Appendix 12: Cumulative Distribution, Percentile Functions, and Random Numbers

The functions described in this appendix are used in more than one procedure. They are grouped into the following categories:

- **Special Functions.** Gamma function, beta function, incomplete gamma function (ratio), incomplete beta function (ratio), and standard normal function
- **Continuous Distributions.** Beta, Cauchy, chi-square, exponential, *F*, gamma, Laplace, logistic, lognormal, normal, Pareto, Student's *t*, uniform, and Weibull
- **Discrete Distributions.** Bernoulli, binomial, geometric, hypergeometric, negative binomial, and Poisson
- **Noncentral Continuous Distributions.** Noncentral beta, noncentral chi-square, noncentral *F*, and noncentral Student's *t*

Notation

The following notation is used throughout this appendix unless otherwise stated:

$f(x)$	Density function of continuous random variable X
$F(x)$	Cumulative distribution function of X
x_p	*100p*th percentile such that $F(x_p) = p,\ \text{for } 0 \le p \le 1$
Prob$(X = x)$	Probability density (or mass) function of discrete random variable X
Prob$(X \le x)$	Cumulative probability density function of X

Special Functions

Gamma Function

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx \qquad a > 0$$

Properties

- $\Gamma(1) = 1$
- $\Gamma(1/2) = \sqrt{\pi}$
- $\Gamma(a) = (a-1)\Gamma(a-1) \qquad a > 1$
- $\Gamma(a) = (a-1)!$ when a is a positive integer

Note. Since $\Gamma(a)$ can be very large even for a moderate value of a, the (natural) logarithm of $\Gamma(a)$ is computed instead.

References. The $\ln(\Gamma(a))$ function: CACM 291 (1966) and AS 245 (1989).

Beta Function

$$\mathrm{B}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx \qquad a > 0,\ b > 0$$

Properties

- $\mathrm{B}(a,1) = 1/a$
- $\mathrm{B}\left(\frac{1}{2},\frac{1}{2}\right) = \pi$
- $\mathrm{B}(a,b) = \Gamma(a)\Gamma(b)/\Gamma(a+b)$
- $\mathrm{B}(b,a) = \mathrm{B}(a,b)$

- $\mathrm{B}(a,b) = (b-1)\mathrm{B}(a+1,b-1)/a$
- $\mathrm{B}(a,b) = (a+b)\mathrm{B}(a+1,b)/a$

Note. Usually, B(*x*, *y*) is calculated as:

$$\mathrm{B}(x,y) = \exp\Big(\ln\big(\Gamma(x)\big) + \ln\big(\Gamma(y)\big) - \ln\big(\Gamma(x+y)\big)\Big)$$

Incomplete Gamma Function (Ratio)

$$\mathrm{IG}(x;a) = \int_0^x \frac{1}{\Gamma(a)} t^{a-1} e^{-t} dt \qquad x \ge 0$$

$$x_p = \mathrm{IG}^{-1}(p;a) \Leftrightarrow p = \mathrm{IG}(x_p;a) \qquad 0 \le p \le 1$$

for $a > 0$

Properties

- $\mathrm{IG}(x;1) = 1 - e^{-x}$
- Using integration by parts, for $a > 1$,

$$\mathrm{IG}(x;a) = \frac{1}{\Gamma(a+1)} x^a e^{-x} + \mathrm{IG}(x;a+1)$$

Note. $\mathrm{IG}^{-1}(1,a) = \infty$.

References. AS 32 (1970), AS 147 (1980), and AS 239 (1988).

Incomplete Beta Function (Ratio)

$$\mathrm{IB}(x;a,b) = \int_0^x \frac{1}{\mathrm{B}(a,b)} t^{a-1}(1-t)^{b-1} dt \qquad 0 \le x \le 1$$

$$x_p = \mathrm{IB}^{-1}(p;a,b) \Leftrightarrow p = \mathrm{IB}(x_p;a,b) \qquad 0 \le p \le 1$$

for $a > 0$ and $b > 0$

Properties

- $\mathrm{IB}(x;a,1) = x^a$
- Using the transformation $t = \sin^2\theta$, we get $\mathrm{IB}\left(x;\frac{1}{2},\frac{1}{2}\right) = \frac{2}{\pi}\sin^{-1}\sqrt{x}$
- $\mathrm{IB}(x;a,b) = 1 - \mathrm{IB}(1-x;b,a)$
- Using integration by parts, we get, for $b > 1$,

  $$\mathrm{IB}(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^a (1-x)^{b-1} + \mathrm{IB}(x;a+1,b-1)$$

- Using the fact that $\frac{d}{dx} x^a (1-x)^b = ax^{a-1}(1-x)^{b-1} - (a+b)x^a(1-x)^{b-1}$

  we have

  $$\mathrm{IB}(x;a,b) = \frac{\Gamma(a+b)}{\Gamma(a+1)\Gamma(b)} x^a (1-x)^b + \mathrm{IB}(x;a+1,b)$$

References. AS 63 (1973); Inverse: AS 64 (1973), AS 109 (1977).

Standard Normal Function

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du \qquad -\infty < x < \infty$$

$$x_p = \Phi^{-1}(p) \Leftrightarrow \Phi(x) = p$$

For Φ^{-1}, the Abramowitz and Stegun method is used.

References. AS 66 (1973); Inverse: AS 111 (1977) and AS 241 (1988). See Patel and Read (1982) for related distributions.

Continuous Distributions

Beta $(x;a,b)$. $0 \le x \le 1$, $a > 0$ and $b > 0$

$$f(x) = \frac{1}{\mathrm{B}(a,b)} x^{a-1}(1-x)^{b-1}$$

$$F(x) = \mathrm{IB}(x;a,b)$$

$$x_p = \mathrm{IB}^{-1}(p;a,b) \qquad 0 \le p \le 1$$

Note. Uniform(0,1) is a special case of beta for $a = 1$ and $b = 1$.

Random Number

Special case

1. Generate U from Uniform(0,1).
2. If $a = 1$, set $X = 1-(1-U)^{1/b}$.
3. If $b = 1$, set $X = U^{1/a}$.
4. If both $a = 1$ and $b = 1$, set $X = U$.

Algorithm BN due to Ahrens and Dieter (1974) for a > 1 and b > 1

1. Set $e = a-1$, $f = b-1$, $c = e+f$, $g = c\ln(c)$, $u = e/c$, and $s = 0.5/\sqrt{c}$.
2. Generate Y from N(0,1) and set $X = sY + u$.
3. If $X < 0$ or $X > 1$, go to step 2.
4. Generate U from Uniform(0,1).
5. If $\ln(U) \le \left(e\ln(X/e) + f\ln((1-X)/f) + g + 0.5Y^2\right)$, finish; otherwise go to step 2.

References. CDF: AS 63 (1973); ICDF: AS 64 (1973) and AS 109 (1977); RV: AS 134 (1979), Ahrens and Dieter (1974), and Cheng (1978).

Cauchy $(x; a, b)$. $-\infty < x < \infty, \quad -\infty < a < \infty$ and $b > 0$

$$f(x) = \frac{1}{\pi b}\left(1 + \left(\frac{x-a}{b}\right)^2\right)^{-1}$$

$$F(x) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{x-a}{b}\right)$$

$$x_p = a + b\tan\left(\pi(p - 1/2)\right) \qquad 0 \le p \le 1$$

Random Number

Inverse CDF algorithm

1. Generate U from Uniform(0,1).
2. Set $X = a + b\tan\left(\pi(U - 1/2)\right)$.

Chi-Square $(x; a)$. $x \ge 0$ and $a > 0$

$$f(x) = \frac{1}{2^{a/2}\Gamma(a/2)} x^{(a/2)-1} e^{-x/2}$$

$$F(x) = \mathrm{IG}\left(\tfrac{x}{2}; \tfrac{a}{2}\right)$$

by the incomplete gamma function on p. 4.

$$x_p = 2\mathrm{IG}^{-1}\left(p; \tfrac{a}{2}\right) \qquad 0 \le p \le 1$$

Note. It is the Gamma(x; a/2, 1/2) distribution.

Random Number

Generate X from the Gamma(a/2, 1/2) distribution.

References. CDF: CACM 299 (1967); ICDF: AS 91 (1975), AS R85(1991), and CACM 451 (1973).

Exponential $(x;a)$. $x \ge 0, \quad a > 0$

$$f(x) = ae^{-ax}$$
$$F(x) = 1 - e^{-ax}$$
$$x_p = -\frac{1}{a}\ln(1-p) \qquad 0 \le p \le 1$$

Note. This is the Gamma(x; 1, a) distribution.

Random Number

Inverse CDF algorithm

Generate U from Uniform(0,1); $X = -\ln(1-U)/a$.

F $(x;a,b)$. $x \ge 0$ and $b > 0$

$$f(x) = \frac{1}{\mathrm{B}(a/2, b/2)}\left(\frac{a}{b}\right)^{a/2} x^{(a/2)-1}\left(1+\frac{a}{b}x\right)^{-(a+b)/2}$$
$$F(x) = \mathrm{IB}\left(\frac{ax}{b+ax}; \frac{a}{2}, \frac{b}{2}\right)$$

$$x_p = \frac{b}{a}\left(\frac{\mathrm{IB}^{-1}(p; a/2, b/2)}{1-\mathrm{IB}^{-1}(p; a/2, b/2)}\right) \qquad 0 \le p \le 1$$

Random Number

Using the chi-square distribution

1. Generate Y and Z independently from chi-square(a) and chi-square(b), respectively.
2. Set $X = (Y/a)/(Z/b)$.

References. CDF: CACM 332 (1968). ICDF: use inverse of incomplete beta function.

Gamma $(x; a, b)$. $x \ge 0$, $a > 0$, and $b > 0$

$$f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$$

$$F(x) = \mathrm{IG}(bx; a)$$

$$x_p = \frac{1}{b} \mathrm{IG}^{-1}(p; a) \qquad 0 \le p \le 1$$

Note. The Erlang distribution is a gamma distribution with integers a and $b = 1$.

Random Number

Special case

If $a = 1$ and $b > 0$, generate X from an exponential distribution with parameter b.

Algorithm GS due to Ahrens and Dieter (1974) for $0 < a < 1$ and $b = 1$

1. Generate U from Uniform(0,1). Set $c = (e + a)/e$, where $e = \exp(1)$.
2. Set $P = cU$. If $P > 1$, go to step 4.
3. $(P \le 1)$ Set $X = P^{1/a}$. Generate V from Uniform(0,1). If $V > \exp(-x)$, go to step 1; otherwise finish.
4. $(P > 1)$ Set $X = -\ln((c - P)/a)$. If $X < 0$, go to step 1; otherwise go to step 5.
5. Generate V from Uniform(0,1). If $V > X^{a-1}$, go to step 1; otherwise finish.

Algorithm due to Fishman (1976) for $a > 1$ and $b = 1$

1. Generate Y from Exponential (1).
2. Generate U from Uniform(0,1).
3. If $\ln U \le (a-1)(1 - Y + \ln Y)$, $X = aY$; otherwise go to Step 1.

References. CDF: AS 32 (1970) and AS 239 (1988); ICDF: Use the relationship between gamma and chi-square distributions. RV: Ahrens and Dieter (1974), Fishman (1976), and Tadikamalla (1978).

Laplace $(x;a,b)$. $-\infty < x < \infty,\ -\infty < a < \infty,$ and $b > 0$

$$f(x) = \frac{1}{2b} e^{-|x-a|/b}$$

$$F(x) = \begin{cases} \frac{1}{2} e^{(x-a)/b} & x \le a \\ 1 - \frac{1}{2} e^{(a-x)/b} & x > a \end{cases}$$

$$x_p = \begin{cases} a + b\ln(2p) & 0 \le p \le \frac{1}{2} \\ a - b\ln\big(2(1-p)\big) & \frac{1}{2} < p \le 1 \end{cases}$$

Note. The Laplace distribution is also known as the *double exponential* distribution.

Random Number

Inverse CDF algorithm

1. Generate Y and U independently from Exponential(1/ b) and Uniform(0,1), respectively.
2. If $U \ge \frac{1}{2}$, set $X = a + Y$; otherwise set $X = a - Y$.

Logistic $(x;a,b)$. $-\infty < x < \infty,\ -\infty < a < \infty,$ and $b > 0$

$$f(x) = \frac{1}{b} e^{-(x-a)/b} \left(1 + e^{-(x-a)/b}\right)^{-2}$$

$$F(x) = \frac{1}{1 + e^{-(x-a)/b}}$$

$$x_p = a + b\ln\left(\frac{p}{1-p}\right) \qquad 0 \le p \le 1$$

Random Number

Inverse CDF algorithm

1. Generate U from Uniform(0,1).
2. Set $X = a + b\ln\big(U/(1-U)\big)$.

Lognormal $(x;a,b)$. $x \ge 0$, $a > 0$, and b > 0

$$f(x) = \frac{1}{xb\sqrt{2\pi}} e^{-(\ln(x/a))^2/(2b^2)}$$

$$F(x) = \Phi\left(\frac{1}{b}\ln\left(\frac{x}{a}\right)\right)$$

$$x_p = ae^{b\Phi^{-1}(p)} \qquad 0 \le p \le 1$$

Random Number

Inverse CDF algorithm

1. Generate Z from N(0,1).
2. Set $X = a\exp(bZ)$.

Normal $(x;a,b)$. $-\infty < x < \infty$, $-\infty < a < \infty$, and $b > 0$

$$f(x) = \frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2/(2b^2)}$$

$$F(x) = \Phi\left(\frac{x-a}{b}\right)$$

$$x_p = a + b\Phi^{-1}(p) \qquad 0 \le p \le 1$$

For Φ and Φ^{-1}, see “Standard Normal Function” on p. 5.

Random Number

Kinderman and Ramage (1976) method

1. Generate as $X = a + bz$, where z is an $N(0,1)$ random number.

References. CDF: AS 66 (1973); ICDF: AS 111 (1977) and AS 241 (1988); RV: CACM 488 (1974) and Kinderman and Ramage (1976).

Pareto $(x; a, b)$. $x \geq a > 0$ and $b > 0$

$$f(x) = \frac{b}{a}\left(\frac{a}{x}\right)^{b+1}$$

$$F(x) = 1 - \left(\frac{a}{x}\right)^{b}$$

$$x_p = a(1-p)^{-1/b} \qquad 0 \leq p \leq 1$$

Random Number

Inverse CDF

1. Generate U from Uniform(0,1).
2. Set $X = a(1-U)^{-1/b}$.

Student's *t* $(x; a)$. $-\infty < x < \infty$ and $a > 0$

$$f(x) = \frac{1}{\sqrt{a}\,\mathrm{B}(a/2, 1/2)}\left(1 + \frac{x^2}{a}\right)^{-(a+1)/2}$$

$$F(x) = \begin{cases} \frac{1}{2}\,\mathrm{IB}\left(\frac{a}{a+x^2}; \frac{a}{2}, \frac{1}{2}\right) & \mathrm{x} \leq 0 \\ 1 - \frac{1}{2}\,\mathrm{IB}\left(\frac{a}{a+x^2}; \frac{a}{2}, \frac{1}{2}\right) & x > 0 \end{cases}$$

$$x_p = \begin{cases} -\sqrt{a\left(1/\left(\mathrm{IB}^{-1}\left(2p; \frac{a}{2}, \frac{1}{2}\right)\right) - 1\right)} & 0 \leq p \leq \frac{1}{2} \\ \sqrt{a\left(1/\left(\mathrm{IB}^{-1}\left(2(1-p); \frac{a}{2}, \frac{1}{2}\right)\right) - 1\right)} & \frac{1}{2} < p \leq 1 \end{cases}$$

When $a = 1$, this is the Cauchy distribution (Cauchy(x; 0, 1)).

Random Number

Special case

If $a = 1$, generate X from a Cauchy (0, 1) distribution.

Using the normal and the chi-square distributions

1. Generate Z from N(0,1) and V from Chi-square(a) independently.
2. Set $X = Z / \sqrt{V / a}$.

References. CDF: AS 3 (1968), AS 27 (1970), and CACM 395 (1970); ICDF: CACM 396 (1970).

Uniform $(x; a, b)$. $a \le x \le b,\ -\infty < a < b < \infty$

$$f(x) = \frac{1}{b-a}$$

$$F(x) = \frac{x-a}{b-a}$$

$$x_p = a + (b-a)p \qquad 0 \le p \le 1$$

Random Number

Inverse CDF algorithm

1. Generate U from Uniform(0,1).
2. Set $X = a + (b-a)U$.

References. Uniform of (0,1) is generated by the method in Schrage (1979).

Weibull $(x; a, b)$. $x \ge 0,\ a > 0,$ and $b > 0$

$$f(x) = \frac{b}{a}\left(\frac{x}{a}\right)^{b-1} e^{-(x/a)^b}$$

$$F(x) = 1 - e^{-(x/a)^b}$$

$$x_p = a(-\ln(1-p))^{1/b} \qquad 0 \le p \le 1$$

Random Number

Inverse CDF algorithm

1. Generate U from Uniform(0,1).
2. Set $X = a(-\ln(1-U))^{1/b}$.

Discrete Distributions

Bernoulli $(x;a)$. $x = 0,1$ and $0 \le \text{a} \le 1$

$$\text{Prob}(X = x) = a^x (1-a)^{1-x}$$
$$\text{Prob}(X \le x) = \begin{cases} 1-a & x = 0 \\ 1 & x = 1 \end{cases}$$

Random Number

Special case

If $a = 0$, $X = 0$. If $a = 1$, $X = 1$.

Direct algorithm for $0 < a < 1$

1. Generate U from Uniform(0,1).
2. Set $X = 1$ if $U \le a$ (a success) and $X = 0$ if $U > a$ (a failure).

Binomial $(x;a,b)$. $x = 0,1,\ldots,a; a = 1,2,\ldots,$ and $0 \le b \le 1$

$$\text{Prob}(X = x) = \binom{a}{x} b^x (1-b)^{a-x}$$
$$\text{Prob}(X \le x) = \begin{cases} 1-\text{IB}(b; x+1, a-x) & x = 0,1,\ldots,a-1 \\ 1 & x = a \end{cases}$$

Random Number

Special case

If $b = 0$, $X = 0$. If $b = 1$, $X = a$.

Algorithm BB due to Ahrens and Dieter (1974) for $0 < b < 1$

1. Set $c = a,\ d = b,\ k = 0,\ y = 0,$ and $h = 1$.
2. If $c < 40$, generate J from Binomial(c, d) using algorithm BU. $X = k + J$.
3. If c is odd, go to step 4. If c is even, set $c = c - 1$ and generate U from Uniform(0,1). If $U \le d$, set $k = k + 1$.
4. Set $s = (c + 1)/2$ and generate S from Beta(s, s). Set $G = hS$ and $Z = y + G$.
5. If $Z \le b$, set $y = Z,\ h = h - d,\ d = (b - Z)/h,$ and $k = k + s$; otherwise set $h = G$ and $d = (b - y)/h$.
6. Set $c = s - 1$ and go to step 2.

Computation time for algorithm BB is O(log a).

References. RV: Ahrens and Dieter (1974).

Geometric $(x; a).\ x = 1, 2, \ldots$ and $0 < a \le 1$

$$\text{Prob}(X = x) = a(1 - a)^{x-1}$$

$$\text{Prob}(X \le x) = 1 - (1 - a)^{x}$$

Note. Geometric is a special case of the negative binomial (x; 1, a).

Random Number

Special case

If $a = 1$, $X = 1$.

Direct algorithm for $0 < a < 1$

1. Set $X = 1$.
2. Generate U from Uniform(0,1).
3. If $U > a$, set $X = X + 1$ and go to step 2; otherwise finish.

Hypergeometric $(x; a, b, c)$. $x = \max(0, b + c - a), \ldots, \min(c, b); a = 1,2,\ldots; b = 1,2,\ldots, a$ and $c = 1,2,\ldots, a$

$$\text{Prob}(X = x) = \frac{\binom{c}{x}\binom{a-c}{b-x}}{\binom{a}{b}}$$

$$\text{Prob}(X \le x) = \sum_{k=\max(0,b+c-a)}^{x} \text{Prob}(X = k)$$

Random Number

Special case

If $b = a,\ X = c.$ If $c = a,\ X = b$.

Direct algorithm

1. (Initialization) $X = 0, g = c, h = b, t = a$.
2. Do the following loop exactly b times:

   Begin Loop

   i. Generate U from Uniform(0,1).

   ii. If $U \le (g / t)$, set $X = X + 1,\ g = g - 1,$ else $h = h - 1$.

   iii. If $g = 0$, go to step 3.

   iv. If $h = 0$, set $X = X + b - i$, where i (from 1 to b) is the loop index. Go to step 3.

   v. Set $t = t - 1$.

   End Loop
3. Finish.

References. CDF: AS 152 (1989), AS R77 (1989), and AS R86 (1991).

Negative Binomial $(x; a, b).\ x = a,\ a+1, \ldots;\ a = 1, 2, \ldots$ and $0 < b \le 1$

$$\text{Prob}(X = x) = \binom{x-1}{a-1} b^a (1-b)^{x-a}$$

$$\text{Prob}(X \le x) = \text{IB}(b; a, x-a+1)$$

Random Number

Special case

If $b = 1,\ X = a$.

Direct algorithm

1. Generate G from $\text{Gamma}(a, b/(1-b))$.
2. If $G = 0$, go to step 1. Otherwise generate P from Poisson (G).
3. Compute $X = P + a$.

Poisson $(x; a).\ x = 0, 1, 2, \ldots$ and $a > 0$

$$\text{Prob}(X = x) = \frac{a^x}{x!} e^{-a}$$

$$\text{Prob}(X \le x) = 1 - \text{IG}(a; x+1)$$

Random Number

Algorithm PG due to Ahrens and Dieter (1974)

1. (Initialization) Set $X = 0$ and $w = a$.
2. If $w > 16$, go to step 6.
3. Set $c = \exp(-w)$ and $p = 1$.
4. Generate U from Uniform(0,1). Set $p = pU$.
5. If $p < c$, continue with step 6; otherwise set $X = X + 1$ and go to step 4.
6. Set $n = [7w/8]$. Generate G from Gamma(n, 1).
7. If $G > w$, generate Y from $\text{Binomial}(n-1, w/G)$, set $X = X + Y$.
8. If $G \le w$, set $X = X + n,\ w = w - G,$ and go to step 2.

Notes. [y] means the integer part of y.

Steps 3 to 5 of Algorithm PG are in fact the direct algorithm.

References. RV: Ahrens and Dieter (1974).

Noncentral Continuous Distributions

Noncentral Beta $(x; a, b, c)$. $0 \le x \le 1$, $a > 0$, $b > 0$, and $c \ge 0$

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{c}{2}\right)^j e^{-c/2} \frac{x^{a+j-1}(1-x)^{b-1}}{\mathrm{B}(a+j;b)}$$

$$F(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{c}{2}\right)^j e^{-c/2} \mathrm{IB}(x; a = j, b)$$

Note. c is the noncentrality parameter. If $c = 0$, $F(x)$ is the (central) beta distribution function.

References. CDF: Abramowitz and Stegun (1965, Chapter 26), AS 226 (1987), and AS R84 (1990).

Noncentral Chi-Square $(x; a, c)$. $x \ge 0$, $a > 0$, and $c \ge 0$

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{c}{2}\right)^j e^{-c/2} \frac{x^{a/2+j-1} e^{-x/2}}{2^{a/2+j}\,\Gamma(a/2+j)}$$

$$F(x) = \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{c}{2}\right)^j e^{-c/2} \mathrm{IG}\left(\frac{x}{2}; \frac{a}{2} + j\right)$$

The noncentral chi-square random variable is generated as the sum of squares of a independent normal random variates each with mean μ_i and variance 1. Then $c = \sum \mu_i{}^2$.

Note. c is the noncentrality parameter. If $c = 0$, $F(x)$ is the (central) chi-square distribution function.

References. CDF: Abramowitz and Stegun (1965, Chapter 26), AS 170 (1981), AS 231 (1987). Density: AS 275 (1992).

Noncentral *F* $(x;a,b,c)$. $x \geq 0$, $a > 0$, $b > 0$, and $c \geq 0$

$$f(x)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{c}{2}\right)^{j} e^{-c/2}\,\frac{(a/b)^{a/2+j}}{\mathrm{B}(a/2+j,b/2)}\,x^{a/2+j-1}\left(1+\frac{a}{b}x\right)^{-((a+b)/2+j)}$$

$$F(x)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(\frac{c}{2}\right)^{j} e^{-c/2}\,\mathrm{IB}\left(\frac{ax}{b+ax};\frac{a}{2}+j,\frac{b}{2}\right)$$

The noncentral *F* random variable is generated as the ratio between two independent chi-square random variables. The numerator is a noncentral chi-square random variable (with parameters *a* and *c*) divided by *a*. The denominator is a central chi-square random variable (with parameter *b*) divided by *b*.

Note. *c* is the noncentrality parameter. If $c=0$, $F(x)$ is the (central) *F* distribution function.

References. CDF: Abramowitz and Stegun (1965, Chapter 26).

Noncentral Student's *t* $(x;a,c)$. $-\infty < x < \infty$, $a > 0$, and $-\infty < c < \infty$

$$f(x)=\sum_{j=0}^{\infty}\frac{1}{j!}\left(c\sqrt{2}\right)^{j} e^{-c^2/2}\,\frac{\Gamma\left((a+j+1)/2\right)}{\Gamma(a/2)\Gamma(1/2)}\,\frac{x^{j}}{a^{(j+1)/2}}\left(1+\frac{x^2}{a}\right)^{-(a+j+1)/2}$$

$$F(x)=\begin{cases}\dfrac{1}{2}\displaystyle\sum_{j=0}^{\infty}\frac{1}{j!}\left(-c\sqrt{2}\right)^{j} e^{-c^2/2}\,\frac{\Gamma\left((j+1)/2\right)}{\Gamma(1/2)}\,\mathrm{IB}\left(\frac{a}{a+x^2};\frac{a}{2},\frac{j+1}{2}\right) & x\leq 0\\ 1-\dfrac{1}{2}\displaystyle\sum_{j=0}^{\infty}\frac{1}{j!}\left(c\sqrt{2}\right)^{j} e^{-c^2/2}\,\frac{\Gamma\left((j+1)/2\right)}{\Gamma(1/2)}\,\mathrm{IB}\left(\frac{a}{a+x^2};\frac{a}{2},\frac{j+1}{2}\right) & x>0\end{cases}$$

Special case

$$F(0)=1-\Phi(c)$$

The noncentral Student's *t* random variable is generated as the ratio between two independent random variables. The numerator is a normal random variable with mean *c* and variance 1. The denominator is a central chi-square random variable (with parameter *a*) divided by *a*.

Note. c is the noncentrality parameter. If $c = 0$, $F(x)$ is the (central) Student's t distribution function.

References. CDF: Abramowitz and Stegun (1965, Chapter 26), AS 5 (1968), AS 76 (1974), and AS 243 (1989).

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AS 3:	Cooper (1968a)
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CACM 488: Brent (1974)

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