

IV. *A most compendious and facile Method for Constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given.* By E. Halley.

THE Invention of the Logarithms is justly esteemed one of the most Useful Discoveries in the Art of Numbers, and accordingly has had an Universal Reception and Applause; and the great Geometricians of this Age have not been wanting to cultivate this Subject with all the Accuracy and Subtily a matter of that consequence doth require; and they have demonstrated several very admirable Properties of these Artificial Numbers, which have rendred their Construction much more facile than by those operose Methods at first used by their truly Noble Inventer the Lord *Napeir*, and our worthy Country-man Mr. *Briggs*.

But notwithstanding all their Endeavours, I find very few of those who make constant use of Logarithms, to have attained an adequate Notion of them, to know how to make or examine them; or to understand the extent of the use of them: Contenting themselves with the Tables of them as they find them, without daring to question them, or caring to know how to rectifie them, should they be found amiss, being I suppose under the apprehension of some great difficulty therein. For the sake of such the following Tract is principally intended, but not without hopes however to produce something that may be acceptable to the most knowing in these matters.

But first, it may be requisite to premise a definition of Logarithms, in order to render the ensuing Discourse more clear, the rather because the old one *Numerorum proportionalium æqui differentes comites*, seems too scanty to define them fully. They may much more properly be said to be *Numeri Rationum Exponentes*: Wherein we consider ratio as a *Quantitas sui generis*, beginning from the ratio of equality, or 1 to 1 = 0; being
Affix-

Affirmative when the *ratio* is increasing, as of Unity to a greater Number, but Negative when decreasing; and these *rationes* we suppose to be measured by the Number of *ratiunculae* contained in each. Now these *ratiunculae* are so to be understood as in a continued Scale of Proportionals infinite in Number between the two terms of the *ratio*, which infinite Number of mean Proportionals is to that infinite Number of the like and equal *ratiunculae* between any other two terms, as the Logarithm of the one *ratio* is to the Logarithm of the other. Thus if there be supposed between 1 and 10 an infinite Scale of mean Proportionals, whose Number is 100000 &c. *in infinitum*; between 1 and 2 there shall be 30102 &c. of such Proportionals, and between 1 and 3 there will be 47712 &c. of them, which Numbers therefore are the Logarithms of the *rationes* of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the Logarithms of 10, 2 and 3.

This being laid down, it is obvious that if between Unity and any Number proposed, there be taken any infinity of mean Proportionals, the infinitely little augment or decrement of the first of those means from Unity, will be a *ratiuncula*, that is, the *momentum* or *Fluxion* of the *ratio* of Unity to the said Number: And seeing that in these continual Proportionals all the *ratiunculae* are equal, their Sum, or the whole *ratio* will be as the said *momentum* is directly; that is, the Logarithm of each *ratio* will be as the Fluxion thereof. Wherefore if the Root of any Infinite Power be extracted out of any Number, the *differentiola* of the said Root from Unity, shall be as the Logarithm of that Number. So that Logarithms thus produced may be of as many forms as you please to assume infinite *Indices* of the Power whose Root you seek: as if the *Index* be supposed 100000 &c. infinitely, the Roots shall be the Logarithms invented by the Lord Napier; but if the said *Index* were 2302585 &c. Mr. Briggs's Logarithms would immediately be produced. And if you please to stop at any number of Figures, and not to continue them on, it will suffice to assume an *Index* of a Figure or two more than your intended Logarithm is to have, as Mr. Briggs did, who to have his Logarithms true to 14 places, by continual extraction of the Square Root, at last came to have the Root of the 140737488355328 *th.* Power; but how operose that Extraction was, will be easily judged by whose shall undertake to examine his *Calculus*. Now

Now, though the Notion of an Infinite Power may seem very strange, and to those that know the difficulty of the Extraction of the Roots of High Powers, perhaps impracticable; yet by the help of that admirable Invention of Mr. *Newton*, whereby he determines the *Unica* or Numbers prefix to the Members composing Powers (on which chiefly depends the Doctrine of Series) the Infinity of the Index contributes to render the Expression much more easie: For if the Infinite Power to be resolv'd be put (after Mr. *Newton's* Method)

$$p + p q, p + p q |^m \text{ or } 1 + q |^m, \text{ instead of } 1 + \frac{1}{m} q + \frac{1}{2 m m} q q + \frac{1 - 3 m + 2 m m}{6 m^3} q^3 + \frac{1 - 6 m + 11 m m - 6 m^2}{24 m^4} q^4$$

&c. (which is the Root when *m* is finite,) becomes $1 + \frac{1}{m} q - \frac{1}{2 m} q q + \frac{1}{3 m} q^3 + \frac{1}{4 m} q^4 + \frac{1}{5 m} q^5, \&c.$ *m* being infinite infinite, and consequently whatever is divided thereby vanishing. Hence it follows that $\frac{1}{m}$ multiplied into

$q - \frac{1}{2} q q + \frac{1}{3} q q q - \frac{1}{4} q^4 + \frac{1}{5} q^5 \&c.$ is the augment of the first of our mean Proportionals between Unity and $1 + q$, and is therefore the Logarithm of the ratio of 1 to $1 + q$; and whereas the Infinite Index *m* may be taken at pleasure, the several Scales of Logarithms to such Indices will be as $\frac{1}{m}$ or reciprocally as the Indices. And if the Index be taken 10000 &c. as in the case of *Napier's* Logarithms, they will be simply $q - \frac{1}{2} q q + \frac{1}{3} q q q - \frac{1}{4} q^4 + \frac{1}{5} q^5 - \frac{1}{6} q^6 \&c.$

Again, if the Logarithm of a decreasing ratio be sought, the infinite Root of $1 - q$ or $1 - q |^m$ is $1 - \frac{1}{m} q - \frac{1}{2 m} q^2 - \frac{1}{3 m} q^3 - \frac{1}{4 m} q^4 - \frac{1}{5 m} q^5 - \frac{1}{6 m} q^6 \&c.$ whence the decrement of the first of our infinite Number of Proportionals will be $\frac{1}{m}$ into $q + \frac{1}{2} q q + \frac{1}{3} q^3 + \frac{1}{4} q^4 + \frac{1}{5} q^5 + \frac{1}{6} q^6 \&c.$ which therefore will be as the Logarithm of the ratio of Unity to $1 - q$. But if *m* be put 10000 &c. then the said Logarithm will be $q + \frac{1}{2} q q + \frac{1}{3} q^3 + \frac{1}{4} q^4 + \frac{1}{5} q^5 + \frac{1}{6} q^6 \&c.$

Hence

Hence the terms of any *ratio* being a and b , q becomes $\frac{b-a}{a}$ or the difference divided by the lesser term, when 'tis an increasing *ratio*; or $\frac{b-a}{b}$ when 'tis decreasing or as b to a . Whence the Logarithm of the same *ratio* may be doubly exprest, for putting x for the difference of the terms a and b , it will be either

$$\frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c. or}$$

$$\frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.}$$

But if the *ratio* of a to b be supposed divided into two parts, *viz.* into the *ratio* of a to the Arithmetical Mean between the terms, and the *ratio* of the said Arithmetical Mean to the other term b , then will the Sum of the Logarithms of those two *rationes* be the Logarithm of the *ratio* of a to b ; and substituting $\frac{1}{2}x$ instead of $\frac{1}{2}a + \frac{1}{2}b$ the said Arithmetical Mean, the Logarithms of those *rationes* will be by the foregoing Rule,

$$\frac{1}{m} \text{ in } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c. and}$$

$$\frac{1}{m} \text{ in } \frac{x}{z} - \frac{xx}{2zz} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c.}$$

the Sum $\frac{1}{m} \text{ in } \frac{2x}{z} * + \frac{2x^3}{3z^3} * + \frac{2x^5}{5z^5} * \frac{2x^7}{7z^7} \text{ \&c. will}$

be the Logarithm of the *ratio* of a to b , whose difference is x and Sum z . And this *Series* converges twice as swift as the former, and therefore is more proper for the Practice of making of Logarithms: Which it performs with that expedition, that where x the difference is but the hundredth part of the

Sum, the first step $\frac{2x}{z}$ suffices to seven places of the Logarithm, and the second step to twelve; But if *Briggs's* first Twenty Chiliads of Logarithms be supposed made, as he has very carefully computed them, to fourteen places, the first step alone is capable to give the Logarithm of any intermediate Number true to all the places of those Tables.

After the same manner may the difference of the said two Logarithms be very fitly applyed to find the Logarithms of Prime Numbers, having the Logarithms of the two next Numbers above and below them: For the difference of the *ratio* of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b is the *ratio* of ab to $\frac{1}{4}zz$, and the half of that *ratio* is that of \sqrt{ab} to $\frac{1}{2}z$, or of the Geometrical Mean to the Arithmetical. And consequently the Logarithm thereof will be the half difference of the Logarithms of those *rationes*, *viz.*

$$\frac{1}{m} \text{ into } \frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} \ \&c.$$

Which is a Theorem of good dispatch to find the Logarithm of $\frac{1}{2}z$. But the same is yet much more advantageously performed by a Rule derived from the foregoing, and beyond which in my Opinion nothing better can be hoped. For the *ratio* of ab to $\frac{1}{4}zz$ or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$ or the Square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which in the present case of finding the Logarithms of Prime Numbers is always Unity, and calling the Sum of the terms $\frac{1}{4}z + ab = yy$, the Logarithm of the *ratio* of \sqrt{ab} to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$ will be found

$$\frac{1}{m} \text{ in } \frac{1}{yy} + \frac{1}{3y^3} + \frac{1}{5y^5} + \frac{1}{7y^7} + \frac{1}{9y^9} \ \&c.$$

which converges very much faster than any Theorem hitherto published for this purpose.

Here note that $\frac{1}{m}$ is all along applyed to adapt these Rules to all sorts of Logarithms. If m be 10000 &c. it may be neglected, and you will have *Napier's* Logarithms, as was hinted before; but if you desire *Briggs's* Logarithms, which are now generally received, you must divide your Series by 2,302585092994045684017991454684364207601101488628772976033328 or multiply it by the reciprocal thereof, *viz.*

0,434294481903251827651128918916605082294397005803666566114454

But to save so operose a Multiplication (which is more than all the rest of the Work) it is expedient to Divide this Multiplicator by the Powers of z or y continually, according to the direction of the Theorem, especially where x is small and Integer, reserving the proper Quotes to be added together, when you have produced your Logarithm to as many

many Figures as you desire, of which Method I will give a Specimen.

If the Curiosity of any Gentleman that has leisure would prompt him to undertake to do the Logarithms of all Prime Numbers under 100000 to 25 or 30 Figures, I dare assure him that the facility of this Method will invite him thereto, nor can any thing more easie be desired. And to encourage him, I here give the Logarithms of the first Prime Numbers under 20 to sixty places, computed by the accurate Pen of Mr. *Abraham Sharp*, (from whose Industry and Capacity the World may in time expect great Performances) as they were communicated to me by our common Friend Mr. *Euclid Speidall*.

Numb.	Logarithm.
2	0,301029995663981195213738894724493026768189881462108541310427
3	0,477121254719662437295027903255115309200128864190695864829866
5	0,845098040014256830712216258592646193483572396323965406503835
11	1,041392685158225040750199971243024241706702190466453094596539
13	1,113943352306837769206541895026246254561189005053673288598083
17	1,230448921378273028540169894328337030007567378425046397380368
19	1,278753600952828961536333475756929317951129337394497598906819

The next Prime Number is 23, which I will take for an Example of the foregoing Doctrine; and by the first Rules, the Logarithm of the *ratio* of 22 to 23 will be found to be either

$$\frac{1}{22} - \frac{1}{968} + \frac{1}{31944} - \frac{1}{937024} + \frac{1}{25768160} \&c. \text{ or}$$

$$\frac{1}{23} + \frac{1}{1058} + \frac{1}{36501} + \frac{1}{1119364} + \frac{1}{32181715} \&c.$$

As likewise that of the *ratio* of 23 to 24 by a like Process.

$$\frac{1}{23} - \frac{1}{1058} + \frac{1}{36501} - \frac{1}{1119364} + \frac{1}{32181715} \&c. \text{ or}$$

$$\frac{1}{24} + \frac{1}{1152} + \frac{1}{41472} + \frac{1}{1327104} + \frac{1}{39813120} \&c.$$

And this is the Result of the Doctrine of *Mercator*, as improved by the Learned Dr. *Wallis*. But by the second Theorem, *viz.* $\frac{2^x}{2} + \frac{2^x}{3 \cdot 2^x} + \frac{2^x}{5 \cdot 2^x}$ &c. the same Logarithms are obtained by fewer Steps. To wit,

$$\frac{2}{45} + \frac{2}{273375} + \frac{2}{922640625} + \frac{2}{2615686171875} \text{ \&c. and}$$

$$\frac{2}{47} + \frac{2}{311469} + \frac{2}{1146725035} + \frac{2}{3546361843241} \text{ \&c.}$$

which was invented and demonstrated in the Hyperbolick Spaces Analogous to the Logarithms, by the Excellent Mr. *James Gregory*, in his *Exercitationes Geometricæ*, and since further prosecuted by the aforesaid Mr. *Speidall*, in a late Treatise in *English* by him published on this Subject. But the Demonstration as I conceive was never till now perfected without the consideration of the Hyperbola, which in a matter purely Arithmetical as this is, cannot so properly be applied. But what follows I think I may more justly claim as my own, *viz.* That the Logarithm of the *ratio* of the Geometrical Mean to the Arithmetical between 22 and 24, or of $\sqrt{528}$ to 23 will be found to be either

$$\frac{1}{1058} + \frac{1}{1119364} + \frac{1}{888215334} + \frac{1}{626487882248} \text{ \&c. or}$$

$$\frac{1}{1057} + \frac{1}{3542796579} + \frac{1}{659676558485285} \text{ \&c.}$$

All these *Series* being to be multiplied into 0,4342944819 &c. if you design to make the Logarithm of *Briggs*. But with great Advantage in respect of the Work, the said 4342944819 &c. is divided by 1057, and the Quotient thereof again divided by three times the Square of 1057, and that Quotient again by $\frac{1}{3}$ of that Square, and that Quotient by $\frac{1}{3}$ thereof, and so forth, till you have as many Figures of your Logarithm as you desire. As for Example, The Logarithm of the Geometrical Mean between 22 and 24 is found by the Logarithms of 2, 3 and 11 to be

	1.36131696126690612945009172669805
1057)43429 &c.(41087462810146814347315886368
3 in 1117249)41087 &c.(12258521544181829460074
$\frac{2}{3}$ in 1117249)12258 &c.(6583235184376175
$\frac{7}{5}$ in 1117249)65832 &c.(4208829765
$\frac{2}{7}$ in 1117249)42088 &c.(2930
Summa	1.36172783601759287886777711225117

Which is the Logarithm of 23 to thirty two places, and obtained by five Divisions with very small *Divisors*, all which is much less work than simply multiplying the *Series* into the said Multiplicator 43429 &c.

Before I pass on to the converse of this Problem, or to shew how to find the Number appertaining to a Logarithm assigned, it will be requisite to advertise the Reader, that there is a small mistake in the afore said Mr. *James Gregory's Vera Quadratura Circuli & Hyperbolæ*, published at Padua Anno 1667, wherein he applies his Quadrature of the Hyperbola to the making the Logarithms; In pag. 48. he gives the Computation of the Lord *Napeir's* Logarithm of 10, to five and twenty places, and finds it 2302585092994045624017870 instead of 2302585092994045684017991, erring in the eighteenth Figure, as I was assured upon my own Examination of the Number I here give you, and by comparison thereof with the same wrought by another hand, agreeing therewith to 57. of the 60 places. Being desirous to be satisfied how this difference arose, I took the no small trouble of examining Mr. *Gregory's* Work, and at length found that in the inscribed *Polygon* of 512 Sides, in the eighteenth Figure was a 0 instead of 9, which being rectified, and the subsequent Work corrected therefrom, the result did agree to a Unite with our Number. And this I propose not to Cavil at an easie mistake in managing of so vast Numbers, especially by a Hand that has so well deserved of the Mathematical Sciences, but to shew the exact coincidence of two so very differing Methods to make Logarithms, which might otherwise have been questioned.

From the Logarithm given to find what *ratio* it expresses, is a Problem that has not been so much considered as the former, but which is solved with the like ease, and demonstrated by a like Process, from the same general Theorem of Mr. *Newton*: For as the Logarithm of the *ratio* of 1 to $1+q$ was proved to be $\frac{1}{1+q} \frac{1}{m} - 1$, and that of the *ratio* of 1 to $1-q$ to be $1 - \frac{1}{1-q} \frac{1}{m}$: so the Logarithm, which we will from henceforth call L, being given, $1+L$ will be equal to $\frac{1}{1+q} \frac{1}{m}$ in the one case; and $1-L$ will be equal to $\frac{1}{1-q}$

$\frac{1}{1-q} | \frac{1}{m}$ in the other: Consequently $\overline{1-L}^m$ will be equal

to $1+q$, and $\overline{1-L}^m$ to $1-q$; that is, according to Mr. Newton's said Rule, $1 + mL + \frac{1}{2} m^2 L^2 + \frac{1}{6} m^3 L^3 + \frac{1}{24} m^4 L^4 + \frac{1}{120} m^5 L^5$ &c. will be $= 1+q$, and $1 - mL + \frac{1}{2} m^2 L^2 - \frac{1}{6} m^3 L^3 + \frac{1}{24} m^4 L^4 - \frac{1}{120} m^5 L^5$ &c. will be equal to $1-q$, m being any infinite Index whatsoever, which is a full and general Proposition from the Logarithm given to find the Number, be the *Species* of Logarithm what it will. But if *Napier's* Logarithm be given, the Multiplication by m is saved, (which Multiplication is indeed no other than the reducing the other *Species* to his) and the *Series* will be more simple, *viz.* $1 + L + \frac{1}{2} LL + \frac{1}{6} L^3 + \frac{1}{24} L^4 + \frac{1}{120} L^5$ &c. or $1 - L + \frac{1}{2} LL - \frac{1}{6} L^3 + \frac{1}{24} L^4 - \frac{1}{120} L^5$ &c. This *Series*, especially in great Numbers, converges so slowly, that it were to be wished it could be contracted.

If one term of the *ratio*, whereof L is the Logarithm, be given, the other term will be had easily by the same Rule: For if L were *Napier's* Logarithm of the *ratio* of a the lesser to b the greater term, b would be the Product of a into $1 + L + \frac{1}{2} LL + \frac{1}{6} LLL$ &c. $= a + aL + \frac{1}{2} aLL + \frac{1}{6} aL^3$ &c. But if b were given, a would be $= b - bL + \frac{1}{2} bLL - \frac{1}{6} bL^3$ &c. Whence by the help of the *Cbiliads*, the Number appertaining to any Logarithm will be exactly had to the utmost extent of the Tables. If you seek the nearest next Logarithm, whether greater or lesser, and call its Number a if lesser, or b if greater than the given L , and the difference thereof from the said nearest Logarithm you call l ; it will follow that the Number answering to the Logarithm L will be either a into $1 + l + \frac{1}{2} ll + \frac{1}{6} lll + \frac{1}{24} ll^4 + \frac{1}{120} ll^5$ &c. or else b into $1 - l + \frac{1}{2} ll - \frac{1}{6} lll + \frac{1}{24} ll^4 - \frac{1}{120} ll^5$ &c. wherein as l is less, the *Series* will converge the swifter. And if the first 20000 Logarithms be given to fourteen places, there is rarely occasion for the three first steps of this *Series* to find the Number to as many places. But for *Ulaq's* great Canon of 100000 Logarithms, which is made but to ten places, there is scarce ever need for more than the first step $a + al$ or $a + mal$ in one case, or else $b - bl$ or $b - mbl$ in the other, to have the Number true to as many Figures as those Logarithms consist of.

If future Industry shall ever produce Logarithmick Tables to many more places, than now we have them; the aforefaid Theorems will be of more ufe to deduce the correspondent Natural Numbers to all the places thereof. In order to make the first *Cibiliad* ferve all Ufes, I was defirous to contract this *Series*, wherein all the Powers of l are prefent, into one, wherein each alternate Power might be wanting; but found it neither fo fimple or uniform as the other. Yet the firft ftep thereof is I conceive moft commodious for Practice, and withal exact enough for Numbers not exceeding fourteen places, fuch as are Mr. *Briggs's* large Table of Logarithms; and therefore I recommend it to common Ufe. It is thus: $a + \frac{al}{1 - \frac{1}{2}l}$

or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the Number anfwering to the Logarithm given, differing from the truth by but one half of the third ftep of the former *Series*. But that which renders it yet more eligible is, that with equal facility it ferves for *Briggs's* or any other fort of Logarithms, with the only variation of writing $\frac{1}{m}$ inftead of 1, that is, $a + \frac{al}{1 - \frac{1}{m}l}$ and $b - \frac{bl}{1 + \frac{1}{m}l}$, or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$ and $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$, which are eafily refolved into

Analogies, *viz.*

As 43429 &c. $- \frac{1}{2}l$ to 43429 $+ \frac{1}{2}l$:: So is a } to the Num-
 or As 43429 &c. $+ \frac{1}{2}l$ to 43429 $- \frac{1}{2}l$:: So is b } ber fought.

If more fteps of this *Series* be defired, it will be found as follows,

$a + \frac{al}{1 - \frac{1}{2}l} - \frac{\frac{1}{2}al^2}{1 - l} + \frac{\frac{1}{3}al^3}{1 - 2l}$ &c. as may eafily be demonftrated by working out the Divifions in each ftep, and collecing the Quotes, whole Sum will be found to agree with our former *Series*.

Thus I hope I have cleared up the Doctrin of Logarithms, and fhewn their Conffruktion and Ufe independent from the *Hyperbola*, whole Affections have hitherto been made ufe of for this purpofe, though this be a matter purely Arithmetical, nor properly demonffrable from the Principles of Geometry. Nor have I been obliged to have recourfe to the Method of Indivifibles, or the Arithmetick of Infinities, the whole being no other than an eafie Corollary to Mr. *Newton's* General Theorem for forming Roots and Powers. V. A