AdS/CFT Correspondence and Quark-AntiQuark Potential

By

Tambe Ebai Norbert

Supervisor: Prof. Kumar S. Narain

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Contents

Chapter 1

Introduction

Understanding Large N behavior of $SU(N)$ gauge theories has always been a long standing problem[1]. It is Related perhaps to understanding strongly coupled systems like QCD. Indeed in [2] Polyakov showed that large N behavior of a strongly coupled gauge theory, if accessible, should be described by string Theory. There are several contending ways of realizing field theories from strings. In This essay, we will focus on field theories that are obtained by decoupling the theory on a brane from gravity. In fact not un till recently has exploring the near Horizon structure of certain Black hole metrics reveal such a possibility. In [3] Maldacena suggested that large N behavior of rather conformal $SU(N)$ gauge theories in d dimensions can be equivalently described mathematically by super gravity (string Theory) on a $d+1$-dimensional AdS times Compact $d+1$ dimensional manifold which in the maximally super symmetric case is: $AdS_{d+1} \times S^{d+1}$. The Conformal Field Theory is said to be living on the boundary of this AdS space which is a d dimensional Conformal space. A very good practical example to which this argument applies is between the Type2B superstring theory compactified on an $AdS_5 \times S^5$ background whose boundary is a conformal 4-dimensional "Minkowski" space carrying a conformal $\aleph = 4$ Super Yang-Mills gauge theory with $SU(N)$ gauge group and coupling constant g_{YM}. Our Main aim in this

Thesis is to understand the derivation of this conjecture, Test it for the very simplest case and apply it to calculate the Quark -anti-Quark Potential.By This Conjecture, the string coupling constant g_s is proportional to g_{YM}^2, and radius of curvature of AdS_5 and S_5 is $(g_{YM}^2 N)^{1/4}$, where N is the N units of five form flux on S^5 which is also the number of D-branes and as well the rank of the U(N) gauge group. Actually this conjecture has been tested and is valid [4, 5, 6] only in certain limits. For example, in the t'hooft limit: $\lambda = g_{YM} N$ fixed but large, the string theory is weakly coupled and the super gravity is a very reliable solution. The hope is that for large N and so λ but fixed, $\aleph = 4$ $SU(N)$ Super Yang Mills theory in 4-dimensions should be governed by a tree approximation to super gravity. Our Motivation for considering Extremal black holes or black P-branes with near Horizon geometry is because in the low energy limit, they actually behave like or are [7, 8, 10] D-branes which are solitonic solutions of close string theories on which Open strings end. In Their near Horizon geometry they give the $AdS \times S$ geometry with a boundary space which is conformal. For our Particular case, in the large N limit, N parallel D3-brane solution is exactly an extremal 3-brane solution to Classical Supergravity whose near horizon limit is an $AdS_5 \times S^5$ having a 4-dimensional Minkowski space on its boundary. It turns out that the isometry of the gauge theory which lives on the boundary space and the super gravity in the bulk agree and is the super conformal group, $PSU(2, 2/4)$. This group has twice the amount of super symmetry of the corresponding super-Poincare group. The $AdS_5 \times S^5$ has a isometry group of $SO(2, 4) \times SO(6) \subset SU(2, 2/4)$ which actually matches with the symmetry of the conformal Minkowski space at the boundary of the AdS_5. Apart from matching the Isometry group and its representation in both sides, We also check this equivalence,by calculating some two point correlation functions in the conformal field and show that they are equally given in the super gravity side. Considering that the supergravity action and its contents fields have an asymptotic behavior at infinity; at the boundary of AdS, a special proposal by [4, 11]is that,the dimension of gauge invariant(traceless symmetric) Operators in the conformal field theory is determined by the masses of the dual particles(fields) in the super gravity (string Theory). Indeed these guys conjectured that there exists a one-to-one map between gauge invariant operators in the CFT and the fields in the AdS side.The Holographic principle in use highly whereby, the partition functions of the fields in the 5dimensional AdS i.e the bulk is equivalent to the 4-dimensional generating functional of the vacuum expectation values of operators of the CFT in the boundary. Upon this the boundary values of the fields acts as sources.[4, 11].

It is quite non trivial to show this correspondence completely because of our present insufficient techniques to handle strongly curved space time string Theory and strongly coupled gauge theory.

In This thesis, we have for the Kaluza-Klein states on $AdS_5 \times S^5$, showed the exact behavior at least for the conformal dimension of the two point function in both sides. We have not considered Non-Chiral or Non-BPS operators in this work. Therefore, we have only matched Kaluza-Klein states in the AdS in the low energy limit to their dual Chiral or BPS Operators in the Super Yang Mills side at least in comformal dimension. We have also use this same approach to determine the dual to the Wilson loop operator in the Supergravity side which is this case is just the Minimal world sheet surface area in the Superstring side. Calculating this Minimal Surface area actually gives us a coulomb-like potential between a quark-antiquark pair. This case extends the conjecture to rather superstring limit instead of the supergravity as observe when this same calculations are done in the Super Yang Mills Theory.

Chapter 2

Superstring Quantization and D-branes

2.1 SuperString Quantisation and D-branes

2.1.1 Superstring in flat back ground

Starting with a superstring world sheet action:

$$\mathcal{S} = -\frac{T}{2}\int d^2\sigma(\partial_\alpha X^\mu \partial^\alpha X_\mu - i\bar{\Psi^\mu}\gamma^\alpha\partial_\alpha\Psi_\mu) \tag{2.1}$$

where γ^α are the dirac matrices on the world sheet such that

$$\{\gamma\alpha, \gamma\beta\} = 2\eta^{\alpha\beta} \tag{2.2}$$

$$T = \frac{1}{2\pi\alpha'} and \tag{2.3}$$

$$\ell_s^2 = 2\pi\alpha' \tag{2.4}$$

with $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$

$X^\mu(\tau,\sigma)$ is a Bosonic vector field and

$\Psi^\mu(\tau,\sigma)$ two component Weyl-Majorana world sheet. spinor[7, 12] Action has been written under consideration of Supersymmetry,Poincare global invariance, local deffeomophism and Weyl invariance choosing a conformal gauge. The most convenient coordinate for quantizing the superstring is the light cone coordinates [13, 14, 7]. Where the fermionic part becomes

$$\mathcal{S} = iT \int d^2\sigma(\Psi_-\partial_+\Psi_- + \Psi_+\partial_-\Psi_+) \tag{2.5}$$

Our interest is in Closed strings with two boundary conditions; Periodic and anti Periodic boundary conditions.We have two possibilities;

Periodic boundary condition

$$\Psi^\mu_A(\sigma) = \Psi^\mu_A(\sigma + 2\pi)$$

giving Ramond (R) fields and

Anti-Periodic Boundary Condition

$$\Psi^\mu_A(\sigma) = -\Psi^\mu_A(\sigma + 2\pi)$$

A, labeling the two components, giving Neveu-Schwarz(NS) fields.

Closed strings have independent left and right movers, so we apply this boundary Conditions to them separately. Finally we get 4 Possibilities:

The R Sector

With mode expansion

$$\Psi^{\mu}_{+}(\tau,\sigma) = (\sum_{n\in\mathbb{Z}} \tilde{d}^{\mu}_{n} e^{-2in(\tau+\sigma)})_{Left} \tag{2.6}$$

$$\Psi^{\mu}_{-}(\tau,\sigma) = (\sum_{n\in\mathbb{Z}} d^{\mu}_{n} e^{-2in(\tau-\sigma)})_{right} \tag{2.7}$$

Quantizing along with the Bosonic sector gives us creation and annihilation operators and a spectrum of both left and right moving strings. We Observe that the ground state operators obey the Dirac-Clifford algebra

$$\{d^{\mu}_{0}, d^{\nu}_{0}\} = \eta^{\mu\nu} \tag{2.8}$$

. So zero modes must act on the ground state as dirac matrices.

This tells us that the R-Sector gives spacetime spinors.

State

$|0\rangle^{a}_{R}$

where $a = 1, 2, ..., 32$ is spinor with 32 Components.

In D-dimension, we have $2^{\frac{D}{2}}$ components of a dirac spinor, with $D = 10$ we have 32 components.

We conclude that the ground state $|0\rangle^{a}_{R}$ is a 32 component Majorana spinor and from supersymmetry, SuperString has 32 supercharges.

If we introduce the Chirality Operator

$\Gamma_{11} = \prod_{i=0}^{9} \Gamma_i$

projecting in chiral directions gives 16 component Majorana-Weyl spinors with definite chirality.

Our State

$|0\rangle^{a}_{R} = |0\rangle^{+}_{R} \oplus |0\rangle^{-}_{R}$ means we have

$32 = 16 \oplus 16$ components.Therefore states $|0\rangle^{\pm}_{R}$ are spacetime *fermions*.

The NS Sector(Anti Periodic)

The mode expansion is

$$\Psi^{\mu}_{+}(\tau,\sigma) = (\sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^{\mu}_{r} e^{-2in(\tau+\sigma)})_{left} \tag{2.9}$$

$$\Psi^{\mu}_{-}(\tau,\sigma) = (\sum_{r\in\mathbb{Z}+\frac{1}{2}} b^{\mu}_{r} e^{-2in(\tau+\sigma)})_{Right} \tag{2.10}$$

Quantization with Bosonic sector, the NS sector gives a spectrum of spacetime *Bosons*. Thus our most general space time state is a tensor product of the left and right moving sector of either NS or R sector. This gives us 4 major possibilities

1. If $|\Psi\rangle = |NS\rangle_{left} \otimes |NS\rangle_{right}$ State is a spacetime *Boson*
2. If $|\Psi\rangle = |R\rangle_{left} \otimes |R\rangle_{right}$ Again a spacetime boson(Bispinor) since it is constructed from two spinor.
3. If $|\Psi\rangle = |R\rangle_{left} \otimes |NS\rangle_{right}$ or
4. $|\Psi\rangle = |NS\rangle_{left} \otimes |R\rangle_{right}$ These states are spacetime *fermions*

By defining a 32 component $SO(10)$ spin field operator which act like a super charge taking bosonic states into fermionic states.

$$|o\rangle^{a}_{R} = S^{a}|o\rangle_{NS}$$

We see clearly that the Original Superstring theory has 32 Supercharges. We can guest that superstring in a $D = (9+1)$-spacetime (eg Flat Minkowski space time) will have 32 superchrages and so $\aleph = 1$ supersymmetry.

Aside If Left and right movers have the same GSO projection,then they belong to same Equivalence Class and by this Tachyonic particles are removed from the spectrum so that states are $|0\rangle_R^{left} \otimes b^\mu_{-\frac{1}{2}}|0\rangle_{NS}^{right}$ and $b^\mu_{-\frac{1}{2}}|0\rangle_{NS}^{left} \otimes |0\rangle_R^{right}$ and have the same chirality. The GSO projection takes a 32 component spinor into a 16 component Majorana-Weyl Spinor removing states with the same chirality in the left and right sectors.

2.1.2 SuperString Spectrum

For simplicity we will consider only the massless spectrum of the different spectra in the superstring.[12, 13]

First The $|NS\rangle \oplus |NS\rangle$ sector has

- The Scalar field ϕ or dilaton with single state
- The Asymmetric gauge field $B_{\mu\nu}$ with 28 states.
- The Symmetric traceless 2nd rank tensor; graviton $G_{\mu\nu}$ with 35 states.

Secondly

The $|NS\rangle \oplus |R\rangle$,$|R\rangle \oplus |NS\rangle$ sector. These states are super partners of the dilaton and graviton

- The Dilatino,λ^i spin $\frac{1}{2}$ super partner of the dilaton with 28 states
- The Gravitino,Ψ^i_M of positive chirality spin $\frac{3}{2}$ fermion and super partner of graviton.

where $i = 1, 2$. They have the same chirality for Type2B.

Finally

For the $|R\rangle \oplus |R\rangle$ Sector

. These are Bosonic states from the tensor product of 2 Majorana-Weyl spinors.The spectra include

- A scalar field C_0
- A two form gauge field $C_{\mu\nu}$
- A four form gauge field $C_{\mu\nu\rho\sigma}$ whose $F_5 = dC_{\mu\nu\rho\sigma}$ is self dual $F_5 = *\tilde{F}_5$.

Type2B strings have only odd form potentials(R-R- forms) while Type2A have even form.

2.2 D-Branes

Considering the bosonic part of a Polyakov action;

$$S_p = -\frac{T}{2}\int d^2\sigma \partial_\alpha X_\mu \partial^\alpha X^\mu \tag{2.11}$$

Varying the action, we obtain equations of motion for X^μ

$$\delta S_p = -T\{\int \partial^\alpha \partial_\alpha X_\mu \delta X^\mu - n^\alpha \partial_\alpha X_\mu \delta X^\mu|_{boundary}\} \tag{2.12}$$

By Requiring that the boundary terms vanish for Open strings we find that

$$\delta X^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0, \textbf{Dirichlit boundary conditions} \tag{2.13}$$

. The open strings also satisfy

$$\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=o,\pi} = 0, \ \textbf{Neuman Boundary Conditions} \tag{2.14}$$

Since any number of components of X^μ can satisfy these conditions, we guess that open strings can end on higher dimensional extended objects(branes) through Dirichlet boundary conditions. [7, 10, 12]. These objects are now called D-branes.

Branes are classified by the number of p-spatial dimensions. The number, type and arrangement of Dp-branes restrict the open strings states that can exists(since we quantize the Open strings based on the Dirichlit or Neuman conditions). A convenient mathematical description of a Dp-brane is to use light cone coordinates in which for simplicity, we consider the Bosonic sector, and choose coordinates
$X^+(\sigma,\tau)$, $X^-(\sigma,\tau)$ and $X^i(\sigma,\tau)$ with $i = 2,...,p$ to satisfy the Neuman Boundary Conditions: $\partial_\sigma X^\mu(\tau,\sigma)|_{\sigma=o,\pi} = 0$ with $\mu = +,-, i = 2,3,...,p$
while the coordinates X^a $a = p+1, p+2, ..., p$ satisfy the Dirichlit Boundary Conditions: $X^a(\tau,\sigma = 0) = X^a(\tau,\sigma = \pi) = \bar{X}^a$ which also specify the location of the D-brane.
The index coordinates μ define the world Volume and are called the NN coordinates while the index coordinates a describe the bulk and are called the DD coordinates.
Quantizing the open strings considering the Neuman boundary conditions for X^i $i = 2,3...,P$ and Dirichlit boundary conditions for the X^a, $a = p+1,...,d$, we get the full spectrum for states for the D-branes.
The states are labeled by momentum $|P^+P^i\rangle$ while all the P^a vanish because of Dirichlet boundary conditions. We have Lorentz invariance only in the brane world Volume since these states transform under Lorentz transformation.
We can conclude that with states $|P^+P^i\rangle$, $i = 2,...p$ only depend on transverse momentum; P^i. A field d must be function of momentum P^i so that string states only have momentum along the D-brane world volume. If we write the fourier transform, we see that,these are functions of the coordinates X^i with $i = 2,3,...p$ and so depends only on the world Volume coordinates X^i with no dependence on the bulk coordinates X^a for $a = p+1,...,d$. Therefore these fields are lorentz vectors living on the DP-brane world Volume.

Massless Dp-brane spectrum

The NS-sector massless states are those defined by

$$(b^i_{-1/2})^+|P^+P^i\rangle$$

and since $i = 2, 3, ..., p$, there is a total of $(p+1-2)$ massless states which transform as vectors under lorentz transformation on the brane while in the transverse coordinates,

$$(b^a_{-1/2})^+|P^+P^i\rangle,$$

these transform as scalars and their expectation values gives the position of the D-branes. Since the states

$$(b^i_{-1/2})^+|P^+P^i\rangle$$

are massless vectors, We claim that they must give rise to gauge fields living on the D-brane. On the Other hand,gravity is not living on the brane but rather in the Bulk. In Type2B strings, solitonic D-branes have odd P-spartial coordinates. For example, in a D3-brane, the gauge fields will suggest the presence of a $U(1)$ gauge group. If we have N D3-branes which are coincident , then we have N^2 massless gauge fields which will characterize a $U(N)$ Yang-Mills theory in the world-volume of the N coincident D-branes. The states are in the fundamental representation of the $U(N)$ group.
However we also do have $d - P$ massless Scalars as shown in [12, 13, 14, 15, 16]
The Quantization of the open strings on a Dp-brane gives a massless spectrum that is of Maximal Yang-Mills supermultiplet in the (P+1)-dimensions. Its Massless spectrum consist of :
A Lorentz Vector, $d - P$ Scalars and the associated fermions.
While the R-sector, with zero modes operators obeying the Dirac- Clifford algebra act on the ground state giving massless spinors(fermions) in 10-dimensions.

2.3 Effective string and D-Brane Action

2.3.1 Type2B Supergravity

From studying the World sheet action and requiring world sheet conformal symmetry, we can write an effective action for the Type2B Superstring as,

$$S_\sigma = \frac{1}{4\pi\alpha\prime}\int_M d^2\sigma\sqrt{g}[(g^{\alpha\beta}G_{\mu\nu}(X) + i\epsilon^{\alpha\beta}B_{\mu\nu}(X))\partial_\alpha X^\mu\partial_\beta X^\nu + \alpha' R\Phi(X)] \quad (2.15)$$

where the fields $B_{\mu\nu}(X)$ is the antisymmetric tensor, and $\Phi(X)$ trace of symmetric tensor $G_{\mu\nu}$ the graviton. This gives an effective Type2B string action[17]:

$$S_{2B} = S_{NS} + S_R + S_{CS} \quad (2.16)$$

$$S_{NS} = \frac{1}{2(\kappa_{10})^2}\int d^{10}x\sqrt{-G}e^{-2\Phi}(R + 4\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}(|H_3|)^2) \quad (2.17)$$

$$S_R = -\frac{1}{4(\kappa_{10})^2}\int d^{10}x\sqrt{-G}((|F_1|)^2 + (|\tilde{F_3}|)^2 + \frac{1}{2}(|\tilde{F_5}|)^2) \quad (2.18)$$

$$S_{CS} = -\frac{1}{4(\kappa_{10})^2}\int C_4 \wedge H_3 \wedge F_3 \quad (2.19)$$

$$where\tilde{F_3} = F_3 - C_0 \wedge H_3 \quad (2.20)$$

$$\tilde{F_5} = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \quad (2.21)$$

C_p ,F_{p+1} are the R-R fields for potential and field strengths respectively. B_2 and H_3 are the NS-NS fields. The Chern-Simon (CS) action contains both.

Remark

In writing the effective action, we made used of the fact that; the Radius of Curvature is far greater than the string length meaning $\sqrt{\alpha'}R^{-1} <<< 1$ neglecting higher order terms in action.

2.3.2 D-Brane Dynamics-DBI Action

From the Dp-brane spectrum the massless fields in the world-volume give a $U(1)$ vector plus $9-p$ scalars. This describes the low energy dynamics of the D-brane. The effective action in the low energy limit(low energy Dynamics) is described only by the massless modes of Open strings completely decoupled from the closed strings in the bulk. Let ξ^a , $a = 0, 1, ..., p$ be coordinates of the world Volume of the brane .Then we write the p-brane Bosonic sector of the action as

$$S_p = -T_p \int d^{p+1}\xi T_r\{e^{-\Phi}\sqrt{[-det(G_{ab} + B_{ab} + 2\pi\alpha\prime F_{ab})]}\} \tag{2.22}$$

. We can in principle write this action by considering the world sheet action of an open string and require conformal invariance. From the equations of motions of the open strings in the D-brane background we arrive at above action.

The Dilaton dependence $e^{-\Phi}$ arises because this is an open string tree-level action and the topology is a disc on the world sheet. The F_{ab} dependence is because of T-duality and B_{ab} dependence because of imposing gauge invariance on the world sheet action.From the world sheet action of a closed string,

$$S = \frac{i}{4\pi\alpha'}\int_M d^2\sigma\sqrt{g}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu} + i\int_{\partial M} dx^\mu A_\mu \tag{2.23}$$

. Through imposing on each field tensor gauge invariance

$$\delta B_{\mu\nu} = \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

and

$$\delta A_\mu = -\frac{\xi_\mu}{2\pi\alpha'}$$

, Only the combination

$$B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu} \equiv 2\pi\alpha' \digamma_{\mu\nu}$$

appearing in the effective action satisfies our invariance conditions as seen in [7, 10, 18]. We simply write

$$S_p = T_p \int d^{p+1}\xi T_r\{e^{-\Phi}\sqrt{-detG}\sqrt{[det(\delta_{ab} + 2\pi\alpha' \digamma_{ab})]}\} \tag{2.24}$$

giving the effective action of the brane dynamics, at least for the Bosonic sector. In the Limit where α' tends to zero $\alpha' \to 0$ we realize

$$S_p = T_p \int d^{p+1}\xi T_r\{e^{-\Phi}\sqrt{-detG}(det(1 - \frac{(2\pi\alpha')^2}{2} T_r \digamma^2) + ...\} \tag{2.25}$$

where we have used the fact that $\digamma^a_a = 0$ and

$$det(I - \epsilon A) \simeq 1 + \epsilon T_r A + \frac{\epsilon^2}{2}(T_r(A)^2 - T_r A^2)$$

with

$$\frac{T_p(2\pi\alpha')^2 e^{-\phi}}{2} = \frac{1}{4g^2_{YM,p}}$$

Suppose we are in the flat Minkowski space with $\xi^a = X^a$, $a = 0, 1, 2, ...p$(world Volume) and X^I, $I = p+1, ..., d-p+1$ transverse coordinates. X^I are scalars and the induced metric on the brane is

$$G_{ab} = \eta_{ab} + \partial_a X^I \partial_b X^I$$

Then action becomes

$$S_{eff} = T_p \int d^{p+1}x e^{-\Phi} Tr[1 + F_{ab}F^{ab} + \partial_a X^I \partial_b X^I] + ... \tag{2.26}$$

$$S_{eff} = T_p \int d^{p+1}x e^{-\Phi} T_r[1 + \partial_a X^I \partial_b X^I - \frac{(2\pi\alpha')^2}{2} T_r F^2] + ... \tag{2.27}$$

. From the Euler characteristic of the annulus diagram describing the interaction between two parallel branes, we can find that the tension of a D-brane is proportional to its charge density and goes like

$$T_p = \rho(p) = \frac{\sqrt{\pi}}{g_s \ell_s^{p+1}} (4\pi^2\alpha')^{3-p}$$

. Indeed through this same calculations but considering that the D-branes interact by exchanging closed strings, we find that the force between two D-branes actually vanishes.and in particular for D3-brane .The reason is because, there is the cancelation between attraction due to the graviton and dilaton and the repulsion due to the R-R p-forms. For a D3-brane,

$$T_3 = \rho(3) = \frac{\sqrt{\pi}}{g_s \ell_s^4}$$

Secondly, D-brane are BPS states since from the Neumann and Dirichlet boundary conditions, they break $\frac{1}{2}$ in this case 16 of the 32 super symmetric charges.
.Thirdly they carry Ramond-Ramond(RR) charges with non vanishing charge density. SO they coupled to $p + 1$-form potential as follows

$$\int_{V_{p+1}} C_{p+1}$$

with the integral running over the D-brane world volume. In fact,[7, 8] D-branes are known to be extremal P-branes being reliable solutions to Supergravity as we will see in the chapter 3.1.

Remark

WE have learn that, the low energy Physics of D-branes living in a flat space can be described by the dimensional reductions of $\aleph = 1$ SU(N) Super Yang Mills in 10-dimensions to $p+1$ dimensions. The space time fermionic part of this action have a dirac form of

$$-i\int d^{p+1}\xi T_r(\bar{\lambda}\Gamma^a D_a\lambda) \tag{2.28}$$

For N Dp-branes of small separations, the Coordinates $X^{\mu}(\xi)$, the gauge fields $A_a(\xi)$ and the fermionic partners $\lambda(\xi)$ all become $n\times n$ matrices or adjoint representation of the $SU(N)$ giving rise to the $\aleph = 4$ Super Yang-Mills.

In the N-coincident D3-brane configuration, each brane is labeled by Chan-Paton (charges) (i,j)indices of the open strings on the brane. We have for $N = 2$ D3-branes, we have 4 possible open strings attached to the branes with Chan-Paton indices: $1-2$,$2-2$,$1-2$, and $2-1$. Each Open string will give rise to a massless Yang-Mills Multiplet in the Adjoint representation of the SU(N) gauge group . These gauge fields can interact with each other, giving a non-abelian gauge group.For 2 coincident D3-brane; a $U(2)$ gauge symmetry appears. The scalars X^I are also in the adjoint representation of this gauge group and give an interesting idea of space time, since X^I is now non-commutative $[X^I, X^J] \neq o$ for $I \neq J$.

If we separate the two branes by a length $\bar{x_2^a} - \bar{x_1^a}$ in the transverse space,(see Diagram). The two strings $1-2$ and $2-1$ are now stretched by tension σ in length $|\bar{x_2^a} - \bar{x_1^a}|$ and so acquire energy $\sigma|\bar{x_2^a} - \bar{x_1^a}|$ therefore have mass. This is just the ordinary Higgs-effect. Further more, D-branes carry gauge fields because the massless spectrum of the open strings living in a D-pbrane is that of a maximally supersymmetric $U(1)$ gauge theory in the $p+1$ dimensions. The $9-p$ massless scaler fields present in this supermultiplet are expected Golstone Modes associated with the transverse

Oscillations of the Dp-brane. The photons and fermions provide the unique supersymetric completion. For N Parallel D-branes then we have N^2 different species of the open strings because the Chan-Paton indices $i, j = 1, 2, ..., N$, charged end on the different branes. But N^2 is the dimension of the adjoint representation of U(N) so indeed we find the maximally supersymmetric U(N) gauge theory living in the $p + 1$ world Volume of the N parallel Dp-branes.The expectation values of the Scalar fields determine the relative separations of the Dp-branes in the $9 - p$ transverse dimensions. We are interested in the case where all the scalar expectation values vanish so that the N Dp-branes are stacked on top of each other(Coincident branes)[3].For Large N, this stack is a heavy object embedded into a theory of closed strings which contains gravity. Naturally, this macroscopic object will curve peacetime so could be described by some classical metric and other background fields. In particular if $p = 3$ we have N parallel D3-branes, the gauge group is $U(N)$ with N^2 sectors since $i, j = 1, 2, ...N$.Separating any one of the branes from the array of branes will lead spontaneous breaking of the symmetry group where

$$U(N) \longrightarrow U(1) \times SU(N)$$

because one of the stretch strings will now acquire mass and becomes a "W-Boson". The overall $U(1)$ corresponds to the center of mass position of the stack D3-branes while $SU(N)$ describes the internal dynamics.

We conclude that the spacetime world-volume of the N-coincident D3-branes is described by an $\aleph = 4$ $SU(N)$ Super Yang-Mills gauge theory living on the branes. As we earlier comment, through dimensional reduction we can get [9, 32],$p+1$-dimensions from $d = 10$ dimensions $\aleph = 4$ from $\aleph = 1$ $U(N)$ Super symmetric Yang Mills.

Chapter 3

Supergravity Solutions and AdS geometry

3.1 Supergravity and P-branes

Dp-branes are solitonic solutions to closed string theory. However, in the low energy limit, $\alpha' \longrightarrow 0$, the $Type2B$ Superstring theory becomes supergravity whose solutions are well known [19] to be extremal p-branes.

We saw that,the effective action involve the dilaton ϕ, the symmetric metric $G_{\mu\nu}$ and the R-R$p+1$-form fields;

$$\mathcal{S}_{2B} = \int d^{10}x\sqrt{-g_s}\{e^{-2\phi}[R(g_s)+4(d\phi)^2 - \frac{1}{12}(dB)^2] - \frac{1}{2}\sum_p^{\infty}\frac{1}{(p+2)!}(F^{(p+2)})^2\} \quad (3.1)$$

where P even or odd means type 2 A or B string theory respectively. B is the NS-NS two form, $F^{(p+2)}$ is the R-R field strength,where $F^{(p+2)} = dC^{(p+1)}$

The Newton's constants

$$G_N \simeq g_s^2 \ell_s^8$$

, $g_s = e^{\phi_o} = constant$ where $\ell_s^2 = 2\pi\alpha'$ is string Length related to the string tension through as $(2\pi\alpha')^{-1}$

Simplifying our equation of motions by going to the Einstein frame.

$$G_{MN,s} = e^{\frac{\phi}{2}} G_{MN,e}$$

we get

$$\mathcal{S}_{2B} = \frac{1}{G_N} \int d^{10}x\sqrt{-g_e}[R(g_e) - \frac{1}{2}(d\phi)^2 - \frac{1}{12}e^{-\phi}(dB)^2] - \frac{1}{2}\sum_p^{\infty} e^{\frac{(3-p)}{2}\phi}\frac{1}{(p+2)!}(F^{(p+2)})^2] \tag{3.2}$$

R-R fields "couple" to the dilaton. Our interest is only on the fields $G_{\mu\nu}$, ϕ , C^{p+1}-forms. An acceptable action [20]is

$$\mathcal{S}_{2B} = \frac{1}{G_N} \int d^{10}x\sqrt{-g_e}[R(g_e) - \frac{1}{2}(d\phi)^2 - \frac{1}{2}e^{a_p\phi}\frac{1}{(p+2)!}(dC^{(p+1)})^2] \tag{3.3}$$

where $a_p = \frac{3-p}{2}$

We obtain equations of motion with (L,M,N $= 0, 1, 2, ..., 9$)

$$\begin{aligned}
R_{MN} &= \frac{1}{2}\partial_M\phi\partial_N\phi + S_{MN} \\
S_{MN} &= \frac{1}{2(p+1)!}e^{a_p\phi}(F_{ML_1}, ..., L_{p+1}F_N^{L_1,...,L_{P+1}} - \frac{p+1}{8(p+2)}g_{MN}F^2) \\
0 &= \nabla_M(e^{a_p\phi}F^{ML_1,...,L_{P+1}}) \\
\Box\phi &= \frac{a_p}{2(p+2)}e^{a_p\phi}F^2
\end{aligned}$$

We require the solution to have the following properties: Have Poincare invariance in $p+1$-dimensions and rotational $SO(q-p)$ in the transverse directions. To carry Charge with non vanishing charge density. To be BPS i.e preserve half of the supersymmetry. To be asymptotically Minkowskian with asymptotic value of the dilaton being a constant g_s.

Of course,the 10-dimensional Minkowski metric is also a solution.

P-Branes(Extremal Classical Solutions). In 1991 Horowizt and Strominger [19, 20]put forward the following solution to Type2B supergravity.These p-dim Black holes are electrically charged [21, 22, 23]through coupling with Ramond-Ramond(R-R) $(p+1)$-form, A_{p+1}

$$ds^2 = H^{-\frac{1}{2}}(r)[-f(r)dt^2 + d\vec{x}^2] + H^{\frac{1}{2}}(r)[f^{-1}(r)dr^2 + r^2 d\Omega_5^2]$$

$$H(r) \equiv 1 + \frac{R^4}{r^4}, f(r) \equiv 1 - \frac{r_0^4}{r^4}$$

with $r_0 = 0$ the Horizon

where in the near when $r_0 = 0$ we get an extremal P-brane as

$$ds_e^2 = H_p^{p-7/8} d\vec{x}^2 + H_p^{p+1/8} d\vec{y}^2 \tag{3.4}$$

$$C^{p+1} = -(H_p^{-1} - 1) g_s^{-1} dx^0 \wedge ... \wedge dx^p \tag{3.5}$$

$$H_p = 1 + (\frac{r_p}{r})^{(7-p)} \tag{3.6}$$

$$r_p^{7-p} = d_p (2\pi)^{p-2} g_s N (\alpha')^{\frac{7-p}{2}} \tag{3.7}$$

$$d_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma(\frac{7-p}{2}) \tag{3.8}$$

$$e^{2\phi} = g_s^2 H_p^{3-p/2} \tag{3.9}$$

Still in the string coordinates where

$$e^{\phi/2} = g_s^{1/2} H_p^{3-p/8}$$

we get for $p = 3$

$$ds_s^2 = H_3^{-1/2}(r)d\vec{x}^2 + H_3^{1/2}(r)d\vec{y}^2$$

with y^m, $m = p + 1, ..., 9$ transverse coordinates $r = \sqrt{\vec{y}^2}$ and

$$d\vec{y}^2 = dr^2 + r^2 d\Omega_{8-p}^2$$

x^μ, $\mu = 0, 1, 2, ...p$ and

$$d\vec{x}^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

, in general H_p being a harmonic function.

The R-R p-branes have string tension which goes as [24, 25]

$$T_p \sim \frac{1}{g_s \ell_s^{p+1}} \sim \rho(p),$$

equal to their charge density.

ASIDE

When $g_s \to 0$, their gravitational interaction vanishes since $G_N \sim T_p g_s \to 0$ and we have flat space descriptions of these objects involving open strings, as with D-branes. We are interested in the solution[26] of supergravity with N Coincident D3-branes in

$d = 10$ dimensions:

$$ds^2 = H(r)^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H(r)^{1/2}(dr^2 + r^2 d\Omega_5^2) \tag{3.10}$$

$$H(r) = 1 + \frac{4\pi g_s N \ell_s^4}{r^4} \tag{3.11}$$

$$g_s = e^\phi \tag{3.12}$$

$$B_{\mu\nu} = A_{2\mu\nu} = 0 \tag{3.13}$$

$$F^+_{5\mu\nu\rho\sigma\tau} = \epsilon_{\mu\nu\rho\sigma\tau\lambda}\partial^\lambda H \tag{3.14}$$

$$T_3 = \frac{2\pi}{(2\pi\ell_s)^4 g_s} \tag{3.15}$$

$$16\pi\kappa_\mu = \frac{(2\pi\ell_s)^8}{2\pi} g_s^2 \tag{3.16}$$

This solution satisfy our requirements of a valid solution. There is a $p+1$dimensional Poincare invariance on the Dp-brane World volume, while its transverse coordinates has an $SO(d-p)$ symmetry. It also has a constant Axion χ and Dilaton Φ fields.Has a true horizon at $r = 0$ and a Singularity at $r = R$. In the limit; alpha approaches zero, $\alpha' \to 0$, the metric is flat everywhere except on the 4dimensional hyperplane characterized by $\vec{y} = r = 0$. They are Charged with $p+1$ form field $A_{\mu_1\mu_2\cdots\mu_{p+1}}$. given by the coupling

$$\int d^d j^{\mu_1\cdots\mu_{p+1}} A_{\mu_1\cdots,\mu_{p+1}} \to \int j^{012\ldots p} A_{012,\ldots,p}$$

with the source of the $A_{012,\ldots,p}$ field being of type $j^{012,\ldots,p} = Q\delta^{(d-p-1)}(x)$.

Remark

The stack of N D3-branes, will curve space time and we can infer a classical description with a metric and other background fields. Therefore, we have two separate description of the stack of Dp-branes. One in terms of the U(N) Supersymmetric gauge theory on its world volume and the other in terms of classically charged p-brane background of type2 Closed string theory. The relation between these two description is at the heart of the connection between gauge filed theory and strings which is the subject of my thesis.

In 1995 Polchinski, showed that Dp-branes preserve $\frac{1}{2}$ of the bulk supersymmetry which was in the superstring theory and that they carry an elementary unit of charge with respect to a $p+1$ form gauge potential from the Ramond-Ramond (R-R)sector of Type 2 Superstring. In fact, he showed that the D-branes and the extremal p-branes are one and the same thing.Therefore, the dynamical end points of Open strings correspond to extremal solutions of supergravity. The Prove involves commputing, p-brane charge and tension of the endpoints of open strings and matching them with the supergravity solutions. In particular in [27] for $p = 3$-brane, the solution corresponds to the stack of N D3-brane with:

$$ds^2 = H^{-\frac{1}{2}}(r)d\vec{x_{||}}^2 + H^{\frac{1}{2}}(r)(dr^2 + r^2 d\Omega_5^2) \tag{3.17}$$

$$F_5 = (1 + *)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge (dH^{-1}) \tag{3.18}$$

$$F_5 = F_{\mu_1\mu_2...\mu_5} dx^{\mu_1} \wedge ... \wedge dx^{\mu_5} \tag{3.19}$$

$$H(r) = 1 + \frac{R^4}{r^4} \tag{3.20}$$

$$R^4 = 4\pi g_s N(\alpha')^2 \tag{3.21}$$

$$Q = g_s N \tag{3.22}$$

We know that if supergravity solution(extremal p-brane) is a D-brane then one should be able to derive the Hawking radiation in the D-brane picture by allowing two open strings living in the D-brane to collide to form a closed string which then is not bound to the D-brane anymore and can peel off away as the Hawking radiation. This has been shown [8]and indeed, Dp-brane as extremal p-brane is a $(p + 1)$-dimensional hyperplane in spacetime where Open strings can end. By the T-duality, a D-brane is also a source for open strings. We know that D-brane carry R-R Charges and if in

Particular a stack N Dp-brane, the resulting $(p+1)$-dimensional hyperplane carries exactly N-units of the $(p+1)$-form flux.

On the world sheet of type 2 string, the left mover and right movers carry separate space time supercharges. Consequently, Open string boundary condition identifies the left and right movers, and so the D-brane breaks at least one half of the spacetime supercharges. In Type2B string theory, precisely one half of the supersymmetry is preserved for odd P. This is consistent with the type of R-R charges that appear in the theory. Therefore Dp-branes are BPS objects in string theory which carry exactly the same charge as the black (classical)p-brane solution in Supergravity.

Since supersymmetry invariance requires

$\delta\Psi_{\mu\alpha} \sim (...)(1-\Gamma_0\Gamma_1....\Gamma_p)\epsilon = 0$ and $\Gamma_0\Gamma_1...\Gamma_p\epsilon = \epsilon($ selects half of the spinors $\epsilon)$. For $p=3$, our 4-dimensional World Volume of D3-brane will contain $\aleph = 4$ supersymmetric Yang Mills conformal Field theory with SU(N) gauge group. On the other hand, since extremal P-branes also have BPS bound (Mass less or equal to charge $(M \leqslant Q)$) ,So the solution is also left invariant by one half supersymmetric charges).D-branes are also stable like extremal p-branes. Moreover, by Calculating, the annulus diagram of the tree level interaction of a two D3-branes, which vanish, we can check that there is no force between the two parallel D-branes. The diagram can be seen as the exchange of a closed string from one D-brane to the other.The string graph is an annulus with no vertex operators. The poles from graviton and dilaton then give the coupling T_p of the closed string states to the D-brane. One finds that the tension of the D-brane is proportional to g_s^{-1} and from the R-R part of the String effective action which comes with the factor of $e^{-\phi} = g_s^{-1}$ i.e from amplitude calculation of the exchange of closed string between two Parallel D-brane in the Low energy Limit(massless Modes) we finally arrive at

$$T_p = \frac{\sqrt{\pi}}{16\kappa}(4\pi^2\alpha')^{\frac{(11-p)}{2}}$$

where $\kappa = \kappa_0 e^{\phi}$. Which is the same for the classical p-brane solutions. Finally we conclude that the D3-brane is an extremal 3-brane solution to Supergravity. Since D-branes have no force between them, for our N coincident D3-branes, the effective loop expansion parameter for the open string is $g_s N$ rather than g_s; after all each open string boundary loop ending on the D-branes comes with a chan-Paton factor N as well as the string coupling g_s. Therefore, the D-brane description is good when $g_s N >> 1$. This is exactly the regime where supergravity description is appropriate. However we also have that the low energy effective theory of open strings in Dp-brane is the U(N) gauge theory in $(p+1)$ dimensions with 16 supercharges and $(9-p)$ scalar fields in the adjoin representation of U(N). We should find the relation between the theory of the open strings living on the D3-brane; $\aleph = 4$ Super Yang- Mills, and the gravity theory of the fields living in the space curved by the D3-brane (the Bulk).

3.2 Maldacena Limit or Near Horizon Limit

Our motivation for examining superstring dynamics in the presence of D3-Branes has two important points of view,[25, 28, 29]

3.2.1 View Point One

Now that D3-branes are end points on which open strings are attached, we see that superstrings with such boundaries have three ingredients:

First the open strings living on the D3-branes in the low energy limit will describe the dynamics of the D3-brane,by a certain Yang Mills gauge theory. Secondly, the closed strings living in the bulk is described by a Type2B closed superstring. In the low energy limit, the supergravity remains. Finally, there are interactions between the open and open strings on the D-branes and the closed strings in the bulk. The final action for these strings can be written as:

$$S = S_{bulk} + S_{brane} + S_{interaction}$$

where

S_{bulk} = close strings in bulk

S_{brane} = Open strings on the D-brane and

$S_{Interaction}$ = is the interaction between the open strings on the D-brane and the close string in the bulk

In the low energy limit; $\alpha' \to 0$, massive modes of strings drop out and

$S_{bulk} \to$ Supergravity and

$S_{brane} \to \aleph = 4$ $S(U)N$ Super Yang Mills field theory.

Because $S_{interaction}$ α' $\kappa_{10} \sim g_s(\alpha')^2$

where κ_{10} is the Newton's constant in 10-dimensions, as $\alpha' \to 0$ with g_s fixed, means

$S_{interaction} \to 0$ decouples

as $\kappa_{10} \to 0$. The gravity becomes entirely free and we are left with; free gravity in the bulk of spacetime and a 4-dimensional $\aleph = 4$ gauge field theory on the D3-brane.

3.2.2 View point Two

Suppose our D3-branes are now extremal 3-branes of supergravity, then the energy measured at a point P at r is E_p and the energy at infinity E are related through:

$$E_p \sim \frac{d}{d\tau} = \frac{1}{\sqrt{-g_{oo}}}\frac{d}{dt} \sim \frac{1}{\sqrt{-g_{oo}}}E$$

This means

$$E = H(r)^{-1/4}E_p \sim rE_P$$

Therefore for E_p fixed, as the radial distance from D-brane reduces i.e $r \to 0$, the energy observed at infinity vanishes, $E \to 0$, which gives the low energy limit or regime.

We conclude that, from this point of view, we also have two decoupled low energy systems of excitation. The First is that, at Large distances or low energies (Energy $\alpha \ \frac{1}{length}$), the gravity in bulk is free(away from the D3-brane). Secondly, for small distances, we have low energy excitations and the D3-brane becomes and $\aleph = 4$ SYM theory.So in low energy limit, we have two decoupled systems; gravity which is in the bulk and a SYM field theory on the D3-branes. Naively, we say that,near the D-brane, $r \to 0$, $\aleph = 4$ SYM with gauge group SU(N) for large N is equivalent to a gravity theory in the bulk when alpha goes to zero; $\alpha' \to o$.

3.2.3 The limit

Let us consider the following limits:

First,

when $r << R$, i.e $r \to 0$, the harmonic function behaves as: $H(r) \backsimeq \frac{R^4}{r_4}$ such that the metric becomes:

$$ds^2 \backsimeq \frac{r^2}{R^2}(-dt^2 + d\vec{x}^2) + \frac{R^2}{r^2}dr^2 + R^2 d\Omega_5^2 \tag{3.23}$$

. If we change coordinates through

$$\frac{r}{R} \equiv \frac{R}{x_o}$$

We get:

$$ds^2 = R^2 \frac{-dt^2 + d\vec{x}^2 + dx_o^2}{x_o^2} + R^2 d\Omega_5^2 \tag{3.24}$$

This metric is $AdS_5 \times S^5$ in Poincare coordinates with the 5-dimensional Anti-de Sitter space and 5-sphere having the same radius R.

In the limit where x_0 tends to zero, $x_o \longrightarrow o$, we have a 4-dimensional space which is isomorphic to Minkowski space living on the boundary of the AdS_5 space i.e an $\Re^1 \times S^4$ isomorphic to Minkowski.The $\Re^1$ is for the time coordinate.

Suppose we change coordinates with $U \equiv \frac{r}{\alpha'}$ fixed and find out what happens when alpha approaches zero; $\alpha' \longrightarrow o$ and $r \longrightarrow o$. We can think of U as the energy scale in the gauge theory and demand that $\frac{E}{U}$ be fixed. With $r \equiv U \times \alpha'$, our metric becomes

$$ds^2 = \alpha'[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + d\vec{x}^2) + \sqrt{4\pi g_s N}(\frac{dU^2}{U^2} + d\Omega_5^2] \tag{3.25}$$

where as $\alpha' \rightarrow 0$ everything inside bracket is finite.

$$R^4 = (\alpha')^2 g_s 4\pi N \tag{3.26}$$

Here N is the number of D3-branes and g_s is the string coupling. While in the Super Yang-Mills gauge theory, N is the rank of the $SU(N)$ gauge group.

The Limit, where r tends to zero, $r \rightarrow o$ is called the Near Horizon limit. In this limit,

$$D3 - brane \longrightarrow AdS_5 \times S^5$$

with the curvature radii $R_{AdS_5} = R_{S^5}$ given as

$$R_{AdS} = R_S = \ell_s(\frac{g_s N}{\pi})^{1/4}$$

Second

On the other hand, when $r >> R$, in other words if we take $r \rightarrow \infty$, Radius of curvature becomes very small, we recover our flat spacetime $(9+1)$Minkowski or$\Re_{10}$.

In a very heuristic manner, we have seen how we get $\aleph = 4$ SYM from D3-brane in the low energy limit($\alpha' \to o$). Now we try to understand this limit in string theory side.

Supergravity approximation of string means the curvature of the background metric

$$ds^2 = \alpha'[\frac{U^2}{\sqrt{4\pi g_s N}}(-dt^2 + dx^2) + \sqrt{4\pi g_s N}(\frac{dU^2}{U^2} + d\Omega_5^2)$$

must be large compared to the string length, ie $R = \sqrt{\alpha'}(4\pi g_s N)^{1/4} >> \sqrt{\alpha'} = \ell_s$ That means,from the action

$$S_{2B} = \frac{1}{G_{10}}\int d^{10}x\sqrt{g}e^{-2\phi}[\Re + ... + \frac{\ell_s^2}{R^2}\Re^2 + ...]$$

where the Ricci scalar $\Re \sim \frac{1}{R^2}$; R being the radius of curvature, we can neglect higher derivatives terms of $\Re$. From ND3-branes, $g_s = g_{YM}^2$, therefore for $R >> \ell_s$ which means $g_s N >> 1$ or $g_{YM}^2 N >> 1$, Quantum string corrections governed by g_s come in powers of g_s. When g_s small i.e $g_s \to 0$ ($g_s N >>> 1$,large N), we can neglect these higher order terms and concentrate on the tree level which is well described by supergravity. Therefore for supergravity to be a valid approximation of Type2B superstring, we need to have $g_s \to 0$, N$\to \infty$ but with t'Hooft coupling $\lambda = g_s N = g_{YM}^2 N$ remaining finite and large; $g_s N = \lambda >> 1$.

On the other hand as t'Hooft showed in [1] at large N, effective coupling is $\lambda \equiv g_{YM}^2 N$ which means that when $\lambda >> 1$, perturbation theory does not longer work, which is supergravity while for for $\lambda << 1$ perturbation theory is reliable but supergravity is replaced with the full sting theory. This makes the AdS/CFT duality very hard to check since for $\lambda >> 1$ we cannot perform calculations in the strongly interacting gauge theory (but Supergravity reliable) while $\lambda << 1$ we still cannot handle strongly coupled (Large Curvature) string theory (but we can handle very well the weakly coupled gauge theory). We resort to a conjecture of AdS/CFT duality in the following versions:

3.3 Maldacena's Conjecture

In [3, 4, 24],versions of Maldacena's Conjecture:

The Weakest Version: $\aleph = 4$ SU(N) Super Yang Mills in 4-dimension is dual to supergravity; the low energy limit of Type2B superstring theory on $AdS_5 \times S^5$ in the limit when t'Hooft coupling $\lambda = g_{YM}^2 N >> 1$, $g_s \to 0$, $N \to \infty$. In this case, there would be order α' and g_s corrections to supergravity which might not agree with order$\frac{1}{N^2}$ and $\frac{1}{\sqrt{g^2 N}}$ correction to $\aleph = 4$ supersymmetric Yang-Mills theory.

The stronger Version: AdS/CFT duality is valid at any finite $g_s N$, but only if $N \to \infty$ and $g_s \to 0$, which means that α' corrections, given by $\frac{\alpha'}{R^2} = \frac{1}{\sqrt{g_s N}}$ agree, but g_s corrections might not.

The Strongest Version: AdS/CFT duality is valid at any g_s and N even if we cannot make calculations in certain limits.

This also contains the conjecture that the $AdS_5 \times S^5$ background is an exact solution of Type 2B superstring theory. An obvious example we will be using through out is that Type2B string Theory compactified on $AdS_5 \times S^5$ plus some appropriate boundary conditions (and possibly also some degree of freedom) is dual to $N = 4$; $d = 3 + 1$ U(N) Super Yang-Mills Theory.

3.4 AdS Spacetime

Low Energy effective action of supergravity

$$S_{gravity} = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g}(R - \Lambda) \tag{3.27}$$

AdS spaces are maximally symmetric solutions to Einstein equations with a constant energy momentum tensor known as cosmological constant $T_{\mu\nu} = \Lambda g_{\mu\nu}$. Varying the action with respect to the metric we have equation of motion

$$R_{\mu\nu} - 1/2 g_{\mu\nu} R = 8\pi G_N \lambda g_{\mu\nu} \tag{3.28}$$

Anti-de sitter spaces(AdS) is a space-time of negative curvature. We denote a $(d+1)$dimensional AdS space by AdS_{d+1}. Constant curvature means its curvature can be expressed in terms of the metric G_{MN}, $M, N = 0, 1, 2...d$ as:

$$R_{KLMN} = \frac{R}{d(d+1)}(G_{KM}G_{LN} - G_{KN}G_{LM})$$

and Ricci Scalar $\Re$ as

$$\Re = -\frac{d(d+1)}{R^2_{AdS}}$$

A convenient way of realizing AdS_{d+1} is that it is a hypersurface

$$(X^0)^2 - \delta_{ij} X^i X^j + (X^{d+1})^2 = R^2_{AdS} \tag{3.29}$$

$i, j = 1, 2, ...d$ in flat space $\mathbb{R}^{2,d}$ with signature $(2, d)$ and the metric induced from the flat space.

$$dS^2_{d+2} = -(dX^o)^2 + \delta_{ij} dX^i dX^j - (dX^{d+1})^2 \tag{3.30}$$

This space has closed-time-like circles since

$$(X^o)^2 + (X^{d+1})^2 = R_{AdS}^2 + \delta_{ij} X^i X^j \tag{3.31}$$

From equation $(3.29, 30)$, we see that AdS_{d+1} is by construction a maximally symmetric space of isometry group $SO(2,d)$. There are several useful coordinates of AdS spacetime. One of these useful coordinates is the Poincare coordinates. As we saw earlier, this system of coordinates arises naturally in the context of supergravity or D3-branes. Radial coordinates other than world volume coordinates $x^\mu = (t, x^a,$ $a = 1, 2, ...d-1)$ are related through change of coordinates

$$X^o = R_{AdS}\frac{t}{z} \text{ and } X^a = R_{AdS}\frac{x^a}{z} \text{ and we get}$$

$$X^{d+1} - X^d = R_{AdS}(z + \frac{1}{z}(x^2 - t^2)) \tag{3.32}$$

Metric becomes

$$ds^2 = \frac{R_{AdS}^2}{z^2}(-dt^2 + \vec{X}^2 + dz^2) \tag{3.33}$$

making manifest, the invariance under $SO(1,d)$ of the isometry group $SO(2,d)$
In terms of the new radial coordinates; $r = \frac{R_{AdS}^2}{z}$ our metric becomes

$$ds^2 = \frac{R_{AdS}^2}{r^2}dr^2 + \frac{r^2}{R_{AdS}^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{3.34}$$

A Euclidean version of AdS_{d+1} in [4, 5, 26] can be realized as follows: We consider a Euclidean space $\mathbb{R}^{d+1}$ with coordinates $y_o, ..., y_d$ and let B_{d+1} be the open ball

$\sum_{i=1}^{d} |\vec{y}|^2 < 1$ then the AdS_{d+1} can be identified as B_{d+1} with metric

$$ds^2 = \frac{4\sum_{i=0}^{d} dy_i^2}{(1-|y|^2)^2}$$

. In the above Poincare coordinates in Euclidean space, the AdS_{d+1} is the upper half space $x^o \in (0, \infty)$ with coordinate $x^0, x^1, ... x^d$ and the metric

$$ds^2 = \frac{1}{x_o^2}(\sum_{i=0}^{d}(dx^i)^2) \tag{3.35}$$

In Poincare coordinates, AdS_{d+1} has a d-dimensional Minkowski space as its boundary while for Euclidean coordinates $\mathbb{R}^d$ is its boundary as x_o tends 0; $x_0 \longrightarrow 0$.
From our above definitions of AdS_{d+1} and its boundary we deduce that the isometry group of AdS_{d+1} is $SO(2,d)$ for Poincare Coordinates ($SO(1, d+1)$ for Euclidean). This isometry group is a conformal group and has dimension as: For a d-dimensional Euclidean space $\mathbb{E}^d$, the conformal group $SO(1, d+1)$ has same number of generators as $SO(d+2)$. This is the number of linearly independent antisymmetric $(n+2)\times(n+2)$ matrices. dim-$SO(1, d+1) = \frac{1}{2}(d+2)(d+1)$. By comparison, the Poincare group in d-dimensions has d translational generators and $\frac{1}{2}d(d-1)$ rotation generators. Dim-Poincare is $(E^d) = \frac{1}{2}d(d+1)$ The difference between the Poincare and conformal group space is just $d+1$. This is just the following extra possible conformal transformations of Dilaton $\vec{x} \rightarrow \lambda\vec{x}$, $\lambda \in \mathbb{R}$ The special Conformal Transformations $\vec{x} \rightarrow \vec{x'}$ such that

$$\frac{x^{\mu\prime}}{x^{2\prime}} = \frac{x^{\mu}}{x^2} + \alpha^{\mu}\alpha^{\mu}, \mu = 1, ..., n$$

and gives rise to additional d generators. which may equivalently write

$$(x^{\mu})' = \frac{x^{\mu} + \alpha^{\mu}x^2}{1 + 2\vec{\alpha}\cdot\vec{x} + \alpha^2 x^2}$$

The d generators for special conformal transformations with the 1 from the dilaton gives the $d + 1$ dimensions.

Chapter 4

Test of AdS/CFT and Wilson Loop

4.1 Global Symmetry

Naively, we begin verifying the AdS/CFT Correspondence with the global symmetry; $SU(2,2/4)$ of both theories. The $\aleph = 4$ Super Yang-Mills has a vanishing beta function so has a conformal symmetry with bosonic isometry group $SO(2,4) \sim SU(2,2)$ generated by the translations P_μ, Lorentz transformations $L_{\mu\nu}$, dilatons D and special conformal transformations K^μ.An R-symmetry group $SO(6)_R \simeq Su(4)_R$ generated by T^a,$a = 1, 2, 3, ..., 15$. The Poincare Supersymmetry generated by supercharges Q^a_α and $\bar{Q}^a_{\dot{\alpha}}$, $a = 1, 2, ...4$ which gives $\aleph = 4$ Poincare Super symmetry; and finally Conformal Supersymmetries generated by the supercharges S^a_α and $\bar{S}^a_{\dot{\alpha}}$; resulting from the fact that$[Q^a_\alpha, K_\mu] \sim S^a_\alpha$ do not commute. Putting all these together realizes the global Continuous symmetry group $PSU(2,2/4)$ for $\aleph = 4$ Super Yang-Mills.

On the other hand, in Supergravity, the $AdS_5 \times S^5$, have a Bosonic symmetry group $SO(2,4)$ for the AdS_5 side and $SO(6)$ for S^5 based on their geometry. Hence $AdS_5 \times S^5$ gives the Bosonic isometry group $SO(2,4) \times SO(6) \sim SU(2,2) \times SU(4)_R$. Although only 16 of the 32 Poincare Super symmetry charges are preserved in our BPS N Parallel D3-coincident branes, yet in the AdS limit, they are supplemented by the enhanced extra 16 supercharges because of its geometry making 32 Super charges. All together realizes the global symmetry $PSU(2,2/4)$ in Supergravity matching exactly the global symmetry in both theories [30]

4.2 Global Symmetry Representation

$SU(2,2/4)$ has multiplet representation in Super Yang-Mills and Supergravity. The correspondence is between Fields in Type2B String to gauge invariant operators in super Yang-Mills. Single traceless symmetric tensors of $SO(6)$; short representation, operators in the Super Yang Mills side correspond to Kaluza-Klein modes on the AdS side. Fields in AdS in Multiplet of SU(2,2/4) correspond to Unitary Super Conformal Multiplets of SU(2,2/4) in the Super Yang Mills side. Meaning Kaluza-Klein Modes on AdS with short multiplet representation (BPS) of SO(6) is dual to BPS operators(Short Multiplets) Traceless symmetric tensors of SO(6) in Super Yang-Mills(SYM). [30, 31].

BPS Operators are given by the single traceless symmetric tensors of $SO(6)$;

$$\vartheta_k(x) \equiv \frac{1}{n_k} ST_r(X^{i_1}(x), X^{i_2}(x), ..., X^{i_k}(x))$$

Aside $\frac{1}{2}$ BPS for $\aleph = 4$ SYM means half of the 16 supercharges leaves the primary or chiral Operator invariant. The constant n_k is for the overall normalization of the operator. The dimension of this operator is unrenormalized and thus $\Delta = k$ For more on this check [24]

In the Type2B super gravity, using Kaluza-Kelin compactification of $d = 10$ on S^5. First for a sphere of radius L of S^5, the Newton constant in $d = 5$ becomes

$$G_N^5 = \frac{G_N^{10}}{Vol(S^5)} = \frac{8\pi^3 g_s^2 (\alpha')^4}{L^5}$$

Decomposing the $d = 10$ fields in spherical harmonics $Y_\ell^I(\hat{y})$ on S^5 with I a multi-index running from over the possible projections of $\bar{\ell}$. We linearize as in [31]. The field equations in AdS_5 vacuum configuration.

For simplicity we consider the case of the dilaton. The spectrum of fluctuations of the dilaton can be deduced from Kaluza-Klein (K-K) ansatz.

$$\Phi = \sum_{l=0}^{\infty} \sum_{I=1}^{\frac{(l+1)(l+2)^2(l+3)}{12}} \Phi_I^\ell(z) Y_\ell^I(\hat{y}) \tag{4.1}$$

Using

$$\nabla_{S^5}^2 Y_\ell^I = -\frac{\ell(\ell+4)}{L^2} Y_\ell^I \text{for all I} \tag{4.2}$$

We conclude that the Φ_ℓ^I component field transforms in the irreducible representation $r = [0, \ell, 0]$ of $SO(6) \simeq SU(4)$ and has an AdS mass $(M_\ell L)^2 = \ell(\ell+4)$ .For the other fields we will have to resolve the intricate mixing with the other fields. Therefor, each Φ_ℓ^I satisfies the equation of motion of the form

$$(\Box_{AdS_5} - M^2)\Phi_\ell^I = \text{ Non Linear terms} \tag{4.3}$$

with $M^2 = \frac{m^2}{L^2}$

The whole spectrum of the linearized fluctuations assembles into represenetations of the super isometry group

$$SU(2, 2/4) \supset SO(4, 2) \times SO(6)$$

. We observe that the particles in the same multiplet have different AdS mass since $Mass^2$ is not a casimir operator of the $SU(2, 2/4)$.

These K-K modes classified in $SU(2, 2/4)$ representations give exactly the set of short representations for composite Operators in the Super Yang- Mills field theory. There is a one-to-one correspondence between the K-K fields of Type2B $D = 10$ supergravity and the composite operators(in the short representations) of the $\aleph = 4$ Super Yang-Mills theory.

Finally we observed that the:

- Φ^I_ℓ is dual to chiral primary Operators(CPO) $T_r X^\ell$, in the $\ell = 2$ multiplet,
- 15 bulk gauge fields A_μ from the $A_M = (A_\mu, \phi_i)$ is dual to the conserved currents J_i of $SO(6)_R$ symmetry group and the
- AdS_5 Metric fluctuation $h_{\mu\nu}$ is dual to the field theory stress Energy tensor T_{ij} [31]

We must note that, this correspondence is not as a result of the dynamics but due to the symmetry especially the $SO(6) \simeq SU(4)$ group. It turns out that the Unitary Irreducible Representations(UIR'S) have all been classified completely[30, 31, 32].

4.3 AdS/CFT Correlation Functions

Observed quantities in $\aleph = 4$ Super-Yang-Mills theory are gauge invariant. They are composite operators living on the boundary of AdS_5 including Correlations functions and Wilson loops. From [4] using a standard prescription for computing correlation functions of gauge invariant local composite operators, we consider massive scalar field (dilaton). In Euclidean Coordinates of AdS_5, its equation

$$(\nabla^2_{AdS_5} - m^2)\Phi = 0 \tag{4.4}$$

with metric

$$dS^2_{AdS_5} = \frac{R^2}{\rho^2}(d\rho^2 + d\vec{x}^2) \tag{4.5}$$

where $d\vec{x^2} = \delta_{ij} dx^i dx^j$. The generalized Dirichlit boundary conditions is

$$\Phi(x, \rho) \to \rho^{\nu} \phi(x) \tag{4.6}$$

where as ρ tends to 0 (Boundary of AdS_5 being conformal), the Laplace equation

$$\nabla^2_{AdS_5} \phi(\vec{x}, \rho) = \frac{1}{\sqrt{G}} \partial_\mu (\sqrt{G} G^{\mu\nu} \partial_\nu \Phi) \tag{4.7}$$

$$\nabla^2_{AdS_5} \phi(\vec{x}, \rho) = \rho^5 \partial_\rho (\rho^{-3} \partial_\rho \Phi) + \rho^2 \partial \cdot \partial \Phi \tag{4.8}$$

becomes

$$(\rho^2 \partial^2 \rho - 3\rho \partial \rho + \rho^2 \nabla^2 - m^2) \Phi(x, \rho) = 0 \tag{4.9}$$

Using the green's function method such that a solution

$$\Phi(x, \rho) = \int_{\partial AdS_5} d^4\vec{x} K(\rho, x; x_0) \phi_0(x) \tag{4.10}$$

$$\Box_{\vec{x},\rho} K(\vec{x}, \vec{x'}, \rho) = \rho^\beta \delta^4(\vec{x} - \vec{x'}) \tag{4.11}$$

we are interested in the behavior that

$$K(\rho, x; x_0) \longrightarrow \rho^{4-\beta} \delta(x - x_0) \tag{4.12}$$

at the boundary so that we obtain the solution to above equation expressed in terms of Bulk-to-Boundary Propagator as

$$K(\rho, x, x_0) = C\frac{\rho^{\alpha}}{(\rho^2 + (x - x_0^2))^{\beta}} \tag{4.13}$$

where C_β is the Normalisation factor given as

$$C_\beta = \frac{\Gamma(\beta)}{\pi^2\Gamma(\beta - 2)} \tag{4.14}$$

putting K into the differential equation and solving for β we obtain the mass-to-dimension relation for scalars

$$(m)^2 = \Delta(\Delta - 4) \tag{4.15}$$

where $m^2 = \frac{m^2}{R^2}$ and the inverse of equation is

$$\Delta = 2 + \sqrt{4 + (m)^2} \text{ and } \beta = \Delta \tag{4.16}$$

For any dimension d

$$\Delta = \frac{d}{2} + \frac{1}{2}\sqrt{d^2 + m^2} \tag{4.17}$$

where m^2 = Mass of fields from spherical correction of radius R while M^2 = Mass of field in AdS_5

REMARKS

First we observed that Δ is real as expected for any Unitary theory once the Breitenlohmer-Freedman bound $(m)^2 \geq -4$ required for stability of the AdS_5 is enforced.

Second,

$$(\nabla^2_{AdS_5} - m^2)\Phi(\rho, x) = 0 \tag{4.18}$$

gives two solutions and as

$$\rho \to 0$$

,

$$\Phi(x,\rho) \simeq \rho^{\mu}\phi(x) = \begin{cases} \rho^{4-\Delta}\phi(x), \\ \rho^{\Delta}\phi(x), \end{cases}$$

$\Phi(x,\rho) \simeq \rho^{\beta}\phi(x,)$ is non-recompensable and in [25] that correspondence to studying the theory in a background with non-vanishing Vacuum Expectation Values(VEV) of operators for the field ie determining the VEV of operators of associated dimensions. While $\Phi(x,\rho) \simeq \rho^{4-\beta}\phi(x)$ do not correspond to bulk excitations but rather they represent the coupling of external sources to the supergravity or string theory. This is like operator deformations of the fixed point action required to compute correlation functions. The Mass-to-dimension relations and their inverses for other fields includes:

1. Symmetric Tensors $(ML)^2 = \Delta(\Delta - 4)$ and $\Delta = 2 \pm \sqrt{4 + (ML)^2}$
2. P-forms $(ML)^2 = (\Delta + p)(\Delta + P - 4)$ and $\Delta = 2 \pm \sqrt{(2-p)^2} + (ML)^2$
3. fermions $(ML)^2 = (\Delta - 2)^2$ and $\Delta = 2 \pm |ML|$

For Our scalers above reading the masses from Table in [31]

$$m^2 = k(k-4), (k \geq 2) \text{ and } m^2 = (k+4)(k+8), (k \geq o)$$

we get that

$$\Delta = k, \text{for scaler } h^{\mu}_{\mu} \text{ and}$$

$$\Delta = k + 8, \text{ for field } a_{\alpha\beta\gamma\delta}$$

They all have the $[0, k, 0]$ representation of the $SO(6)$ group as seen on the table on presentation slide. Vectors and scalars correspond to P-forms with $P = 1, 0$ respectively.

2 **Point Functions**

We consider calculating the 2 point function for a Massive Scalar in the AdS side. The map between AdS and the CFT quantities is given by relation proposed in [4, 33, 34]

$$exp(-\Gamma_{Sugra}(\phi_i)) = \langle exp(\int d^4x \phi_i^0 \vartheta_i) \rangle$$

where the Left hand side (LHS) is the supergravity action evaluated on the classical solution given by ϕ_i and the right hand side(RHS) is a generating function for the correlations functions in the super Yang-Mills theory. Therefore, for every field ϕ_i in the AdS side, there is a unique operator; traceless and symmetric tensor of $SO(6)$ ϑ_i in the Yang-Mills side, so that this operator has a conformal dimension Δ_i.

Consider a two point function of a scalar operator whose corresponding field in the AdS_{5+1} is a massive scalar field h^μ_μ of mass m, then the action is:

$$S = \frac{1}{2} \int d^{4+1}x \sqrt{g}[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2]$$

in the AdS_{4+1} this action becomes

$$S = \frac{1}{2} \int d^5x d\rho \rho^{-4+1}[(\partial_\rho\phi)^2 + (\partial_i\phi)^2 + \frac{m^2}{\rho^2}\phi^2]$$

where we have set from the AdS metric $L = 1$ as in [35]

For $\rho \to 0$ we find that

$$\Phi(\rho, x) \to \rho^{d-\beta}(\phi_0(\vec{x}) + O(\rho^2) + \rho^\Delta[A(\vec{x}) + O(\rho^2)$$

where Δ are root to $\Delta(\Delta - d) = m^2$

when $\phi_0(\vec{x})$ is boundary values of $\phi(\rho, \vec{x})$ and as source of the operator ϑ_β

With our bulk-to-boundary propagator

$$K(\rho, \vec{x}; x_0) = C_\beta \frac{\rho^\alpha}{(\rho^2 + (\vec{x} - \vec{x_0})^2)^\beta}$$

and

$$\Phi(\vec{x}, \rho) = \int_{\partial AdS_{4+1}} d^4\vec{x'} K(\rho, \vec{x}, \vec{x'}) \phi_0(\vec{x'})$$

where

$$C_\beta = \frac{\Gamma(\beta)}{\pi^{\frac{d}{2}} \Gamma(\beta - \frac{d}{2})}$$

then

$$S(\phi_0) = I(\phi_0) = -\frac{(\beta - \frac{d}{2})\Gamma(\beta)}{\pi^{\frac{d}{2}} \Gamma(\beta - \frac{d}{2})} \int d^d x' \int d^d x'' \frac{\phi_0(x')\phi_0(x'')}{|\vec{x} - \vec{x'}|^{2\beta}}$$

then the two point function is

$$\langle \vartheta(\vec{x}) \vartheta(\vec{y}) \rangle = \frac{\delta^2 I(\phi_0)}{\delta\phi_0(\vec{x}) \delta\phi_0(\vec{y})} |_{\phi_0 = 0}$$

$$\langle \vartheta(\vec{x}) \vartheta(\vec{y}) \rangle = \frac{(\Delta - \frac{d}{2})\Gamma(\beta)}{\pi^{\frac{d}{2}} \Gamma(\beta - \frac{d}{2})} \frac{1}{|\vec{x} - \vec{y}|^{2\beta}}$$

While in the SYM side, our composite operators

$$\vartheta(x)_\Delta \equiv \frac{1}{n_k} ST_r\{X^{i_1}(x), ..., X^{i_\Delta}(x)\}$$

by Poincare symmetry $\langle \vartheta_{\Delta_1}(x_1), \vartheta_{\Delta_2}(x_2) \rangle$ must depend on $(x_1 - x_2)^2)$ By inversion symmetry it must vanish unless $\Delta_1 = \Delta_2$ by scaling symmetry one fixes the exponent and by properly normalizing the operators we obtain

$$\langle \vartheta_{\Delta_1}(x_1), \vartheta_{\Delta_2}(x_2) \rangle = \frac{C_\Delta \delta_{\Delta_1, \Delta_2}}{|x_1 - x_2|^{2\Delta_1}}$$

where

$$C_\beta = \frac{(2\beta-4)\Gamma(\beta)}{\pi^{\frac{d}{2}}\Gamma(\beta-\frac{d}{2})}\frac{1}{|\vec{x_1}-\vec{x_2}|^{2\beta_1}}$$

which is the same as obtained above.

Another corresponding field in the AdS side from the Born-Infeld D-brane action is the anti symmetric tensor.

$$S_{int} = \frac{\sqrt{\pi}}{K}\int d^4x[t_r(\frac{1}{4}\phi F^2_{\alpha\beta} - \frac{1}{4}CF_{\alpha\beta}\tilde{F^{\alpha\beta}}) + \frac{1}{2}h^{\alpha\beta}T_{\alpha\beta}]$$

Consider, the coupling of the Dilatons Φ to the $\frac{1}{4g^2_{YM}}t_rF^2_{\alpha\beta}$. In Yangs-Mills. Since $\aleph = 4$ Super Yang-Mills $SU(N)$ is conformal invariant quantum theory and $F^2_{\alpha\beta}$ has dimension 4 the 2-point functions of

$$\langle F^2(x), F^2(y)\rangle\ \frac{N^2}{|\vec{x}-\vec{y}|^2}$$

where N is the rank of he $SU(N)$ group.

While in the AdS_5 Volume of $S^5 = \pi^3L^3 = \pi^3R^3$ then

$$S = \frac{\pi^3R^3}{4\kappa^2_{10}}\int d^5x\sqrt{g}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi$$

with $g_{\mu\nu} = \frac{R^2}{\rho^2}\delta_{\mu\nu}$ is the metric of AdS_5

For $\lambda >>> 1$ we have a classical sugra solution, so from solving the equation

$$\partial_\mu[\sqrt{g}g^{\mu\nu}\partial_\nu\Phi] = 0$$

with boundary condition

$$\phi(\rho,\vec{x}) \to \rho^{4-\Delta}\phi_0(\vec{x})$$

$$\Phi(\rho,\vec{x}) = \int d^4x' K_\Delta(\rho,\vec{x},\vec{x'})\phi_0(x')$$

with

$$K_\Delta(\rho, \vec{x}, \vec{x'}) \sim \frac{\rho^4}{[\rho^2 + (\vec{x} - \vec{x'})^2]^4}$$

. Substituting in $S(\Phi)$ we get

$$S(\phi_0) = \frac{\pi^3 R^3}{4\kappa_{10}^2} \int d^4\vec{x'} \rho^{-3} \Phi \partial_0 \Phi$$

$$S(\phi_0) \ \frac{\pi^3 R^8}{4\kappa_{10}^2} \int d^4x' d^4x'' \frac{\phi_0(x')\phi_0(x'')}{(\vec{x} - \vec{x'})^8}$$

so that our generating function

$$Z(\phi_0) = exp[\frac{\pi^3 R^8}{4\kappa_{10}^2} \int d^4x' d^4x'' \frac{\phi_0(x')\phi_0(x'')}{(\vec{x} - \vec{x'})^8}]$$

with

$$R^2 = \alpha' \sqrt{4\pi N g_s}$$

and

$$2\kappa_{10}^2 = (2\pi)^7 g_s^2 (\alpha')^4$$

Therefore our two point function is

$$\langle F^2(x) F^2(y) \rangle = \frac{\partial^2 Z(\phi_0)}{\partial\phi_0(x)\partial\phi_0(y)}|_{\phi_0=0}$$

$$\langle F^2(x) F^2(y) \rangle \ \frac{N^2}{(\vec{x} - \vec{y})^8}$$

Some of the most stricking results of this AdS/CFT conjecture obtained in the same way for other fields are in [36]

4.3.1 Wilson Loop

The Wilson loop operator is a phase factor associated with the trajectory of a heavy quark in the fundamental representation of the gauge group which in this case is U(N).

$$W(C) = \frac{1}{N} Tr Pexp \oint_C d\tau (iA_\mu(x)\dot{x_\mu} + \Phi_i(x)(y_i))$$

where x_μ is a parametrization of the loop $y_i = \sqrt{\dot{x^2}\theta_i}$ and θ_i is a point on a five dimensional unit sphere ($\theta^2 = 1$). In the AdS/CFT corresponds , the Wilson loop is computed by finding the area of the world sheet of the classical string on $AdS_5 \times S^5$ whose boundary is the loop C, which in turn lies on the boundary of AdS_5.

In gauge theory,[37] the Wilson loop operator is

$$W(C) = T_r[Pexp(i \oint_c A)]$$

depends on the loop C embedded in 4 dimension space and P path ordered integral of the gauge correction along the contour. The trace is taken over some representation of the gauge group(here in the fundamental representation).

From the expectation Value of the Wilson loop operator $\langle W(C) \rangle$ we can calculate the Quark-AntiQuark Potential. We will consider a rectangular loop with sides length T and width L in Euclidean space. Then if we view T as the time direction, we can say that for large T, we can show that [37, 18] the expectation value will behave as e^{-TE} where E is the lowest possible energy of the quark-anti quark configuration. Therefore we have

$$\langle W \rangle \sim e^{-TV(L)}$$

where $V(L)$ is the quark-anti quark potential. For large N and $g_{YM}^2 N$ large(in the strong coupling limit), The AdS/CFT correspondence maps the computation of $\langle W(C) \rangle$ in the CFT into a problem of finding a Minimum Surface area in the AdS

as in [38].

DISCUSSION

AdS/CFT realises a U(N) gauge group from the large number of D3-branes (Large N) situated at the same point-Concident. The strings with two ends on different branes in such a concident configuration are massless, since there is no physical seperation between the D-branes and they correspond to gauge fields,

$$A^a_\mu = (\Lambda^a)_{ij}|i\rangle \otimes |j\rangle \otimes |\mu\rangle$$

. We consider a simple case with $N+1$ Coinsident D3-branes giving a $U(N+1)$ gauge group. Let us take one of the D-branes and seperate it from the rest. This is like breaking the gauge group spontaneously via a Higgs-like mechanism;

$$U(N+1) \longrightarrow U(N) \times U(1),$$

where U(N) corresponds to the remaining coincident N D3-branes. The strings that have one end on the N D3-branes and the other end on the extra D3-brane will become massive with mass Mass = string Tensions $\times$ D3-branes seperations. They also have state

$$|i_0\rangle \times |i\rangle = |N+1\rangle \times |i\rangle$$

with the i being the fundamental index. Thus they have the fundamental representation of the U(N) gauge group. Its Mass is therefore

$$M = \frac{1}{2\pi\alpha'}r = \frac{U}{2\pi}$$

for $U = \frac{r}{\alpha'}$. This string behaves as a " W-Boson" and acts on the U(N) as a quark. If we take $U \to \infty$,($\alpha' \to 0$) we obtain an infinitely massive external quark which is a string stretched in the AdS space from some U_0 and some finite part of U.

Since we are taking $U \to \infty$ where the $\aleph = 4$ Super Yang-Mills(SYM) gauge theory lives, our Wilson loop contour C at infinity is a boundary of the string. This is our string world sheet stretched between the contour C at infinity down to a finite point in the AdS forming a smooth surface.

Supersymmetry of the $\aleph = 4$ SYM dictates that we have to generalised the Wilson loop as

$$W(C) = \frac{1}{N} T_r Pexp[\oint_C (iA_\mu \dot{x^\mu} + \theta^I X^I(x^\mu)\sqrt{\dot{x^2}})d\tau]$$

Where $x^\mu(\tau)$ are parameters of the loop and θ^I is a unit vector giving the position of the string in S^5 of the $AdS_5 \times S^5$ and corresponds to the scalar X^I of the $\aleph = 4$ SYM.

In order to calculate $\langle W(C) \rangle$ we use the prescription for finding the partition function for the string with boundary on C [4].

In the Supergravity limit, $g_s \to 0$ or $g_s N$ large but fixed, then

$$\langle W(C) \rangle = Z_{string}[C] = e^{-S_{string}[C]}$$

where S_{string} = string World Sheet action = $\frac{1}{2\pi\alpha'}\times$Area of world sheet.

Since the string has tension and gravity in the $AdS_5 \times S^5$, its will want to minimize its "gravitational potential" which is the minimum of U. Therefore the string between $U = \infty$ at boundary and some $U = U_0$ by is tension. But the area of this world sheet will be divergent and $\langle W(C) \rangle = 0$

Although the string is stretched between the $|i\rangle$ and $|N+1\rangle$ D-branes thus between $U = \infty$ and $U = U_0$, it will represent an infinitely massive " W-boson" whose mass ϕ we must subtract. Then from the action we must subtract the term $\phi\ell$, where ℓ is the length of the loop C and ϕ is the free "W-boson"(free string) mass= $\frac{U}{2\pi}$ as from [39]. Then

$$\langle W(C) \rangle = e^{-(S_\phi - \ell\phi)}$$

Quark-AntiQuark Potential Calculation. Our Contour as we said earlier is an infinitely thin rectangle [40]. $T \to \infty$ with a quark at $x = -L/2$ and Anti quark at $x = L/2$ see figure.
with $U = \frac{r}{\alpha'}$, our $AdS_5 \times S^5$ metric is

$$dS^2 = \alpha'[\frac{U^2}{R^2}(dt^2 + d\vec{x}^2) + R^2\frac{dU^2}{U^2} + R^2 d\Omega_5^2]$$

where $R^2 = \sqrt{4\pi g_s N}$
Therefore, Our Nambu-Goto Action for the string will be

$$S_{string} = \frac{1}{2\pi\alpha'}\int d\tau d\sigma\sqrt{det(G_{MN}\partial_\alpha X^M \partial_\beta X^N)}$$

Now since, we want a static solution, let us use the static gauge where $\tau = t$ and $\sigma = x$.
Then for $T \to \infty$, meaning $\frac{T}{L} \to \infty$ we assume that the world sheet is translation invariant in [41](Otherwise the curvature of the world sheet near corner becomes important). For Static configuration we neglect time coordinates and so $U = U(\sigma) = U(x)$. Calculating the induced world sheet metric

$$h_{\alpha\beta} = G_{MN}\partial_\alpha X^M \partial_\beta X^N$$

with

$$h_{11} = \alpha'\frac{U^2}{R^2}(\frac{dt}{d\tau})^2 = (\alpha')^2$$

and

$$h_{22} = \alpha'\frac{U^2}{R^2}(\frac{dx}{d\sigma})^2 + \alpha'\frac{R^2}{U^2}(\frac{dU}{d\sigma})^2$$

and so

$$h_{22} = \alpha'(\frac{U^2}{R^2} + \frac{R^2}{U^2}(U')^2)$$

and $h_{12} = 0 = h_{21}$ and $U' = (\frac{dU}{d\sigma})$ Thus

$$S_{string} = \frac{1}{2\pi}T\int dx\sqrt{(\partial_x U)^2 + \frac{U^4}{R^4}}$$

Let U_0 be the minimum of the $U(x)$ and $g = \frac{U}{U_0}$ Then the Euler equation of Motion is

$$\frac{1}{U^2\sqrt{(\partial_x U)^2 + 1}} = constant$$

whose solution is

$$x = \frac{R^2}{U^2}\int_1^{\frac{U}{U_0}} \frac{dy}{y^2\sqrt{y^4 - 1}}$$

By inverting $X(U, U_0)$ to $U(x, U_0)$ we can find U_0. Now, because At $U = \infty$ at $x = L/2$ then we

$$\frac{L}{2} = \frac{R^2}{U_0}\int_1^{\infty} \frac{dy}{y^2\sqrt{y^4 - 1}} = \frac{R^2\sqrt{2}\pi^{\frac{3}{2}}}{U_0\Gamma(\frac{1}{4})^2}$$

From Wilson loop, prescription

$$S_\phi - \ell\phi = TV_{q\bar{q}}(L)$$

we regularize this formaular by intergrating only up to U_{max}. Then $\ell \simeq 2T$ and mass of string is

$$\phi = \frac{U_{max} - U_0}{2\pi} + \frac{U_0}{2\pi} = \frac{U_0}{2\pi}\int_1^{y_{max}} dy + \frac{U_0}{2\pi}$$

Integrating from U_{max} to U_0 and then U_0 to $Umax$ we get

$$TV_{q\bar{q}}(L) = T\frac{U_0}{2\pi}[\int_1^{\infty} dy(\frac{y^2}{\sqrt{y^4 - 1}} - 1) - 1]$$

and substituting from

$$\frac{L}{2} = \frac{R^2\sqrt{2}\pi^{\frac{3}{2}}}{U_0\Gamma(\frac{1}{4})^2}$$

we finally obtain

$$V_{q\bar{q}}(L) = -\frac{4\pi^2\sqrt{2g_{YM}^2 N}}{\Gamma(\frac{1}{4})^4 L}$$

Observation

Finally that the potential [41, 42] $V_{q\bar{q}}(L)\ \alpha\ \frac{1}{L}$ determined by conformal invariance as in the gauge filed case

$$\langle W(C)\rangle_{ladder} = exp[(\frac{\sqrt{g_{YM}^2 N}}{\pi} - 1 + O(\frac{1}{\sqrt{g_{YM}^2 N}})\frac{T}{L})]$$

for $g_{YM}^2 N >> 1$ (strong Coupling limit.

The Potential is also a $V_{q\bar{q}}(L)\ \alpha\ (g_{YM}^2 N)^{\frac{1}{2}}$ as opposed to $g_{YM}^2 N$, Therefore we conclude that it is a non-perturbative result. This only indicates, the screening of the charges at strong coupling. The calculations still make sense for all distances L when $g_s N$ is large independent of g_s. Also, subleading corrections from quantum fluctuations of the world sheet were calculated in [41, 42, 43]

4.3.2 CONCLUSION

The AdS/CFT Conjecture depends entirely on D-branes being extremal branes, Carrying Ramond-Ramond Charges.

Low energy D3-brane dynamics is entirely described by N=4, Super Yang-Mills SU(N) in 4-dimesion

The AdS/CFT conjecture holds well only when we consider BPS(Tracless Symmetric) Operators in the N=4, Super Yang-Mills whose dimensions are protected by Supersymmetry. This BPS Operators in the short multiplets of SU(2,2/4) have same dimension and representation as the Kaluza-Klein modes in Super gravity on AdS_5.

For non-BPS operators we will have to consider excitation modes of the Superstring, whose conformal dimension is not protected.

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