

# NPAR TESTS

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If a WEIGHT variable is specified, it is used to replicate a case as many times as indicated by the weight value rounded to the nearest integer. If the workspace requirements are exceeded and sampling has been selected, a random sample of cases is chosen for analysis using the algorithm described in SAMPLE. For the RUNS test, if sampling is specified, it is ignored. The tests are described in Siegel (1956).

*Note:* Before SPSS version 10.1.3, the WEIGHT variable was used to replicate a case as many times as indicated by the integer portion of the weight value. The case then had a probability of additional inclusion equal to the fractional portion of the weight value.

## One-Sample Chi-Square Test

### Cell Specification

If the (lo, hi) specification is used, each integer value in the lo to hi range is designated a cell. Otherwise, each distinct value encountered is considered a cell.

### Observed Frequencies ( $O_i$ )

If (lo, hi) has been selected, every observed value is truncated to an integer and, if it is in the lo to hi range, it is included in the frequency count for the corresponding cell. Otherwise, a count of the frequency of occurrence of the distinct values is obtained.

### Expected Frequencies ( $EXP_i$ )

If none or EQUAL is specified,

$$EXP_i = \frac{\text{number of observations } (N) \text{ [in range]}}{\text{number of cells } (k)}$$

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When the expected values ( $E_i$ ) are specified either as counts, percentages, or proportions,

$$EXP_i = \left( \frac{E_i}{\sum_{i=1}^k E_i} \right) N$$

If there are cells with expected values less than 5, the number of such cells and the minimum expected value are printed.

If the number of user-supplied expected frequencies is not equal to the number of cells generated, or if an expected value is less than or equal to zero, the test terminates with an error message.

### Chi-Square and Its Degrees of Freedom

$$\chi^2 = \sum_{i=1}^k \frac{(O_i - EXP_i)^2}{EXP_i}$$
$$df = k - 1$$

The significance level is from the chi-square distribution with  $k - 1$  degrees of freedom.

## Kolmogorov-Smirnov One-Sample Test

### Calculation of Empirical Cumulative Distribution Function

The observations are sorted into ascending order  $X_{(1)}$  to  $X_{(N)}$ . The empirical cdf,  $\hat{F}(X)$ , is

$$\hat{F}(X) = \begin{cases} 0 & -\infty < X < X_{(1)} \\ i/N & X_{(i)} \leq X < X_{(i+1)} \\ 1 & X_{(N)} \leq X < \infty \end{cases} \quad i = 1, \dots, N-1$$

### Estimation of Parameters for Theoretical Distribution

It is possible to test that the underlying distribution is either uniform, normal, or Poisson. If the parameters are not specified, they are estimated from the data.

#### Uniform

$$\text{minimum} = X_{(1)}$$

$$\text{maximum} = X_{(N)}$$

#### Normal

$$\text{mean} (\bar{X}) = \sum_{i=1}^N X_i / N$$

$$\text{standard deviation} (S) = \sqrt{\left( \sum_{i=1}^N X_i^2 - \left( \sum_{i=1}^N X_i / N \right) \left( \sum_{i=1}^N X_i \right) \right) / (N-1)}$$

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### Poisson

$$\text{mean}(\lambda) = \sum_{i=1}^N X_i / N$$

The test is not done if, for the uniform, all data are not within the user-specified range or, for the Poisson, the data are not non-negative integers. If the variance of the normal or the mean of the Poisson is zero, the test is also not done.

### Calculation of Theoretical Cumulative Distribution Functions

#### For Uniform

$$F_0(X_i) = \frac{X_i - \min}{\max - \min}$$

#### For Poisson

$$F_0(X_i) = \sum_{l=0}^{X_i} \frac{e^{-\lambda} \lambda^l}{l!}$$

If  $\lambda \geq 100,000$ , the normal approximation is used.

#### For Normal

$$F_0(X_i) = F_{0,1}\left(\frac{X_i - \bar{X}}{S}\right)$$

where the algorithm for the generation of  $F_{0,1}(Z)$  is described in Appendix 1.

## Calculation of Differences

For the Uniform and Normal, two differences are computed:

$$\begin{aligned} D_i &= \hat{F}(X_{i-1}) - F_0(X_i) \\ \tilde{D}_i &= \hat{F}(X_i) - F_0(X_i) \quad i = 1, \dots, N \end{aligned}$$

For the Poisson:<sup>1</sup>

$$D_i = \begin{cases} \hat{F}(X_i - 1) - F(X_i - 1) & X_i > 0 \\ 0 & X_i = 0 \end{cases} \quad i = 1, 2, \dots, N.$$

$$\tilde{D}_i = \hat{F}(X_i) - F(X_i)$$

The maximum positive, negative, and absolute differences are printed.

## Test Statistic and Significance

The test statistic is

$$Z = \sqrt{N} \max_i (|D_i|, |\tilde{D}_i|)$$

The two-tailed probability level is estimated using the first three terms of the Smirnov (1948) formula.

$$\text{if } 0 \leq Z < 0.27, \quad p = 1$$

$$\text{if } 0.27 \leq Z < 1, \quad p = 1 - \frac{2.506628}{Z} (Q + Q^9 + Q^{25})$$

$$\text{where } Q = e^{-1.233701Z^2}.$$

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<sup>1</sup> This algorithm applies to SPSS 7.0 and later releases.

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$$\text{if } 1 \leq Z < 3.1, \quad p = 2(Q - Q^4 + Q^9 - Q^{16})$$

$$\text{where } Q = e^{-2Z^2}.$$

$$\text{if } Z \geq 3.1, \quad p = 0.$$

## Runs Test

### Computation of Cutting Point

The cutting point which is used to dichotomize the data can be specified as a particular number, or the value of a statistic which is to be calculated. The possible statistics are

$$\text{Mean} = \sum_{i=1}^N X_i / N$$

$$\text{Median} = \begin{cases} (X_{(N/2+1)} + X_{(N/2)}) / 2 & \text{if } N \text{ is even} \\ X_{((N+1)/2)} & \text{if } N \text{ is odd} \end{cases}$$

**where the data are sorted in ascending order from  $X_{(1)}$ , the smallest, to  $X_{(N)}$ , the largest.**

**Mode = most frequently occurring value**

If there are multiple modes, the one largest in value is selected and a warning printed.

## Number of Runs

For each of the data points, in the sequence in the file, the difference

$$D_i = X_i - \text{CUTPOINT}$$

is computed. If  $D_i \geq 0$ , the difference is considered positive, otherwise negative. The number of times the sign changes, that is,  $D_i \geq 0$  and  $D_{i+1} < 0$ , or  $D_i < 0$  and  $D_{i+1} \geq 0$ , as well as the number of positive ( $n_p$ ) and ( $n_a$ ) signs, are determined. The number of runs ( $R$ ) is the number of sign changes plus one.

## Significance Level

The sampling distribution of the number of runs ( $R$ ) is approximately normal with

$$\mu_r = \frac{2n_p n_a}{n_p + n_a} + 1$$

$$\sigma_r = \sqrt{\frac{2n_p n_a (2n_p n_a - n_a - n_p)}{(n_p + n_a)^2 (n_p + n_a - 1)}}$$

The two-sided significance level is based on

$$Z = \frac{R - \mu_r}{\sigma_r}$$

unless  $n < 50$ ; then

$$Z_c = \begin{cases} (R - \mu_r + 0.5)/\sigma_r & \text{if } R - \mu_r \leq 0.5 \\ (R - \mu_r - 0.5)/\sigma_r & \text{if } R - \mu_r \geq 0.5 \\ 0 & \text{if } |R - \mu_r| < 0.5 \end{cases}$$

## Binomial Test

$n_1$	Number of observations in category 1
$n_2$	Number of observations in category 2
$p$	Test probability
$m$	$\min(n_1, n_2)$
$N$	$n_1 + n_2$
$p^*$	$p$ if $m = n_1$ , $1 - p$ if $m = n_2$

Two-tailed exact probability is

$$2 \left( \sum_{i=0}^m \binom{N}{i} p^{*i} (1-p^*)^{N-i} \right) - \binom{N}{m} p^{*m} (1-p^*)^{N-m}$$

If an approximate probability is reported, the following algorithm is used:

$$Z_1 = \frac{n_1 + 0.5 - Np}{\sqrt{Np(1-p)}} \quad Z_2 = \frac{n_1 - 0.5 - Np}{\sqrt{Np(1-p)}}$$

$P(Z_i)$  = probability from standard normal distribution

Then,

$$P(X = n_1) = P(Z_2) - P(Z_1)$$

$$P(X \leq n_1) = P(Z_1)$$

and the two-tailed approximate probability is

$$2P(X \leq n_1) - P(X = n_1)$$



## McNemar's Test

### Table Construction

The data values are searched to determine the two unique response categories. If the variables  $X$  and  $Y$  take on more than two values, or only one value, a message is printed and the test is not done. The number of cases that have  $X_i < Y_i(n_1)$  or  $X_i > Y_i(n_2)$  are counted.

### Test Statistic and Significance Level

If  $n_1 + n_2 \leq 25$ , the exact probability of  $r$  or fewer "successes" occurring in  $n_1 + n_2$  trials when  $p = 0.5$  and  $r = \min(n_1, n_2)$  is calculated recursively from the binomial.

$$p(X \leq r) = \sum_{i=0}^r \binom{n_1 + n_2}{i} (0.5)^{n_1 + n_2}$$

The two-tailed probability level is obtained by doubling the computed value. If  $n_1 + n_2 > 25$ , a  $\chi^2$  approximation with a correction for continuity is used.

$$\chi_c^2 = \frac{(|n_1 - n_2| - 1)^2}{n_1 + n_2}, \quad df = 1$$

## Sign Test

### Count of Signs

For each case, the difference

$$D_i = X_i - Y_i$$

is computed and the number of positive ( $n_p$ ) and negative ( $n_n$ ) differences counted. Cases in which  $X_i = Y_i$  are ignored.

### Test Statistic and Significance Level

If  $n_p + n_n \leq 25$ , the exact probability of  $r$  or fewer “successes” occurring in  $n_p + n_n$  trials, when  $p = 0.5$  and  $r = \min(n_p, n_n)$ , is calculated recursively from the binomial

$$p(X \leq r) = \sum_{i=0}^r \binom{n_p + n_n}{i} (0.5)^{n_p + n_n}$$

If  $n_p + n_n > 25$ , the significance level is based on the normal approximation

$$Z_c = \frac{\max(n_p, n_n) - 0.5(n_p + n_n) - 0.5}{0.5\sqrt{n_p + n_n}}$$

A two-tailed significance level is printed.

## Wilcoxon Matched-Pairs Signed-Rank Test

### Computation of Ranked Differences

For each case, the difference

$$D_i = X_i - Y_i$$

is computed, as well as the absolute value of  $D_i$ . All nonzero absolute differences are then sorted into ascending order, and ranks are assigned. In the case of ties, the average rank is used. The sums of the ranks corresponding to positive differences ( $S_p$ ) and negative differences ( $S_n$ ) are calculated. The average positive rank is

$$\bar{X}_p = S_p / n_p$$

and the average negative rank is

$$\bar{X}_n = S_n/n_n$$

where  $n_p$  is the number of cases with positive differences and  $n_n$  the number with negative differences.

## Test Statistic and Significance Level

The test statistic is<sup>2</sup>

$$Z = \frac{\min(S_p, S_n) - (n(n+1)/4)}{\sqrt{n(n+1)(2n+1)/24 - \sum_{j=1}^l (t_j^3 - t_j)/48}}$$

where

- $n$  Number of cases with non-zero differences
- $l$  Number of ties
- $t_j$  Number of elements in the  $j$ -th tie,  
 $j = 1, \dots, l$

For large sample sizes the distribution of  $Z$  is approximately standard normal. A two-tailed probability level is printed.

## Cochran's $Q$ Test

### Computation of Basic Statistics

For each of the  $N$  cases, the  $k$  variables specified may take on only one of two possible values. If more than two values, or only one, are encountered, a message is printed and the test is not done. The first value encountered is designated a "success" and for each case the number of variables that are "successes" are

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<sup>2</sup> This algorithm applies to SPSS 7.0 and later releases.

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counted. The number of “successes” for case  $i$  will be designated  $R_i$  and the total number of “successes” for variable  $l$  will be designated  $C_l$ .

### Test Statistic and Level of Significance

Cochran’s  $Q$  is calculated as

$$Q = \frac{(k-1) \left[ k \sum_{l=1}^k C_l^2 - \left( \sum_{l=1}^k C_l \right)^2 \right]}{k \sum_{l=1}^k C_l - \sum_{i=1}^N R_i^2}$$

The significance level of  $Q$  is from the  $\chi^2$  distribution with  $k-1$  degrees of freedom.

## Friedman’s Two-Way Analysis of Variance by Ranks

### Sum of Ranks

For each of the  $N$  cases, the  $k$  variables are sorted and ranked, with average rank being assigned in the case of ties. For each of the  $k$  variables, the sum of ranks over the cases is calculated. This will be denoted as  $C_l$ . The average rank for each variable is

$$\bar{R}_l = C_l / N$$

### Test Statistic and Significance Level

The test statistic is<sup>3</sup>

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<sup>3</sup> This algorithm applies to SPSS 7.0 and later releases.

$$\chi^2 = \frac{(12/Nk(k+1)) \sum_{l=1}^k C_l^2 - 3N(k+1)}{1 - \sum T/Nk(k^2 - 1)}$$

where  $\sum T$  is the same as in Kendall's coefficient of concordance (see Lehmann, 1985, p. 265).

The significance level is from the  $\chi^2$  distribution with  $k - 1$  degrees of freedom.

## Kendall's Coefficient of Concordance

$N$ ,  $k$ , and  $l$  are the same as in Friedman, above.

### Coefficient of Concordance (W)<sup>4</sup>

$$W = \left( \frac{F}{N(k-1)} \right) \left( \frac{N^2 k(k^2 - 1)/12}{N^2 k(k^2 - 1)/12 - N \sum T/12} \right)$$

where  $F =$  Friedman  $\chi^2$  statistic .

$$\sum T = \sum_{i=1}^N \sum_{l=1}^k (t^3 - t)$$

with  $t =$  number of variables tied at each tied rank for each case.

### Test Statistic and Significance Level

$$\chi^2 = N(k-1)W$$

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<sup>4</sup> This algorithm applies to SPSS 7.0 and later releases.

The significance level is from the  $\chi^2$  distribution with  $k - 1$  degrees of freedom.

## The Two-Sample Median Test

### Table Construction

If the median value is not specified by the user, the combined data from both samples are sorted and the median calculated.

$$Md = \begin{cases} (X_{[N/2]} + X_{[N/2+1]})/2 & \text{if } N \text{ is even} \\ X_{[(N+1)/2]} & \text{otherwise} \end{cases}$$

where  $X_{[N]}$  is the largest value and  $X_{[1]}$  the smallest. The number of cases in each of the two groups which exceed the median are counted. These will be denoted as  $g_1$  and  $g_2$ , and the corresponding sample sizes as  $n_1$  and  $n_2$ .

### Test Statistic and Significance Level

- If  $N \leq 30$ , the significance level is from Fisher's exact test. (See Appendix 5.)
- If  $N > 30$ , the test statistic is

$$\chi_c^2 = \frac{[|g_1(n_2 - g_2) - g_2(n_1 - g_1)| - N/2]^2 N}{(g_1 + g_2)(n_1 + n_2 - g_1 - g_2)n_1 n_2}$$

which is distributed as a  $\chi^2$  with 1 degree of freedom.

## Mann-Whitney $U$ Test

### Calculation of Sums of Ranks

The combined data from both groups are sorted and ranks assigned to all cases, with average rank being used in the case of ties. The sum of ranks for each of the groups ( $S_1$  and  $S_2$ ) is calculated, as well as, for tied observations,  $T_i = \frac{t^3 - t}{12}$ , where  $t$  is the number of observations tied for rank  $i$ .

The average rank for each group is

$$\bar{S}_i = S_i / n_i$$

where  $n_i$  is the sample size in group  $i$ .

### Test Statistic and Significance Level

The  $U$  statistic for group 1 is

$$U = n_1 n_2 + \frac{n_1(n_1 + 1)}{2} - S_1$$

- If  $U > n_1 n_2 / 2$ , the statistic used is

$$U' = n_1 n_2 - U$$

- If  $n_1 n_2 \leq 400$  and  $n_1 n_2 / 2 + \min(n_1, n_2) \leq 220$  the exact significance level is based on an algorithm of Dineen and Blakesley (1973).
- The test statistic corrected for ties is

$$Z = \frac{(U - n_1 n_2 / 2)}{\sqrt{\frac{n_1 n_2}{N(N-1)} \left( \frac{N^3 - N}{12} - \sum_i T_i \right)}}$$

which is distributed approximately as a standard normal. A two-tailed significance level is printed.

## Kolmogorov-Smirnov Two-Sample Test

### Calculation of the Empirical Cumulative Distribution Functions and Differences

For each of the two groups separately the data sorted into ascending order, from  $X_{[1]}$  to  $X_{[n_i]}$ , and the empirical cdf for group  $i$  is computed as

$$\hat{F}_i(X) = \begin{cases} 0 & -\infty < X < X_{[1]} \\ j/n_i & X_{[j]} \leq X < X_{[j+1]} \\ 1 & X_{[n_i]} \leq X < \infty \end{cases}$$

For all of the  $X_j$  values in the two groups, the difference between the two groups is

$$D_j = \hat{F}_1(X_j) - \hat{F}_2(X_j)$$

where  $\hat{F}_1(X_j)$  is the cdf for the group with the larger sample size. The maximum positive, negative, and absolute differences are also computed.

### Test Statistic and Level of Significance

The test statistic (Smirnov, 1948) is

$$Z = \max_j |D_j| \sqrt{\frac{n_1 n_2}{n_1 + n_2}}$$



and the significance level is calculated using the Smirnov approximation described in the K-S one sample test.

## Wald-Wolfowitz Runs Test

### Calculation of Number of Runs

All observations from the two samples are pooled and sorted into ascending order. The number of changes in the group numbers corresponding to the ordered data are counted. The number of runs ( $R$ ) is the number of group changes plus one.

If there are ties involving observations from the two groups, both the minimum and maximum number of runs possible are calculated.

### Significance Level

If  $n_1 + n_2$ , the total sample size, is less than or equal to 30, the one-sided significance level is exactly calculated from

$$P(r \leq R) = \frac{2}{\binom{n_1 + n_2}{n_1}} \sum_{r=2}^R \binom{n_1 - 1}{r/2 - 1} \binom{n_2 - 1}{r/2 - 1}$$

when  $R$  is even. When  $R$  is odd

$$P(r \leq R) = \frac{1}{\binom{n_1 + n_2}{n_1}} \sum_{r=2}^R \left[ \binom{n_1 - 1}{k - 1} \binom{n_2 - 1}{k - 2} + \binom{n_1 - 1}{k - 2} \binom{n_2 - 1}{k - 1} \right]$$

where

$$r = 2k - 1.$$

For sample sizes greater than 30, the normal approximation is used (see RUNS test described previously).

## Moses Test of Extreme Reaction

### Span Computation

Observation from both groups are jointly sorted and ranked, with the average rank being assigned in the case of ties. The ranks corresponding to the smallest and largest control group (first group) members are determined, and the span is computed as

$$\text{SPAN} = \text{Rank}(\text{Largest Control Value}) - \text{Rank}(\text{Smallest Control Value}) + 1$$

rounded to the nearest integer.

### Significance Level

The exact one-tailed probability level is computed from

$$P(\text{SPAN} \leq n_c - 2h + g) = \frac{\sum_{i=0}^g \left[ \binom{i+n_c-2h-2}{i} \binom{n_e+2h+1-i}{n_e-i} \right]}{\binom{n_c+n_e}{n_c}}$$

where  $h=0$ ,  $n_c$  is the number of cases in the control group, and  $n_e$  is the number of cases in the experimental group. The same formula is used in the next section where  $h$  is not zero.

### Censoring of Range

The test is repeated, dropping the  $h$  lowest and  $h$  highest ranks from the control group. If not specified by the user,  $h$  is taken to be the integer part of  $0.05n_c$  or 1, whichever is greater. If  $h$  is user specified, the integer value is used unless it is less than one. The significance level is determined as above.

## K-Sample Median Test

### Table Construction

If the median value is not specified by the user, the combined data from all groups are sorted and the median is calculated.

$$\text{Md} = \begin{cases} (X_{[N/2]} + X_{[N/2+1]})/2 & \text{if } N \text{ is even} \\ X_{[(N+1)/2]} & \text{if } N \text{ is odd} \end{cases}$$

where  $X_{[N]}$  is the largest value and  $X_{[1]}$  the smallest.

The number of cases in each of the groups that exceed the median are counted and the following table is formed.

		<i>Group</i>					
		1	2	3	...	<i>k</i>	
LE Md	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding: 5px;"><math>O_{11}</math></td> <td style="padding: 5px;"><math>O_{12}</math></td> <td style="padding: 5px;"><math>O_{13}</math></td> <td style="padding: 5px;">...</td> <td style="padding: 5px;"><math>O_{1k}</math></td> </tr> </table>	$O_{11}$	$O_{12}$	$O_{13}$	...	$O_{1k}$	$R_1$
$O_{11}$	$O_{12}$	$O_{13}$	...	$O_{1k}$			
GT Md	<table style="border-collapse: collapse; width: 100%;"> <tr> <td style="padding: 5px;"><math>O_{21}</math></td> <td style="padding: 5px;"><math>O_{22}</math></td> <td style="padding: 5px;"><math>O_{23}</math></td> <td style="padding: 5px;">...</td> <td style="padding: 5px;"><math>O_{2k}</math></td> </tr> </table>	$O_{21}$	$O_{22}$	$O_{23}$	...	$O_{2k}$	$R_2$
$O_{21}$	$O_{22}$	$O_{23}$	...	$O_{2k}$			
		$n_1$	$n_2$	$n_3$	...	$n_k$	$N$

### Test Statistic and Level of Significance

The  $\chi^2$  statistic for all nonempty groups is calculated as

$$\chi^2 = \sum_{j=1}^k \sum_{i=1}^2 (O_{ij} - E_{ij})^2 / E_{ij}$$

where

$$E_{ij} = \frac{R_i n_j}{N}.$$

The significance level is from the  $\chi^2$  distribution with  $k - 1$  degrees of freedom, where  $k$  is the number of nonempty groups. A message is printed if any cell has an expected value less than one, or more than 20% of the cells have expected values less than five.

## Kruskal-Wallis One-Way Analysis of Variance

### Computation of Sums of Ranks

Observations from all  $k$  nonempty groups are jointly sorted and ranked, with the average rank being assigned in the case of ties. The number of tied scores in a set of ties,  $t_i$ , is also found, and the sum of  $T_i = t_i^3 - t_i$  is accumulated. For each group the sum of ranks,  $R_i$ , as well as the number of observations,  $n_i$ , is obtained.

### Test Statistic and Level of Significance

The test statistic unadjusted for ties is

$$H = \frac{12}{N(N+1)} \sum_{i=1}^k R_i^2 / n_i - 3(N+1)$$

where  $N$  is the total number of observations.

Adjusted for ties, the statistic is

$$H' = \frac{H}{1 - \sum_{i=1}^m T_i / (N^3 - N)}$$

where  $m$  is the total number of tied sets.

The significance level is based on the  $\chi^2$  distribution, with  $k - 1$  degrees of freedom.

## References

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Siegel, S. 1956. *Nonparametric statistics for the behavioral sciences*. New York: McGraw-Hill.

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