Appendix 13: SPCHART

SPSS creates nine types of Shewhart control charts. In this appendix, the charts are grouped into five sections:

- X-Bar and R Charts
- X-Bar and s Charts
- Individual and Moving Range Charts
- p and np Charts
- u and c Charts

For each type of control chart, the process, the center line, and the control limits (upper and lower) are described.

Notation

The following notation is used throughout this appendix unless otherwise stated:

σ	Population standard deviation for measurements *X*
A	Number of sigmas specified by the user, $0 \le A \le 9$
K	Number of subgroups
n_i	Number of units (samples) for subgroup *i*
N	Total sample size, equal to $n_1 + \ldots + n_K$
x_{ij}	Measurement (observation) for the *j*th unit (sample) of subgroup *i*
x_i	Mean of measurements for subgroup *i*, $\bar{x}_i = \left(\sum_{i=1}^{n_i} x_{ij} \right) / n_i$
S_i	Sample standard deviation for subgroup *i*, $S_i^2 = \left(\sum_{i=1}^{n_i} \left(x_{ij} - \bar{x}_i \right)^2 \right) / \left(n_i - 1 \right)$
R_i	Sample range for subgroup *i*, $R_i = \max\left(x_{i1}, \ldots, x_{in_i} \right) - \min\left(x_{i1}, \ldots, x_{in_i} \right)$
LCL	Lower Control Limit
UCL	Upper Control Limit

Weight

Weights can be used when the data organization is *Cases are units*.

- Each value for *weight* must be a positive integer.
- Cases with either non-positive or fractional weights are dropped.
- When weight is in effect, n_i is a weighted sum for all the units in subgroup *i* and x_i and *x* are weighted means.

X-Bar and R Charts

When X-Bar and R charts are paired, the sample range statistic *R* is used to construct the control limits for the X-Bar chart.

Note: Subgroups whose sample sizes are less than the specified minimum value are dropped.

Equal Sample Sizes

Assume that $n_i = n$ for $i = 1, \ldots, K$. The process for the X-Bar chart is $\{x_i : i = 1, \ldots, K\}$. The center line for an X-Bar chart is the **grand mean** statistic $\bar{x}$:

$$\bar{x} = \frac{1}{K} \sum_{i=1}^{K} \bar{x}_i$$

and the control limits are

$$\text{LCL} = \bar{x} - A\overline{R} / \left(d_2(n)\sqrt{n}\right)$$

$$\text{UCL} = \bar{x} + A\overline{R} / \left(d_2(n)\sqrt{n}\right)$$

where

$$\overline{R} = \frac{1}{K}\sum_{i=1}^{K} R_i$$

is the **mean range** statistic. The process for an R chart is $\{R_i : i = 1, \ldots, K\}$. The center line for an R chart is $\overline{R}$ and the control limits are

$$\text{LCL} = \max\left(\overline{R}\left(1 - Ad_3(n) / d_2(n)\right), 0\right)$$
$$\text{UCL} = \overline{R}\left(1 + Ad_3(n) / d_2(n)\right)$$

The auxiliary functions are

$$d_2(n) = \int_{-\infty}^{\infty} \left(1 - (1 - \Phi(x))^n - (\Phi(x))^n\right) dx$$

$$d_3(n) = \left(2\int_{-\infty}^{\infty}\int_{-\infty}^{x} \left(1 - (\Phi(x))^n - (1 - \Phi(y))^n + (\Phi(x) - \Phi(y))^n\right) dy dx - (d_2(n))^2\right)^{\frac{1}{2}}$$

$$\Phi(z) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

Unequal Sample Sizes

The processes for X-Bar and R charts are the same as described in the section “Equal Sample Sizes” above. The center line for an X-Bar chart is the grand mean statistic $\overline{x}$ (numerically identical to that in the section “Equal Sample Sizes”):

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{K} n_i \overline{x}_i \qquad \textbf{(1)}$$

and the control limits for subgroup i are

$$\text{LCL} = \bar{x} - A\hat{\sigma} / \sqrt{n_i}$$
$$\text{UCL} = \bar{x} + A\hat{\sigma} / \sqrt{n_i}$$

The center line for an R chart for subgroup i is $R_i = \hat{\sigma} d_2(n_i)$ for $i = 1, \dots, K$ where

$$\hat{\sigma} = \frac{1}{K} \sum_{i=1}^{K} \left(R_i / d_2(n_i) \right)$$

and the control limits for subgroup i are

$$\text{LCL} = \max\left(R_i - A\hat{\sigma} d_3(n_i), 0 \right)$$
$$\text{UCL} = R_i + A\hat{\sigma} d_3(n_i)$$

X-Bar and s Charts

When X-Bar and s charts are paired, the sample sandard deviation is used to construct the control limits for the X-Bar chart.

Equal Sample Sizes

Assume $n_i = n$. The process for the X-Bar chart is $\{x_i : i = 1, \dots, K\}$. The center line for an X-Bar chart is $\bar{x}$ and the control limits are

$$\text{LCL} = \bar{x} - A\bar{S} / \left(c_4(n)\sqrt{n} \right)$$
$$\text{UCL} = \bar{x} + A\bar{S} / \left(c_4(n)\sqrt{n} \right)$$

The process for an s chart is $\{S_i : i = 1, \ldots, K\}$. The center line for an s chart is

$$\bar{S} = \frac{1}{K}\sum_{i=1}^{K} S_i$$

and the control limits are

$$\text{LCL} = \max\left(\bar{S}\left(1 - A\sqrt{\left(1-(c_4(n))^2\right)} / c_4(n)\right), 0\right)$$

$$\text{UCL} = \bar{S}\left(1 + A\sqrt{\left(1-(c_4(n))^2\right)} / c_4(n)\right)$$

The auxiliary function is

$$c_4(n) = \sqrt{\frac{2}{n-1}}\,\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}$$

where $\Gamma(.)$ is the complete Gamma function.

Note: When $n \geq 25$, $c_4(n)\sqrt{n}$ can be approximated by $\sqrt{n-0.5}$, $\sqrt{\left(1-(c_4(n))^2\right)} / c_4(n)$ can be approximated by $1/\sqrt{2n-2.5}$, and $c_4(n)$ can be approximated by $\sqrt{(4n-5)/(4n-3)}$.

Unequal Sample Sizes

The processes for X-Bar and s charts are the same as the processes in the section "Equal Sample Sizes" above. The center line for an X-Bar chart is $\bar{x}$ (as defined in equation (1)) and the control limits are

$$\text{LCL} = \bar{x} - A\hat{\sigma} / \sqrt{n_i}$$

$$\text{UCL} = \bar{x} + A\hat{\sigma} / \sqrt{n_i}$$

or

$$\text{LCL} = \bar{x} - AS_i / \left(c_4(n_i)\sqrt{n_i}\right)$$

$$\text{UCL} = \bar{x} + AS_i / \left(c_4(n_i)\sqrt{n_i}\right)$$

where

$$\hat{\sigma} = \frac{1}{K}\sum_{i=1}^{K} S_i / c_4(n_i)$$

However, the center line for an s chart for subgroup i is $S_i = \hat{\sigma} c_4(n_i)$ for $i = 1, \ldots, K$ and the control limits are

$$\text{LCL} = \max\left(S_i - A\hat{\sigma}\sqrt{\left(1-\left(c_4(n_i)\right)^2\right)},\ 0\right)$$

$$\text{UCL} = S_i + A\hat{\sigma}\sqrt{\left(1-\left(c_4(n_i)\right)^2\right)}$$

or

$$\text{LCL} = \max\left(S_i - AS_i\sqrt{\left(1-\left(c_4(n_i)\right)^2\right)} / c_4(n_i),\ 0\right)$$

$$\text{UCL} = S_i + AS_i\sqrt{\left(1-\left(c_4(n_i)\right)^2\right)} / c_4(n_i)$$

Individual and Moving Range Charts

When a weight variable is specified, each unit of the process is expanded to multiple units based on the case weight associated with this particular unit. The **span** (specified by the user) is associated with the expanded process. If the span is greater than *N* (the total number of units of the expanded process), an error message is displayed and neither an Individual nor a Moving Range chart is generated.

Since each subgroup has only one unit, the process for an Individual chart is $\{y_i : i = 1, \ldots, N\}$ where *yi* is the *i*th unit of the expanded process. For a span of length *m*, the moving ranges, are

$$R_i = \begin{cases} \max(y_{i-m+1}, \ldots, y_i) - \min(y_{i-m+1}, \ldots, y_i) & \text{if } i = m, \ldots, N \\ \text{SYSMIS} & \text{if } i = 1, \ldots, m-1 \end{cases}$$

The average moving range is

$$\overline{R} = \frac{1}{(N-m+1)}\sum_{m}^{N} R_i$$

The center line for an Individual chart is $\overline{x}$ and the control limits for an Individual chart are

$$\text{LCL} = \overline{x} - A\overline{R} / d_2(m)$$
$$\text{UCL} = \overline{x} + A\overline{R} / d_2(m)$$

The process for a moving range chart is $\{Ri,\ i = m,\ldots,\ N\}$. The center line for a moving range chart is $\overline{R}$. The control limits for a moving range chart are

$$\text{LCL} = \max\left(\overline{R}\left(1 - Ad_3(m) / d_2(m)\right), 0\right)$$
$$\text{UCL} = \overline{R}\left(1 + Ad_3(m) / d_2(m)\right)$$

p and np Charts

The data for p and np charts are attribute data. Each measurement x_{ij} is either 0 or 1, where 1 indicates a non-conforming measurement. Therefore,

$$x_{i+} = \sum_{j=1}^{n_i} x_{ij}$$

is the count of non-conforming units for subgroup *i*. When a weight variable is specified, x_{i+} is a weighted sum of non-conforming units. If the data are aggregated and the value of the count variable is greater than the total number of units for any subgroup, this subgroup is dropped.

Equal Sample Sizes

Assume $n_i = n$ The process for a p chart is $\{p_i : i = 1, \ldots, K\}$ where $p_i = x_{i+} / n$. The center line for a p chart is

$$\bar{p} = \frac{1}{K} \sum_{i=1}^{K} p_i$$

and the control limits are

$$\text{LCL} = \max\left(\bar{p} - A\sqrt{\left(\bar{p}(1-\bar{p})\right)/n},\, 0 \right)$$

$$\text{UCL} = \min\left(\bar{p} + A\sqrt{\left(\bar{p}(1-\bar{p})\right)/n},\, 1 \right)$$

The process for an np chart is $\{x_{i+} : i = 1, \ldots, K\}$. The center line for an np chart is

$$\bar{x} = \frac{1}{K} \sum_{i=1}^{K} x_{i+}$$

and the control limits are

$$\text{LCL} = \max\left(\bar{x} - A\sqrt{n\bar{p}(1-\bar{p})},\, 0 \right)$$

$$\text{UCL} = \min\left(\bar{x} + A\sqrt{n\bar{p}(1-\bar{p})},\, n \right)$$

Unequal Sample Sizes

The process for a p chart is $\{p_i : i = 1, \ldots, K\}$ where $p_i = x_{i+} / n_i$. The center line for a p chart is

$$\bar{p} = \frac{1}{N} \sum_{i=1}^{K} x_{i+} = \frac{1}{N} \sum_{i=1}^{K} n_i p_i$$

and the control limits for subgroup *i* are

$$\text{LCL} = \max\left(\bar{p} - A\sqrt{\left(\bar{p}(1-\bar{p})\right)/n_i}\ , 0 \right)$$
$$\text{UCL} = \min\left(\bar{p} + A\sqrt{\left(\bar{p}(1-\bar{p})\right)/n_i}\ , 1 \right)$$

The process for an np chart is $\{x_{i+}: i = 1, \ldots, K\}$. However, the center line for an np chart for subgroup *i* is $n_i \bar{p}$. The control limits for subgroup *i* are

$$\text{LCL} = \max\left(n_i \bar{p} - A\sqrt{\left(n_i \bar{p}(1-\bar{p})\right)}, 0 \right)$$
$$\text{UCL} = \min\left(n_i \bar{p} + A\sqrt{\left(n_i \bar{p}(1-\bar{p})\right)}, n_i \right)$$

Note: A warning message is issued when an np chart is requested for subgroups of unequal sample sizes.

u and c Charts

Measurements x_{ij} show the number of defects for the *j*th unit for subgroup *i*. Hence,

$$x_{i+} = \sum_{j=1}^{n_i} x_{ij}$$

is the total number of defects for subgroup *i*. When a weight variable is used, x_{i+} is a weighted sum of defects.

Equal Sample Sizes

Assume $n_i = n$. The process for a u chart is $\{u_i : i = 1, \ldots, K\}$ where $u_i = x_{i+} / n$. The center line for a u chart is

$$\bar{u} = \frac{1}{K}\sum_{i=1}^{K} u_i$$

and the control limits are

$$\mathrm{LCL} = \max\left(\bar{u} - A\sqrt{\bar{u} / n},\ 0\right)$$
$$\mathrm{UCL} = \bar{u} + A\sqrt{\bar{u} / n}$$

The process for a c chart is $\{x_{i+} : i = 1, \ldots, K\}$. The center line for a c chart is

$$\bar{c} = \frac{1}{K}\sum_{i=1}^{K} x_{i+}$$

and the control limits for a c chart are

$$\mathrm{LCL} = \max\left(\bar{c} - A\sqrt{\bar{c}},\ 0\right)$$
$$\mathrm{UCL} = \bar{c} + A\sqrt{\bar{c}}$$

Unequal Sample Sizes

The process for a u chart is $\{u_i : i = 1, \ldots, K\}$ where $u_i = x_{i+} / n_i$ *ui* = *xi*+ / *ni*. The center line for a u chart is

$$\overline{u} = \frac{1}{N}\sum_{i=1}^{K} x_{i+}$$

and the control limits are

$$\text{LCL} = \max\left(\overline{u} - A\sqrt{\overline{u} / n_i}\,,\, 0\right)$$
$$\text{UCL} = \overline{u} + A\sqrt{\overline{u} / n_i}$$

The process for a c chart is $\{x_{i+} : i = 1, \ldots, K\}$. The center line for subgroup *i* is $n_i\overline{u}$ and the control limits are

$$\text{LCL} = \max\left(n_i\overline{u} - A\sqrt{n_i\overline{u}}\,,\, 0\right)$$
$$\text{UCL} = n_i\overline{u} + A\sqrt{n_i\overline{u}}$$

Note: A warning message is issued when a c chart is requested for subgroups of unequal sample sizes.

References

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