, its area is now equal to that constant, i.e.

$$\int_{-\infty}^{\infty} K\delta\left(t - \tau\right) dt = K \qquad (3.36)$$

The area of the unit impulse is usually indicated by the number shown next to the arrow as seen in Figure 3.2.

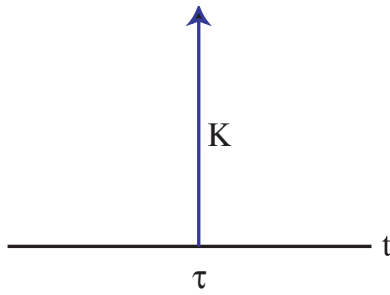[4]This content is available online at <http://cnx.org/content/m32896/1.3/>.

Figure 3.2: $K\delta(t - \tau)$.

Suppose we multiply the signal $x(t)$ with a time-shifted unit impulse, $\delta(t - \tau)$. The product is a unit impulse, having an area of $x(\tau)$. This is illustrated in Figure 3.3.
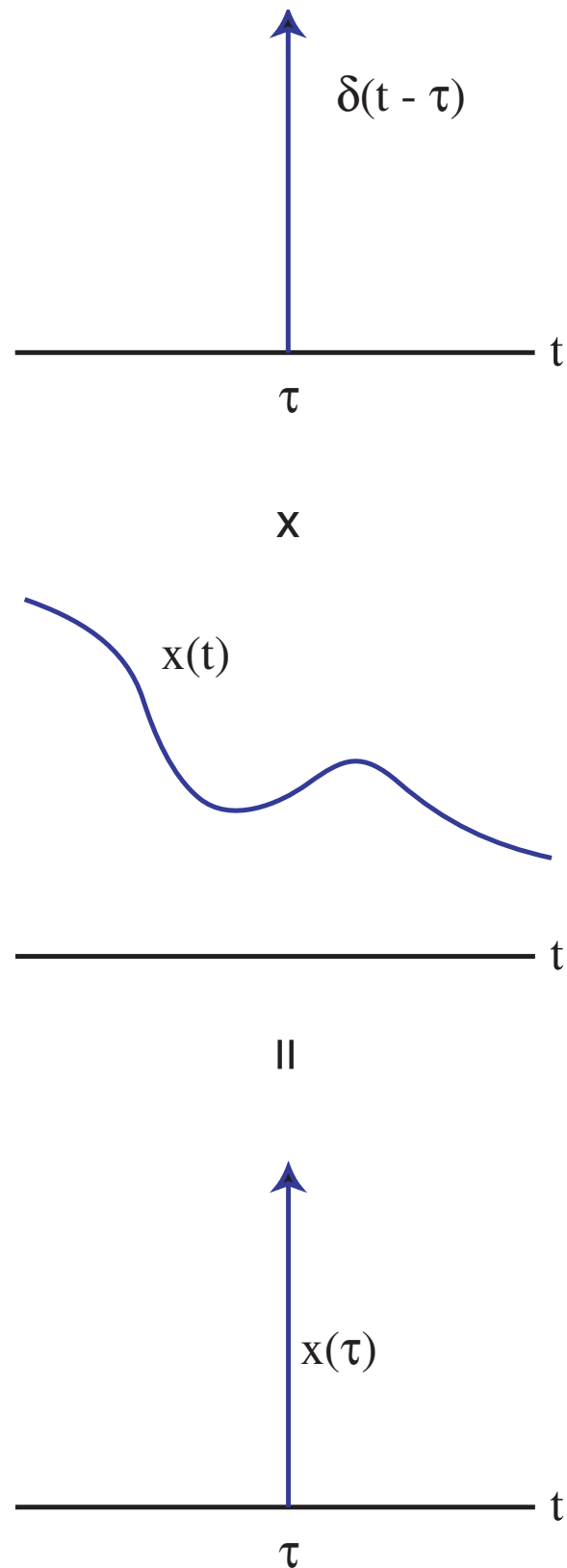
Figure 3.3: Sifting property of unit impulse, the product of the two signals, $x(t)$ and $\delta(t - \tau)$, is $x(\tau)\delta(t - \tau)$. Consequently, the area under $x(\tau)\delta(t - \tau)$ is $x(\tau)$.

In other words,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,dt = x(\tau) \tag{3.37}$$

Equation (3.37) is called the *sifting property* of the unit impulse. As we will see, the sifting property of the unit impulse will be very useful.

3.5 The Unit Step Function[5]

The unit step function is defined as

$$u(t) = \{ \begin{array}{ll} 1, & t \geq 0 \\ 0, & t < 0 \end{array} \tag{3.38}$$

This function is useful for defining signals which we wish to start at $t = 0$. In other words, often, we would like for signals to be zero for negative values of t. We can force this situation by simply multiplying by $u(t)$.

3.6 Fourier Transform of Common Signals[6]

Next, we'll derive the FT of some basic continuous-time signals. Table 3.2 summarizes these transform pairs.

3.6.1 Rectangular pulse

Let's begin with the rectangular pulse

$$\text{rect}(t,\tau) \equiv \{ \begin{array}{ll} 1, & t \leq \tau/2 \\ 0, & t > \tau/2 \end{array} \tag{3.39}$$

The pulse function, $\text{rect}(t,\tau)$ is shown in Figure 3.4. Substituting $x(t) = \text{rect}(t,\tau)$ into (3.7) gives

$$\begin{aligned} X(j\Omega) &= \int_{-\tau/2}^{\tau/2} e^{-j\Omega t}\,dt \\ &= \frac{-1}{j\Omega} e^{-j\Omega t}\Big|_{-\tau/2}^{\tau/2} \\ &= \frac{1}{j\Omega}\left[e^{j\Omega\tau/2} - e^{-j\Omega\tau/2}\right] \\ &= \tau\frac{sin(\Omega\tau/2)}{\Omega\tau/2} \\ &= \tau\text{sinc}(\Omega\tau/2\pi) \end{aligned} \tag{3.40}$$

A plot of $X(j\Omega)$ is shown in Figure 3.4.

---

[5]This content is available online at <http://cnx.org/content/m32898/1.2/>.
[6]This content is available online at <http://cnx.org/content/m32899/1.4/>.

Figure 3.4: Fourier transform pair showing the rectangular pulse signal (left) and its Fourier Transform, the sinc function (right).

Note that when $\tau = 0$, $X(j\Omega) = 1$. We now have the following transform pair:

$$\text{rect}(t, \tau) \leftrightarrow \tau \frac{\sin(\Omega\tau/2)}{\Omega\tau/2} \tag{3.41}$$

3.6.2 Impulse

The unit impulse function was described in Section 3.4. From the sifting property of the impulse function we find that

$$
\begin{aligned}
X\left(j\Omega\right) &= \int_{-\infty}^{\infty} \delta\left(t-\tau\right) e^{-j\Omega t} dt \\
&= e^{-j\Omega\tau}
\end{aligned}
\tag{3.42}
$$

or

$$
\delta\left(t-\tau\right) \leftrightarrow e^{-j\Omega\tau}
\tag{3.43}
$$

3.6.3 Complex Exponential

The complex exponential function, $x\left(t\right) = e^{j\Omega_0 t}$, has a Fourier Transform which is difficult to evaluate directly. It is easier to start with the Fourier Transform itself and work backwards using the inverse Fourier Transform. Suppose we want to find the time-domain signal which has Fourier Transform $X\left(j\Omega\right) = \delta\left(\Omega - \Omega_0\right)$. We can begin by using the inverse Fourier Transform (3.8)

$$
\begin{aligned}
x\left(t\right) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta\left(\Omega - \Omega_0\right) e^{j\Omega t} d\Omega \\
&= \frac{1}{2\pi} e^{\Omega t}
\end{aligned}
\tag{3.44}
$$

This result follows from the sifting property of the impulse function. By linearity, we can then write

$$
e^{j\Omega t} \leftrightarrow 2\pi\delta\left(\Omega - \Omega_0\right)
\tag{3.45}
$$

3.6.4 Cosine

The cosine signal can be expressed in terms of complex exponentials using Euler's Identity

$$
cos\left(\Omega_0 t\right) = \frac{1}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)
\tag{3.46}
$$

Applying linearity and the Fourier Transform of complex exponentials to the right side of (3.46), we quickly get:

$$
cos\left(\Omega_0 t\right) \leftrightarrow \pi\delta\left(\Omega - \Omega_0\right) + \pi\delta\left(\Omega + \Omega_0\right)
\tag{3.47}
$$

3.6.5 Real Exponential

The real exponential function is given by $x\left(t\right) = Ke^{-\alpha t}u\left(t\right)$, where K and α are real constants. To find its FT, we start with the definition

$$
\begin{aligned}
X\left(j\Omega\right) &= K \int_{0}^{\infty} e^{-\alpha t} e^{-j\Omega t} dt \\
&= K \int_{0}^{\infty} e^{-(\alpha + j\Omega)t} dt \\
&= \frac{-K}{\alpha + j\Omega} e^{-(\alpha + j\Omega)t}\Big|_{0}^{\infty} \\
&= \frac{-K}{\alpha + j\Omega}\left(0 - 1\right) \\
&= \frac{K}{\alpha + j\Omega}
\end{aligned}
\tag{3.48}
$$

therefore,

$$Ke^{-\alpha t}u\left(t\right) \leftrightarrow \frac{K}{\alpha + j\Omega} \tag{3.49}$$

$x\left(t\right)$	$X\left(j\Omega\right)$
rect $\left(t, \tau\right)$	$\tau \frac{sin\left(\Omega\tau/2\right)}{\Omega\tau/2}$
$\delta\left(t - \tau\right)$	$e^{-j\Omega\tau}$
$e^{j\Omega_0 t}$	$2\pi\delta\left(\Omega - \Omega_0\right)$
$cos\left(\Omega_0 t\right)$	$\pi\delta\left(\Omega - \Omega_0\right) + \pi\delta\left(\Omega + \Omega_0\right)$
$Ke^{-\alpha t}u\left(t\right)$	$\frac{K}{\alpha + j\Omega}$

Table 3.2: Some common Fourier Transform pairs.

When working problems involving finding the Fourier Transform, it is often preferable to use a table of transform pairs rather than to recalculate the Fourier Transform from scratch. Often, transform pairs in can be combined with known Fourier Transform properties to find new Fourier Transforms.

Example 3.1 Find the Fourier Transform of: $y\left(t\right) = 2e^{5t}u\left(-t\right)$. Clearly, we can write $y\left(t\right) = x\left(-t\right)$ where $x\left(t\right) = 2e^{-5t}u\left(t\right)$. Therefore, we can combine the known transform of $x\left(t\right)$ from Table 3.2, namely,

$$X\left(j\Omega\right) = \frac{2}{5 + j\Omega} \tag{3.50}$$

with the time reversal property found in Table 3.1:

$$x\left(-t\right) \leftrightarrow X\left(j\Omega\right)^* \tag{3.51}$$

to get the answer:

$$Y\left(j\Omega\right) = \frac{2}{5 - j\Omega} \tag{3.52}$$

3.7 Fourier Transform of Periodic Signals[7]

If the signal of interest is periodic with period T, then it has a Fourier Series:

$$x\left(t\right) = \sum_{n=-\infty}^{\infty} c_n e^{j\Omega_0 nt} \tag{3.53}$$

Using the linearity of the Fourier Transform, we have

$$\begin{aligned} X\left(j\Omega\right) &= \sum_{n=-\infty}^{\infty} c_n F\{e^{j\Omega_0 nt}\} \\ &= 2\pi \sum_{n=-\infty}^{\infty} c_n \delta\left(\Omega - n\Omega_0\right) \end{aligned} \tag{3.54}$$

where $F\{\ \}$ corresponds to the Fourier Transform of the signal within the brackets.

---

[7]This content is available online at $<$http://cnx.org/content/m32900/1.2/$>$.

3.8 Filters[8]

Filters are devices which are commonly found in electronic gadgets. When you adjust the *bass* (low frequency) or *treble* (high frequency) settings on your MP3 player, you are adjusting the characteristics of a filter. A more technical name for a filter is a *linear system* . A filter is represented by a box having a single input (usually $x(t)$) and a single output (say, $y(t)$) as seen in Figure 3.5.
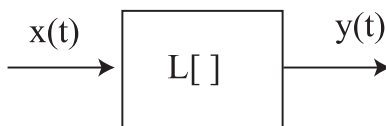

Figure 3.5: Continuous-time filter.

We can denote the operation the filter has on the input using the following notation:

$$y(t) = L[x(t)] \tag{3.55}$$

The types of filters we will consider in this book are *linear* and *time-invariant*. A filter is time-invariant if given that $y(t) = L[x(t)]$, then $y(t - \tau) = L[x(t - \tau)]$. In other words, if the input to the filter is delayed by τ, then the output is also delayed by τ. A filter is linear if given that $y_1(t) = L[x_1(t)]$ and $y_2(t) = L[x_2(t)]$ then

$$\alpha y_1(t) + \beta y_2(t) = L[\alpha x_1(t) + \beta x_2(t)] \tag{3.56}$$

Equation (3.56) is often referred to as the *superposition principle*. We can use linearity and time invariance to derive the mathematical operation which the filter performs on the input, $x(t)$. To do this we begin with the assumption that

$$h(t) = L[\delta(t)] \tag{3.57}$$

The signal $h(t)$ is called the *impulse response* of the filter. From time invariance, we have

$$h(t - \tau) = L[\delta(t - \tau)] \tag{3.58}$$

Now we can use linearity to find the filter output when the input is $x(\tau)\delta(t - \tau)$, where $x(\tau)$ is a constant

$$x(\tau)h(t - \tau) = L[x(\tau)\delta(t - \tau)] \tag{3.59}$$

We can extend the linearity property further by noting that

$$\sum_n x(\tau_n)\Delta_n h(t - \tau_n) = L\left[\sum_n x(\tau_n)\Delta_n\delta(t - \tau_n)\right] \tag{3.60}$$

where we can assume that the constants τ_n are ordered so that $\tau_i < \tau_k, i < k$ and $\Delta_n \equiv \tau_n - \tau_{n-1}$. In (3.60), we are simply multiplying each $\delta(t - \tau_n)$ by the constant $x(\tau_n)\Delta_n$, so once again linearity should prevail. Now if we take the limit $\Delta_n \to 0$, we obtain

$$\int_\infty^\infty x(\tau)h(t - \tau)d\tau = L\left[\int_\infty^\infty x(\tau)\delta(t - \tau)d\tau\right] \tag{3.61}$$

---

[8]This content is available online at <http://cnx.org/content/m32913/1.2/>.

Using the sifting property of the unit impulse in the right side of (3.61) gives

$$\int_{\infty}^{\infty} x\left(\tau\right) h\left(t-\tau\right) d\tau = L\left[x\left(t\right)\right] \tag{3.62}$$

So it follows that the filter performs the following operation on the input, $x\left(t\right)$:

$$\begin{aligned} y\left(t\right) &= L\left[x\left(t\right)\right] \\ &= \int_{-\infty}^{\infty} x\left(\tau\right) h\left(t-\tau\right) d\tau \end{aligned} \tag{3.63}$$

The integral in (3.63) is called the *convolution integral*. A change of variables can be used to show that

$$\int_{-\infty}^{\infty} h\left(\tau\right) x\left(t-\tau\right) d\tau = \int_{-\infty}^{\infty} x\left(\tau\right) h\left(t-\tau\right) d\tau \tag{3.64}$$

which means that the order in which two signals are convolved is unimportant. A short-hand notation for convolution is

$$\int_{-\infty}^{\infty} x\left(\tau\right) h\left(t-\tau\right) d\tau \equiv x\left(t\right) * h\left(t\right) \tag{3.65}$$

3.9 Properties of Convolution Integrals[9]

We list several important properties and their proofs.

1. *Commutative Property:*

$$x\left(t\right) * h\left(t\right) = h\left(t\right) * x\left(t\right) \tag{3.66}$$

Lets start with

$$x\left(t\right) * h\left(t\right) = \int_{-\infty}^{\infty} x\left(\tau\right) h\left(t-\tau\right) d\tau \tag{3.67}$$

and make the substitution $\gamma = t - \tau$. It follows that

$$\begin{aligned} x\left(t\right) * h\left(t\right) &= \int_{-\infty}^{\infty} x\left(t-\gamma\right) h\left(\gamma\right) d\gamma d\tau \\ &= h\left(t\right) * x\left(t\right) \end{aligned} \tag{3.68}$$

2. *Associative Property:*

$$\left[x\left(t\right) * h_1\left(t\right)\right] * h_2\left(t\right) = x\left(t\right) * \left[h_1\left(t\right) * h_2\left(t\right)\right] \tag{3.69}$$

To prove this property we begin with an expression for the left-hand side of (3.69)

$$\int_{-\infty}^{\infty} x\left(\tau\right) h_1\left(t-\tau\right) d\tau * h_2\left(t\right) \tag{3.70}$$

where we have expressed $x\left(t\right) * h_1\left(t\right)$ as a convolution integral. Expanding the second convolution gives

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x\left(\tau\right) h_1\left(\gamma-\tau\right) d\tau\right] h_2\left(t-\gamma\right) d\gamma \tag{3.71}$$

Reversing the order of integration gives

$$\int_{-\infty}^{\infty} x\left(\tau\right) \left[\int_{-\infty}^{\infty} h_1\left(\gamma-\tau\right) h_2\left(t-\gamma\right) d\gamma\right] d\tau \tag{3.72}$$

---

[9]This content is available online at $<$http://cnx.org/content/m32904/1.2/$>$.

Using the variable substitution $\phi = \gamma - \tau$ and integrating over ϕ in the inner integral gives the final result:

$$\int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h_1(\phi) h_2(t - \tau - \phi) \, d\gamma \right] d\tau \tag{3.73}$$

where the inner integral is recognized as $h_1(t) * h_2(t)$ evaluated at $t = t - \tau$, which is required for the convolution with $x(t)$.

3. *Distributive Property:*

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t) \tag{3.74}$$

This property is easily proven from the definition of the convolution integral.

4. *Time-Shift Property:* If $y(t) = x(t) * h(t)$ then $x(t - t_0) * h(t) = y(t - t_0)$ Again, the proof is trivial.

3.10 Evaluation of Convolution Integrals[10]

The key to evaluating a convolution integral such as

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) \, d\tau \tag{3.75}$$

is to realize that as far as the integral is concerned, the variable t is a constant and the integral is over the variable τ. Therefore, for each t, we are finding the area of the product $x(\tau) h(t - \tau)$. Let's look at an example that illustrates how this works.

Example 3.1 Find the convolution of $x(t) = u(t)$ and $h(t) = e^{-t} u(t)$. The convolution integral is given by

$$h(t) * x(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) u(t - \tau) \, d\tau \tag{3.76}$$

Figure 3.6 shows the graph of $e^{-\tau} u(\tau)$, $e^{-t} u(t)$, and their product. From the graph of the product, it is easy to see the the convolution integral becomes

$$\int_0^t e^{-\tau} d\tau = \left\{ \begin{array}{ll} 1 - e^{-t}, & t \geq 0 \\ 0, & t < 0 \end{array} \right. \tag{3.77}$$
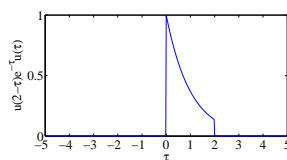
---

[10] This content is available online at $<$http://cnx.org/content/m32909/1.2/$>$.
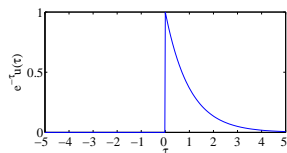
(a)
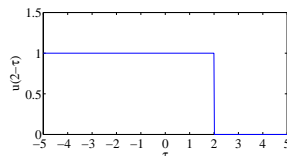
(b)

(c)

Figure 3.6: Graphs of signals used in Example .

Signals which can be expressed in functional form should be convolved as in the above example. Other signals may not have an easy functional representation but rather may be piece-wise linear. In order to convolve such signals, one must evaluate the convolution integral over different intervals on the t-axis so that each distinct interval corresponds to a different expression for $x(t) * h(t)$. The following example illustrates this:

Example 3.2 Suppose we attempt to convolve the unit step function $x(t) = u(t)$ with the trapezoidal function

$$
h(t) = \left\{ \begin{array}{ll} t, & 0 \leq t < 1 \\ 1, & 1 \leq t < 2 \\ 0, & \text{elsewhere} \end{array} \right.
\tag{3.78}
$$

From Figure 3.7, it can be seen that on the interval $0 \leq t < 1$, the product $x(t-\tau) h(\tau)$ is an equilateral triangle with area $t^2/2$. On the interval $1 \leq t < 2$, the area of $x(t-\tau) h(\tau)$ is $t-1/2$. This latter area results by adding the area of an equilateral triangle having a base of 1, and the area of a rectangle having a base of $t-1$ and a height of 1. For all values of t greater than 2, the convolution is 1.5 since $x(t-\tau) h(\tau) = h(\tau)$ and $h(\tau)$ is a trapezoid having an area of 1.5. Finally, for $t < 0$, the convolution is zero since $x(t-\tau) h(\tau) = 0$.

(a)

(b)
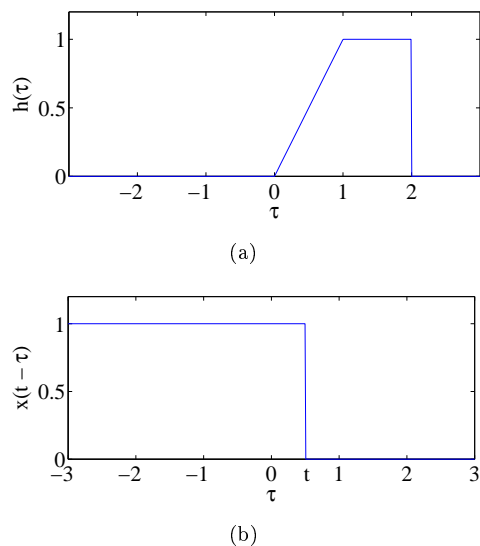
Figure 3.7: Graphs of signals used in Example .

3.11 Frequency Response[11]

Recall from Section 3.2 that the convolution integral

$$y\left(t\right) = \int_{-\infty}^{\infty} x\left(\tau\right) h\left(t - \tau\right) d\tau \tag{3.79}$$

has the Fourier Transform:

$$Y\left(j\Omega\right) = H\left(j\Omega\right) X\left(j\Omega\right) \tag{3.80}$$

where $H\left(j\Omega\right)$ and $X\left(j\Omega\right)$ are the Fourier Transforms of $h\left(t\right)$ and $x\left(t\right)$, respectively. Solving for $H\left(j\Omega\right)$ gives the *frequency response*:

$$H\left(j\Omega\right) = \frac{Y\left(j\Omega\right)}{X\left(j\Omega\right)} \tag{3.81}$$

The frequency response, the Fourier Transform of the impulse response of a filter, is useful since it gives a highly descriptive representation of the properties of the filter. The frequency response can be considered to be the *gain* of the filter, expressed as a function of frequency. The magnitude of the frequency response evaluated at $\Omega = \Omega_0$, $\left|H\left(j\Omega_0\right)\right|$ gives the factor the frequency component of $x\left(t\right)$ at $\Omega = \Omega_0$ would be scaled by. The phase of the frequency response at $\Omega = \Omega_0$, $\angle H\left(j\Omega_0\right)$ gives the phase shift the component of $x\left(t\right)$ at $\Omega = \Omega_0$ would undergo. This idea will be discussed in greater detail in . A *lowpass* filter is a filter which only passes low frequencies, while *attenuating* or filtering out higher frequencies. A *highpass* filter would do just the opposite, it would filter out low frequencies and allow high frequencies to pass. Figure 3.8 shows examples of these various filter types.
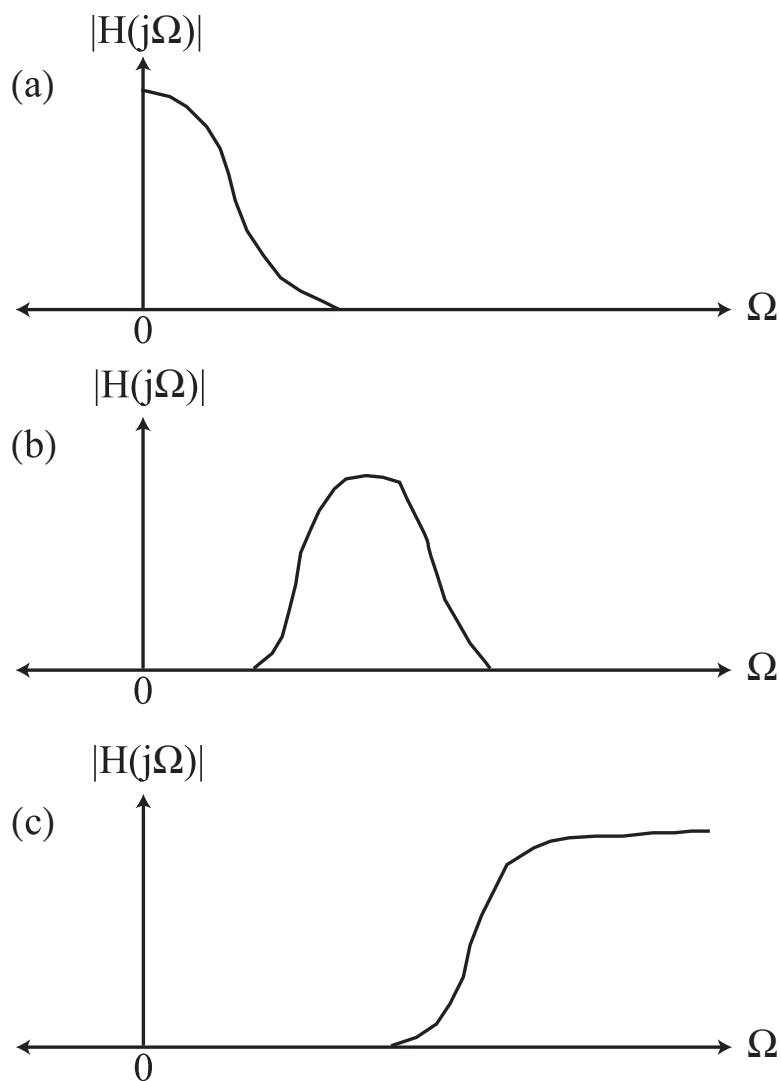
---

[11]This content is available online at <http://cnx.org/content/m32921/1.3/>.

Figure 3.8: Different filter types: (a) lowpass, (b) bandpass, (c) highpass.

3.12 The Sinusoidal Steady State Response[12]

It is useful to see what the effect of the filter is on a sinusoidal signal, say $x(t) = cos(\Omega_0 t)$. If $y(t)$ is the output of the filter, then we can write

$$y(t) = \int_{-\infty}^{\infty} cos(\Omega_0(t-\tau)) h(\tau) d\tau \tag{3.82}$$

---

[12]This content is available online at <http://cnx.org/content/m32916/1.2/>.

Using the Euler formula for $cos\left(\Omega_0 t\right)$, right hand side of (3.82) can be written as:

$$\frac{1}{2}\int_{-\infty}^{\infty}\left(e^{j(\Omega_0(t-\tau))}+e^{-j(\Omega_0(t-\tau))}\right)h\left(\tau\right)d\tau \tag{3.83}$$

This integral can be split into two separate integrals, and written as:

$$\frac{e^{j\Omega_0 t}}{2}\int_{-\infty}^{\infty}e^{-j\Omega_0\tau}h\left(\tau\right)d\tau+\frac{e^{-j\Omega_0 t}}{2}\int_{-\infty}^{\infty}e^{j\Omega_0\tau}h\left(\tau\right)d\tau \tag{3.84}$$

The first of the two integrals can be recognizes as the Fourier Transform of the impulse response evaluated at $\Omega=\Omega_0$. The second integral is just the complex conjugate of the first integral. Therefore (3.84) can be written as:

$$\frac{e^{j\Omega_0 t}}{2}H\left(j\Omega_0\right)+\frac{e^{-j\Omega_0 t}}{2}H^{*}\left(j\Omega_0\right) \tag{3.85}$$

Since the second term in (3.85) is the complex conjugate of the first term, we can express (3.85) as:

$$Re\{e^{j\Omega_0 t}H\left(j\Omega_0\right)\} \tag{3.86}$$

or expressing $H\left(j\Omega_0\right)$ in terms of polar coordinates:

$$Re\{e^{j\Omega_0 t}|H\left(j\Omega_0\right)|e^{j\angle H(j\Omega_0)}\}=Re\{|H\left(j\Omega_0\right)|e^{j(\Omega_0 t+\angle H(j\Omega_0))}\} \tag{3.87}$$

Therefore, we find that the filter output is given by

$$y\left(t\right)=|H\left(j\Omega_0\right)|cos\left(\Omega_0 t+\angle H\left(j\Omega_0\right)\right) \tag{3.88}$$

This is called the *sinusoidal steady state response*. It tells us that when the input to a linear, time-invariant filter is a cosine, the filter output is a cosine whose amplitude has been scaled by $|H\left(j\Omega_0\right)|$ and that has been phase shifted by $\angle H\left(j\Omega_0\right)$. The same result applies to an input that is an arbitrarily phase shifted cosine (e.g. a sine wave).

   Example 3.1 Find the output of a filter whose impulse response is $h\left(t\right)=e^{-5t}u\left(t\right)$ and whose input is given by $x\left(t\right)=cos\left(2t\right)$. It can be readily seen that the frequency response of the filter is

$$H\left(j\Omega\right)=\frac{1}{5+j\Omega} \tag{3.89}$$

and therefore $|H\left(j2\right)|=0.1857$ and $\angle H\left(j2\right)=-0.3805$. Therefore, using (3.88):

$$y\left(t\right)=0.1857cos\left(2t-0.3805\right) \tag{3.90}$$

3.13 Parallel and Cascaded Filters[13]

In some applications, such as graphic equalizers, it is useful to place filters in parallel as shown in Figure 3.9. Can the parallel combination of filters be characterized by a single equivalent filter $h_{eq}\left(t\right)$? The answer is yes and results by noting that

$$\begin{aligned}y\left(t\right)&=&\sum_{i=1}^{N}x\left(t\right)*h_i\left(t\right)\\&=&\sum_{i=1}^{N}\int_{-\infty}^{\infty}x\left(t-\tau\right)h_i\left(\tau\right)d\tau\\&=&\int_{-\infty}^{\infty}x\left(t-\tau\right)\sum_{i=1}^{N}h_i\left(\tau\right)d\tau\end{aligned} \tag{3.91}$$

---

[13]This content is available online at $<$http://cnx.org/content/m32919/1.3/$>$.

Therefore, the last equation in (3.91) shows that

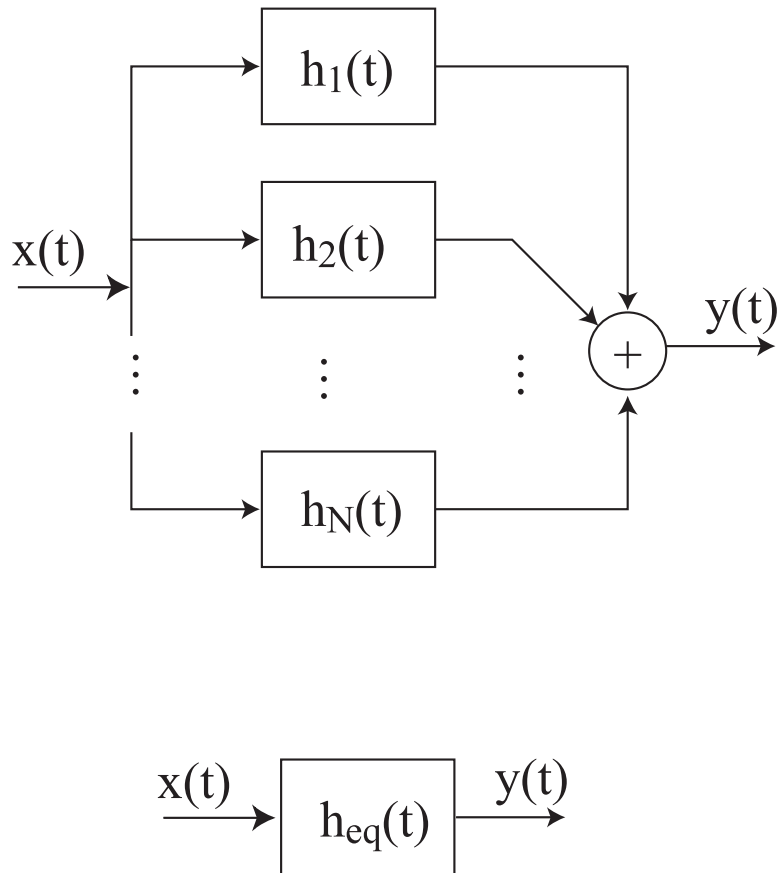$$h_{eq}(t) = \sum_{i=1}^{N} h_i(t) \tag{3.92}$$

Figure 3.9: Parallel filter structure. We wish to find an equivalent filter with impulse response $h_{eq}(t)$.

The equivalent transfer function for the parallel filter structure is given by

$$H_{eq}(j\Omega) = \sum_{i=1}^{N} H_i(j\Omega) \tag{3.93}$$

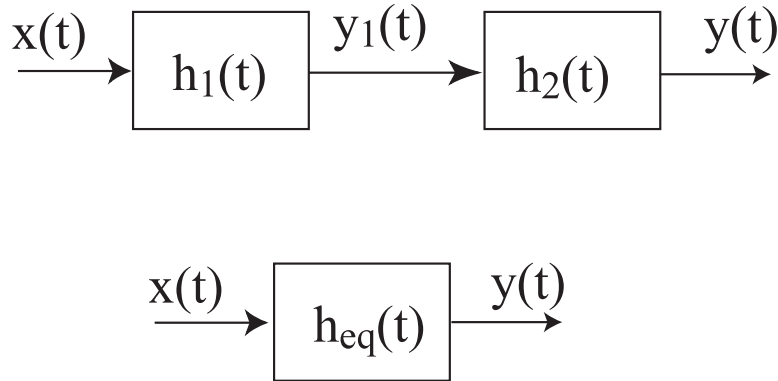Next we wish to find an equivalent filter for the cascaded structure shown in Figure 3.10.

Figure 3.10: Cascaded filter structure. We wish to find an equivalent filter with impulse response $h_{eq}(t)$.

This can be done by finding an expression for the intermediate value $y_1(t)$:

$$y_1(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h_1(\tau)\, d\tau \tag{3.94}$$

The output of the cascaded structure is given by

$$y(t) = \int_{-\infty}^{\infty} y_1(t-\gamma)\, h_2(\gamma)\, d\gamma \tag{3.95}$$

substituting (3.94) into (3.95) gives

$$y(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t-\gamma-\tau)\, h_1(\tau)\, d\tau \right] h_2(\gamma)\, d\gamma \tag{3.96}$$

Reversing the order of integration and rearranging slightly gives

$$y(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t-\gamma-\tau)\, h_1(\tau)\, h_2(\gamma)\, d\gamma d\tau \tag{3.97}$$

Now let $\xi = \gamma + \tau$, solving for τ gives $\tau = \xi - \gamma$ and $d\xi = d\tau$. Substituting these quantities into (3.97) leads to

$$y(t) = \int_{-\infty}^{\infty} x(t-\xi) \left[\int_{-\infty}^{\infty} h_1(\xi-\gamma)\, h_2(\gamma)\, d\gamma \right] d\xi \tag{3.98}$$

Notice that we can factor $x(t-\xi)$ from the inner integral since $x(t-\xi)$ does not depend on γ. The integral in the brackets is recognized as $h_1(t) * h_2(t)$ evaluated at ξ. Therefore for the cascaded system, the equivalent impulse response is given by

$$h_{eq}(t) = \int_{-\infty}^{\infty} h_1(t-\gamma)\, h_2(\gamma)\, d\gamma \tag{3.99}$$

This can be generalized to any number of cascaded filters giving

$$h_{eq}(t) = h_1(t) * h_1(t) * \cdots * h_N(t) \tag{3.100}$$

3.14 First Order Filters[14]

A first-order lowpass filter has the frequency response

$$H_{LP}(j\Omega) = \frac{1}{1 + j\frac{\Omega}{\Omega_c}} \tag{3.101}$$

The frequency at which the frequency response magnitude has dropped to $1/\sqrt{2}$ is called the *corner frequency*[15]. The frequency response magnitude and phase are plotted in Figure 3.11. It is common to express the frequency response magnitude in units of *decibels* (dB) using the formula

$$20log_{10}|H(j\Omega)| \tag{3.102}$$

At the corner frequency for a first order lowpass filter, the frequency response magnitude is $1/\sqrt{2}$ or roughly -3 dB. From Table 3.2, it can easily be seen that the impulse response for the first-order lowpass filter is given by

$$h_{LP}(t) = \Omega_c e^{-\Omega_c t} u(t) \tag{3.103}$$

A first-order highpass filter is given by

$$H_{HP}(j\Omega) = \frac{j\frac{\Omega}{\Omega_c}}{1 + j\frac{\Omega}{\Omega_c}} \tag{3.104}$$

Notice that

$$H_{HP}(j\Omega) = 1 - \frac{1}{1 + j\frac{\Omega}{\Omega_c}} \tag{3.105}$$

This makes sense since a highpass filter can be constructed by taking the filter input $x(t)$ and subtracting from it a lowpass filtered version of $x(t)$. The impulse response of the first-order highpass filter therefore becomes:

$$h_{HP}(t) = \delta(t) - \Omega_c e^{-\Omega_c t} u(t) \tag{3.106}$$

---

[14]This content is available online at <http://cnx.org/content/m32925/1.3/>.

[15]This term most likely originates from it's role in Bode plots, a shortcut method for sketching the graph of a frequency response.
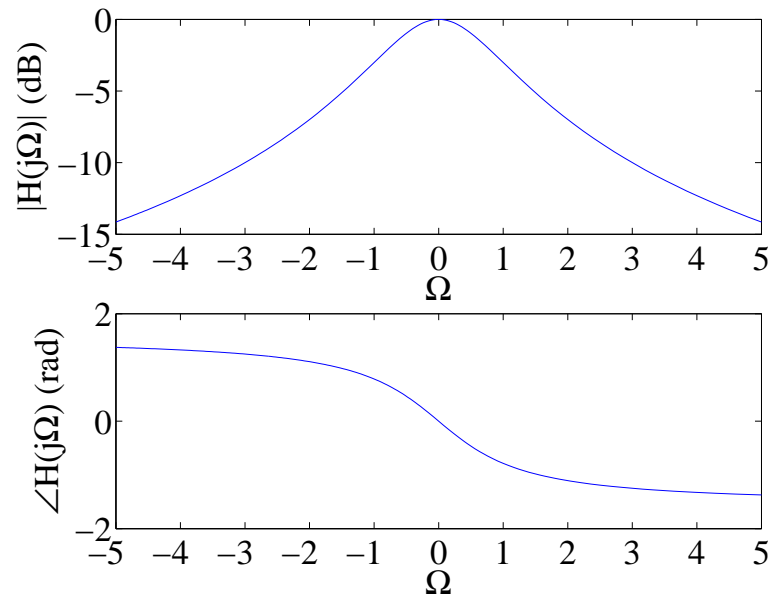
Figure 3.11: Frequency response magnitude and phase for a first-order lowpass filter ($\Omega_c = 1$ rad/sec).
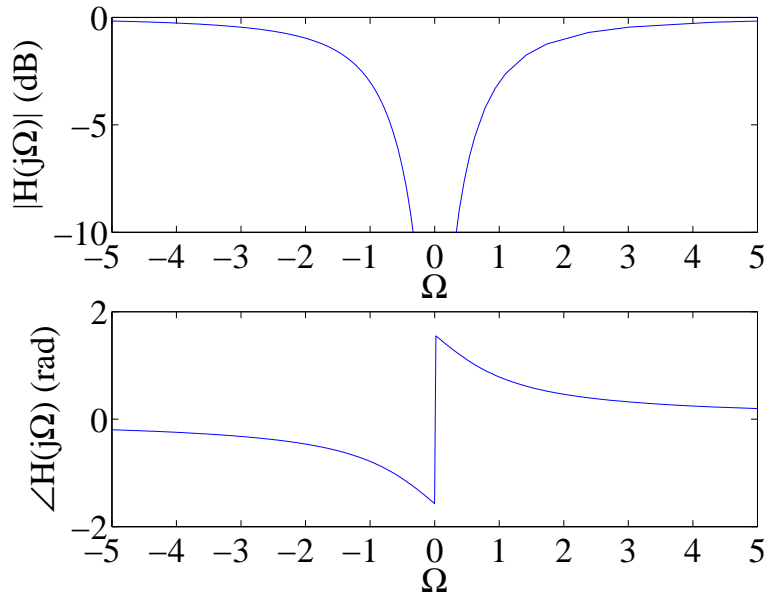
Figure 3.12: Frequency response magnitude and phase for a first-order highpass filter ($\Omega_c = 1$ rad/sec).

First order filters can be easily implemented using linear circuit elements like resistors, capacitors, and inductors. Figure 3.13 shows a first order filter based on a resistor and a capacitor. Since the impedance for a resistor and capacitor are R and $1/j\Omega C$, respectively, voltage division leads to a frequency response of

$$H_{LP}\left(j\Omega\right) = \frac{1}{1 + j\Omega R_1 C_1} \tag{3.107}$$

Therefore the corner frequency for this filter is $\Omega_c = \frac{1}{R_1 C_1}$. Similarly, a first-order highpass filter can be implemented using a resistor and capacitor as shown in Figure 3.14. This filter has a frequency response of

$$H_{HP}\left(j\Omega\right) = \frac{j\Omega R_2 C_2}{1 + j\Omega R_2 C_2} \tag{3.108}$$

The corner frequency for the highpass filter is seen to be $\Omega_c = \frac{1}{R_2 C_2}$.
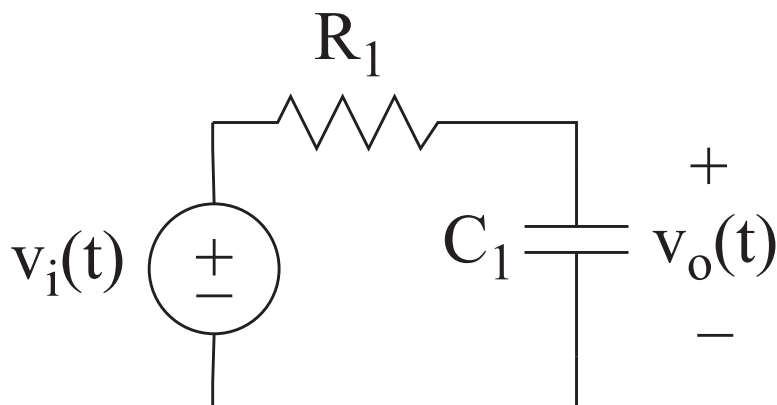
Figure 3.13: Circuit implementation of a first-order lowpass filter having $\Omega_c = 1/R_1 C_1$.
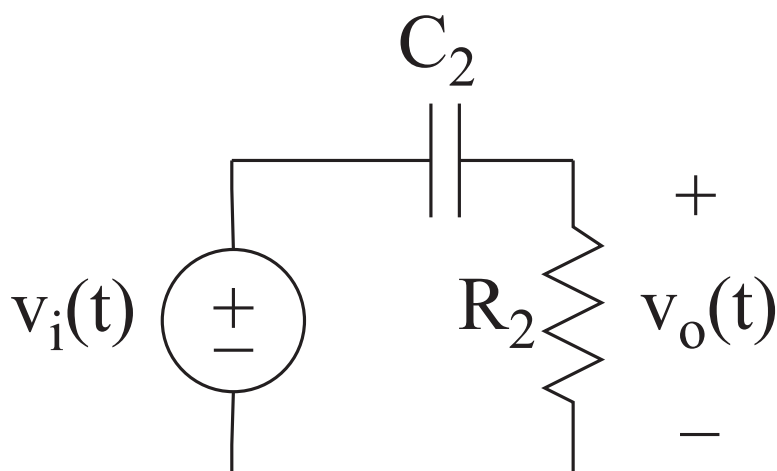
Figure 3.14: Circuit implementation of a first-order highpass filter having $\Omega_c = 1/R_2 C_2$.

Now one might be tempted to apply the results of Section 3.13 to build a bandpass filter by cascading the lowhpass and highpass circuits in Figures Figure 3.13 and Figure 3.14, respectively. Theory would predict that the equivalent frequency response of this circuit is given by

$$H_{eq}(j\Omega) = H_{LP}(j\Omega) H_{HP}(j\Omega) \tag{3.109}$$

Unfortunately, this is not possible since the circuit elements in the lowpass and highpass filters interact with one another and therefore affect the overall behavior of the circuit. This interaction between the two circuits

is called *loading* will be studied in greater detail in the exercises. To get theoretical behavior, it is necessary to use a *voltage follower* circuit, between the lowpass filter from the highpass circuits. The voltage follower circuit is usually an active circuit (requires external power supply) that has very high input impedance and very low output impedance. This eliminates any loading effects which would normally occur between the lowpass and highpass filter circuits.

3.15 Parseval's Theorem for the Fourier Transform[16]

In Chapter 2, we looked at a version of Parseval's theorem for the Fourier series. Here, we will look at a similar version of this theorem for the Fourier transform. Recall that the energy of a signal is given by

$$e_x = \int_{-\infty}^{\infty} x(t)^2 dt \tag{3.110}$$

If the energy is finite then $x(t)$ is an energy signal, as described in Chapter 1. Suppose $x(t)$ is an energy signal, then the *autocorrelation* function is defined as

$$r_x(t) = x(t) * x(-t) \tag{3.111}$$

It can be shown that $r_x(t)$ is an even function of t and that $r_x(0) = e_x$ (see Exercises). The Fourier transform of $r_x(t)$ is given by $X(j\Omega) X(j\Omega)^* = |X(j\Omega)|^2$. If follows that

$$
\begin{aligned}
e_x &= \left. \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 e^{j\Omega t} d\Omega \right|_{t=0} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\Omega)|^2 d\Omega
\end{aligned} \tag{3.112}
$$

Which is Parseval's theorem for the Fourier transform.

3.16 The Fourier Transform: Excercises[17]

1. Find the Fourier Transform of the following signals, for each case sketch the magnitude of the Fourier Transform:

   a. $x(t) = 4e^{-0.2t} u(t)$
   b. $x(t) = 4e^{0.2t} u(-t)$
   c. $x(t) = 4e^{-0.2(t-10)} u(t-10)$
   d. $x(t) = \delta(t-5)$
   e. $x(t) = \begin{cases} 1, |t| \le 0.5 \\ 0, |t| > 0.5 \end{cases}$
   f. $x(t) = 4e^{-j0.2t}$
   g. $x(t) = \cos(10\pi t)$
   h. $x(t) = 6$
   i. $x(t) = \begin{cases} \cos(100t), & |t| \le 0.5 \\ 0, & |t| > 0.5 \end{cases}$

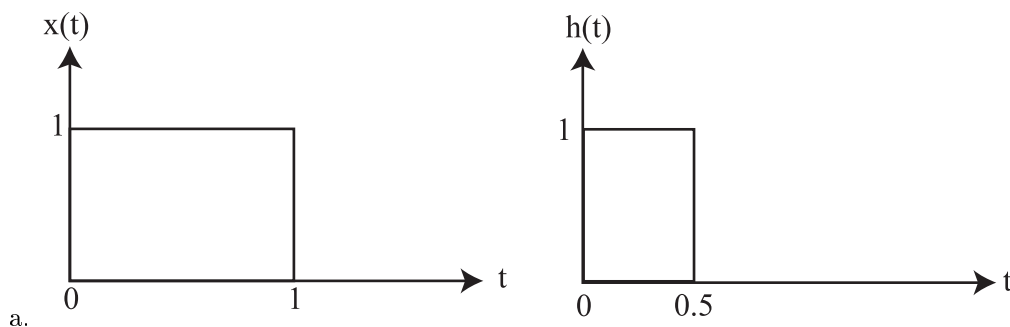2. Find the convolution of the following pairs of signals:
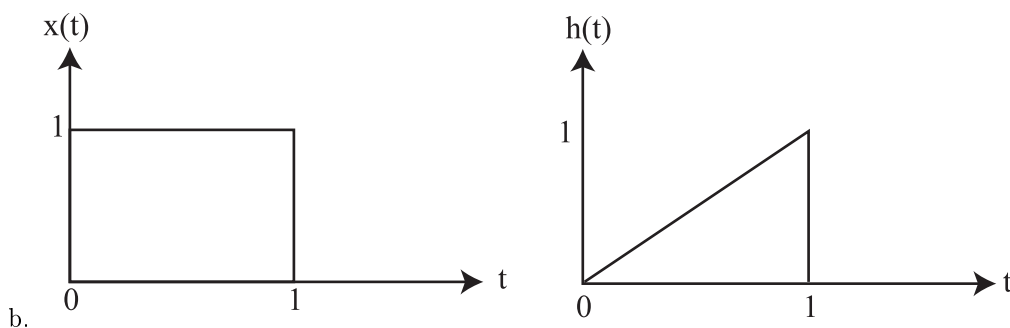

Figure 3.15

Figure 3.16

3. Find the output of the filter whose transfer function is

$$H\left(j\Omega\right) = \frac{2\pi}{2\pi + j\Omega} \tag{3.113}$$

and whose input is $x\left(t\right) = u\left(t\right)$. Hint, find the impulse response $h\left(t\right)$ corresponding to $H\left(j\Omega\right)$ and convolve it with the input.

4. Show that if $v\left(t\right) = L\left[u\left(t\right)\right]$, then

$$\int_{\infty}^{\infty} v\left(t\right) dt = L\left[\int_{\infty}^{\infty} u\left(t\right) dt\right] \tag{3.114}$$

Hint: Integrate both sides of $v\left(t\right) = L\left[u\left(t\right)\right]$. Then express the right hand integral as the limit of a sum (as in a calculus textbook). Then by linearity, you can exchange the sum and the $L\left[\cdot\right]$.

5. Find an expression for the convolution of $x\left(t\right) = u\left(t\right)$ and $h\left(t\right) = sin\left(8t\right) u\left(t\right)$

6. Find an expression for the convolution of $x\left(t\right) = rect\left(t - 0.5, 1\right)$ and $h\left(t\right) = e^{-t} u\left(t\right)$.

7. Find the Fourier transform of the periodic signal in problem 2, Chapter 2.

8. Consider a filter having the impulse response

$$h\left(t\right) = e^{-2t} u\left(t\right) \tag{3.115}$$

Sketch the frequency response (both magnitude and phase) of the filter and find the output of the filter when the input is $x\left(t\right) = cos\left(10t\right)$.

9. Repeat the previous problem for the impulse response given by

$$h(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & \text{otherwise} \end{cases} \tag{3.116}$$

10. Suppose that two filters having impulse responses $h_1(t)$ and $h_2(t)$ are cascaded (i.e. connected in series). Find the impulse response of the equivalent filter assuming $h_1(t) = 10e^{-10t}u(t)$ and $h_2(t) = 5e^{-5t}u(t)$.

11. Design a first-order lowpass filter having a corner frequency of 100 Hz. Use a $100k\Omega$ resistor. Plot both the magnitude and phase of the filter's frequency response.

12. Design a first-order highpass filter having a corner frequency of 1000 Hz. Use a $0.01\mu F$ capacitor. Plot both the magnitude and phase of the filter's frequency response.

13. The following problems are associated with the circuits in Figure 3.17:
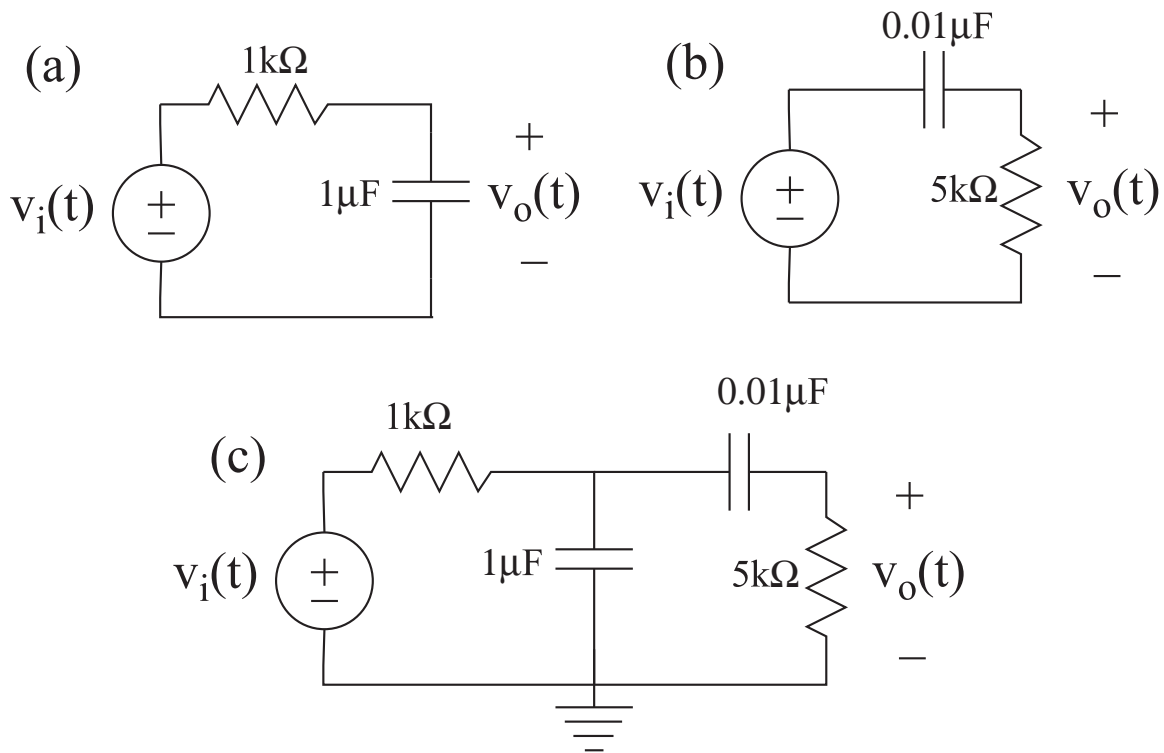
Figure 3.17: Problem list, p. 65.

    a. Find the frequency response of the circuit in Figure 3.17(a), and sketch its magnitude and phase.

    b. Find the frequency response of the circuit in Figure 3.17(b) and sketch its magnitude and phase.

    c. Find the frequency response of the filter in Figure 3.17(c), sketch its magnitude and phase and show that it is not the product of the frequency responses for problems list, p. 65 and list, p. 65.

Chapter 4

The Laplace Transform

4.1 The Laplace Transform: Introduction[1]

4.1.1 The Double-Sided Laplace Transform

You may have noticed that we avoided taking the Fourier transform of a number of signals. For example, we did not try to compute the Fourier Transform of the unit step function, $x(t) = u(t)$, or the ramp function $x(t) = tu(t)$. The reason for this is that these signals do not have a Fourier transform that converges to a finite value for all Ω. Recall that in order to deal with periodic signals such as $cos(\Omega_0 t)$ we had to settle for Fourier transforms having impulse functions in them. The Laplace transform gives us a mechanism for dealing with signals that do not have finite-valued Fourier Transforms.

The double-sided Laplace Transform of $x(t)$ is defined as follows:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt \tag{4.1}$$

where we define

$$s = \sigma + j\Omega \tag{4.2}$$

We observe that the Laplace transform is a generalization of the Fourier transform since

$$X(j\Omega) = X(s)|_{s=j\Omega} \tag{4.3}$$

Therefore, we can write

$$X(s) = F\{e^{-\sigma t} x(t)\} \tag{4.4}$$

The inverse Laplace transform can be derived using this idea. Applying the inverse Fourier transform to $F\{e^{-\sigma t} x(t)\}$ gives

$$e^{-\sigma t} x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F\{e^{-\sigma t} x(t)\} e^{j\Omega t} d\Omega \tag{4.5}$$

Using (4.2) leads to

$$d\Omega = \frac{ds}{j} \tag{4.6}$$

---

[1]This content is available online at <http://cnx.org/content/m32847/1.3/>.

Substituting s for Ω and solving for $x(t)$ in (4.5) gives

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\, e^{st} ds \tag{4.7}$$

Equation (4.7) is called the inverse Laplace transform. The integration is along a straight line in the complex *s-plane* corresponding to a fixed value of σ. This is illustrated in Figure 4.1.
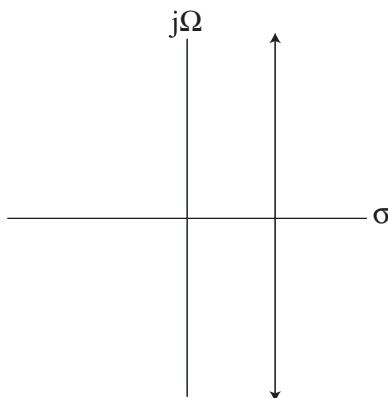

Figure 4.1: The inverse Laplace transform integrates along a line having a constant σ in the complex $s-$plane.

It is important that this line exist in a region of the *s*-plane that corresponds to the region of convergence for the Laplace transform. The region of convergence is defined as that region in the *s*-plane for which

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-st} \right| dt < \infty \tag{4.8}$$

Note that since $s = \sigma + j\Omega$ this is equivalent to

$$\int_{-\infty}^{\infty} \left| x(t)\, e^{-\sigma t} \right| dt < \infty \tag{4.9}$$

4.1.2 The Single-Sided Laplace Transform

We define the single-sided Laplace transform as

$$X(s) = \int_{0^-}^{\infty} x(t)\, e^{-st} dt \tag{4.10}$$

where the lower limit of integration tacitly includes the point $t = 0$. That is, 0^- represents a point just to the negative side of $t = 0$. This allows for the integral to take into account signal features that occur at $t = 0$ such as a step or impulse function. The single-sided Laplace transform is motivated by the fact that most signals are *turned on* at some point. If the region of convergence is not specified then different signals can yield identical bilateral Laplace transforms, as the following example illustrates.

Example 3.1 Consider the signal

$$x_1 (t) = e^{-\alpha t} u (t) \tag{4.11}$$

The bilateral Laplace transform is given by

$$
\begin{aligned}
X_1 (s) &= \int_0^\infty e^{-\alpha t} e^{-st} dt \\
&= \int_0^\infty e^{-(s+\alpha)t} dt \\
&= \left. \frac{-1}{s+\alpha} e^{-(\sigma+j\Omega+\alpha)t} \right|_0^\infty \\
&= \frac{-1}{s+\alpha} \left(e^{-(\sigma+\alpha)\infty} e^{j\Omega\infty} - 1 \right)
\end{aligned}
\tag{4.12}
$$

the magnitude of $e^{-(\sigma+\alpha)\infty}$ is zero only if $\sigma > -\alpha$ which establishes the region of convergence. Therefore we have

$$e^{-\alpha t} u (t) \leftrightarrow \frac{1}{s+\alpha}, \sigma > -\alpha \tag{4.13}$$

Now consider the Laplace transform of the signal

$$x_2 (t) = -e^{-\alpha t} u (-t) \tag{4.14}$$

We have

$$
\begin{aligned}
X_2 (s) &= -\int_{-\infty}^0 e^{-\alpha t} e^{-st} dt \\
&= \int_{-\infty}^0 e^{-(s+\alpha)t} dt \\
&= \left. \frac{-1}{s+\alpha} e^{-(\sigma+j\Omega+\alpha)t} \right|_{-\infty}^0 \\
&= \frac{1}{s+\alpha} \left(1 - e^{(\sigma+\alpha)\infty} e^{j\Omega\infty} \right)
\end{aligned}
\tag{4.15}
$$

Here, the quantity $e^{(\sigma+\alpha)\infty}$ is zero only if $\sigma < -\alpha$ so we have

$$-e^{-\alpha t} u (-t) \leftrightarrow \frac{1}{s+\alpha}, \sigma < -\alpha \tag{4.16}$$

which is identical to $X_1 (s)$ except for the region of convergence.

To avoid scenarios where two different signals have the same bilateral Laplace transform, we restrict our signals to those which are assumed to be zero for $t < 0$, for which the single-sided Laplace transform applies. Such signals are called *causal signals*. This sets the stage for the single-sided Laplace transform to be discussed in the next section.

The region of convergence for the single-sided Laplace transform is a region in the s-plane satisfying $\sigma > \sigma_{min}$ as shown in Figure 4.2.
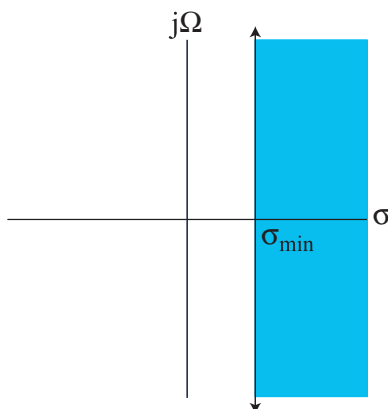
Figure 4.2: The region of convergence of the single-sided Laplace transform in the complex $s-$plane.

To see this, we observe that if

$$\int_{0^-}^{\infty} \left| x\left(t\right) e^{-\sigma_{min}t} \right| dt < \infty \tag{4.17}$$

then it must be the case that

$$\int_{0^-}^{\infty} \left| x\left(t\right) e^{-\sigma t} \right| dt < \infty \tag{4.18}$$

for $\sigma > \sigma_{min}$, since $e^{-\sigma t}$ decreases faster than $e^{-\sigma_{min}t}$. Finally, the inverse single-sided Laplace transform is the same as the inverse double-sided Laplace transform (see (4.7)), since a single sided Laplace transform can be interpreted as the double-sided Laplace transform of a signal satisfying $x\left(t\right) = 0, t < 0$. From here on, we will work exclusively with the single-sided Laplace transform. Unless we need to specifically differentiate between the single or double-sided transforms, we will refer to the single-sided Laplace transform as simply the "Laplace transform".

4.2 Properties of the Laplace Transform[2]

4.2.1 Properties of the Laplace Transform

The properties associated with the Laplace transform are similar to those of the Fourier transform. First, let's set define some notation, we will use the notation $\mathfrak{L}\{\}$ to denote the Laplace transform operation. Therefore we can write $X\left(s\right) = \mathfrak{L}\{x\left(t\right)\}$ and $x\left(t\right) = \mathfrak{L}^{-1}\{X\left(s\right)\}$ for the *forward* and *inverse* Laplace transforms, respectively. We can also use the transform pair notation used earlier:

$$x\left(t\right) \leftrightarrow X\left(s\right) \tag{4.19}$$

With this notation defined, lets now look at some properties.

---
[2]This content is available online at $<$http://cnx.org/content/m32848/1.6/$>$.

4.2.1.1 Linearity

Given that $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ then for any constants α and β, we have

$$\alpha x_1(t) + \beta x_2(t) \leftrightarrow \alpha X_1(s) + \beta X_2(s) \tag{4.20}$$

The linearity property follows easily using the definition of the Laplace transform.

4.2.1.2 Time Delay

The reason we call this the *time delay* property rather than the time shift property is that the time shift must be positive, i.e. if $\tau > 0$, then $x(t - \tau)$ corresponds to a delay. If $\tau < 0$ then we would not be able to use the single-sided Laplace transform because we would have a lower integration limit of τ, which is less than zero. To derive the property, lets evaluate the Laplace transform of the time-delayed signal

$$\mathfrak{L}\{x(t - \tau)\} = \int_0^\infty x(t - \tau) e^{-st} dt \tag{4.21}$$

Letting $\gamma = t - \tau$ leads to $t = \gamma + \tau$ and $dt = d\gamma$. Substituting these quantities into (4.21) gives

$$\begin{aligned}
\mathfrak{L}\{x(t - \tau)\} &= \int_{-\tau}^\infty x(\gamma) e^{-s(\gamma + \tau)} d\gamma \\
&= e^{-s\tau} \int_{-\tau}^\infty x(\gamma) e^{-s\gamma} d\gamma \\
&= e^{-s\tau} \int_{-\tau}^0 x(\gamma) e^{-s\gamma} d\gamma + e^{-s\tau} \int_0^\infty x(\gamma) e^{-s\gamma} d\gamma
\end{aligned} \tag{4.22}$$

where we note that the first integral in the last line is zero since $x(t) = 0, t < 0$. Therefore the time delay property is given by

$$\mathfrak{L}\{x(t - \tau)\} = e^{-s\tau} X(s) \tag{4.23}$$

4.2.1.3 s-Shift

This property is the Laplace transform corresponds to the frequency shift property of the Fourier transform. In fact, the derivation of the s-shift property is virtually identical to that of the frequency shift property.

$$\begin{aligned}
\mathfrak{L}\{e^{-at} x(t)\} &= \int_0^\infty e^{-at} x(t) e^{-st} dt \\
&= \int_0^\infty x(t) e^{-(a+s)t} dt \\
&= \int_0^\infty x(t) e^{-(a+\sigma+j\Omega)t} dt \\
&= X(s + a)
\end{aligned} \tag{4.24}$$

The s-shift property also alters the region of convergence of the Laplace transform. If the region of convergence for $X(s)$ is $\sigma > \sigma_{min}$, then the region of convergence for $\mathfrak{L}\{e^{-at} x(t)\}$ is $\sigma > \sigma_{min} - \text{Re}(a)$.

4.2.1.4 Multiplication by t

Let's begin by taking the derivative of the Laplace transform:

$$\begin{aligned}
\frac{dX(s)}{ds} &= \frac{d}{ds} \int_0^\infty x(t) e^{-st} dt \\
&= \int_0^\infty x(t) \frac{d}{ds} e^{-st} dt \\
&= -\int_0^\infty t x(t) e^{-st} dt
\end{aligned} \tag{4.25}$$

So we can write

$$\mathfrak{L}\{tx\left(t\right)\} = -\frac{dX\left(s\right)}{ds} \tag{4.26}$$

This idea can be extended to multiplication by t^n. Letting $y\left(t\right) = tx\left(t\right)$, if follows that

$$\begin{aligned} ty\left(t\right) &\leftrightarrow & -\frac{dY(s)}{ds} \\ &\leftrightarrow & \frac{d^2X(s)}{ds^2} \end{aligned} \tag{4.27}$$

Proceeding in this manner, we find that

$$t^n x\left(t\right) \leftrightarrow \left(-1\right)^n \frac{d^n X\left(s\right)}{ds^n} \tag{4.28}$$

4.2.1.5 Time Scaling

The time scaling property for the Laplace transform is similar to that of the Fourier transform:

$$\begin{aligned} \mathfrak{L}\{x\left(\alpha t\right)\} &=& \int_0^\infty x\left(\alpha t\right)e^{-st}dt \\ &=& \frac{1}{\alpha}\int_0^\infty x\left(\gamma\right)e^{-\frac{s}{\alpha}\gamma}d\gamma \\ &=& \frac{1}{\alpha}X\left(\frac{s}{\alpha}\right) \end{aligned} \tag{4.29}$$

where in the second equality, we made the substitution $t = \frac{\gamma}{\alpha}$ and $dt = \frac{d\gamma}{\alpha}$.

4.2.1.6 Convolution

The derivation of the convolution property for the Laplace transform is virtually identical to that of the Fourier transform. We begin with

$$\mathfrak{L}\{\int_{-\infty}^\infty x\left(\tau\right)h\left(t-\tau\right)d\tau\} = \int_{-\infty}^\infty x\left(\tau\right)\mathfrak{L}\{h\left(t-\tau\right)\}d\tau \tag{4.30}$$

Applying the time-delay property of the Laplace transform gives

$$\begin{aligned} \int_{-\infty}^\infty x\left(\tau\right)\mathfrak{L}\{h\left(t-\tau\right)\}d\tau &=& H\left(s\right)\int_{-\infty}^\infty x\left(\tau\right)e^{-s\tau}d\tau \\ &=& H\left(s\right)X\left(s\right) \end{aligned} \tag{4.31}$$

If $h\left(t\right)$ is the the impulse response of a linear time-invariant system, then we call $H\left(s\right)$ the *system function* of the system. The frequency response results by setting $s = j\Omega$ in $H\left(s\right)$. The system function provides us with a very powerful means of determining the output of a linear time-invariant filter given the input signal. It will also enable us to determine a means of establishing the stability[3] of a linear-time invariant filter, something which was not possible with the frequency response.

4.2.1.7 Differentiation

The Laplace transform of the derivative of a signal will be used widely. Consider

$$\mathfrak{L}\{\frac{d}{dt}x\left(t\right)\} = \int_{0^-}^\infty x^{'}\left(t\right)e^{-st}dt \tag{4.32}$$

---

[3]We will discuss stability shortly

this can be integrated by parts:

$$
\begin{aligned}
u &= e^{-st} & v^{'} &= x^{'}(t) \\
u^{'} &= -se^{-st} & v &= x(t)
\end{aligned}
\tag{4.33}
$$

which gives

$$
\begin{aligned}
\mathfrak{L}\{\tfrac{d}{dt}x(t)\} &= uv|_{0^-}^{\infty} - \int_{0^-}^{\infty} u^{'}v\,dt \\
&= e^{-st}x(t)|_{0^-}^{\infty} + \int_{0^-}^{\infty} sx(t)e^{-st}dt \\
&= -x(0^-) + sX(s)
\end{aligned}
\tag{4.34}
$$

therefore we have,

$$
\frac{d}{dt}x(t) \leftrightarrow sX(s) - x(0^-)
\tag{4.35}
$$

4.2.1.8 Higher Order Derivatives

The previous derivation can be extended to higher order derivatives. Consider

$$
y(t) = \frac{dx(t)}{dt} \leftrightarrow sX(s) - x(0^-)
\tag{4.36}
$$

it follows that

$$
\frac{dy(t)}{dt} \leftrightarrow sY(s) - y(0^-)
\tag{4.37}
$$

which leads to

$$
\frac{d^2}{dt^2}x(t) \leftrightarrow s^2 X(s) - sx(0^-) - \frac{dx(0^-)}{dt}
\tag{4.38}
$$

This process can be iterated to get the Laplace transform of arbitrary higher order derivatives, giving

$$
\frac{d^n x(t)}{dt^n} \quad \leftrightarrow \quad s^n X(s) - s^{n-1}x(0^-) - \sum_{k=2}^{n} s^{n-k}\frac{d^{k-1}x(0^-)}{dt^{k-1}}
\tag{4.39}
$$

where it should be understood that

$$
\frac{d^m x(0^-)}{dt^m} \equiv \left.\frac{d^m x(t)}{dt^m}\right|_{t=0^-}, m = 1, ..., n-1
\tag{4.40}
$$

4.2.1.9 Integration

Let

$$
g(t) = \int_{0^-}^{t} x(\tau)\,d\tau
\tag{4.41}
$$

it follows that

$$
\frac{dg(t)}{dt} = x(t)
\tag{4.42}
$$

and $g\left(0^-\right) = 0$. Moreover, we have

$$
\begin{aligned}
X\left(s\right) &= \mathfrak{L}\{\tfrac{dg(t)}{dt}\} \\
&= sG\left(s\right) - g\left(0^-\right) \\
&= sG\left(s\right)
\end{aligned}
\tag{4.43}
$$

therefore

$$
G\left(s\right) = \frac{X\left(s\right)}{s}
\tag{4.44}
$$

but since

$$
G\left(s\right) = \mathfrak{L}\{\int_{0^-}^{t} x\left(\tau\right) d\tau\}
\tag{4.45}
$$

we have

$$
\int_{0^-}^{t} x\left(\tau\right) d\tau \leftrightarrow \frac{X\left(s\right)}{s}
\tag{4.46}
$$

Now suppose $x\left(t\right)$ has a non-zero integral over negative values of t. We have

$$
\int_{\infty}^{t} x\left(\tau\right) d\tau = \int_{-\infty}^{0^-} x\left(\tau\right) d\tau + \int_{0^-}^{t} x\left(\tau\right) d\tau
\tag{4.47}
$$

The quantity $\int_{-\infty}^{0^-} x\left(\tau\right) d\tau$ is a constant for positive values of t, and can be expressed as

$$
u\left(t\right) \int_{-\infty}^{0^-} x\left(\tau\right) d\tau
\tag{4.48}
$$

it follows that

$$
\int_{\infty}^{t} x\left(\tau\right) d\tau \leftrightarrow \frac{\int_{-\infty}^{0^-} x\left(\tau\right) d\tau}{s} + \frac{X\left(s\right)}{s}
\tag{4.49}
$$

where we have used the fact that $u\left(t\right) \leftrightarrow \frac{1}{s}$.

4.2.1.10 The Initial Value Theorem

The initial value theorem makes it possible to determine $x\left(t\right)$ at $t = 0^+$ from $X\left(s\right)$. From the derivative property of the Laplace transform, we can write

$$
\mathfrak{L}\{\frac{dx\left(t\right)}{dt}\} = sX\left(s\right) - x\left(0^-\right)
\tag{4.50}
$$

Taking the limit $s \to \infty$

$$
\begin{aligned}
\lim_{s\to\infty} \int_{0^-}^{\infty} \tfrac{dx(t)}{dt} e^{-st} dt &= \lim_{s\to\infty} \left[sX\left(s\right) - x\left(0^-\right)\right] \\
\int_{0^-}^{\infty} \lim_{s\to\infty} \tfrac{dx(t)}{dt} e^{-st} dt &= \lim_{s\to\infty} \left[sX\left(s\right) - x\left(0^-\right)\right]
\end{aligned}
\tag{4.51}
$$

There are two cases, the first is when $x\left(t\right)$ is continuous at $t = 0$. In this case it is clear that $\frac{dx(t)}{dt} e^{-st} \to 0$ as $s \to \infty$, so (4.51) can be written as

$$
0 = \lim_{s\to\infty} \left[sX\left(s\right) - x\left(0^-\right)\right]
\tag{4.52}
$$

Since $x(t)$ is continuous at $t = 0$, $x(0^-) = x(0^+)$, the Initial Value Theorem follows,

$$x(0^+) = \lim_{s \to \infty} sX(s) \tag{4.53}$$

The second case is when $x(t)$ is discontinuous at $t = 0$. In this case, we use the fact that

$$\left. \frac{dx(t)}{dt} \right|_{t=0} = \left[x(0^-) - x(0^+) \right] \delta(t) \tag{4.54}$$

For example, if we integrate the right-hand side of (4.54) with $x(0^-) = 0$ and $x(0^+) = 1$, we get the unit step function, $u(t)$. Proceeding as before, we have

$$\lim_{s \to \infty} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \lim_{s \to \infty} \left[sX(s) - x(0^-) \right] \tag{4.55}$$

The left-hand side of (4.55) can be written as

$$\lim_{s \to \infty} \int_{0^-}^{0^+} \left[x(0^-) - x(0^+) \right] \delta(t) e^{-st} dt + \lim_{s \to \infty} \int_{0^+}^{\infty} \frac{dx(t)}{dt} e^{-st} dt \tag{4.56}$$

From the sifting property of the unit impulse, the first term in (4.56) is

$$\left[x(0^-) - x(0^+) \right] \tag{4.57}$$

while the second term is zero since in the limit, the real part of s goes to infinity. Substituting these results into the left-hand side of (4.55) again leads to the initial value theorem, in (4.53).

4.2.1.11 The Final Value Theorem

The Final Value Theorem allows us to determine

$$\lim_{t \to \infty} x(t) \tag{4.58}$$

from $X(s)$. Taking the limit as s approaches zero in the derivative property gives

$$\lim_{s \to 0} \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \lim_{s \to 0} \left[sX(s) - x(0^-) \right] \tag{4.59}$$

The left-hand-side of (4.59) can be written as

$$\int_{0^-}^{\infty} \lim_{s \to 0} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} \frac{dx(t)}{dt} dt = x(\infty) - x(0^-) \tag{4.60}$$

Substituting this result back into (4.59) leads to the Final Value Theorem

$$x(\infty) = \lim_{s \to 0} sX(s) \tag{4.61}$$

which is only valid as long as the limit $x(\infty)$ exists.

Property	$y(t)$	$Y(s)$
Linearity	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(s) + \beta X_2(s)$
Time Delay	$x(t-\tau)$	$X(s)e^{-s\tau}$
s-Shift	$x(t)e^{-at}$	$X(s+a)$
Multiplication by t	$tx(t)$	$-\frac{dX(s)}{ds}$
Multiplication by t^n	$t^n x(t)$	$(-1)^n \frac{d^n X(s)}{ds^n}$
Convolution	$x(t)*h(t)$	$X(j\Omega)H(j\Omega)$
Differentiation	$\frac{dx(t)}{dt}$	$sX(s) - x(0^-)$
	$\frac{d^2 x(t)}{dt^2}$	$s^2 X(s) - sx(0^-) - \frac{dx(0^-)}{dt}$
	$\frac{d^n x(t)}{dt^n}$	$s^n X(s) - s^{n-1}x(0^-) - \sum_{k=2}^{n} s^{n-k}\frac{d^{k-1}x(0^-)}{dt^{k-1}}$
Integration	$\int_{\infty}^{t} x(\tau)d\tau$	$\frac{\int_{-\infty}^{0^-} x(\tau)d\tau}{s} + \frac{X(s)}{s}$
Initial Value Theorem	$x(0^+) = \lim\limits_{s\to\infty} sX(s)$	
Final Value Theorem	$x(\infty) = \lim\limits_{s\to 0} sX(s)$	

Table 4.1: Laplace Transform properties.

4.3 Laplace Transforms of Common Signals[4]

We'll next build up a collection of Laplace transform pairs which we will include in a table. It's important to keep in mind that once the transform pair has been derived, the focus should be on utilizing the transform pair found in the table rather than in recalculating the transform.

4.3.1 Exponential Signal

Consider the Laplace transform of $x(t) = e^{\alpha t}u(t)$:

$$
\begin{aligned}
\mathfrak{L}\{e^{-\alpha t}u(t)\} &= \int_0^\infty e^{-\alpha t}e^{-st}dt \\
&= \int_0^\infty e^{-(\alpha+s)t}dt \\
&= \frac{-1}{\alpha+s}e^{-(\alpha+\sigma+j\Omega)t}\Big|_0^\infty \\
&= \frac{1}{\alpha+s}, \ \ \sigma > -\alpha
\end{aligned}
\tag{4.62}
$$

where $\sigma > -\alpha$ defines the region of convergence. Notice also that if $\alpha < 0$, $X(s)$ still exists provided $\sigma > -\alpha$. Therefore,

$$
e^{-\alpha t}u(t) \leftrightarrow \frac{1}{\alpha+s} \tag{4.63}
$$

---

[4]This content is available online at $<$http://cnx.org/content/m32849/1.5/$>$.

4.3.2 Unit Step Function

Recall we did not attempt to compute the Fourier transform of $u(t)$ since the Fourier transform does not converge for this signal. Fortunately, the Laplace transform easily converges. In fact, we find that since $u(t)$ is a special case of the exponential function with $\alpha = 0$, the simply have

$$u(t) \leftrightarrow \frac{1}{s} \tag{4.64}$$

The region of convergence is $\sigma > 0$.

4.3.3 Ramp Signal

This signal is given by $x(t) = tu(t)$ and also does not have a Fourier transform. The Laplace transform is given by

$$\mathfrak{L}\{tu(t)\} = \int_0^\infty te^{-st}dt \tag{4.65}$$

Setting $u = t, u^{'} = 1, v^{'} = e^{-st}, v = -\frac{1}{s}e^{-st}$ and integrating by parts gives

$$\begin{aligned} \mathfrak{L}\{tu(t)\} &= \left. -\frac{t}{s}e^{-st}\right|_0^\infty + \frac{1}{s}\int_0^\infty e^{-st}dt \\ &= \left. 0 - \frac{1}{s^2}e^{-st}\right|_0^\infty \\ &= -\frac{1}{s^2}\left[e^{-(\sigma+j\Omega)\infty} - 1\right] \\ &= \frac{1}{s^2}, \;\; \sigma > 0 \end{aligned} \tag{4.66}$$

Here, the region of convergence is $\sigma > 0$, and is referred to as the *right-half plane*.

4.3.4 Cosine Signal

Even though we computed the Fourier transform of the cosine signal, $x(t) = cos(\Omega_0 t)$, the Fourier transform technically does not converge for this signal. That is why $X(j\Omega)$ involves impulse functions. The Laplace transform produces quite a different result. First we use the fact that

$$cos(\Omega_0 t)u(t) = \frac{e^{j\Omega_0 t}u(t) + e^{-j\Omega_0 t}u(t)}{2} \tag{4.67}$$

Since each of the two terms is an exponential function we have

$$\begin{aligned} \mathfrak{L}\{cos(\Omega_0 t)u(t)\} &= \frac{\frac{1}{2}}{s - j\Omega_0} + \frac{\frac{1}{2}}{s + j\Omega_0} \\ &= \frac{s}{s^2 + \Omega_0^2} \end{aligned} \tag{4.68}$$

Here, the region of convergence corresponds to $\sigma > 0$ or right-half plane.

4.3.5 More Transform Pairs

We can use the existing transform pairs along with the properties of the Laplace transform to derive many new transform pairs. Consider the exponentially weighted cosine signal. This signal is given by

$$x(t) = e^{-\alpha t}cos(\Omega_0 t)u(t) \tag{4.69}$$

We can use the $s-$shift property of the Laplace transform along with the Laplace transform of the cosine signal (4.68) to get

$$e^{-\alpha t}cos(\Omega_0 t)u(t) \leftrightarrow \frac{s + \alpha}{(s + \alpha)^2 + \Omega_0^2} \tag{4.70}$$

Another common signal is

$$x\left(t\right) = te^{-\alpha t}u\left(t\right) \tag{4.71}$$

Here, we use the Laplace transform of the exponential signal "Exponential Signal" (Section 4.3.1: Exponential Signal) and the multiplication by t property to get

$$te^{-\alpha t}u\left(t\right) \leftrightarrow \frac{1}{\left(s+\alpha\right)^2} \tag{4.72}$$

Extending this idea one step further, we have

$$x\left(t\right) = t^2 e^{-\alpha t}u\left(t\right) \tag{4.73}$$

So applies, giving

$$t^2 e^{-\alpha t}u\left(t\right) \leftrightarrow \frac{2}{\left(s+\alpha\right)^3} \tag{4.74}$$

Example 3.1 Consider the signal $x\left(t\right) = te^{-2t}u\left(t\right)$. Therefore, we get

$$te^{-2t}u\left(t\right) \leftrightarrow \frac{1}{\left(s+2\right)^2} \tag{4.75}$$

Example 3.2 Consider the signal $x\left(t\right) = e^{-2t}cos\left(5t\right)u\left(t\right)$. As seen in Table 4.2,

$$e^{-2t}cos\left(5t\right)u\left(t\right) \leftrightarrow \frac{s+2}{\left(s+2\right)^2+25} \tag{4.76}$$

$x\left(t\right)$	$X\left(s\right)$
$e^{-\alpha t}u\left(t\right)$	$\frac{1}{s+\alpha}$
$u\left(t\right)$	$\frac{1}{s}$
$\delta\left(t\right)$	1
$tu\left(t\right)$	$\frac{1}{s^2}$
$cos\left(\Omega_0 t\right)u\left(t\right)$	$\frac{s}{s^2+\Omega_0^2}$
$sin\left(\Omega_0 t\right)u\left(t\right)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$
$e^{-\alpha t}cos\left(\Omega_0 t\right)u\left(t\right)$	$\frac{s+\alpha}{\left(s+\alpha\right)^2+\Omega_0^2}$
$e^{-\alpha t}sin\left(\Omega_0 t\right)u\left(t\right)$	$\frac{\Omega_0}{\left(s+\alpha\right)^2+\Omega_0^2}$
$te^{-\alpha t}u\left(t\right)$	$\frac{1}{\left(s+\alpha\right)^2}$
$t^2 e^{-\alpha t}u\left(t\right)$	$\frac{2}{\left(s+\alpha\right)^3}$
$t^n e^{-\alpha t}u\left(t\right)$	$\frac{n!}{\left(s+\alpha\right)^{n+1}}$

Table 4.2: Some common Laplace Transform pairs.

4.4 Finding the Inverse Laplace Transform[5]

4.4.1 Finding the Inverse Laplace Transform

4.4.1.1 Using Transform Tables

The inverse Laplace transform, given by

$$x\left(t\right) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X\left(s\right) e^{st} ds \tag{4.77}$$

can be found by directly evaluating the above integral. However since this requires a background in the theory of complex variables, which is beyond the scope of this book, we will not be directly evaluating the inverse Laplace transform. Instead, we will utilize the Laplace transform pairs (Table 4.2) and properties (4.56). Consider the following examples:

Example 3.1 Find the inverse Laplace transform of

$$X\left(s\right) = \frac{e^{-10s}}{s+5} \tag{4.78}$$

By looking at the table of Laplace transform properties (4.56) we find that multiplication by e^{-10s} corresponds to a time delay of 10 sec. Then from the table of Laplace transform pairs (Table 4.2), we see that

$$\frac{1}{s+5} \tag{4.79}$$

corresponds to the Laplace transform of the exponential signal $e^{-5t}u\left(t\right)$. Therefore we must have

$$x\left(t\right) = e^{-5\left(t-10\right)}u\left(t-10\right) \tag{4.80}$$

Example 3.2 Find the inverse Laplace transform of

$$X\left(s\right) = \frac{1}{\left(s+2\right)^2} \tag{4.81}$$

First we note that from the table of Laplace transform pairs (Table 4.2), the Laplace transform of $tu\left(t\right)$ is

$$\frac{1}{s^2} \tag{4.82}$$

Then using the s-shift property in the table of Laplace transform properties (4.56) gives

$$x\left(t\right) = te^{-2t}u\left(t\right) \tag{4.83}$$

Also, the same answer may be arrived at by combining the Laplace transform of $e^{-2t}u\left(t\right)$ with the multiplication by t property.

4.4.1.2 Partial Fraction Expansions

Partial fraction expansions are useful when we can express the Laplace transform in the form of a *rational function*,

$$\begin{aligned} X\left(s\right) &= \frac{b_q s^q + b_{q-1}s^{q-1} + \cdots + b_1 s + b_0}{a_p s^p + a_{p-1}s^{p-1} + \cdots + a_1 s + a_0} \\ &= \frac{B(s)}{A(s)} \end{aligned} \tag{4.84}$$

---

[5]This content is available online at <http://cnx.org/content/m32850/1.8/>.

A rational function is a ratio of two polynomials. The numerator polynomial $B(s)$ has order q, i.e., the largest power of s in this polynomial is q, while the denominator polynomial has order p. The partial fraction expansion also requires that the Laplace transform be a *proper* rational function, which means that $q < p$. Since $B(s)$ and $A(s)$ can be factored, we can write

$$X(s) = \frac{(s - \beta_1)(s - \beta_2)\cdots(s - \beta_q)}{(s - \alpha_1)(s - \alpha_2)\cdots(s - \alpha_p)} \tag{4.85}$$

The $\beta_i, i = 1, 2, ..., q$ are the roots of $B(s)$, and are called the *zeros* of $X(s)$. The roots of $A(s)$, are $\alpha_i, i = 1, ..., p$ and are called the *poles* of $X(s)$. If we evaluate $X(s)$ at one of the zeros we get $X(\beta_i) = 0, i = 1, ..., q$. Similarly, evaluating $X(s)$ at a pole gives[6] $X(\alpha_i) = \pm\infty, i = 1, ..., p$. The partial fraction expansion of a Laplace transform will usually involve relatively simple terms whose inverse Laplace transforms can be easily determined from a table of Laplace transforms. We must consider several different cases which depend on whether the poles are distinct.

4.4.1.2.1 *Distinct Poles:*

When all of the poles are distinct (i.e. $\alpha_i \neq \alpha_j, i \neq j$) then we can use the following partial fraction expansion:

$$X(s) = \frac{A_1}{s - \alpha_1} + \frac{A_2}{s - \alpha_2} + \cdots + \frac{A_p}{s - \alpha_p} \tag{4.86}$$

The coefficients, $A_i, i = 1, ..., p$ can then be found using the following formula

$$A_i = X(s)(s - \alpha_i)|_{s=\alpha_i}, i = 1, ..., p \tag{4.87}$$

Equation (4.87) is easily derived by clearing fractions in (4.86). The inverse Fourier transform of $X(s)$ can then be easily found since each of the terms in the right-hand side of (4.86) is the Laplace transform of an exponential signal. This method is called the *cover up method*.

Example 3.3 Find the inverse Laplace transform of

$$\begin{aligned} X(s) \quad &= \frac{2s-10}{s^2+3s+2} \\ &= \frac{2s-10}{(s+1)(s+2)} \end{aligned} \tag{4.88}$$

Since the poles are $\alpha_1 = -1$ and $\alpha_2 = -2$ are distinct, we have the expansion

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} \tag{4.89}$$

Using (4.86) then gives:

$$\begin{aligned} A_1 \quad &= X(s)(s+1)|_{s=-1} \\ &= \left.\frac{2s-10}{s+2}\right|_{s=-1} \\ &= -12 \end{aligned} \tag{4.90}$$

and

$$\begin{aligned} A_2 \quad &= X(s)(s+2)|_{s=-2} \\ &= \left.\frac{2s-10}{s+1}\right|_{s=-2} \\ &= 14 \end{aligned} \tag{4.91}$$

---

[6]The actual sign would need to be evaluated at some value of s that is sufficiently close to the pole.

Therefore, we get:

$$X(s) = \frac{-12}{s+1} + \frac{14}{s+2} \tag{4.92}$$

The inverse Laplace transform of $X(s)$ can be found by looking up the inverse transform of each of the terms in the right-hand side of (4.92) giving

$$x(t) = -12e^{-t}u(t) + 14e^{-2t}u(t) \tag{4.93}$$

4.4.1.2.2 Repeated Poles:

Let's consider the case when each pole is repeated,

$$X(s) = \frac{B(s)}{(s-\alpha_1)^{p_1}(s-\alpha_2)^{p_2}\cdots(s-\alpha_r)^{p_r}} \tag{4.94}$$

where $p_1 + p_2 + \cdots + p_r = p$. In this case the partial fraction expansion goes like this:

$$\begin{aligned} X(s) \quad & = \frac{A_{1,1}}{s-\alpha_1} + \frac{A_{1,2}}{(s-\alpha_1)^2} + \cdots + \frac{A_{1,p_1}}{(s-\alpha_1)^{p_1}} \\ & + \frac{A_{2,1}}{s-\alpha_2} + \frac{A_{2,2}}{(s-\alpha_2)^2} + \cdots + \frac{A_{2,p_2}}{(s-\alpha_2)^{p_2}} \\ & \qquad + \cdots \\ & + \frac{A_{r,1}}{s-\alpha_r} + \frac{A_{r,2}}{(s-\alpha_r)^2} + \cdots + \frac{A_{r,p_r}}{(s-\alpha_r)^{p_r}} \end{aligned} \tag{4.95}$$

We'll look at two methods. In the first method, the coefficients can be found using the following formula

$$A_{i,p_i-k} = \frac{1}{k!}\frac{d^k}{ds^k}X_i(s)\bigg|_{s=\alpha_i} \tag{4.96}$$

where $i = 1, 2, ..., r$, $k = 0, 1, ..., p_i - 1$ and

$$X_i(s) = X(s)(s-\alpha_i)^{p_i} \tag{4.97}$$

Note that the computation of A_{i,p_i} does not require any differentiation, since $k = 0$.

Example 3.4 Find the inverse Laplace transform of

$$X(s) = \frac{s-1}{(s+2)^2} \tag{4.98}$$

Here we have a single repeated pole at $s = -2$. The expansion is therefore given by

$$X(s) = \frac{A_{1,1}}{s+2} + \frac{A_{1,2}}{(s+2)^2} \tag{4.99}$$

Using (4.96), we begin with $k = 0$ which corresponds to

$$\begin{aligned} A_{1,2} \quad & = X(s)(s+2)^2\Big|_{s=-2} \\ & = s-1|_{s=-2} \\ & = -3 \end{aligned} \tag{4.100}$$

Next, we set $k = 1$ in (4.96)

$$
\begin{aligned}
A_{1,1} &= \tfrac{d}{ds}\left[X(s)(s+2)^2\right]\Big|_{s=-2} \\
&= \tfrac{d}{ds}\left[s-1\right]\Big|_{s=-2} \\
&= 1
\end{aligned}
\tag{4.101}
$$

The partial fraction expansion is then given by

$$
X(s) = \frac{1}{s+2} - \frac{3}{(s+2)^2}
\tag{4.102}
$$

Therefore,

$$
x(t) = e^{-2t}u(t) - 3te^{-2t}u(t)
\tag{4.103}
$$

In the second method, the coefficients $A_{i,p_i}, i = 1, ..., r$ can be found via the cover up method. The remaining coefficients, $A_{k,p_i}, i = 1, ..., r, k = 1, ..., p_i - 1$ can be found by substituting values of s that are not equal to one of the poles in (4.95). This leads to a system of linear equations which can be used to solve for the remaining coefficients. This method is generally preferable if the order of each repeated pole as well as the number of poles is sufficiently small so that the number of unknown coefficients is at most two for hand calculations.

Example 3.5 Find the inverse Laplace transform of:

$$
\begin{aligned}
X(s) &= \frac{s}{(s+1)^3} \\
&= \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} + \frac{A_{1,3}}{(s+1)^3}
\end{aligned}
\tag{4.104}
$$

Using the cover-up method we can find $A_{1,3}$ as follows

$$
A_{1,3} = s\big|_{s=-1} = -1
\tag{4.105}
$$

So we are left with

$$
\begin{aligned}
X(s) &= \frac{s}{(s+1)^3} \\
&= \frac{A_{1,1}}{s+1} + \frac{A_{1,2}}{(s+1)^2} - \frac{1}{(s+1)^3}
\end{aligned}
\tag{4.106}
$$

Setting $s = 0$ in (4.106) leads to

$$
A_{1,1} + A_{1,2} = 1
\tag{4.107}
$$

and setting $s = -2$ in (4.106) gives

$$
-A_{1,1} + A_{1,2} = 1
\tag{4.108}
$$

These choices of s were used to simplify the linear equations to the greatest extent possible. The solution to (4.107) and (4.108) is easily found to be $A_{1,1} = 0$ and $A_{1,2} = 1$. The partial fraction expansion is given by

$$
X(s) = \frac{1}{(s+1)^2} - \frac{1}{(s+1)^3}
\tag{4.109}
$$

Using the corresponding Laplace transform pairs (Table 4.2) leads to

$$
x(t) = te^{-t}u(t) - \frac{1}{2}t^2e^{-t}u(t)
\tag{4.110}
$$

4.4.1.2.3 Distinct and Repeated Poles:

If a Laplace transform contains both distinct and repeated poles, then we would combine the expansions in (4.86) and (4.95). Perhaps the easiest way to indicate this is by way of an example:

Example 3.6 Find the inverse Laplace transform of

$$
\begin{aligned}
X\left(s\right) \quad &= \frac{s+2}{(s+1)(s+3)(s+5)^2} \\
&= \frac{A_1}{s+1} + \frac{A_2}{s+3} + \frac{A_{3,1}}{s+5} + \frac{A_{3,2}}{(s+5)^2}
\end{aligned}
\tag{4.111}
$$

The coefficients corresponding to the distinct poles can be found using (4.87):

$$
\begin{aligned}
A_1 \quad &= \left. X\left(s\right)\left(s+1\right)\right|_{s=-1} \\
&= \left. \frac{s+2}{(s+3)(s+5)^2}\right|_{s=-1} \\
&= \frac{1}{32}
\end{aligned}
\tag{4.112}
$$

$$
\begin{aligned}
A_2 \quad &= \left. X\left(s\right)\left(s+3\right)\right|_{s=-3} \\
&= \left. \frac{s+2}{(s+1)(s+5)^2}\right|_{s=-3} \\
&= \frac{1}{8}
\end{aligned}
\tag{4.113}
$$

The coefficient $A_{3,2}$ corresponding to the double pole at $s = -5$ can be found using (4.96) with $k = 0$:

$$
\begin{aligned}
A_{3,2} \quad &= \left. X\left(s\right)\left(s+5\right)^2\right|_{s=-5} \\
&= \left. \frac{s+2}{(s+1)(s+3)}\right|_{s=-5} \\
&= \frac{-3}{8}
\end{aligned}
\tag{4.114}
$$

The remaining coefficient, $A_{3,1}$ can be found using (4.96) with $k = 1$:

$$
\begin{aligned}
A_{3,1} \quad &= \left. \frac{d}{ds} X\left(s\right)\left(s+5\right)^2\right|_{s=-5} \\
&= \left. \frac{d}{ds}\left[\frac{s+2}{(s+1)(s+3)}\right]\right|_{s=-5} \\
&= \left. \frac{\left(s^2+4s+3\right)-(s+2)(2s+4)}{\left(s^2+4s+3\right)^2}\right|_{s=-5} \\
&= \frac{-5}{32}
\end{aligned}
\tag{4.115}
$$

Alternately, $A_{3,1}$ can be computed by substituting the values obtained for A_1, A_2 and $A_{3,2}$ back into (4.111) and then substituting an arbitrary value for s that does not equal one of the poles as indicated earlier, like $s = 0$. This leads to a simple equation whose only unknown is $A_{3,1}$. The partial fraction of $X\left(s\right)$ is then given by:

$$
\begin{aligned}
X\left(s\right) \quad &= \frac{s+2}{(s+1)(s+3)(s+5)^2} \\
&= \frac{\frac{1}{32}}{s+1} + \frac{\frac{1}{8}}{s+3} - \frac{\frac{5}{32}}{s+5} - \frac{\frac{3}{8}}{(s+5)^2}
\end{aligned}
\tag{4.116}
$$

Applying the inverse Laplace transform to each of the individual terms in (4.116) and using linearity gives:

$$
x\left(t\right) = \frac{1}{32}e^{-t}u\left(t\right) + \frac{1}{8}e^{-3t}u\left(t\right) - \frac{5}{32}e^{-5t}u\left(t\right) - \frac{3}{8}te^{-5t}u\left(t\right)
\tag{4.117}
$$

The following example looks at a case where $X\left(s\right)$ is a rational function, but is not *proper*.

Example 3.7 Find the inverse Laplace transform of

$$X(s) = \frac{s^2 + 6s + 1}{s^2 + 5s + 6} \tag{4.118}$$

Here since $q = p = 2$, we cannot perform a partial fraction expansion. First we must perform a long division, this leads to:

$$\begin{aligned} X(s) &= 1 + \frac{s-5}{s^2+5s+6} \\ &= 1 + \frac{s-5}{(s+2)(s+3)} \end{aligned} \tag{4.119}$$

where $s - 5$ is the remainder resulting from the long division. The quotient of 1 is called a *direct term*. In general, the direct term corresponds to a polynomial in s. The partial fraction expansion is performed on the quotient term, which is always proper:

$$\frac{s-5}{(s+2)(s+3)} = \frac{A_1}{s+2} + \frac{A_2}{s+3} \tag{4.120}$$

Using (4.86) gives

$$\begin{aligned} A_1 &= \left. \frac{s-5}{s+3} \right|_{s=-2} \\ &= -7 \end{aligned} \tag{4.121}$$

$$\begin{aligned} A_2 &= \left. \frac{s-5}{s+2} \right|_{s=-3} \\ &= 8 \end{aligned} \tag{4.122}$$

So we have

$$X(s) = 1 - \frac{7}{s+2} + \frac{8}{s+3} \tag{4.123}$$

and

$$x(t) = \delta(t) - 7e^{-2t}u(t) + 8e^{-3t}u(t) \tag{4.124}$$

4.4.1.2.4 Complex Conjugate Poles:

Some poles occur in complex conjugate pairs as in the following example:

Example 3.8 Find the output of a filter whose impulse response is $h(t) = e^{-5t}u(t)$ and whose input is given by $x(t) = cos(2t)u(t)$. Since the output is given by $y(t) = x(t) * h(t)$, its Laplace transform is $Y(s) = X(s)H(s)$. Therefore using the table of Laplace transform pairs (Table 4.2) we have

$$X(s) = \frac{s}{s^2 + 4} \tag{4.125}$$

and

$$H(s) = \frac{1}{s+5} \tag{4.126}$$

which leads to

$$\begin{aligned} Y(s) &= \frac{s}{(s^2+4)(s+5)} \\ &= \frac{s}{(s+j2)(s-j2)(s+5)} \\ &= \frac{A_1}{s+j2} + \frac{A_2}{s-j2} + \frac{A_3}{s+5} \end{aligned} \tag{4.127}$$

The poles are at $s = j2, -j2$ and -5, all of which are distinct, so equation (4.86) applies:

$$\begin{aligned} A_1 &= Y(s)(s+j2)|_{s=-j2} \\ &= \frac{s}{(s-j2)(s+5)}\Big|_{s=-j2} \\ &= \frac{-j2}{-j4(5-j2)} \\ &= \frac{5+j2}{58} \end{aligned} \tag{4.128}$$

The second coefficient is

$$\begin{aligned} A_2 &= Y(s)(s-j2)|_{s=j2} \\ &= \frac{5-j2}{58} \end{aligned} \tag{4.129}$$

The calculations for A_2 where omitted but it is easy to see that A_2 will be the complex conjugate of A_1 since all of the terms in A_2 are the complex conjugates of those in A_1. Therefore, when there are a pair of complex conjugate poles, we need only calculate one of the two coefficients and the other will be its complex conjugate. The last coefficient corresponding to the pole at $s = -5$ is found using

$$\begin{aligned} A_3 &= Y(s)(s+5)|_{s=-5} \\ &= \frac{s}{(s^2+4)}\Big|_{s=-5} \\ &= -\frac{5}{29} \end{aligned} \tag{4.130}$$

This gives

$$Y(s) = \frac{\frac{5+j2}{58}}{s+j2} + \frac{\frac{5-j2}{58}}{s-j2} - \frac{\frac{5}{29}}{s+5} \tag{4.131}$$

We can now easily find the inverse Laplace transform of each individual term in the right-hand side of (4.131):

$$y(t) = \frac{5+j2}{58}e^{-j2t}u(t) + \frac{5-j2}{58}e^{j2t}u(t) - \frac{5}{29}e^{-5t}u(t) \tag{4.132}$$

At this point, we are technically done, however the first two terms in $y(t)$ are complex and also happen to be complex conjugates of each other. So we can simplify further by noting that

$$\begin{aligned} \frac{5+j2}{58}e^{-j2t}u(t) + \frac{5-j2}{58}e^{j2t}u(t) &= 2\text{Re}\left(\frac{5-j2}{58}e^{j2t}u(t)\right) \\ &= 2\text{Re}\left(0.0928e^{-j0.3805}e^{j2t}u(t)\right) \\ &= 0.1857cos(2t - 0.3805)u(t) \end{aligned} \tag{4.133}$$

The simplified answer is given by

$$y(t) = 0.1857cos(2t - 0.3805)u(t) - 0.1724e^{-5t}u(t) \tag{4.134}$$

We note that the answer contains a transient term, $-0.1724e^{-10t}u(t)$, and a steady-state term $0.1857cos(2t - 0.3805)$. The steady-state term corresponds to the sinusoidal steady-state response of the filter (see Chapter 3). It can be readily seen that the frequency response of the filter is

$$H(j\Omega) = \frac{1}{5 + j\Omega} \tag{4.135}$$

and therefore $|H(j2)| = 0.1857$ and $\angle H(j2) = -0.3805$.

While the above example provides some insight into the sinusoidal steady-state response, the number of complex arithmetic calculations can be tedious. We repeat the example using an alternative expansion involving complex conjugate poles:

$$\frac{1}{s^2 + bs + c} = \frac{A_1 s + A_2}{s^2 + bs + c} \tag{4.136}$$

where it has been assumed that $b^2 - 4c < 0$ (otherwise, we have distinct or repeated real poles). As mentioned above, the expansion in (4.136) can be combined with expansions for distinct or repeated poles.

Example 3.9

$$\begin{aligned} Y(s) \quad &= \frac{s}{(s^2+4)(s+5)} \\ &= \frac{A_1 s + A_2}{(s^2+4)} + \frac{A_3}{s+5} \end{aligned} \tag{4.137}$$

Using the cover up method gives

$$A_3 = \left. \frac{s}{s^2 + 4} \right|_{s=-5} = -\frac{5}{29} \tag{4.138}$$

Clearing fractions in (4.137) gives:

$$s = (A_1 s + A_2)(s + 5) - \frac{5}{29}\left(s^2 + 4\right) \tag{4.139}$$

Setting $s = 0$ in (4.139) gives $A_2 = \frac{4}{29}$. Substituting this value back into (4.139) and setting $s = 1$ leads to $A_1 = \frac{5}{29}$. The resulting Laplace transform is:

$$Y(s) = \frac{\frac{5}{29}s + \frac{4}{29}}{(s^2 + 4)} - \frac{\frac{5}{29}}{s + 5} \tag{4.140}$$

Using the table of Laplace transforms then leads to

$$y(t) = \frac{5}{29}cos(2t)\,u(t) + \frac{2}{29}sin(2t)\,u(t) - \frac{5}{29}e^{-5t}u(t) \tag{4.141}$$

Comparing this answer with (4.134), we see that the sum of a cosine and a sine having the same frequency is equal to a cosine at the same frequency having a certain phase shift and amplitude. In fact, it can be shown that

$$acos\left(\Omega_o t\right) + bsin\left(\Omega_o t\right) = rcos\left(\Omega_o t - \phi\right) \tag{4.142}$$

with $r = \sqrt{a^2 + b^2}$ and $\phi = arctan\frac{b}{a}$. The following example also involves complex conjugate poles and illustrates some additional tricks to solving the partial fraction expansion.

Example 3.10 Find the output of a filter whose input has Laplace transform $X(s) = \frac{1}{s}$ and whose system function is given by

$$H(s) = \frac{1}{s^2 + 2s + 3} \tag{4.143}$$

Multiplying $X(s)$ and $H(s)$ gives

$$\begin{aligned} Y(s) \quad &= \frac{1}{s(s^2+2s+3)} \\ &= \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+2s+3} \end{aligned} \tag{4.144}$$

Clearing fractions gives:

$$\begin{aligned} 1 \quad &= A_1\left(s^2 + 2s + 3\right) + s\left(A_2 s + A_3\right) \\ &= (A_1 + A_2)s^2 + (2A_1 + A_3)s + 3A_1 \end{aligned} \tag{4.145}$$

Setting $s = 0$ leads to a quick solution for A_1, however two subsequent substitutions are needed to find A_2 and A_3. A slightly faster way of solving for the coefficients in (4.145) is to rearrange the right hand side in terms of different powers of s (see second line). Then equate the coefficients of like powers of s on both sides of the equation to solve for the coefficients. For example equating the constant terms leads to $1 = 3A_1$ which gives $A_1 = \frac{1}{3}$. The coefficients of s on either side of the equation are related by $0 = 2A_1 + A_3$ which leads to $A_3 = -\frac{2}{3}$. Similarly, equating the coefficients of s^2 gives $0 = A_1 + A_2$ which leads to $A_2 = -\frac{1}{3}$. So we have:

$$Y(s) = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}s + \frac{2}{3}}{s^2 + 2s + 3} \tag{4.146}$$

The second term in $Y(s)$ does not appear in most Laplace transform tables, however, we can complete the square of $s^2 + 2s + 3$ by taking one-half the coefficient of s, squaring it, then adding and subtracting it to give:

$$s^2 + 2s + 3 + 1 - 1 = (s+1)^2 + 2 \tag{4.147}$$

After a bit more massaging we get

$$Y(s) = \frac{\frac{1}{3}}{s} - \frac{\frac{1}{3}(s+1)}{(s+1)^2 + 2} - \frac{\frac{1}{3}}{(s+1)^2 + 2} \tag{4.148}$$

whose inverse Laplace transform is readily found from the table of Laplace transforms as

$$y(t) = \frac{1}{3}u(t) - \frac{1}{3}e^{-t}cos\left(\sqrt{2}t\right)u(t) - \frac{1}{3\sqrt{2}}e^{-t}sin\left(\sqrt{2}t\right)u(t) \tag{4.149}$$

4.5 Transfer Functions and Frequency Response[7]

We saw in Section 4.2.1.6 (Convolution) that the transfer function of a linear time-invariant system is given by

$$H(s) = \frac{Y(s)}{X(s)} \tag{4.150}$$

If we assume that $H(s)$ is a rational function of s then we can write

$$H(s) = \frac{(s - \beta_1)(s - \beta_2)\cdots(s - \beta_q)}{(s - \alpha_1)(s - \alpha_2)\cdots(s - \alpha_p)} \tag{4.151}$$

where $\beta_1, \beta_2, ..., \beta_q$ are the zeros, and $\alpha_1, \alpha_2, ..., \alpha_p$ are the poles of $H(s)$. The poles and zeros are points in the s-plane where the transfer function is either non-existent (for a pole) or zero (for a zero). These points can be plotted in the s-plane with "$\times$" representing the location of a pole and a "$\circ$" representing the location of a zero. Since

$$|H(j\Omega)| = H(s)|_{s=j\Omega} \tag{4.152}$$

we have

$$|H(j\Omega)| = \frac{|j\Omega - \beta_1| |j\Omega - \beta_2| \cdots |j\Omega - \beta_q|}{|j\Omega - \alpha_1| |j\Omega - \alpha_2| \cdots |j\Omega - \alpha_p|} \tag{4.153}$$

---

[7]This content is available online at $<$http://cnx.org/content/m32851/1.4/$>$.

and

$$\angle H\left(j\Omega\right) = \sum_{k=1}^{q} \angle\left(j\Omega_k - \beta_k\right) - \sum_{l=1}^{p} \angle\left(j\Omega_l - \alpha_l\right) \tag{4.154}$$

Each of the quantities $j\Omega - \beta_k$ have the same magnitude and phase as a vector from the zero β_k to the $j\Omega$ axis in the complex s-plane. Likewise, the quantities $j\Omega - \alpha_k$ have the same magnitude and phase as vectors from the pole α_k to the $j\Omega$ axis.

Example 3.1 Consider a first-order lowpass filter with transfer function

$$H\left(s\right) = \frac{1}{s+2} \tag{4.155}$$

Then

$$\left|H\left(j\Omega\right)\right| = \frac{1}{\left|j\Omega + 2\right|} \tag{4.156}$$

The quantity $j\Omega + 2$ has the same magnitude and phase as a vector from the pole at $s = -2$ to the $j\Omega$ axis in the complex plane. The magnitude of the frequency response is the inverse of the magnitude of this vector. The length of $j\Omega + 2$ increases as Ω increases thereby making $\left|H\left(j\Omega\right)\right|$ decrease as one would expect of a lowpass filter.

Example 3.2 Consider a first-order highpass filter with transfer function

$$H\left(s\right) = \frac{s}{s+2} \tag{4.157}$$

Then

$$\left|H\left(j\Omega\right)\right| = \frac{\left|j\Omega\right|}{\left|j\Omega + 2\right|} \tag{4.158}$$

When $\Omega = 0$, $\left|H\left(j\Omega\right)\right| = 0$, but as Ω approaches infinity, the two vectors $j\Omega$ and $j\Omega + 2$ have equal lengths, so the magnitude of the frequency response approaches unity.

Example 3.3 Let's now look at the transfer function corresponding to a second-order filter

$$H\left(s\right) = \frac{s+1}{\left(s+1+j5\right)\left(s+1-j5\right)} \tag{4.159}$$

Or

$$\left|H\left(j\Omega\right)\right| = \frac{\left|j\Omega + 1\right|}{\left|j\Omega + 1 + j5\right|\left|j\Omega + 1 - j5\right|} \tag{4.160}$$

The pole-zero plot for this transfer function is shown in Figure 4.3. The corresponding magnitude and phase of the frequency response are shown in Figure 4.4.
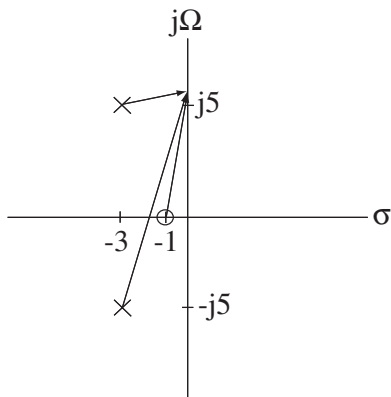
Figure 4.3: Pole-zero diagram for Example .
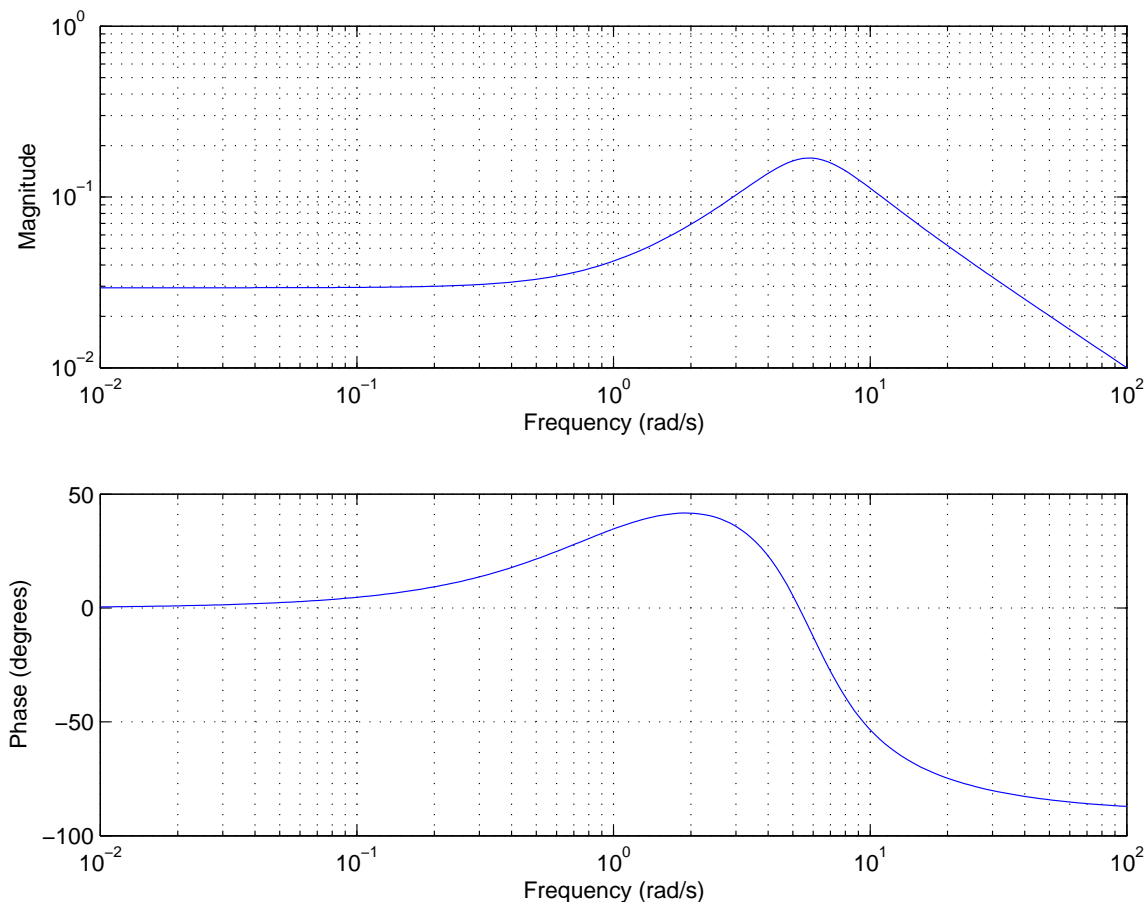
Figure 4.4: Magnitude and phase of frequency response for Example .

The two poles are at $s = -3 \pm j5$ and the zero is at $s = -1$. When the frequency gets close to either one of the poles, the frequency response magnitude increases since the lengths of one of the vectors $j\Omega \pm j5$ is small.

In the previous example we saw that as the poles get close to the $j\Omega$ axis in the s-plane, the frequency response magnitude increases at frequencies that are close to the poles. In fact, when a pole is located directly on the $j\Omega$ axis, the filter's frequency response magnitude becomes infinite, and will begin to oscillate. If a zero is located on the $j\Omega$ axis, the filters frequency response magnitude will be zero.

Example 3.4 Suppose a filter has transfer function

$$H(s) = \frac{s}{s^2 + 25} \tag{4.161}$$

The two poles are at $s = \pm j5$ and there is a zero at $s = 0$. From Table 4.2, the impulse response of the filter is $h(t) = \cos(5t)\, u(t)$. This impulse response doesn't die out like most of the impulse responses we've seen. Instead, it oscillates at a fixed frequency. The filter is called an *oscillator*. Oscillators are useful for generating high-frequency sinusoids used in wireless communications.

4.6 Stability[8]

4.6.1 Filter Stability

We define a filter as being *stable* if a bounded input produces a bounded output. This is called *BIBO* stability. Consider the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, d\tau \tag{4.162}$$

We wish to find a condition that is necessary for the filter output $y(t)$ to be bounded whenever the input $x(t)$ is bounded. Let's take the absolute value of both sides of (4.162)

$$
\begin{aligned}
|y(t)| \quad &= \left| \int_{-\infty}^{\infty} h(\tau) x(t - \tau) \, d\tau \right| \\
&\leq \int_{-\infty}^{\infty} |h(\tau) x(t - \tau)| \, d\tau \\
&\leq x_{max} \int_{-\infty}^{\infty} |h(\tau)| \, d\tau < \infty
\end{aligned}
\tag{4.163}
$$

where the first inequality is due to the *triangle inequality* and the second inequality results from replacing $|x(t - \tau)|$ by its upper bound x_{max}. Therefore, the condition for BIBO stability is that the impulse response the *absolutely integrable*.

$$\int_{-\infty}^{\infty} |h(t)| \, dt < \infty \tag{4.164}$$

Example 3.1 The impulse response of a filter is $h(t) = cos(5t) u(t)$. If the input to this filter is $x(t) = cos(5t) u(t)$, then the output is given by

$$
\begin{aligned}
y(t) \quad &= \int_0^{\infty} cos(5\tau) cos(5(t - \tau)) u(t - \tau) \, d\tau \\
&= \int_0^t cos(5\tau) cos(5(t - \tau)) \, d\tau \\
&= \frac{1}{2} \int_0^t cos(10\tau - 5t) \, d\tau + \frac{1}{2} cos(5t) \int_0^t d\tau \\
&= \frac{1}{2} \int_0^t cos(10\tau - 5t) \, d\tau + \frac{1}{2} t cos(5t)
\end{aligned}
\tag{4.165}
$$

where in the third line, we have used the trigonometric identity

$$cos(\theta_1) cos(\theta_2) = \frac{1}{2} (cos(\theta_1 - \theta_2) + cos(\theta_1 + \theta_2)) \tag{4.166}$$

As t approaches infinity, then it is clear that $y(t)$ also becomes unbounded. Therefore the filter is unstable. Moreover, it is clear that $h(t)$ is not absolutely integrable.

If a transfer function is rational, it can be expressed as a sum of any direct term that may be present plus a proper rational function $(q < p)$.

$$H(s) = c_m s^m + c_{m-1} s^{m-1} + \cdots + c_1 s + c_0 + \frac{b_q s^q + b_{q-1} s^{q-1} + \cdots + b_1 s + b_0}{a_p s^p + a_{p-1} s^{p-1} + \cdots + a_1 s + a_0} \tag{4.167}$$

The direct terms can be shown to produce unbounded outputs when the input is a step function (which of course, is bounded). The proper rational function produces output terms that depend on whether poles are distinct or repeated and whether these are real or complex:

- Distinct real poles, $s = \sigma_k$ lead to an impulse response with terms:

$$K e^{\alpha_k t} u(t) \tag{4.168}$$

- Distinct complex conjugate poles $s = \sigma_k \pm j\Omega_k$ produce impulse response terms:

$$Ke^{\sigma_k t}cos\left(\Omega_k t + \theta\right)u\left(t\right) \tag{4.169}$$

- Repeated real poles $s = \sigma_k$ produce impulse response terms having the form:

$$Kt^n e^{\alpha_k t}u\left(t\right) \tag{4.170}$$

The quantities K and θ are constants while n is a positive integer. In all of these cases the filter impulse response dies out with time if the poles have negative real parts, i.e. it is absolutely integrable and therefore leads to a stable filter. If the poles have zero or positive real parts, then the impulse response terms either oscillate or grow with time, and are not absolutely integrable. When this happens then one can always find a bounded input that produces an unbounded output. Therefore, in order for a filter with a rational transfer function to have BIBO stability, the transfer function should be proper and the poles of the transfer function should have negative real parts.

4.7 Second-Order Filters[9]

4.7.1 Second-Order Lowpass Filters

The second-order lowpass filter has system function

$$H\left(s\right) = \frac{\Omega_n^2}{s^2 + 2\zeta\Omega_n s + \Omega_n^2} \tag{4.171}$$

where Ω_n is the *undamped natural frequency*, and ζ is the *damping ratio*. The undamped natural frequency and damping ratio are properties of the physical devices used to implement the second-order filter (capacitors, inductors, and resistors, in the case of an electrical circuit). It so happens that the damping ratio satisfies $\zeta \geq 0$. Using the formula for the roots of a quadratic polynomial, the two poles of $H\left(s\right)$ are easily found to be

$$\begin{aligned} s_1 &= -\zeta\Omega_n + \Omega_n\sqrt{\zeta^2 - 1} \\ s_2 &= -\zeta\Omega_n - \Omega_n\sqrt{\zeta^2 - 1} \end{aligned} \tag{4.172}$$

There are three possible sets of poles that are categorized as follows:

1. *Overdamped:* the poles are real and distinct. This occurs if $\zeta > 1$. In this case the impulse response is given by:

$$h\left(t\right) = \frac{\Omega_n}{2\sqrt{\zeta^2 - 1}}e^{s_1 t}u\left(t\right) - \frac{\Omega_n}{2\sqrt{\zeta^2 - 1}}e^{s_2 t}u\left(t\right) \tag{4.173}$$

2. *Critically damped:* corresponds to $\zeta = 1$. The two poles are repeated with,

$$s_1 = s_2 = -\zeta\Omega_n = -\Omega_n \tag{4.174}$$

and the impulse response is given by

$$h\left(t\right) = \Omega_n^2 te^{-\zeta\Omega_n t}u\left(t\right) \tag{4.175}$$

---

[9]This content is available online at $<$http://cnx.org/content/m34269/1.4/$>$.

3. *Underdamped:* corresponds to $0 \leq \zeta < 1$, giving a pair of complex conjugate poles. In this case, the impulse response is given by

$$h(t) = \frac{\Omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta \Omega_n t} sin\left(\sqrt{1 - \zeta^2} \Omega_n t\right) u(t) \tag{4.176}$$

Note that in the underdamped case, the magnitude of the poles is $|s_1| = |s_2| = \Omega_n$, and when $\zeta = 0$, the two poles are on the imaginary axis which corresponds to an impulse response that is a pure sinusoid and the system is unstable.

A root locus diagram shows the paths that the poles of $H(s)$ would take as the damping ratio is decreased from some number greater than 1 down to 0, and is shown in Figure 4.5.
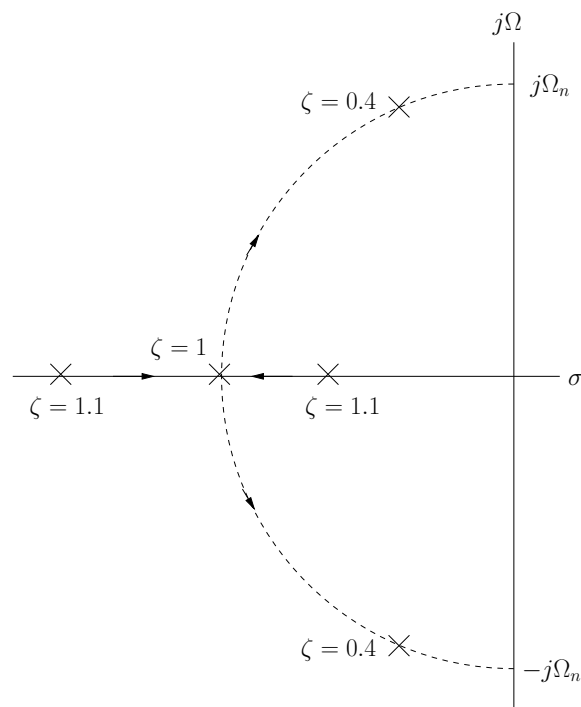
Figure 4.5: Root locus of second-order lowpass filter having $\Omega_n = 10$.

The frequency response of the second-order lowpass filter can be found using the substitution $H(j\Omega) = H(s)|_{s=j\Omega}$, giving

$$H(j\Omega) = \frac{\Omega_n^2}{\Omega_n^2 - \Omega^2 + j2\zeta\Omega_n\Omega} \tag{4.177}$$

The frequency response magnitude is shown in Figure 4.6 for $\Omega_n = 10$ and several values of ζ. Note that for the underdamped case, there is a resonance or peak that would be expected to maximize at the undamped natural frequency Ω_n, when $\zeta = 0$.
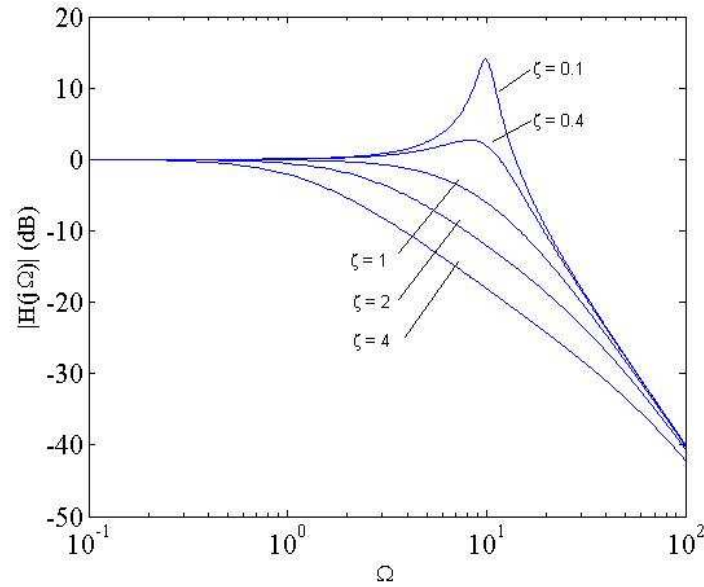
Figure 4.6: Frequency response of second-order lowpass filter with $\Omega_n = 10$ and several values of ζ.

We also observe that for the critically damped case, since

$$H\left(s\right) = \frac{\Omega_n^2}{\left(s + \Omega_n\right)^2} \tag{4.178}$$

setting $s = j\Omega$ gives $\left|H\left(\Omega_n\right)\right| = \frac{1}{2}$. Moreover, it is clear that for $\Omega < \Omega_n$, $\left|H\left(\Omega_n\right)\right| > \frac{1}{2}$ and for $\Omega > \Omega_n$, $\left|H\left(\Omega_n\right)\right| < \frac{1}{2}$. These ideas will be used later when we discuss Bode plots.

4.7.2 Second-Order Filter Implementation
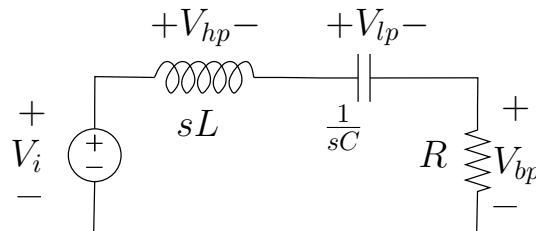
Figure 4.7: Series RLC circuit.

Second order filters can be implemented using either passive or active circuit elements. We will consider the series RLC circuit shown in Figure 4.7. A lowpass filter results by taking the filter output to be the voltage across the capacitor. Using voltage division, the system function is easily found to be

$$
\begin{aligned}
H_{lp}(s) \quad &= \frac{V_{lp}(s)}{V_i(s)} \\
&= \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
\end{aligned}
\tag{4.179}
$$

Comparing (4.179) with (4.171) we find that the undamped natural frequency is $\Omega_n = \frac{1}{\sqrt{LC}}$, while $2\zeta\Omega_n = \frac{R}{L}$, giving the attenuation coefficient:

$$
\alpha \equiv \zeta\Omega_n = \frac{R}{2L}
\tag{4.180}
$$

A second-order highpass filter is implemented by taking the output of the filter to be the inductor voltage in the series RLC circuit. The resulting system function is

$$
\begin{aligned}
H_{hp}(s) \quad &= \frac{V_{hp}(s)}{V_i(s)} \\
&= \frac{s^2}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
\end{aligned}
\tag{4.181}
$$

So in terms of the undamped natural frequency and damping ratio, the second-order highpass filter is given by

$$
H_{hp}(s) = \frac{s^2}{s^2 + 2\zeta\Omega_n s + \Omega_n^2}
\tag{4.182}
$$

The graph of the frequency response magnitude of this filter is shown in Figure 4.8 for several damping ratios and $\Omega_n = 10$.
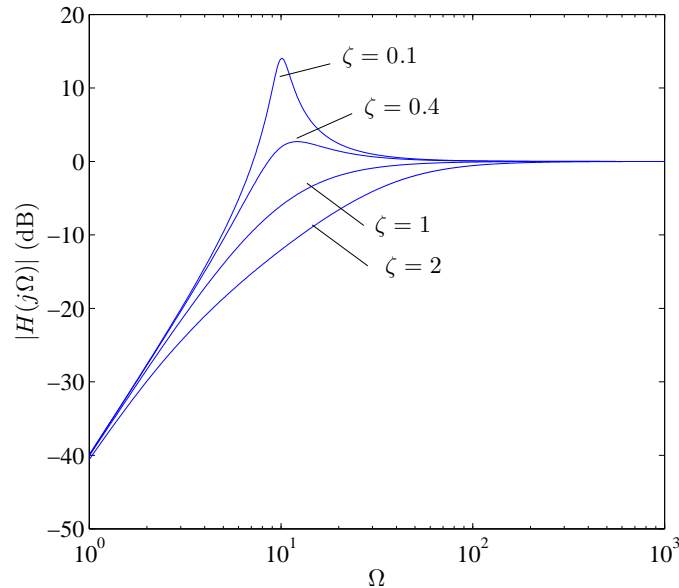
Figure 4.8: Frequency response of second-order highpass filter with $\Omega_n = 10$ and several values of ζ.

Not surprisingly, a second-order bandpass filter results by taking the output of the filter be the resistor voltage in the series RLC circuit. The system function is then

$$
\begin{aligned}
H_{bp}\left(s\right) \quad &= \frac{V_{bp}(s)}{V_i(s)} \\
&= \frac{\frac{R}{L}s}{s^2 + \frac{R}{L}s + \frac{1}{LC}}
\end{aligned}
\tag{4.183}
$$

This can be expressed in terms of ζ and Ω_n as

$$
H_{bp}\left(s\right) = \frac{2\zeta\Omega_n s}{s^2 + 2\zeta\Omega_n s + \Omega_n^2}
\tag{4.184}
$$

4.7.3 Frequency Response of Second-Order Bandpass Filter

Setting $s = j\Omega$ in (4.184) gives the frequency response of the second-order bandpass filter

$$
H_{bp}\left(j\Omega\right) = \frac{j2\zeta\Omega_n\Omega}{\Omega_n^2 - \Omega^2 + j2\zeta\Omega_n\Omega}
\tag{4.185}
$$

The magnitude of this frequency response is shown in Figure 4.9 where $\Omega_n = 10$.
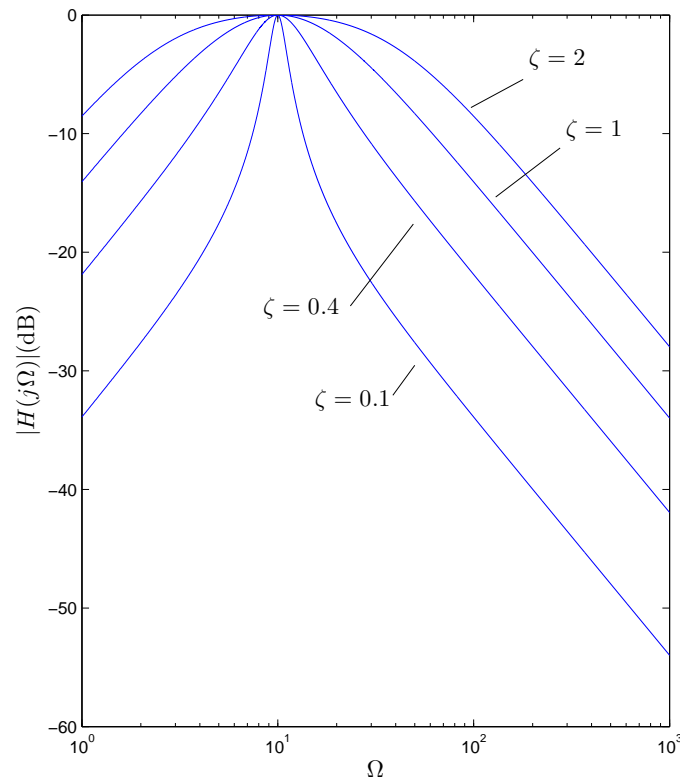
Figure 4.9: Frequency response of second-order bandpass filter with $\Omega_n = 10$ and several values of ζ.

Evidently, the frequency response peaks at $\Omega = \Omega_n$. To prove this, we can divide the numerator and denominator of (4.185) by $j2\zeta\Omega_n\Omega$, giving

$$
\begin{aligned}
H_{bp}(j\Omega) &= \frac{1}{1 - j\frac{\Omega_n^2 - \Omega^2}{2\zeta\Omega_n\Omega}} \\
&= \frac{1}{1 + j\frac{\Omega^2 - \Omega_n^2}{2\zeta\Omega_n\Omega}} \\
&= \frac{1}{1 + jA(\Omega)}
\end{aligned}
\tag{4.186}
$$

where

$$
A(\Omega) = \frac{\Omega^2 - \Omega_n^2}{2\zeta\Omega_n\Omega}
\tag{4.187}
$$

It is easy to check that $|A(\Omega_n)| = 0$ and $|A(\Omega)| > 0, \Omega \neq \Omega_n$. Therefore $|H_{bp}(j\Omega)|$ does in fact peak at $\Omega = \Omega_n$.

The bandwidth of bandpass filters is typically measured as the difference between the two frequencies, Ω_2 and Ω_1, at which the gain has dropped by 3 decibels from its peak of 0 dB. A diagram of this is shown in Figure 4.10.
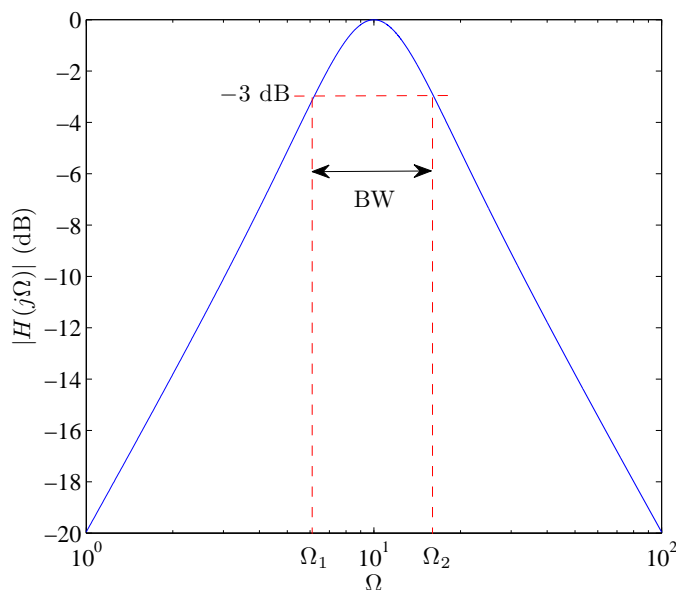
Figure 4.10: Definition of bandpass filter bandwidth, $BW = \Omega_2 - \Omega_1$.

Formulas for Ω_1 and Ω_2 can be found by solving

$$A(\Omega) = \pm 1 \tag{4.188}$$

since if $A(\Omega) = \pm 1$, we get $|H(j\Omega)| = \frac{1}{\sqrt{2}}$. Solving the quadratic equation $A(\Omega) = 1$ gives two solutions:

$$\tilde{\Omega}_1 = \zeta\Omega_n + \Omega_n\sqrt{1 + \zeta^2} \tag{4.189}$$

$$\tilde{\Omega}_2 = \zeta\Omega_n - \Omega_n\sqrt{1 + \zeta^2} \tag{4.190}$$

while solving $A(\Omega) = -1$ gives two additional solutions

$$\tilde{\Omega}_3 = -\zeta\Omega_n + \Omega_n\sqrt{1 + \zeta^2} \tag{4.191}$$

$$\tilde{\Omega}_4 = -\zeta\Omega_n - \Omega_n\sqrt{1 + \zeta^2} \tag{4.192}$$

Since $\Omega_n\sqrt{1 + \zeta^2} > \zeta\Omega_n$, we pick the two positive solutions as the "corner" frequencies:

$$\Omega_1 = -\zeta\Omega_n + \Omega_n\sqrt{1 + \zeta^2} \tag{4.193}$$

$$\Omega_2 = \zeta\Omega_n + \Omega_n\sqrt{1 + \zeta^2} \tag{4.194}$$

The two negative frequencies are just the negatives of the two positive frequencies (recall that $|H(j\Omega)|$ has even symmetry). So the bandwidth is $BW = \Omega_2 - \Omega_1 = 2\zeta\Omega_n$. It can also be readily verified that $\Omega_n = \sqrt{\Omega_1\Omega_2}$, that is, the center frequency of the bandpass filter is the geometric mean of the two corner frequencies. The *qualify factor*, Q_o is defined as

$$Q_o = \frac{\Omega_n}{BW} \tag{4.195}$$

and is a measure of the narrowness of the filter bandwidth, with respect to its center frequency. It can be seen that $Q_o = \frac{1}{2\zeta}$.

4.8 Bode Plots[10]

4.8.1 Bode Plots

Plotting the magnitude and phase of the frequency response is most easily accomplished with a computer, provided you have the right software (for example, the Matlab function "freqs"). However if there is no computer handy, a classical method for quickly sketching the magnitude or phase response of a filter is using a *Bode plot*. Consider a general system function given by

$$H(s) = K_1 \frac{\prod_{k=1}^{q}(s-\beta_k)^{\gamma_k}}{\prod_{k=1}^{p}(s-\alpha_k)^{\delta_k}} \tag{4.196}$$

where we assume the γ_k and δ_k are integers. The corresponding frequency response is therefore

$$H(j\Omega) = K_1 \frac{\prod_{k=1}^{q}(j\Omega-\beta_k)^{\gamma_k}}{\prod_{k=1}^{p}(j\Omega-\alpha_k)^{\delta_k}} \tag{4.197}$$

(4.197) can be written as

$$H(j\Omega) = K \frac{\prod_{k=1}^{q}\left(1-j\frac{\Omega}{\beta_k}\right)^{\gamma_k}}{\prod_{k=1}^{p}\left(1-j\frac{\Omega}{\alpha_k}\right)^{\delta_k}} \tag{4.198}$$

where

$$K = K_1 \frac{\prod_{k=1}^{q}\beta_k^{\gamma_k}(-1)^{\gamma_k}}{\prod_{k=1}^{p}\alpha_k^{\delta_k}(-1)^{\delta_k}} \tag{4.199}$$

Since most frequency response plots are expressed in units of decibels, we have

$$
\begin{aligned}
20log_{10}|H(j\Omega)| \qquad &= 20log_{10}\left|K\frac{\prod_{k=1}^{q}\left(1-j\frac{\Omega}{\beta_k}\right)^{\gamma_k}}{\prod_{k=1}^{p}\left(1-j\frac{\Omega}{\alpha_k}\right)^{\delta_k}}\right| \\
&= 20log_{10}|K| \\
&+ \sum_{k=1}^{q}\gamma_k 20log_{10}\left|1-j\frac{\Omega}{\beta_k}\right| - \sum_{k=1}^{p}\delta_k 20log_{10}\left|1-j\frac{\Omega}{\alpha_k}\right|
\end{aligned}
\tag{4.200}
$$

where we have used basic properties of logarithms. The individual terms in the sums can be plotted, to within a reasonable approximation, with relative ease. So the Bode magnitude plot involves summing the graphs of each individual term in (4.200). Lets consider first a single positive term having the form

$$M \equiv \gamma 20log_{10}\left|1-j\frac{\Omega}{\beta}\right| \tag{4.201}$$

It is clear that when $\Omega \ll \beta$ then $M \approx 0$. On the other hand if $\Omega \gg \beta$, $M \approx \gamma 20 log_{10} \left| \frac{\Omega}{\beta} \right| = \gamma 20 log_{10} |\Omega| - \gamma 20 log_{10} |\beta|$. If we plot this as a function of $log_{10} |\Omega|$, this represents a straight line having a slope of $\gamma 20$ that crosses the $log_{10} |\Omega|$-axis at $log_{10} |\Omega| = log_{10} |\beta|$. These approximations are illustrated in Figure 4.11.
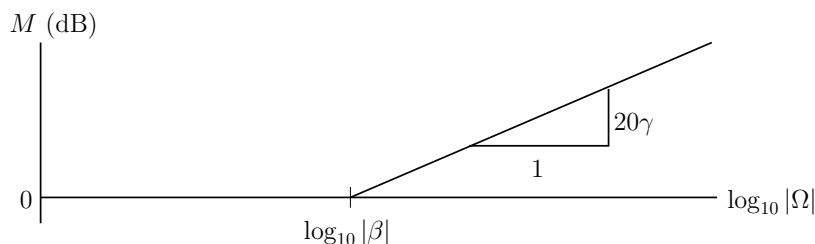
Figure 4.11: Straight-line approximations to $\gamma 20 log_{10} \left| 1 - j\frac{\Omega}{\beta} \right|$.

Instead of plotting M as a function of $log_{10} |\Omega|$, it is more common to plot it as a function of Ω with the frequency axis on a logarithmic scale. In this case the non-zero slope is 20γ decibels per decade, where one decade represents an increase in Ω by a factor of 10. The modified graph is shown in Figure 4.12.
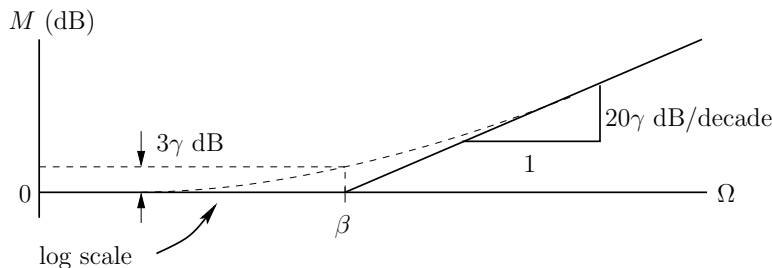
Figure 4.12: Straight-line approximations to $\gamma 20 log_{10} \left| 1 - j\frac{\Omega}{\beta} \right|$ using logarithmic frequency axis.

We also note that when $\Omega = \beta$, the straight-line approximations are not valid, however the true value is easily found to be $\gamma 20 log_{10} |1 - j| = \gamma 20 log_{10} \sqrt{2} \approx 3\gamma$ dB. Negative terms in (4.200) are approximated in a similar manner, but the non-zero slope is now -20γ dB/decade. The resulting approximation is shown in Figure 4.13.
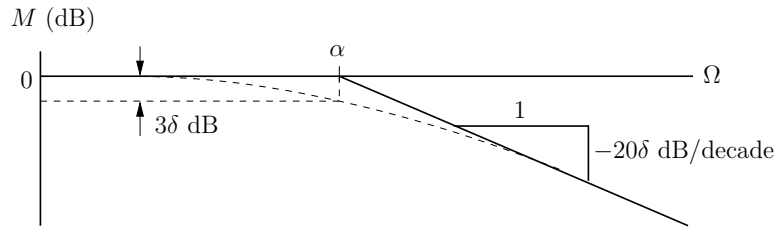
Figure 4.13: Straight-line approximations to $-\delta 20 log_{10} \left| 1 - j\frac{\Omega}{\alpha} \right|$.

(4.200) did not take into account cases where the poles or zeros occur at $s = 0$. For example, if $H(s) = s^{\gamma}$, then $M = 20 log_{10} |H(j\Omega)| = \gamma 20 log_{10} |\Omega|$, which is a line having a slope of 20γ dB/decade passing through the Ω-axis at $\Omega = 1$ (see Figure 4.14). When there is a single or repeated pole at the origin, the graph appears just as in Figure 4.14 but with a negative slope.
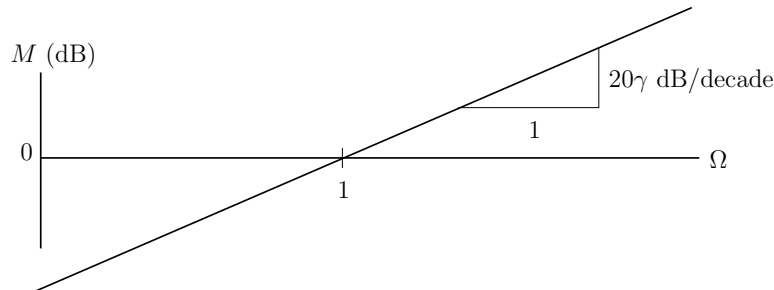
Figure 4.14: Straight-line approximations to $\gamma 20 log_{10} |\Omega|$.

Next we'll look at approximations to the phase response. Here we'll begin with $H(j\Omega)$ as shown in (4.198). Taking the phase of both sides gives

$$
\begin{aligned}
\angle H(j\Omega) &= \angle \{ K \frac{\prod_{k=1}^{q} \left(1 - j\frac{\Omega}{\beta_k} \right)^{\gamma_k}}{\prod_{k=1}^{p} \left(1 - j\frac{\Omega}{\alpha_k} \right)^{\delta_k}} \} \\
&= \angle K + \sum_{k=1}^{q} \gamma_k \angle \left(1 - j\frac{\Omega}{\beta_k} \right) - \sum_{k=1}^{p} \delta_k \angle \left(1 - j\frac{\Omega}{\alpha_k} \right)
\end{aligned}
\tag{4.202}
$$

where we have used the fact $\angle \{ Z_1 Z_2 \} = \angle Z_1 + \angle Z_2$ and $\angle \{ Z^{\gamma} \} = \gamma \angle Z$. Lets now consider how we approximate each of the terms in (4.202). Consider the single term

$$
\angle \left(1 - j\frac{\Omega}{\beta} \right) = -arctan \left(\frac{\Omega}{\beta} \right)
\tag{4.203}
$$

The graph of this function is shown in Figure 4.15.
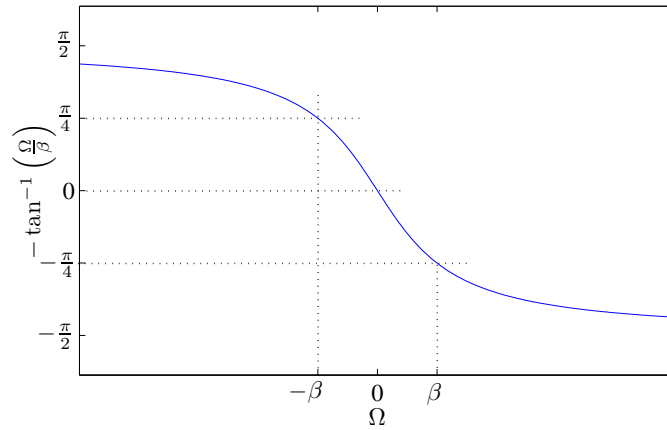
Figure 4.15: Graph of $-arctan\left(\frac{\Omega}{\beta}\right)$, shown on a linear frequency scale.

Since the magnitude response plots are on a logarithmic frequency axis, it would be desirable to do the same for the phase response plots. If we restrict Ω to positive values and plot $-arctan\left(\frac{\Omega}{\beta}\right)$ on a logarithmic frequency scale, we get the graph shown in Figure 4.16.
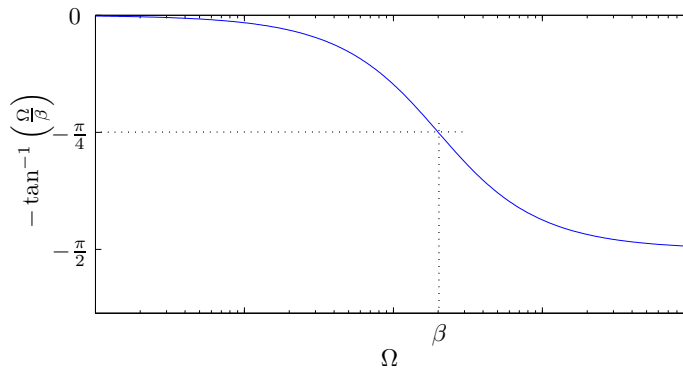
Figure 4.16: Graph of $-arctan\left(\frac{\Omega}{\beta}\right)$, shown on a logarithmic frequency scale, for $\Omega > 0$.

This graph can be approximated with straight lines as shown in Figure 4.17. The approximation is a straight line having a slope of $-\pi/4$ rad/decade passing through a phase of $-\pi/4$ at $\Omega = \beta$. The line then levels off at $\Omega = 0.1\beta$ and $\Omega = 10\beta$. The resulting approximation is shown in Figure 4.17.
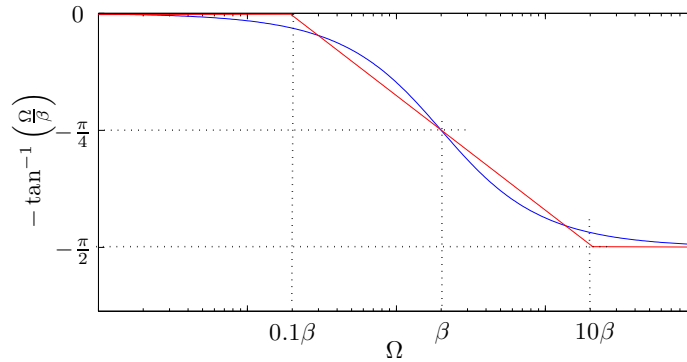
Figure 4.17: Approximation (in red) to graph of $-arctan\left(\frac{\Omega}{\beta}\right)$ on a logarithmic frequency scale.

Constant terms in (4.202) have a phase of 0 or $\pm\pi$, depending on their sign, while poles or zeros of $H(s)$ at the origin produce phase terms of $\gamma\pi/2$ for zeros or order γ or $-\delta\pi/2$ for poles of order δ. Next we'll illustrate these techniques with a few examples.

Example 3.1 Sketch the Bode magnitude and phase response plot for the following filter:

$$H(s) = \frac{10^4}{(s+10)(s+10^3)} \tag{4.204}$$

We begin by finding the corresponding frequency response by setting $s = j\Omega$:

$$H(j\Omega) = \frac{1}{\left(1 + j\frac{\Omega}{10}\right)\left(1 + j\frac{\Omega}{10^3}\right)} \tag{4.205}$$

Lets find the magnitude response first:

$$20log_{10}|H(j\Omega)| = -20log_{10}\left|1 + j\frac{\Omega}{10}\right| - 20log_{10}\left|1 + j\frac{\Omega}{10^3}\right| \tag{4.206}$$

The resulting straight-line approximations to the two terms in (4.206) along with their sum are shown in Figure 4.18.
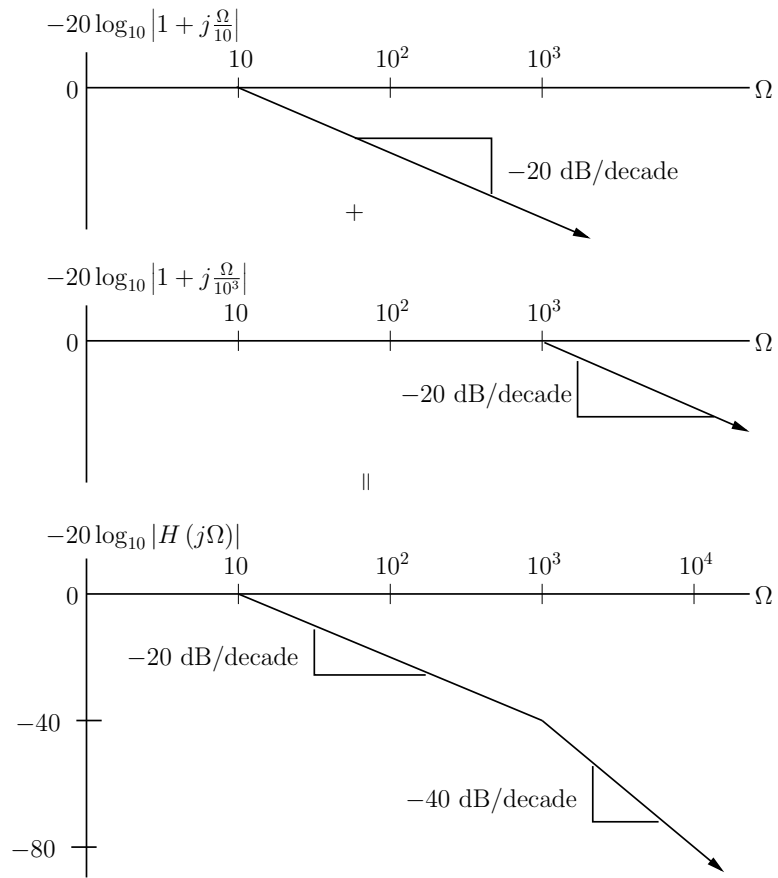
Figure 4.18: Bode magnitude response plot derivation for Example 1.

The phase is given by

$$\angle H\left(j\Omega\right) = -\angle \left(1 + j\frac{\Omega}{10}\right) - \angle \left(1 + j\frac{\Omega}{10^3}\right) \tag{4.207}$$

The straight-line approximations to the two phase terms, along with their sum are shown in Figure 4.19.
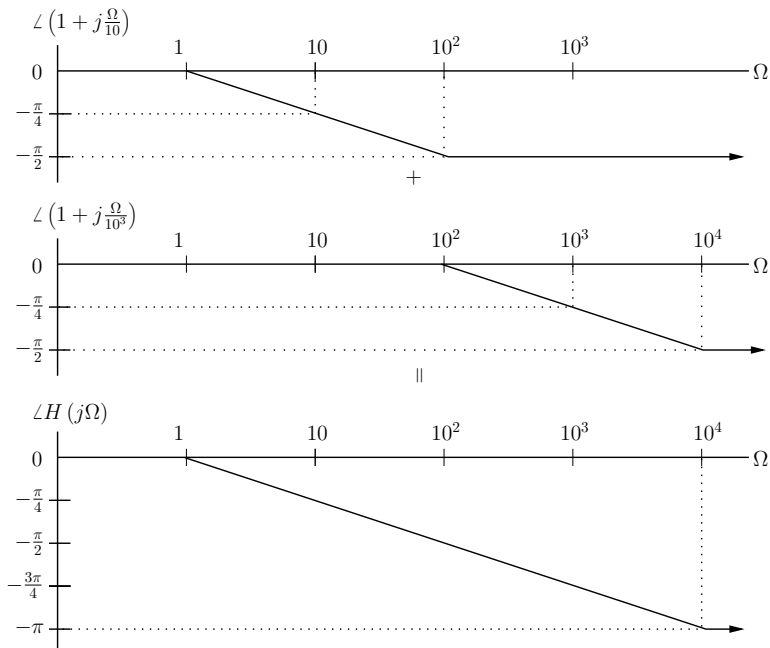
Figure 4.19: Bode phase response plot derivation for Example 1.

Next we'll look at an example of a second-order highpass filter.

Example 3.2 Find the Bode magnitude and phase response of the following filter

$$H\left(s\right) = \frac{s^2}{\left(s + 10\right)^2} \tag{4.208}$$

Substituting $s = j\Omega$ in (4.208) gives

$$H\left(j\Omega\right) = \frac{-10^{-2}\Omega^2}{\left(1 + j\frac{\Omega}{10}\right)^2} \tag{4.209}$$

The magnitude response, expressed in decibels becomes

$$20log_{10}\left|H\left(j\Omega\right)\right| = 20log_{10}\left|10^{-2}\right| + 40log_{10}\left|\Omega\right| - 40log_{10}\left|1 + \frac{j\Omega}{10}\right| \tag{4.210}$$

The graphs of the straight-line approximations for the three terms in the right-hand side of (4.210) along with their sum are shown in Figure 4.20.

$$20 \log_{10} \left| 10^{-2} \right| = -40 \text{ dB}$$
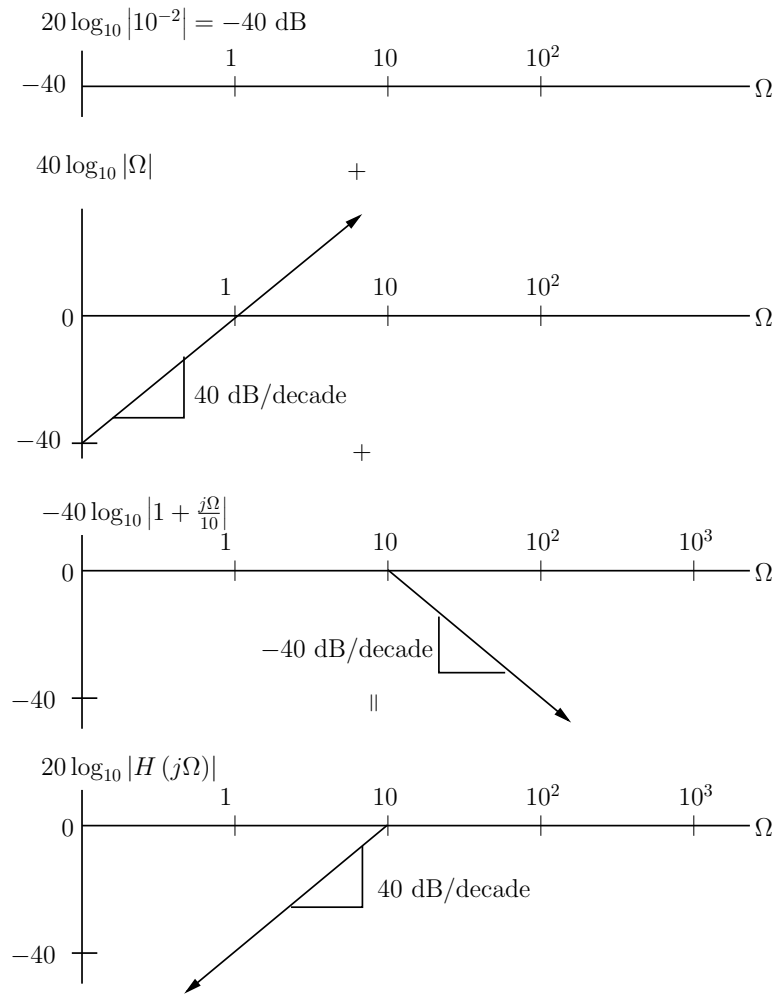
Figure 4.20: Bode magnitude response plot derivation for Example 3.2.

The phase of the frequency response is found to be

$$
\begin{aligned}
\angle H\left(j\Omega\right) \quad &= \angle \left(-10^{-2}\Omega^2\right) - \angle \left(1 + \tfrac{j\Omega}{10}\right)^2 \\
&= \pi - 2\angle \left(1 + \tfrac{j\Omega}{10}\right)
\end{aligned}
\tag{4.211}
$$

The resulting Bode phase response plot is found in Figure 4.21.

$$\angle H\left(j\Omega\right) = \pi - 2\angle\left(1 + \frac{j\Omega}{10}\right)$$
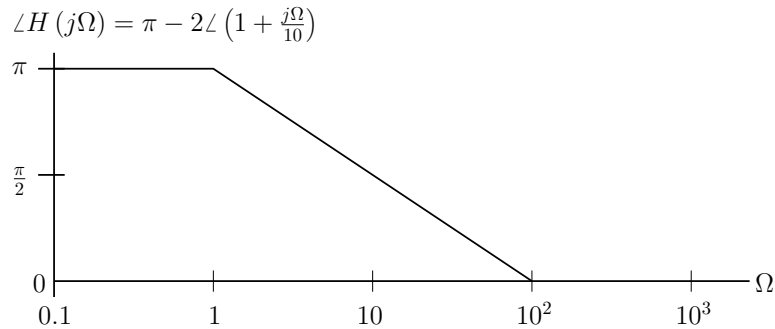
Figure 4.21: Bode phase response plot derivation for Example 3.2.

4.9 The Laplace Transform: Excercises[11]

4.9.1 Exercises

1. Find the Laplace Transform of the following signals, for each case indicate the Laplace transform property that was used:

   a. $x\left(t\right) = 4e^{-0.2t}u\left(t\right)$
   b. $x\left(t\right) = 4te^{-0.2t}u\left(t\right)$
   c. $x\left(t\right) = 4e^{-0.2(t-10)}u\left(t - 10\right)$
   d. $x\left(t\right) = \delta\left(t - 5\right)$
   e. $x\left(t\right) = 10tu\left(t\right)$
   f. $x\left(t\right) = sin\left(10\pi t\right)u\left(t\right)$
   g. $x\left(t\right) = e^{-3t}sin\left(10\pi t\right)u\left(t\right)$
   h. $x\left(t\right) = \mathrm{rect}\left(t - 0.5, 1\right)$

2. Suppose that two filters having impulse responses $h_1\left(t\right)$ and $h_2\left(t\right)$ are cascaded (i.e. connected in series). Find the transfer function of the equivalent filter assuming $h_1\left(t\right) = 10e^{-10t}u\left(t\right)$ and $h_2\left(t\right) = 5e^{-5t}u\left(t\right)$.

3. Find the inverse Laplace transforms of the following:

   a. $X\left(s\right) = \frac{e^{-2s}}{s+5}$
   b. $X\left(s\right) = \frac{se^{-s}}{s^2+9}$
   c. $X\left(s\right) = \frac{1}{(s+3)^2}$
   d. $X\left(s\right) = 10$
   e. $X\left(s\right) = \frac{10}{s^2}$
   f. $X\left(s\right) = \frac{e^{-s}}{s}$

4. Use partial fraction expansions to find the inverse Laplace transforms of the following:

   a. $X\left(s\right) = \frac{s+2}{(s+5)(s+2)(s+1)}$
   b. $X\left(s\right) = \frac{s+1}{(s+2)^3(s+3)}$
   c. $X\left(s\right) = \frac{s}{(s^2+9)(s+2)}$

---

[11]This content is available online at $<$http://cnx.org/content/m32853/1.7/$>$.

   d. $X(s) = \frac{s^2 - 3s + 1}{(s+1)(s+2)}$

5. Consider a filter having impulse response $h(t) = e^{-2t}u(t)$. Use Laplace transforms to find the output of the filter when the input is given by:

   a. $x(t) = u(t)$
   b. $x(t) = tu(t)$
   c. $x(t) = e^{-4t}u(t)$
   d. $x(t) = \cos(10t)u(t)$

6. Indicate whether the following impulse responses correspond to stable or unstable filters:

   a. $h(t) = u(t)$
   b. $h(t) = e^{-3t}u(t)$
   c. $h(t) = e^{-3t}\cos(4tu(t)$
   d. $h(t) = \cos(10t)u(t)$

7. Use Laplace transform tables to find the impulse response of the second-order lowpass filter in terms of ζ and Ω_n for the overdamped, critically damped, and underdamped case.

8. Use a series RLC circuit to design a critically damped second-order lowpass filter with a corner frequency of 100 rad/sec. Use a $R = 6.8$ kΩ resistor in your design.

9. Using a 10 kΩ resistor, design a critically damped bandpass filter, having a center frequency of 100 rad/sec and indicate the resulting bandwidth of the filter. What is the quality factor of the filter?

10. Use bode plots to find the magnitude and phase response of the following filters

   a.

$$H(s) = \frac{1}{(s+1)^3} \tag{4.212}$$

   b.

$$H(s) = \frac{10^3 s}{(s+10)(s+10^3)} \tag{4.213}$$

Chapter 5

References for Signals, Systems, and Society[1]

1. W.H. Hayt, Jr. and J.E. Kemmerly, *Engineering Circuit Analysis*, McGraw Hill, 1978.
2. B.P. Lathi, *Linear Systems and Signals* , Oxford University Press, 2004.
3. A.V. Oppenheim and A.S. Willsky, *Signals and Systems* , Prentice Hall, 1997.
4. M.J. Roberts, *Signals and Systems*, McGraw Hill, 2004.

Index of Keywords and Terms

Keywords are listed by the section with that keyword (page numbers are in parentheses). Keywords do not necessarily appear in the text of the page. They are merely associated with that section. *Ex.* apples, § 1.1 (1) **Terms** are referenced by the page they appear on. *Ex.* apples, 1

Attributions

Module: "The Laplace Transform: Introduction"
By: Carlos E. Davila
URL: http://cnx.org/content/m32847/1.3/
Pages: 67-70
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Properties of the Laplace Transform"
By: Carlos E. Davila
URL: http://cnx.org/content/m32848/1.6/
Pages: 70-76
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Laplace Transforms of Common Signals"
By: Carlos E. Davila
URL: http://cnx.org/content/m32849/1.5/
Pages: 76-78
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Finding the Inverse Laplace Transform"
By: Carlos E. Davila
URL: http://cnx.org/content/m32850/1.8/
Pages: 79-87
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Transfer Functions and Frequency Response"
By: Carlos E. Davila
URL: http://cnx.org/content/m32851/1.4/
Pages: 87-90
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Stability"
By: Carlos E. Davila
URL: http://cnx.org/content/m32852/1.5/
Pages: 91-92
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Second-Order Filters"
By: Carlos E. Davila
URL: http://cnx.org/content/m34269/1.4/
Pages: 92-99
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Module: "Bode Plots"
By: Carlos E. Davila
URL: http://cnx.org/content/m34302/1.3/
Pages: 99-107
Copyright: Carlos E. Davila
License: http://creativecommons.org/licenses/by/3.0/

Signals, Systems, and Society
This textbook gives an introduction to signals, transforms, systems, and strives to present the material in a societal context.

About Connexions
Since 1999, Connexions has been pioneering a global system where anyone can create course materials and make them fully accessible and easily reusable free of charge. We are a Web-based authoring, teaching and learning environment open to anyone interested in education, including students, teachers, professors and lifelong learners. We connect ideas and facilitate educational communities.

Connexions's modular, interactive courses are in use worldwide by universities, community colleges, K-12 schools, distance learners, and lifelong learners. Connexions materials are in many languages, including English, Spanish, Chinese, Japanese, Italian, Vietnamese, French, Portuguese, and Thai. Connexions is part of an exciting new information distribution system that allows for **Print on Demand Books**. Connexions has partnered with innovative on-demand publisher QOOP to accelerate the delivery of printed course materials and textbooks into classrooms worldwide at lower prices than traditional academic publishers.