Technical Analysis

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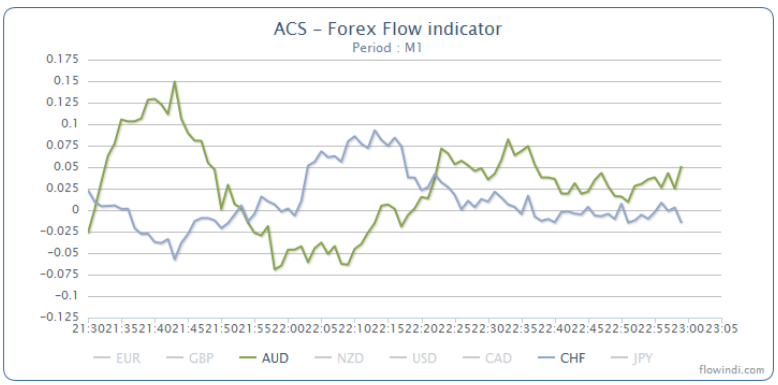
Absolute currency strength

The **Absolute currency strength (ACS)** is a technical indicator used in the technical analysis of forex market.It is intended to chart the current and historical gain or loss of a currency based on the closing prices of a recent trading period.It's based on mathematical decorrelation of 28 cross currency pairs.It shows absolute strength momentum of selected major currency.(EUR,GBP,AUD,NZD,USD,CAD,CHF,JPY)



Absolute currency strength for AUD/CHF pair

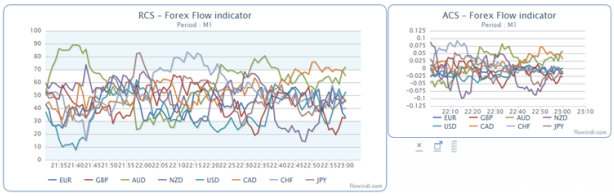
The ACS is typically used on a 15*period timeframe,calculated as a percentage gain or loss.This indicator is not measured on a scale like Relative currency strength. Shorter or longer timeframes are used for alternately shorter or longer outlooks.Extreme high and low percentage values occur less frequently but indicate stronger momentum of currency.

ACS is in most cases used as support indicator for Relative currency strength indicator.But it can be used by itself for currency trading. You can use Absolute currency strength for pattern trading as well.Combination of RCS and ACS indicators gives you entry and exit signals for currency trading.

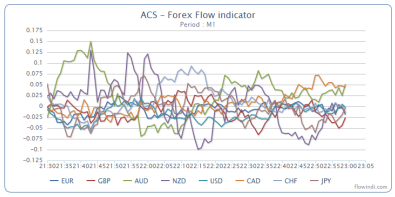
Basic idea

Indicator basic idea is "**buy strong currency and sell weak currency**". If is X/Y currency pair in uptrend,it shows you if it's due to X strength or Y weakness. On this signals you can choose the most worth pair to trade.

Indicator



Combination of Relative currency strength and Absolute currency strength



Absolute currency strength

Advantageous for trading strategies

- Support indicator for technical analysis as combination with Relative currency strength
- information indicator to realize which currencies are being demanded, this is ideal indicator for trend follow traders
- help for scalpers looking for strength trend (trader can see both absolute and relative strength)
- instrument for correlation/spread traders to see reactions of each currencies on moves in correlated instruments (for example CAD/OIL or AUD/GOLD)

References

External links

- How to trade with RCS and ACS indicators (http://www.articlesbase.com/currency-trading-articles/how-to-trade-with-relative-rcs-and-absolute-acs-currency-strength-3100840.html)

Accumulation/distribution index

Accumulation/distribution index is a technical analysis indicator intended to relate price and volume in the stock market.

Formula

$$CLV = \frac{(close - low) - (high - close)}{high - low}$$

This ranges from -1 when the close is the low of the day, to +1 when it's the high. For instance if the close is 3/4 the way up the range then CLV is +0.5. The accumulation/distribution index adds up volume multiplied by the CLV factor, i.e.

$$accdist = accdist_{prev} + volume \times CLV$$

The starting point for the acc/dist total, i.e. the zero point, is arbitrary, only the shape of the resulting indicator is used, not the actual level of the total.

The name accumulation/distribution comes from the idea that during accumulation buyers are in control and the price will be bid up through the day, or will make a recovery if sold down, in either case more often finishing near the day's high than the low. The opposite applies during distribution.

The accumulation/distribution index is similar to on balance volume, but acc/dist is based on the close within the day's range, instead of the close-to-close up or down that the latter uses.

Chaikin oscillator

A Chaikin oscillator is formed by subtracting a 10-day exponential moving average from a 3-day exponential moving average of the accumulation/distribution index. Being an indicator of an indicator, it can give various sell or buy signals, depending on the context and other indicators.

Similar indicators

Other Price × Volume indicators:

- Money Flow
- On-balance Volume
- Price and Volume Trend

Average directional movement index

The **average directional movement index** (**ADX**) was developed in 1978 by J. Welles Wilder as an indicator of trend strength in a series of prices of a financial instrument.[1] ADX has become a widely used indicator for technical analysts, and is provided as a standard in collections of indicators offered by various trading platforms.

Calculation

The ADX is a combination of two other indicators developed by Wilder, the positive directional indicator (abbreviated +DI) and negative directional indicator (-DI).[2] The ADX combines them and smooths the result with an exponential moving average.

To calculate +DI and −DI, one needs price data consisting of high, low, and closing prices each period (typically each day). One first calculates the directional movement (+DM and −DM):

> UpMove = today's high − yesterday's high
>
> DownMove = yesterday's low − today's low
>
> if UpMove > DownMove and UpMove > 0, then +DM = UpMove, else +DM = 0
>
> if DownMove > UpMove and DownMove > 0, then −DM = DownMove, else −DM = 0

After selecting the number of periods (Wilder used 14 days originally), +DI and −DI are:

> +DI = 100 times exponential moving average of +DM divided by average true range
>
> −DI = 100 times exponential moving average of −DM divided by average true range

The exponential moving average is calculated over the number of periods selected, and the average true range is an exponential average of the true ranges. Then:

> ADX = 100 times the exponential moving average of the absolute value of (+DI − −DI) divided by (+DI + −DI)

Variations of this calculation typically involve using different types of moving averages, such as a weighted moving average or an adaptive moving average.

Interpretation

The ADX does not indicate trend direction, only trend strength. It is a lagging indicator; that is, a trend must have established itself before the ADX will generate a signal that a trend is under way. ADX will range between 0 and 100. Generally, ADX readings below 20 indicate trend weakness, and readings above 40 indicate trend strength. An extremely strong trend is indicated by readings above 50.

References

[1] J. Welles Wilder, Jr. (June 1978). *New Concepts in Technical Trading Systems*. Greensboro, NC: Trend Research. ISBN 978-0894590276.

[2] Michael D. Sheimo (1998). *Cashing in on the Dow: using Dow theory to trade and determine trends in today's markets* (http://books.google.com/books?id=jDlZNrcs1RkC&pg=PA87). CRC Press. p. 87. ISBN 9780910944069. .

Average true range

Average true range (**ATR**) is a technical analysis volatility indicator originally developed by J. Welles Wilder, Jr. for commodities.[1] The indicator does not provide an indication of price trend, simply the degree of price volatility.[2][3] The average true range is an N-day exponential moving average of the **true range** values. Wilder recommended a 14-period smoothing.[4]

Calculation

The range of a day's trading is simply $\text{high} - \text{low}$. The **true range** extends it to yesterday's closing price if it was outside of today's range.

$$\text{true range} = \max[(\text{high} - \text{low}), \text{abs}(\text{high} - \text{close}_{\text{prev}}), \text{abs}(\text{low} - \text{close}_{\text{prev}})]$$

The **true range** is the largest of the:

- Most recent period's high minus the most recent period's low
- Absolute value of the most recent period's high minus the previous close
- Absolute value of the most recent period's low minus the previous close

The idea of ranges is that they show the commitment or enthusiasm of traders. Large or increasing ranges suggest traders prepared to continue to bid up or sell down a stock through the course of the day. Decreasing range suggests waning interest.

Applicability to futures contracts vs. stocks

Since true range and ATR are calculated by subtracting prices, the volatility they compute does not change when historical prices are backadjusted by adding or subtracting a constant to every price. Backadjustments are often employed when splicing together individual monthly futures contracts to form a continuous futures contract spanning a long period of time. However the standard procedures used to compute volatility of stock prices, such as the standard deviation of logarithmic price ratios, are not invariant to (addition of a constant). Thus futures traders and analysts typically use one method (ATR) to calculate volatility, while stock traders and analysts typically use another (SD of log price ratios).

References

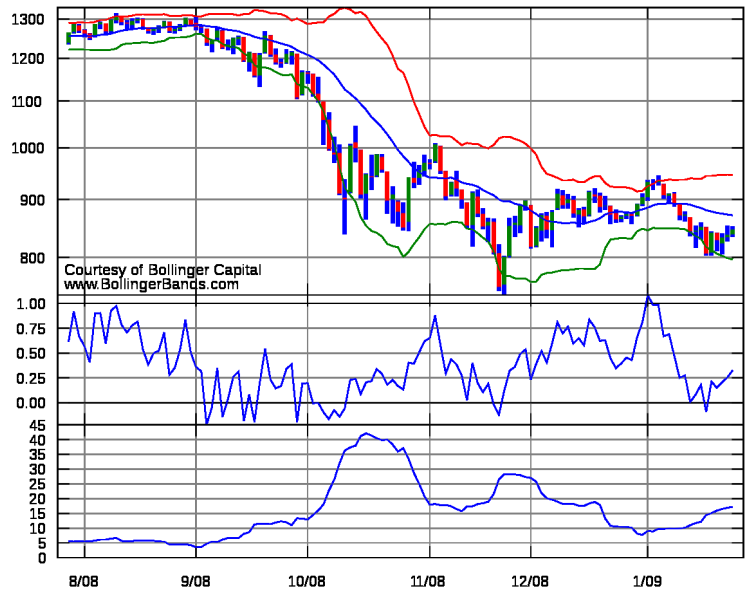
[1] J. Welles Wilder, Jr. (June 1978). *New Concepts in Technical Trading Systems*. Greensboro, NC: Trend Research. ISBN 978-0-89459-027-6.
[2] ATR Definition - investopedia.com (http://www.investopedia.com/terms/a/atr.asp)
[3] Joel G. Siegel (2000). *International encyclopedia of technical analysis* (http://books.google.com/books?id=IjuoD5yJs8YC&pg=PA341). Global Professional Publishing. p. 341. ISBN 978-1-888998-88-7. .
[4] This is by his reckoning of EMA periods, meaning an $\alpha=2/(1+14)=0.1333$.

External links

- *Measure Volatility With Average True Range* (http://www.investopedia.com/articles/trading/08/average-true-range.asp) at investopedia.com
- *Enter Profitable Territory With Average True Range* (http://www.investopedia.com/articles/trading/08/ATR.asp) at investopedia.com
- Average True Range (ATR) (http://stockcharts.com/help/doku.php?id=chart_school:technical_indicators:average_true_range_a) at stockcharts.com
- Average True Range (ATR) (http://www.surefiretradingchallenge.com/average_true_range.html) at surefiretradingchallenge.com
- ATR-Filtered SMA System (http://www.surefiretradingchallenge.com/average_true_range.html) at surefiretradingchallenge.com

Bollinger Bands

Bollinger Bands is a technical analysis tool invented by John Bollinger in the 1980s, and a term trademarked by him in 2011.[1] Having evolved from the concept of trading bands, Bollinger Bands and the related indicators *%b* and *bandwidth* can be used to measure the highness or lowness of the price relative to previous trades.



S&P 500 with 20-day, two-standard-deviation Bollinger Bands, %b and bandwidth.

Bollinger Bands consist of:

- an *N*-period moving average (MA)
- an upper band at *K* times an *N*-period standard deviation above the moving average (MA + $K\sigma$)
- a lower band at *K* times an *N*-period standard deviation below the moving average (MA − $K\sigma$)

Typical values for *N* and *K* are 20 and 2, respectively. The default choice for the average is a simple moving average, but other types of averages can be employed as needed. Exponential moving averages are a common second choice.[2] Usually the same period is used for both the middle band and the calculation of standard deviation.[3]

Purpose

The purpose of Bollinger Bands is to provide a relative definition of high and low. By definition, prices are high at the upper band and low at the lower band. This definition can aid in rigorous pattern recognition and is useful in comparing price action to the action of indicators to arrive at systematic trading decisions.[4]

Indicators derived from Bollinger Bands

In Spring of 2010, John Bollinger introduced three new indicators based on Bollinger Bands. They are BBImpulse, which measures price change as a function of the bands, percent bandwidth (%b), which normalizes the width of the bands over time, and bandwidth delta, which quantifies the changing width of the bands.

%b (pronounced "percent b") is derived from the formula for Stochastics and shows where price is in relation to the bands. *%b* equals 1 at the upper band and 0 at the lower band. Writing *upperBB* for the upper Bollinger Band, *lowerBB* for the lower Bollinger Band, and *last* for the last (price) value:

$$\%b = (\text{last} - \text{lowerBB}) / (\text{upperBB} - \text{lowerBB})$$

Bandwidth tells how wide the Bollinger Bands are on a normalized basis. Writing the same symbols as before, and *middleBB* for the moving average, or middle Bollinger Band:

$$\text{Bandwidth} = (\text{upperBB} - \text{lowerBB}) / \text{middleBB}$$

Using the default parameters of a 20-period look back and plus/minus two standard deviations, *bandwidth* is equal to four times the 20-period coefficient of variation.

Uses for *%b* include system building and pattern recognition. Uses for *bandwidth* include identification of opportunities arising from relative extremes in volatility and trend identification.

Interpretation

The use of Bollinger Bands varies widely among traders. Some traders buy when price touches the lower Bollinger Band and exit when price touches the moving average in the center of the bands. Other traders buy when price breaks above the upper Bollinger Band or sell when price falls below the lower Bollinger Band.[5] Moreover, the use of Bollinger Bands is not confined to stock traders; options traders, most notably implied volatility traders, often sell options when Bollinger Bands are historically far apart or buy options when the Bollinger Bands are historically close together, in both instances, expecting volatility to revert back towards the average historical volatility level for the stock.

When the bands lie close together a period of low volatility in stock price is indicated. When they are far apart a period of high volatility in price is indicated. When the bands have only a slight slope and lie approximately parallel for an extended time the price of a stock will be found to oscillate up and down between the bands as though in a channel.

Traders are often inclined to use Bollinger Bands with other indicators to see if there is confirmation. In particular, the use of an oscillator like Bollinger Bands will often be coupled with a non-oscillator indicator like chart patterns or a trendline; if these indicators confirm the recommendation of the Bollinger Bands, the trader will have greater evidence that what the bands forecast is correct.

Effectiveness

A recent study concluded that Bollinger Band trading strategies may be effective in the Chinese marketplace, stating: "Finally, we find significant positive returns on buy trades generated by the contrarian version of the moving-average crossover rule, the channel breakout rule, and the Bollinger Band trading rule, after accounting for transaction costs of 0.50 percent." Nauzer J. Balsara, Gary Chen and Lin Zheng *The Chinese Stock Market: An Examination of the Random Walk Model and Technical Trading Rules*.[6] (By "the contrarian version", they mean buying when the conventional rule mandates selling, and vice versa.)

A paper by Rostan, Pierre, Théoret, Raymond and El moussadek, Abdeljalil from 2008 at SSRN uses Bollinger Bands in forecasting the yield curve.[7]

Companies like Forbes suggest that the use of Bollinger Bands is a simple and often an effective strategy but stop-loss orders should be used to mitigate losses from market pressure.[8]

Statistical properties

Security price returns have no known statistical distribution, normal or otherwise; they are known to have fat tails, compared to a normal distribution.[9] The sample size typically used, 20, is too small for conclusions derived from statistical techniques like the central limit theorem to be reliable. Such techniques usually require the sample to be independent and identically distributed which is not the case for a time series like security prices. In point of fact, just the opposite is true; it is well recognized by practitioners that such price series are very commonly serially correlated – that is, it is the case that the next price will be closely related to its ancestor 'most of the time'.

For these three principal reasons, it is incorrect to assume that the percentage of the data that will be observed in the future outside the Bollinger Bands range will always be constrained to a certain amount. Instead of finding about 95% of the data inside the bands, as would be the expectation with the default parameters if the data were normally distributed, studies have found that only about 88% of security prices remain within the bands.[10] Practitioners may also use related measures such as the Keltner channels, or the related Stoller average range channels, which base their band widths on different measures of price volatility, such as the difference between daily high and low prices, rather than on standard deviation, which is a statistical measure more appropriate to normal distributions.[10]

Bollinger bands outside of finance

In a paper published in 2006 by the Society of Photo-Optical Engineers, "Novel method for patterned fabric inspection using Bollinger bands", Henry Y. T. Ngan and Grantham K. H. Pang present a method of using Bollinger bands to detect defects in patterned fabrics. From the abstract: "In this paper, the upper band and lower band of Bollinger Bands, which are sensitive to any subtle change in the input data, have been developed for use to indicate the defective areas in patterned fabric."[11]

The International Civil Aviation Organization is using Bollinger bands to measure the accident rate as a safety indicator to measure efficiency of global safety initiatives.[12] %b and bandwidth are also used in this analysis.

Notes

[1] "Bollinger Bands – Trademark Details" (http://trademarks.justia.com/852/32/bollinger-bands-85232573.html). Justia.com. 2011-12-20. .

[2] When the average used in the calculation of Bollinger Bands is changed from a simple moving average to an exponential or weighted moving average, it must be changed for both the calculation of the middle band and the calculation of standard deviation.Bollinger On Bollinger Bands – The Seminar, DVD I ISBN 978-0-9726111-0-7

[3] Bollinger Bands use the population method of calculating standard deviation, thus the proper divisor for the sigma calculation is n, not $n - 1$.

[4] (http://www.bollingerbands.com) second paragraph, center column

[5] Technical Analysis: The Complete Resource for Financial Market Technicians by Charles D. Kirkpatrick and Julie R. Dahlquist Chapter 14

[6] (http://findarticles.com/p/articles/mi_qa5466/is_200704/ai_n21292807/pg_1?tag=artBody;col1) The Quarterly Journal of Business and Economics, Spring 2007

[7] (http://papers.ssrn.com/sol3/papers.cfm?abstract_id=671581) Forecasting the Interest-Rate Term Structure at SSRN

[8] Bollinger Band Trading John Devcic 05.11.07 (http://www.forbes.com/2007/05/11/bollinger-intel-yahoo-pf-education-in_jd_0511chartroom_inl.html)

[9] Rachev; Svetlozar T., Menn, Christian; Fabozzi, Frank J. (2005), Fat Tailed and Skewed Asset Return Distributions, Implications for Risk Management, Portfolio Selection, and Option Pricing, John Wiley, New York

[10] Adam Grimes (2012). *The Art & Science of Technical Analysis: Market Structure, Price Action & Trading Strategies* (http://books.google.com/books?id=Yzs_0ZkXnyAC&pg=PA196). John Wiley & Sons. p. 196. ISBN 9781118224274. .

[11] (http://spiedl.aip.org/getabs/servlet/GetabsServlet?prog=normal&id=OPEGAR000045000008087202000001&idtype=cvips&gifs=yes) Optical Engineering, Volume 45, Issue 8

[12] (http://www.skybrary.aero/index.php/ICAO_Methodology_for_Accident_Rate_Calculation_and_Trending) ICAO Methodology for Accident Rate Calculation and Trending on SKYbary

References

Further reading

- Achelis, Steve. *Technical Analysis from A to Z* (pp. 71–73). Irwin, 1995. ISBN 978-0-07-136348-8
- Bollinger, John. *Bollinger on Bollinger Bands*. McGraw Hill, 2002. ISBN 978-0-07-137368-5
- Cahen, Philippe. *Dynamic Technical Analysis*. Wiley, 2001. ISBN 978-0-471-89947-1
- Kirkpatrick, Charles D. II; Dahlquist, Julie R. *Technical Analysis: The Complete Resource for Financial Market Technicians*, FT Press, 2006. ISBN 0-13-153113-1
- Murphy, John J. *Technical Analysis of the Financial Markets* (pp. 209–211). New York Institute of Finance, 1999. ISBN 0-7352-0066-1

External links

- John Bollinger's website (http://www.bollingerbands.com)
- John Bollinger's website on Bollinger Band analysis (http://www.bollingeronbollingerbands.com)
- December 2008 Los Angeles Times profile of John Bollinger (http://www.latimes.com/business/la-fi-himi7-2008dec07,0,1338099.story)

Commodity channel index

The **commodity channel index** (**CCI**) is an oscillator originally introduced by Donald Lambert in an article published in the October 1980 issue of *Commodities* magazine (now known as *Futures* magazine).

Since its introduction, the indicator has grown in popularity and is now a very common tool for traders in identifying cyclical trends not only in commodities, but also equities and currencies. The CCI can be adjusted to the timeframe of the market traded on by changing the averaging period.

Calculation

CCI measures a security's variation from the statistical mean.[1]

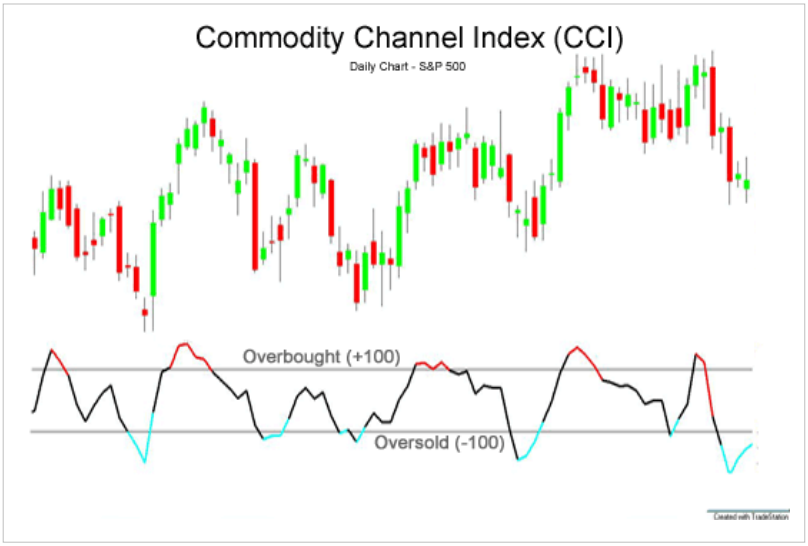
The CCI is calculated as the difference between the typical price of a commodity and its simple moving average, divided by the mean absolute deviation of the typical price. The index is usually scaled by an inverse factor of 0.015 to provide more readable numbers:

$$CCI = \frac{1}{0.015}\frac{p_t - SMA(p_t)}{\sigma(p_t)},$$

where the p_t is the $\text{Typical Price} = \frac{H+L+C}{3}$, SMA is the simple moving average, and σ is the mean absolute deviation.

For scaling purposes, Lambert set the constant at 0.015 to ensure that approximately 70 to 80 percent of CCI values would fall between −100 and +100. The CCI fluctuates above and below zero. The percentage of CCI values that fall between +100 and −100 will depend on the number of periods used. A shorter CCI will be more volatile with a smaller percentage of values between +100 and −100. Conversely, the more periods used to calculate the CCI, the higher the percentage of values between +100 and −100.

Interpretation



Traders and investors use the commodity channel index to help identify price reversals, price extremes and trend strength. As with most indicators, the CCI should be used in conjunction with other aspects of technical analysis. CCI fits into the momentum category of oscillators. In addition to momentum, volume indicators and the price chart may also influence a technical assessment. It is often used for detecting divergences from price trends as an overbought/oversold indicator, and to draw patterns on it and trade according to those patterns. In this respect, it is similar to bollinger bands, but is presented as an indicator rather than as overbought/oversold levels.

The CCI typically oscillates above and below a zero line. Normal oscillations will occur within the range of +100 and −100. Readings above +100 imply an overbought condition, while readings below −100 imply an oversold condition. As with other overbought/oversold indicators, this means that there is a large probability that the price will

correct to more representative levels.

The CCI has seen substantial growth in popularity amongst technical investors; today's traders often use the indicator to determine cyclical trends in not only commodities, but also equities and currencies.[2]

The CCI, when used in conjunction with other oscillators, can be a valuable tool to identify potential peaks and valleys in the asset's price, and thus provide investors with reasonable evidence to estimate changes in the direction of price movement of the asset.[2]

Lambert's trading guidelines for the CCI focused on movements above +100 and below −100 to generate buy and sell signals. Because about 70 to 80 percent of the CCI values are between +100 and −100, a buy or sell signal will be in force only 20 to 30 percent of the time. When the CCI moves above +100, a security is considered to be entering into a strong uptrend and a buy signal is given. The position should be closed when the CCI moves back below +100. When the CCI moves below −100, the security is considered to be in a strong downtrend and a sell signal is given. The position should be closed when the CCI moves back above −100.

Since Lambert's original guidelines, traders have also found the CCI valuable for identifying reversals. The CCI is a versatile indicator capable of producing a wide array of buy and sell signals.

- CCI can be used to identify overbought and oversold levels. A security would be deemed oversold when the CCI dips below −100 and overbought when it exceeds +100. From oversold levels, a buy signal might be given when the CCI moves back above −100. From overbought levels, a sell signal might be given when the CCI moved back below +100.
- As with most oscillators, divergences can also be applied to increase the robustness of signals. A positive divergence below −100 would increase the robustness of a signal based on a move back above −100. A negative divergence above +100 would increase the robustness of a signal based on a move back below +100.
- Trend line breaks can be used to generate signals. Trend lines can be drawn connecting the peaks and troughs. From oversold levels, an advance above −100 and trend line breakout could be considered bullish. From overbought levels, a decline below +100 and a trend line break could be considered bearish.[3]

References

[1] AsiaPacFinance.com Trading Indicator Glossary (http://www.asiapacfinance.com/trading-strategies/technicalindicators/CommodityChannelIndex)

[2] Commodity channel index (http://www.investopedia.com/terms/c/commoditychannelindex.asp) on Investopedia

[3] Commodity Channel Index (CCI) (http://stockcharts.com/school/doku.php?id=chart_school:technical_indicators:commodity_channel_in) on StockCharts.com − ChartSchool

External links

- Commodities & Charts Blog post on CCI (http://cmd-chart.blogspot.com/2007/04/cci-index.html)
- Commodities & Charts Blog images on CCI (http://cmd-chart.blogspot.com/2007/06/cci-commodities-index.html)

Detrended price oscillator

MACD

The **detrended price oscillator** (**DPO**) is an indicator in technical analysis that attempts to eliminate the long-term trends in prices by using a displaced moving average so it does not react to the most current price action. This allows the indicator to show intermediate overbought and oversold levels effectively.

The detrended price oscillator is a form of price oscillator, like the "percentage price oscillator" (PPO) and the "absolute price oscillator" (APO) both of which are forms of Gerald Appel's MACD indicator. The APO is an equivalent to the MACD indicator while the PPO is an improved alternative to the APO or the MACD for use when a stock's price change has been large, or when comparing the oscillator behavior for different stocks which have significantly different prices.

Although these are not so commonly used with the DPO, for the other price oscillators, as for the MACD, a signal line is frequently generated for the price oscillators by taking an exponential moving average (EMA) of the price oscillator values and plotting the two lines together. A histogram can also be generated for the price oscillators, if desired, just as is done for the MACD indicator.

The DPO is calculated by subtracting the simple moving average over an "n" day period and shifted n/2+1 days back from the price.

To calculate the detrended price oscillator:

Decide on the time frame that you wish to analyze. Set "n" as half of that cycle period.

Calculate a simple moving average for n periods.

Calculate (n / 2 + 1)

Subtract the moving average, from (n / 2 + 1) days ago, from the closing price:

DPO = Close - Simple moving average [from (n / 2 + 1) days ago

Donchian channel

The **Donchian channel** is an indicator used in market trading developed by Richard Donchian. It is formed by taking the highest high and the lowest low of the last *n* periods. The area between the high and the low is the channel for the period chosen.

It is commonly available on most trading platforms. On a charting program, a line is marked for the high and low values visually demonstrating the channel on the markets price (or other) values.

The Donchian channel is a useful indicator for seeing the volatility of a market price. If a price is stable the Donchian channel will be relatively narrow. If the price fluctuates a lot the Donchian channel will be wider. Its primary use, however, is for providing signals for long and short positions. If a security trades above its highest *n* periods high, then a long is established. If it trades below its lowest *n* periods low, then a short is established.

Originally the *n* periods were based upon daily values. With today's trading platforms, the period may be of the value desired by the investor. i.e.: day, hour, minute, ticks, etc.

External links

- Capture Profits using Bands and Channels [1]

References

[1] http://www.investopedia.com/articles/forex/06/BandsChannels.asp

Ease of movement

Ease of movement (**EMV**) is an indicator used in technical analysis to relate an asset's price change to its volume. Ease of Movement was developed by Richard W. Arms, Jr and highlights the relationship between volume and price changes and is particularly useful for assessing the strength of a trend.[1] High positive values indicate the price is increasing on low volume: strong negative values indicate the price is dropping on low volume. The moving average of the indicator can be added to act as a trigger line, which is similar to other indicators like the MACD.

References

[1] *Arms Ease of Movement* (http://www.oxfordfutures.com/futures-education/tech/ease-of-movement.htm), , retrieved 17 February 2008

Force index

The **force index** (**FI**) is an indicator used in technical analysis to illustrate how strong the actual buying or selling pressure is. High positive values mean there is a strong rising trend, and low values signify a strong downward trend.

The FI is calculated by multiplying the difference between the last and previous closing prices by the volume of the commodity, yielding a momentum scaled by the volume. The strength of the force is determined by a larger price change or by a larger volume.[1]

The FI was created by Alexander Elder.[2]

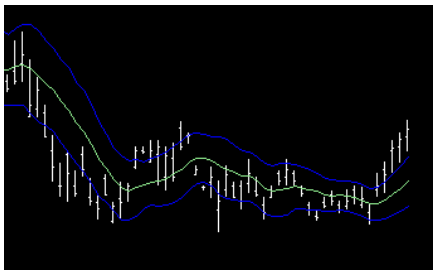
References

[1] Logue, Ann C. (2011). *Day Trading For Dummies* (http://books.google.com/books?id=tpOdM9LsAtcC&pg=PT196). Wiley. p. 196. ISBN 9781118051818. .

[2] Elder, Alexander (1993). *Trading for a Living: Psychology, Trading Tactics, Money Management*. Wiley. ISBN 0-4715-9224-2.

Keltner channel

Keltner channel is a technical analysis indicator showing a central moving average line plus channel lines at a distance above and below. The indicator is named after Chester W. Keltner (1909–1998) who described it in his 1960 book *How To Make Money in Commodities*. This name was applied by those who heard about it from him, but Keltner called it the **ten-day moving average trading rule** and indeed made no claim to any originality for the idea.



Keltner channel example

In Keltner's description the centre line is a 10-day simple moving average of *typical price*, where typical price each day is the average of high, low and close,

$$typical\ price = \frac{high + low + close}{3}$$

The lines above and below are drawn a distance from that centre line, a distance which is the simple moving average of the past 10 days' trading ranges (i.e. range high to low on each day).

The trading strategy is to regard a close above the upper line as a strong bullish signal, or a close below the lower line as strong bearish sentiment, and buy or sell with the trend accordingly, but perhaps with other indicators to confirm.

The origin of this idea is uncertain. Keltner was a Chicago grain trader and perhaps it was common knowledge among traders of the day. Or in the 1930s as a young man Keltner worked for Ralph Ainsworth (1884–1965) backtesting trading systems submitted when Ainsworth offered a substantial prize for a winning strategy, so it could have been among those. But ideas of channels with fixed widths go back to the earliest days of charting, so perhaps applying some averaging is not an enormous leap in any case.

Later authors, such as Linda Bradford Raschke, have published modifications for the Keltner channel, such as different averaging periods; or an exponential moving average; or using a multiple of Wilder's average true range (ATR) for the bands. These variations have merit, but are often still just called Keltner channel, creating some confusion as to what exactly one gets from an indicator called that.

References

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KST oscillator

The **know sure thing (KST) oscillator** is a complex, smoothed price velocity indicator developed by Martin J. Pring.[1][2]

A rate of change (ROC) indicator is the foundation of KST indicator. KST indicator is useful to identify major stock market cycle junctures because its formula is weighed to be more greatly influenced by the longer and more dominant time spans, in order to better reflect the primary swings of stock market cycle.[3] The concept behind the oscillator is that price trends are determined by the interaction of many different time cycles and that important trend reversals take place when a number of price trends are simultaneously changing direction.

Formula

Four different rates of change are calculated, smoothed, multiplied by weights and then summed to form one indicator.[4]

$$ROC1 = (1 - Price/Price(X1)) * 100;$$
$$ROC2 = (1 - Price/Price(X2)) * 100;$$
$$ROC3 = (1 - Price/Price(X3)) * 100;$$
$$ROC4 = (1 - Price/Price(X4)) * 100;$$

Where price refers to current closing price and price(X1) refers to the closing price X1 bars ago.

$$KST = MOV(ROC1, AVG1) * W1 + MOV(ROC2, AVG2) * W2 + MOV(ROC3, AVG3) * W3 + MOV(ROC4, AVG4) * W4$$

Where MOV(ROC1,AVG1) refers to the AVG1 day moving average for ROC1

For short-term trend, Martin J Pring suggest the following parameters:

X1 = 10

X2 = 15

X3 = 20

X4 = 30

AVG1 = 10

AVG2 = 10

AVG3 = 10

AVG4 = 15

W1 = 1

W2 = 2

W3 = 3

W4 = 4

The formula is built into, or can be included into various technical analysis softwares like MetaStock[5] or OmniTrader.

Implications

Entry rules KST Indicator

When KST crosses below its 9 day exponential average, short at the next day opening price.

Exit rules KST indicator

When KST crosses above its 9 day exponential average, close short position at the next day opening price.[4]

Variations

It can be calculated on **daily**[6] or **long term**[7] basis.

The dominant time frame in the (KST)'s construction is a 24-month period, which is half of the 4-year business cycle. This means that the KST will work best when the security in question is experiencing a primary up- and downtrend based on the business cycle.[4]

KST interpretation

KST can be interpreted in the ways mentioned below.[6]

The dominant time frame in the Know Sure Thing (KST)'s construction is a 24-month period, which is half of the 4-year business cycle. This means that the Know Sure Thing (KST) will work best when the security in question is experiencing a primary up- and downtrend based on the business cycle.

Directional changes and moving average crossovers

You've discovered how changes in direction are the way the KST triggers signals, but also that moving-average crossovers offer less timely, but more reliable signals. The average to use is a simple 10-day moving average. It is possible to anticipate a moving average crossover if the KST has already turned and the price violates a trendline. The KST started to reverse to the downside before the up trendline was violated. Since either a reversal or a trading range follow a valid trendline violation, it's evident that upside momentum has temporarily dissipated, causing the KST to cross below its moving average.

Traditionally, the MACD gives buy and sell signals when it crosses above and below its exponential moving average, known as the "signal line". This approach isn't perfect; the ellipses on the chart highlight all the whipsaws. As said earlier, the KST can also give false or misleading signals, as you can see from the April 2005 buy signal. It comes close to a couple of whipsaws, but by and large, it's more accurate, even though the MACD often turns faster than the KST.

Overbought/oversold and divergences

The concept is that when the indicator crosses above and below the overbought/oversold zones, momentum buy and sell signals are triggered. Even so, you must wait for some kind of trend reversal signal in the price, such as a price pattern completion, trendline violation, or similar.

The KST often diverges positively and negatively with the price.

Trendline violations and price pattern completions

It is possible to construct a trendline on the KST and see when it's been violated, but not very often. When it does though, it usually results in a powerful signal.

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[4] "KST(Know sure thing) indicator in stock market" (http://abc-stock-market.blogspot.com/2009/06/kst-know-sure-thing-indicator-in-stock.html). http://abc-stock-market.blogspot.com. . Retrieved 2010-11-10.

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MACD

MACD (moving average convergence/divergence) is a technical analysis indicator created by Gerald Appel in the late 1970s.[1] It is used to spot changes in the strength, direction, momentum, and duration of a trend in a stock's price.

The MACD "oscillator" or "indicator" is a collection of three signals (or computed data-series), calculated from historical price data, most often the closing price. These three signal lines are: the MACD line, the signal line (or average line), and the difference (or divergence). The term "MACD" may be used to refer to the indicator as a whole, or specifically to the MACD line itself. The first line, called the "MACD line", equals the difference between a "fast" (short period) exponential moving average (EMA), and a "slow" (longer period) EMA. The MACD line is charted over time, along with an EMA of the MACD line, termed the "signal line" or "average line". The difference (or divergence) between the MACD line and the signal line is shown as a bar graph called the "histogram" time series (which should not be confused with the normal usage of histogram as an approximation of a probability distribution in statistics - the commonality is just in the visualization using a bar graph).

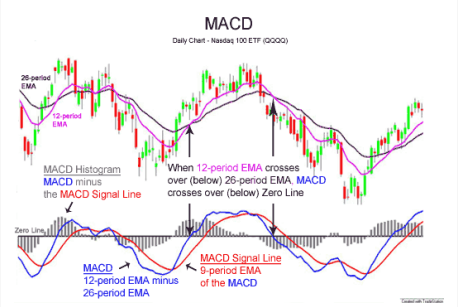
A fast EMA responds more quickly than a slow EMA to recent changes in a stock's price. By comparing EMAs of different periods, the MACD line can indicate changes in the trend of a stock. By comparing that difference to an average, an analyst can detect subtle shifts in the stock's trend.

Since the MACD is based on moving averages, it is inherently a lagging indicator. However, in this regard the MACD does not lag as much as a basic moving average crossing indicator, since the signal cross can be anticipated by noting the convergence far in advance of the actual crossing. As a metric of price trends, the MACD is less useful for stocks that are not trending (trading in a range) or are trading with erratic price action.

History

The MACD was invented by Gerald Appel in the 1970s. Thomas Aspray added a histogram to the MACD in 1986, as a means to anticipate MACD crossovers, an indicator of important moves in the underlying security.

Basic components



MACD 12,26,9

The graph above shows a stock with a MACD indicator underneath it. The indicator shows a blue line, a red line, and a histogram or bar chart which calculates the difference between the two lines. Values are calculated from the price of the stock in the main part of the graph.

For the example above this means:

- MACD line (blue line): difference between the 12 and 26 days EMAs
- signal (red line): 9 day EMA of the blue line
- histogram (bar graph): difference between the blue and red lines

Mathematically:

- MACD = EMA[stockPrices,12] − EMA[stockPrices,26]
- signal = EMA[MACD,9]
- histogram = MACD − signal

The period for the moving averages on which an MACD is based can vary, but the most commonly used parameters involve a faster EMA of 12 days, a slower EMA of 26 days, and the signal line as a 9 day EMA of the difference between the two. It is written in the form, MACD (faster, slower, signal) or in this case, MACD(12,26,9).

Interpretation

Exponential moving averages highlight recent changes in a stock's price. By comparing EMAs of different lengths, the MACD line gauges changes in the trend of a stock. By then comparing differences in the change of that line to an average, an analyst can identify subtle shifts in the strength and direction of a stock's trend.

Traders recognize three meaningful signals generated by the MACD indicator.

When:

- the MACD line crosses the signal line
- the MACD line crosses zero
- there is a divergence between the MACD line and the price of the stock or between the histogram and the price of the stock

Graphically this corresponds to:

- the blue line crossing the red line
- the blue line crossing the x-axis (the straight black line in the middle of the indicator)

- higher highs (lower lows) on the price graph but not on the blue line, or higher highs (lower lows) on the price graph but not on the bar graph

And mathematically:

- $\text{MACD} - \text{signal} = 0$
- $\text{EMA}[\text{fast},12] - \text{EMA}[\text{slow},26] = 0$
- $\text{Sign}\,(\text{relative price extremum}_{\text{final}} - \text{relative price extremum}_{\text{initial}}) \neq \text{Sign}\,(\text{relative MACD extremum}_{\text{final}} - \text{MACD extremum}_{\text{initial}})$

Signal–line crossover

Signal–line crossovers are the primary cues provided by the MACD. The standard interpretation is to buy when the MACD line crosses up through the signal line, or sell when it crosses down through the signal line.

The upwards move is called a bullish crossover and the downwards move a bearish crossover. Respectively, they indicate that the trend in the stock is about to accelerate in the direction of the crossover.

The histogram shows when a crossing occurs. Since the histogram is the difference between the MACD line and the signal line, when they cross there is no difference between them.

The histogram can also help in visualizing when the two lines are approaching a crossover. Though it may show a difference, the changing size of the difference can indicate the acceleration of a trend. A narrowing histogram suggests a crossover may be approaching, and a widening histogram suggests that an ongoing trend is likely to get even stronger.

While it is theoretically possible for a trend to increase indefinitely, under normal circumstances, even stocks moving drastically will eventually slow down, lest they go up to infinity or down to nothing.

Zero crossover

A crossing of the MACD line through zero happens when there is no difference between the fast and slow EMAs. A move from positive to negative is bearish and from negative to positive, bullish. Zero crossovers provide evidence of a change in the direction of a trend but less confirmation of its momentum than a signal line crossover.

Timing

The MACD is only as useful as the context in which it is applied. An analyst might apply the MACD to a weekly scale before looking at a daily scale, in order to avoid making short term trades against the direction of the intermediate trend.[2] Analysts will also vary the parameters of the MACD to track trends of varying duration. One popular short-term set-up, for example, is the (5,35,5).

False signals

Like any stock market forecast, the MACD can generate false signals. A false positive, for example, would be a bullish crossover followed by a sudden decline in a stock. A false negative would be a situation where there was no bullish crossover, yet the stock accelerated suddenly upwards.

A prudent strategy would be to apply a filter to signal line crossovers to ensure that they will hold. An example of a price filter would be to buy if the MACD line breaks above the signal line and then remains above it for three days. As with any filtering strategy, this reduces the probability of false signals but increases the frequency of missed profit.

Analysts use a variety of approaches to filter out false signals and confirm true ones.

Oscillator classification

The MACD is an absolute price oscillator (APO), because it deals with the actual prices of moving averages rather than percentage changes. A percentage price oscillator (PPO), on the other hand, computes the difference between two moving averages of price divided by the longer moving average value.

While an APO will show greater levels for higher priced securities and smaller levels for lower priced securities, a PPO calculates changes relative to price. Subsequently, a PPO is preferred when: comparing oscillator values between different securities, especially those with substantially different prices; or comparing oscillator values for the same security at significantly different times, especially a security whose value has changed greatly.

A third member of the price oscillator family is the detrended price oscillator (DPO), which ignores long term trends while emphasizing short term patterns.

Signal processing theory

In signal processing terms, the MACD is a filtered measure of the derivative of the input (price) with respect to time. (The derivative is called "velocity" in technical stock analysis). MACD estimates the derivative as if it were calculated and then filtered by the two low-pass filters in series, multiplied by a "gain" equal to the difference in their time constants. It also can be seen to approximate the derivative as if it were calculated and then filtered by a single low pass exponential filter (EMA) with time constant equal to the sum of time constants of the two filters, multiplied by the same gain.[3] So, for the standard MACD filter time constants of 12 and 26 days, the MACD derivative estimate is filtered approximately by the equivalent of a low-pass EMA filter of 38 days. The time derivative estimate (per day) is the MACD value divided by 14.

The signal line is also a derivative estimate, with an additional low-pass filter in series for further smoothing (and additional lag). The difference between the MACD line and the signal (the "histogram") represents a measure of the second derivative of price with respect to time ("acceleration" in technical stock analysis). This estimate has the additional lag of the signal filter and an additional gain factor equal to the signal filter constant.

A MACD crossover of the signal line indicates that the direction of the acceleration is changing. The MACD line crossing zero suggests that the average velocity is changing direction.

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[2] Murphy, John (1999). *Technical Analysis of the Financial Markets*. Prentice Hall Press. pp. 252–255. ISBN 0-7352-0066-1.

[3] The MACD Approach to Derivative (Rate of Change) Estimation (http://gregstanleyandassociates.com/whitepapers/FaultDiagnosis/Filtering/MACD-approach/macd-approach.htm)

Mass index

The **mass index** is an indicator, developed by Donald Dorsey, used in technical analysis to predict trend reversals. It is based on the notion that there is a tendency for reversal when the price range widens, and therefore compares previous trading ranges (highs minus lows).

Mass index for a commodity is obtained[1] by calculating its exponential moving average over a 9 day period and the exponential moving average of this average (a "double" average), and summing the ratio of these two over a given amount of days (usually 25).

$$Mass = Sum[25]\ of\ \frac{EMA[9]\ of\ (high - low)}{EMA[9]\ of\ EMA[9]\ of\ (high - low)}$$

Generally the EMA and the re-smoothed EMA of EMA are fairly close, making their ratio is roughly 1 and the sum around 25.

According to Dorsey, a so-called "reversal bulge" is a probable signal of trend reversal (regardless of the trend's direction).[2] Such a bulge takes place when a 25-day mass index reaches 27.0 and then falls to below 26 (or 26.5). A 9-day prime moving average is usually used to determine whether the bulge is a buy or sell signal.

This formula uses intraday range values: not the "true range," which adjusts for full and partial gaps. Also, the "bulge" does not indicate direction.

References

[1] Mass Index construction (http://www.incrediblecharts.com/technical/mi_construction.htm) at IncredibleCharts.com
[2] Mass Index (http://www.incrediblecharts.com/technical/mass_index.htm) at IncredibleCharts.com

Momentum (technical analysis)

Momentum and **rate of change** (ROC) are simple technical analysis indicators showing the difference between today's closing price and the close N days ago. Momentum is the absolute difference in stock, commodity:

$$momentum = close_{today} - close_{N\ days\ ago}$$

Rate of change scales by the old close, so as to represent the increase as a fraction,

$$rate\ of\ change = \frac{close_{today} - close_{N\ days\ ago}}{close_{N\ days\ ago}}$$

"Momentum" in general refers to prices continuing to trend. The momentum and ROC indicators show **trend** by remaining positive while an uptrend is sustained, or negative while a downtrend is sustained.

A crossing up through zero may be used as a signal to buy, or a crossing down through zero as a signal to sell. How high (or how low when negative) the indicators get shows how strong the trend is.

The way momentum shows an absolute change means it shows for instance a $3 rise over 20 days, whereas ROC might show that as 0.25 for a 25% rise over the same period. One can choose between looking at a move in dollar terms, relative point terms, or proportional terms. The zero crossings are the same in each, of course, but the highs or lows showing strength are on the respective different bases.

The conventional interpretation is to use momentum as a trend-following indicator. This means that when the indicator peaks and begins to descend, it can be considered a sell signal. The opposite conditions can be interpreted when the indicator bottoms out and begins to rise.[1]

SMA

Momentum is the change in an N-day simple moving average (SMA) between yesterday and today, with a scale factor N+1, i.e.

$$\frac{momentum}{N+1} = SMA_{today} - SMA_{yesterday}$$

This is the slope or steepness of the SMA line, like a derivative. This relationship is not much discussed generally, but it's of interest in understanding the signals from the indicator.

When momentum crosses up through zero it corresponds to a trough in the SMA, and when it crosses down through zero it's a peak. How high (or low) momentum gets represents how steeply the SMA is rising (or falling).

The TRIX indicator is similarly based on changes in a moving average (a triple exponential in that case).

References

[1] AsiaPacFinance.com Trading Indicator Glossary (http://www.asiapacfinance.com/trading-strategies/technicalindicators/RateofChange)

Money flow index

Money flow index (MFI) is an oscillator calculated over an N-day period, ranging from 0 to 100, showing *money flow* on up days as a percentage of the total of up and down days. Money flow in technical analysis is typical price multiplied by volume, a kind of approximation to the dollar value of a day's trading.

The calculations are as follows. The *typical price* for each day is the average of high, low and close,

$$typical\ price = \frac{high + low + close}{3}$$

Money flow is the product of typical price and the volume on that day.

$$money\ flow = typical\ price \times volume$$

Totals of the money flow amounts over the given N days are then formed. *Positive money flow* is the total for those days where the typical price is higher than the previous day's typical price, and *negative money flow* where below. (If typical price is unchanged then that day is discarded.) A *money ratio* is then formed

$$money\ ratio = \frac{positive\ money\ flow}{negative\ money\ flow}$$

From which a *money flow index* ranging from 0 to 100 is formed,

$$MFI = 100 - \frac{100}{1 + money\ ratio}$$

This can be expressed equivalently as follows. This form makes it clearer how the MFI is a percentage,

$$MFI = 100 \times \frac{positive\ money\ flow}{positive\ money flow + negative\ money\ flow}$$

MFI is used as an oscillator. A value of 80 is generally considered overbought, or a value of 20 oversold. Divergences between MFI and price action are also considered significant, for instance if price makes a new rally high but the MFI high is less than its previous high then that may indicate a weak advance, likely to reverse.

It will be noted the MFI is constructed in a similar fashion to the relative strength index. Both look at up days against total up plus down days, but the scale, i.e. what is accumulated on those days, is volume (or dollar volume approximation rather) for the MFI, as opposed to price change amounts for the RSI.

It's important to be clear about what "money flow" means. It refers to dollar volume, i.e. the total value of shares traded. Sometimes finance commentators speak of money "flowing into" a stock, but that expression only refers to

the enthusiasm of buyers (obviously there's never any net money in or out, because for every buyer there's a seller of the same amount).

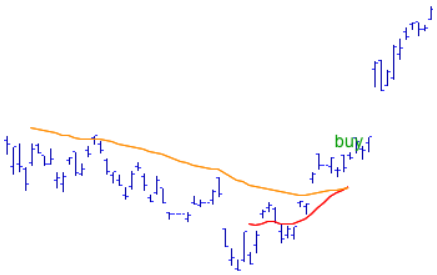
For the purposes of the MFI, "money flow", i.e. dollar volume, on an up day is taken to represent the enthusiasm of buyers, and on a down day to represent the enthusiasm of sellers. An excessive proportion in one direction or the other is interpreted as an extreme, likely to result in a price reversal.

Similar indicators

Other price × volume indicators:

- On-balance volume
- Price and volume trend
- Accumulation/distribution index

Moving average crossover



Moving average crossover of a 15-day exponential close-price MA crossing over a 50-day exponential close-price MA.

In the statistics of time series, and in particular the analysis of financial time series for stock trading purposes, a **moving-average crossover** occurs when, on plotting two moving averages each based on different degrees of smoothing, the traces of these moving averages cross. It does not predict future direction but shows trends. This indicator uses two (or more) moving averages, a slower moving average and a faster moving average. The faster moving average is a short term moving average. For end-of-day stock markets, for example, it may be 5, 10 or 25 day period while the slower moving average is medium or long term moving average (e.g. 50, 100 or 200 day period). A short term moving average is **faster** because it only considers prices over short period of time and is thus more reactive to daily price changes. On the other hand, a long term moving average is deemed **slower** as it encapsulates prices over a longer period and is more lethargic. However, it tends to smoothen out price noises which are often reflected in short term moving averages.

A moving average, as a line by itself, is often overlaid in price charts to indicate price trends. A crossover occurs when a faster moving average (i.e., a shorter period moving average) crosses a slower moving average (i.e. a longer period moving average). In other words, this is when the shorter period moving average line crosses a longer period moving average line. In stock investing, this meeting point is used either to enter (buy) or exit (sell) the market. In stock market investing, the profitability of the moving average crossover system is higher than most other stable forms of investment (e.g., bonds, bank interest).

The particular case where simple equally-weighted moving-averages are used in sometimes called a **simple moving-average (SMA) crossover**. Such a crossover can be used to signal a change in trend and can be used to trigger a trade in a Black Box trading system.

Notes

Negative volume index

Nearly 75 years have passed since Paul L. Dysart, Jr. invented the Negative Volume Index and Positive Volume Index indicators. The indicators remain useful to identify primary market trends and reversals.

In 1936, Paul L. Dysart, Jr. began accumulating two series of advances and declines distinguished by whether volume was greater or lesser than the prior day's volume. He called the cumulative series for the days when volume had been greater than the prior day's volume the Positive Volume Index (PVI), and the series for the days when volume had been lesser the Negative Volume Index (NVI).

A native of Iowa, Dysart worked in Chicago's LaSalle Street during the 1920s. After giving up his Chicago Board of Trade membership, he published an advisory letter geared to short-term trading using advance-decline data. In 1933, he launched the *Trendway* weekly stock market letter and published it until 1969 when he died. Dysart also developed the 25-day Plurality Index, the 25-day total of the absolute difference between the number of advancing issues and the number of declining issues, and was a pioneer in using several types of volume of trading studies. Richard Russell, editor of Dow Theory Letters, in his January 7, 1976 letter called Dysart "one of the most brilliant of the pioneer market technicians."

Dysart's NVI and PVI

The daily volume of the New York Stock Exchange and the NYSE Composite Index's advances and declines drove Dysart's indicators. Dysart believed that "volume is the driving force in the market." He began studying market breadth numbers in 1931, and was familiar with the work of Leonard P. Ayres and James F. Hughes, who pioneered the tabulation of advances and declines to interpret stock market movements.

Dysart calculated NVI as follows: 1) if today's volume is less than yesterday's volume, subtract declines from advances, 2) add the difference to the cumulative NVI beginning at zero, and 3) retain the current NVI reading for the days when volume is greater than the prior day's volume. He calculated PVI in the same manner but for the days when volume was greater than the prior day's volume. NVI and PVI can be calculated daily or weekly.

Initially, Dysart believed that PVI would be the more useful series, but in 1967, he wrote that NVI had "proved to be the most valuable of all the breadth indexes." He relied most on NVI, naming it AMOMET, the acronym of "**A** **M**easure **O**f **M**ajor **E**conomic **T**rend."

Dysart's theory, expressed in his 1967 Barron's article, was that "if volume advances and prices move up or down in accordance [with volume], the move is assumed to be a good movement - if it is sustained when the volume subsides." In other words, after prices have moved up on positive volume days, "if prices stay up when the volume subsides for a number of days, we can say that such a move is 'good'." If the market "holds its own on negative volume days after advancing on positive volume, the market is in a strong position."

He called PVI the "majority" curve. Dysart distinguished between the actions of the "majority" and those of the "minority." The majority tends to emulate the minority, but its timing is not as sharp as that of the minority. When the majority showed an appetite for stocks, the PVI was usually "into new high ground" as happened in 1961.

It is said that the two indicators assume that "smart" money is traded on quiet days (low volume) and that the crowd trades on very active days. Therefore, the negative volume index picks out days when the volume is lower than on the previous day, and the positive index picks out days with a higher volume.

Dysart's Interpretation of NVI and PVI

Besides an article he wrote for Barron's in 1967, not many of Dysart's writings are available. What can be interpreted about Dysart's NVI is that whenever it rises above a prior high, and the DJIA is trending up, a "Bull Market Signal" is given. When the NVI falls below a prior low, and the DJIA is trending down, a "Bear Market Signal" is given. The PVI is interpreted in reverse. However, not all movements above or below a prior NVI or PVI level generate signals, as Dysart also designated "bullish" and "bearish penetrations." These penetrations could occur before or after a Bull or Bear Market Signal, and at times were called "reaffirmations" of a signal. In 1969, he articulated one rule: "signals are most authentic when the NVI has moved sideways for a number of months in a relatively narrow range." Dysart cautioned that "there is no mathematical system devoid of judgment which will continuously work without error in the stock market."

According to Dysart, between 1946 and 1967, the NVI "rendered 17 significant signals," of which 14 proved to be right (an average of 4.32% from the final high or low) and 3 wrong (average loss of 6.33%). However, NVI "seriously erred" in 1963-1964 and in 1968, which concerned him. In 1969, Dysart reduced the weight he had previously given to the NVI in his analyses because NVI was no longer a "decisive" indicator of the primary trend, although it retained an "excellent ability to give us 'leading' indications of short-term trend reversals."

A probable reason for the NVI losing its efficacy during the mid-1960s may have been the steadily higher NYSE daily volume due to the dramatic increase in the number of issues traded so that prices rose on declining volume. Dysart's NVI topped out in 1955 and trended down until at least 1968, although the DJIA moved higher during that period. Norman G. Fosback has attributed the "long term increase in the number of issues traded" as a reason for a downward bias in a cumulative advance-decline line. Fosback was the next influential technician in the story of NVI and PVI.

Fosback's Variations

Fosback studied NVI and PVI and in 1976 reported his findings in his classic Stock Market Logic. He did not elucidate on the indicators' background or mentioned Dysart except for saying that "in the past Negative Volume Indexes have always [his emphasis] been constructed using advance-decline data…." He posited, "There is no good reason for this fixation on the A/D Line. In truth, a Negative Volume Index can be calculated with any market index - the Dow Jones Industrial Average, the S&P 500, or even 'unweighted' market measures…. Somehow this point has escaped the attention of technicians to date."

The point had not been lost on Dysart, who wrote in Barron's, "we prefer to use the issues-traded data [advances and declines] rather than the price data of any average because it is more all-encompassing, and more truly represents what's happening in the entire market." Dysart was a staunch proponent of using advances and declines.

Fosback made three variations to NVI and PVI:

1. He cumulated the daily percent change in the market index rather than the difference between advances and declines. On negative volume days, he calculated the price change in the index from the prior day and added it to the most recent NVI. His calculations are as follows:

If C_t and C_y denote the closing prices of today and yesterday, respectively, the NVI for today is calculated by

- adding $NVI_{yesterday}$ $(C_t - C_y) / C_y$ to yesterday's NVI if today's volume is lower than yesterday's, adding zero otherwise,

and the PVI is calculated by:

- adding $PVI_{yesterday}$ $(C_t - C_y) / C_y$ to yesterday's PVI if today's volume is higher than yesterday's, adding zero otherwise.

2. He suggested starting the cumulative count at a base index level such as 100.

3. He derived buy or sell signals by whether the NVI or PVI was above or below its one-year moving average.

Fosback's versions of NVI and PVI are what are popularly described in books and posted on Internet financial sites. Often reported are his findings that whenever NVI is above its one-year moving average there is a 96% (PVI - 79%) probability that a bull market is in progress, and when it is below its one-year moving average, there is a 53% (PVI - 67%) probability that a bear market is in place. These results were derived using a 1941-1975 test period. Modern tests might reveal different probabilities.

Today, NVI and PVI are commonly associated with Fosback's versions, and Dysart, their inventor, is forgotten. It cannot be said that one version is better than the other. While Fosback provided a more objective interpretation of these indicators, Dysart's versions offer value to identify primary trends and short-term trend reversals.

Although some traders use Fosback's NVI and PVI to analyze individual stocks, the indicators were created to track, and have been tested, on major market indexes. NVI was Dysart's most invaluable breadth index, and Fosback found that his version of "the Negative Volume Index is an excellent indicator of the primary market trend." Traders can benefit from both innovations.

References

- Appel, Gerald, Winning Market Systems, pp. 42–44, Windsor Books, Bridgewaters, New York (1989)
- Dysart, Jr., Paul L., *Bear Market Signal?: A Sensitive Breadth Index Has Just Flashed One*, Barron's (newspaper) (Sept. 4, 1967)
- Fosback, Norman G., Stock Market Logic: A Sophisticated Approach to Profits on Wall Street, pp. 120–124, Deaborn Financial Printing, Chicago, Illinois (1993)
- Market Technicians Association, Paul L. Dysart, Jr. Annual Award (1990, ed. James E. Alphier)
- Russell, Richard, Dow Theory Letters, Jan. 7, 1976 (no. 652); see also Dow Theory Letters nos. 638, 642, and 646
- Schade, Jr., George A., *Traders Adjust the Volume Indicators*, Stocks Futures and Options Magazine (Nov. 2005)

On-balance volume

On-balance volume (**OBV**) is a technical analysis indicator intended to relate price and volume in the stock market. OBV is based on a cumulative total volume.[1]

The formula

$$OBV = OBV_{prev} + \begin{cases} volume & \text{if } close > close_{prev} \\ 0 & \text{if } close = close_{prev} \\ -volume & \text{if } close < close_{prev} \end{cases}$$

Application

Total volume for each day is assigned a positive or negative value depending on prices being higher or lower that day. A higher close results in the volume for that day to get a positive value, while a lower close results in negative value.[2] So, when prices are going up, OBV should be going up too, and when prices make a new rally high, then OBV should too. If OBV fails to go past its previous rally high, then this is a negative divergence, suggesting a weak move.[3]

The technique, originally called "continuous volume" by Woods and Vignolia, was later named "on-balance volume" by Joseph Granville who popularized the technique in his 1963 book *Granville's New Key to Stock Market Profits*.[1] The index can be applied to stocks individually based upon their daily up or down close, or to the market as a whole, using breadth of market data, i.e. the advance/decline ratio.[1]

OBV is generally used to confirm price moves.[4] The idea is that volume is higher on days where the price move is in the dominant direction, for example in a strong uptrend more volume on up days than down days.[5]

Similar indicators

Other price × volume indicators:

- Money flow
- Price and volume trend
- Accumulation/distribution index

References

[1] Joseph E. Granville, *Granville's New Strategy of Daily Stock Market Timing for Maximum Profit*, Prentice-Hall, Inc., 1976. ISBN 0-13-363432-9

[2] *On Balance Volume (OBV)* (http://www.oxfordfutures.com/futures-education/tech/on-balance-volume.htm). 22 September 2007.

[3] OBV Behaviorial Limitations and Formulas (http://financial-edu.com/on-balance-volume-obv.php) at Financial-edu.com.

[4] What Does On-Balance Volume Mean (http://www.investopedia.com/terms/o/onbalancevolume.asp)

[5] StockCharts.com article on On Balance Volume (http://stockcharts.com/education/IndicatorAnalysis/indic-OBV.htm)

Oscillator (technical analysis)

An **oscillator** is a technical analysis indicator that varies over time within a band (above and below a center line, or between set levels). Oscillators are used to discover short-term overbought or oversold conditions.

Common oscillators are MACD, ROC, RSI, CCI.

References

- Investopedia - Oscillator [1]
- StockCharts.com - Introduction to Technical Indicators and Oscillators [2]

References

[1] http://www.investopedia.com/terms/o/oscillator.asp
[2] http://stockcharts.com/school/doku.php?id=chart_school:technical_indicators:introduction_to_tech

Put/call ratio

Put/call ratio (or put–call ratio, PCR) is a technical indicator demonstrating investors' sentiment.[1] The ratio represents a proportion between all the put options and all the call options purchased on any given day. The put/call ratio can be calculated for any individual stock, as well as for any index, or can be aggregated. The ratio may be calculated using the numbers of puts and calls or on a dollar-weighted basis.

Generally, a lower reading (~0.6) of the ratio reflects a bullish sentiment among investors as they buy more calls, anticipating an uptrend. Conversely, a higher reading (~1.02) of the ratio indicates a bearish sentiment in the market. However, the ratio is considered to be a contrarian indicator, so that an extreme reading above 1.0 is actually a bullish signal, and vice versa.[2]

Moving averages are used to smooth and normalize the series of ratios.

References

[1] Put–Call Ratio (http://www.investopedia.com/terms/p/putcallratio.asp)
[2] The Put/Call Ratio: A Useful Indicator of Sentiment (http://www.discoveroptions.com/mixed/content/education/articles/putcallratio.html)

External links

- CBOE Volume & Put/Call Ratios (http://www.cboe.com/data/PutCallRatio.aspx)
- Options offer traders lots of… well, options! (http://www.tradingtips.com/the-putcall-ratio/)
- Put/Call Ratio Soaring (http://www.zealllc.com/2004/pcrsoar.htm)

Rahul Mohindar oscillator

The **Rahul Mohindar oscillator** (RMO) is a type of technical analysis "developed" by Rahul Mohindar of Viratech India. It detects trends in financial markets, and is designed to work on open-high-low-close charts for a wide variety of securities including stocks, commodities and FX.

This analysis is most notably included in version 10 of the MetaStock technical analysis program.

Range expansion index

The **range expansion index** (**REI**) is a technical indicator used in the technical analysis of financial markets. It is intended to chart the relative strength or weakness of a trading vehicle based on the comparison of the recent price changes and the overall price changes for the period.

The REI can be classified as a momentum oscillator, measuring the velocity and magnitude of directional price movements. The REI shows overbought/oversold price conditions by measuring the relation between the sum of "strong" price changes (such that form a trend) and all price changes for the period.

The REI is most typically used on a 8 day timeframe. It changes on a scale from −100 to +100, with the overbought and oversold levels marked at +60 and −60, respectively.

The range expansion index was developed by Thomas DeMark and published in his 1994 book, *The New Science of Technical Analysis*.[1]

Calculation

Two sums are calculated for each day. One is the conditional sum of the "strong" price changes:

$$S1 = \sum_{j=1}^{k} n_j m_j s_j$$

where k is the period of calculation (usually, 8), n_j is a first condition:

```
if ((High[j − 2] < Close[j - 7]) && (High[j - 2] < Close[j - 8]) && (High[j] < High[j - 5]) && (High[j] < High[j - 6]))
```
$n_j = 0$
```
 else
```
$n_j = 1$

m_j is a second condition:

```
if ((Low[j − 2] > Close[j − 7]) && (Low[j − 2] > Close[j − 8]) && (Low[j] > Low[j − 5]) && (Low[j] > Low[j − 6]))
```
$m_j = 0$
```
 else
```
$m_j = 1$

and s_j is the price change parameter:

$$s_j = High[j] - High[j-2] + Low[j] - Low[j-2],$$

The second sum is calculated as following:

$$S2 = \sum_{j=1}^{k} |High[j] - High[j-2]| + |Low[j] - Low[j-2]|$$

For each trading day the value of the indicator is calculated:

$$REI = \frac{S1}{S2} 100$$

Interpretation

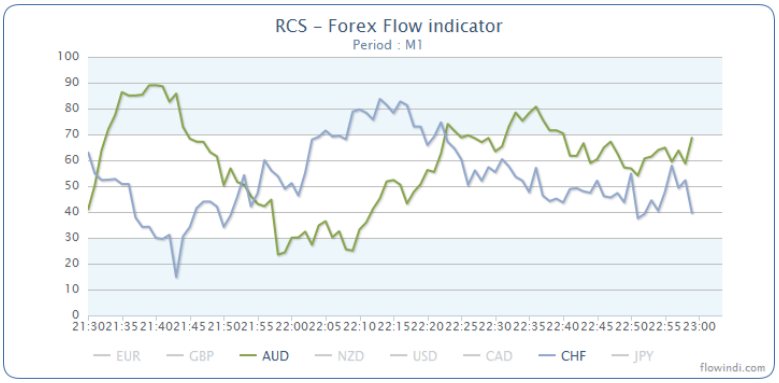
According to Thomas DeMark, price weakness is shown by the indicator when its value rises above level 60 and then declines below it. Price strength is shown when the REI goes below −60 and then rises above that level.

References

[1] Thomas R. DeMark, *The New Science of Technical Analysis*, ISBN 0-471-03548-3

Relative currency strength

The **Relative currency strength (RCS)** is a technical indicator used in the technical analysis of forex market. It is intended to chart the current and historical strength or weakness of a currency based on the closing prices of a recent trading period.It's based on Relative Strength Index and mathematical decorrelation of 28 cross currency pairs.It shows relative strength momentum of selected major currency. (EUR, GBP, AUD, NZD, USD, CAD, CHF, JPY)



Relative currency strength for AUD/CHF pair

The RCS is typically used on a 14*period timeframe, measured on a scale from 0 to 100 like RSI, with high and low levels marked at 70 and 30, respectively. Shorter or longer timeframes are used for alternately shorter or longer outlooks. More extreme high and low levels—80 and 20, or 90 and 10—occur less frequently but indicate stronger momentum of currency.

Combination of Relative currency strength and Absolute currency strength indicators gives you entry and exit signals for currency trading.

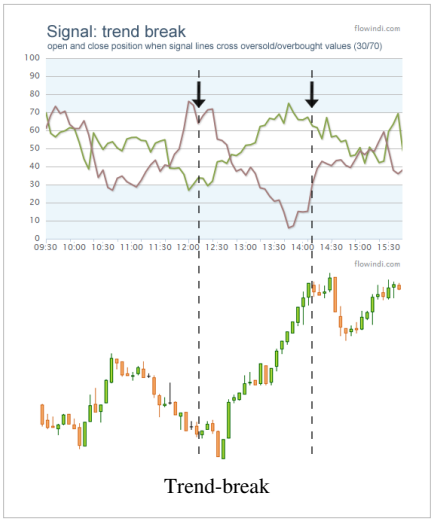
Basic idea

Indicator basic idea is "**buy strong currency and sell weak currency**".
If X/Y currency pair is in an uptrend, it shows you if it's due to X strength or Y weakness.
On these signals you can choose the most worth pair to trade.

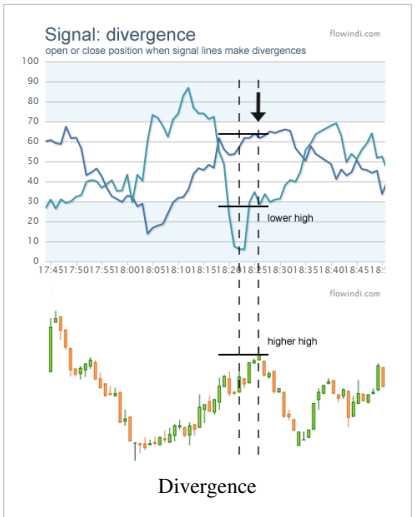
Signals

You can use Relative currency strength for pattern trading as well, among basic patterns which can be used are: cross, trend break, trend follow,divergencies divergencies.

Cross

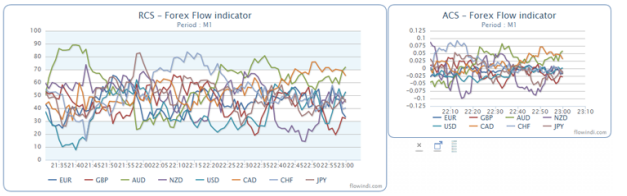


Trend-break

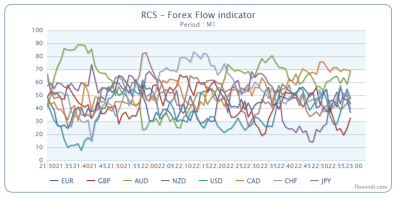


Divergence

Indicator



Combination of Relative currency strength and Absolute currency strength



Absolute currency strength

Advantageous for trading strategies

- Most commonly used as combination with Absolute currency strength
- information indicator to realize which currencies are being demanded, this is ideal indicator for trend follow traders
- help for scalpers looking for strength trend (trader can see both absolute and relative strength)
- instrument for correlation/spread traders to see reactions of each currencies on moves in correlated instruments (for example CAD/OIL or AUD/GOLD)

References

External links

- How to trade with RCS and ACS indicators (http://www.articlesbase.com/currency-trading-articles/how-to-trade-with-relative-rcs-and-absolute-acs-currency-strength-3100840.html)

Relative strength index

The **relative strength index** (**RSI**) is a technical indicator used in the analysis of financial markets. It is intended to chart the current and historical strength or weakness of a stock or market based on the closing prices of a recent trading period. The indicator should not be confused with relative strength.

The RSI is classified as a momentum oscillator, measuring the velocity and magnitude of directional price movements. Momentum is the rate of the rise or fall in price. The RSI computes momentum as the ratio of higher closes to lower closes: stocks which have had more or stronger positive changes have a higher RSI than stocks which have had more or stronger negative changes.

The RSI is most typically used on a 14 day timeframe, measured on a scale from 0 to 100, with high and low levels marked at 70 and 30, respectively. Shorter or longer timeframes are used for alternately shorter or longer outlooks. More extreme high and low levels—80 and 20, or 90 and 10—occur less frequently but indicate stronger momentum.

The relative strength index was developed by J. Welles Wilder and published in a 1978 book, *New Concepts in Technical Trading Systems*, and in *Commodities* magazine (now *Futures* magazine) in the June 1978 issue.[1] It has become one of the most popular oscillator indices.[2]

Calculation

For each trading period an upward change *U* or downward change *D* is calculated. Up periods are characterized by the close being higher than the previous close:

$$U = \text{close}_{\text{now}} - \text{close}_{\text{previous}}$$

$$D = 0$$

Conversely, a down period is characterized by the close being lower than the previous period's (note that D is nonetheless a positive number),

$$U = 0$$

$$D = \text{close}_{\text{previous}} - \text{close}_{\text{now}}$$

If the last close is the same as the previous, both *U* and *D* are zero. The average *U* and *D* are calculated using an *n*-period exponential moving average (EMA) in the AIQ version (but with an equal-weighted moving average in Wilder's original version).[3] The ratio of these averages is the *relative strength* or *relative strength factor*:

$$RS = \frac{\text{EMA}(U, n)}{\text{EMA}(D, n)}$$

If the average of *D* values is zero, then the RSI value is defined as 100.

The relative strength factor is then converted to a relative strength index between 0 and 100:[3]

$$RSI = 100 - \frac{100}{1 + RS}$$

The exponential moving averages should be appropriately initialized with a simple average using the first *n* values in the price series.

Interpretation

Basic configuration

The RSI is presented on a graph above or below the price chart. The indicator has an upper line, typically at 70, a lower line at 30, and a dashed mid-line at 50. Wilder recommended a smoothing period of 14 (see EMA smoothing, i.e. $\alpha = 1/14$ or $N = 27$).

Relative strength index 14-period

Principles

Wilder posited[1] that when price moves up very rapidly, at some point it is considered overbought. Likewise, when price falls very rapidly, at some point it is considered oversold. In either case, Wilder deemed a reaction or reversal imminent.

The level of the RSI is a measure of the stock's recent trading strength. The slope of the RSI is directly proportional to the velocity of a change in the trend. The distance traveled by the RSI is proportional to the magnitude of the move.

Wilder believed that tops and bottoms are indicated when RSI goes above 70 or drops below 30. Traditionally, RSI readings greater than the 70 level are considered to be in overbought territory, and RSI readings lower than the 30 level are considered to be in oversold territory. In between the 30 and 70 level is considered neutral, with the 50 level a sign of no trend.

Divergence

Wilder further believed that divergence between RSI and price action is a very strong indication that a market turning point is imminent. Bearish divergence occurs when price makes a new high but the RSI makes a lower high, thus failing to confirm. Bullish divergence occurs when price makes a new low but RSI makes a higher low.

Overbought and oversold conditions

Wilder thought that "failure swings" above 70 and below 30 on the RSI are strong indications of market reversals. For example, assume the RSI hits 76, pulls back to 72, then rises to 77. If it falls below 72, Wilder would consider this a "failure swing" above 70.

Finally, Wilder wrote that chart formations and areas of support and resistance could sometimes be more easily seen on the RSI chart as opposed to the price chart. The center line for the relative strength index is 50, which is often seen as both the support and resistance line for the indicator.

If the relative strength index is below 50, it generally means that the stock's losses are greater than the gains. When the relative strength index is above 50, it generally means that the gains are greater than the losses.

Uptrends and downtrends

In addition to Wilder's original theories of RSI interpretation, Andrew Cardwell has developed several new interpretations of RSI to help determine and confirm trend. First, Cardwell noticed that uptrends generally traded between RSI 40 and 80, while downtrends usually traded between RSI 60 and 20. Cardwell observed when securities change from uptrend to downtrend and vice versa, the RSI will undergo a "range shift."

Next, Cardwell noted that bearish divergence: 1) only occurs in uptrends, and 2) mostly only leads to a brief correction instead of a reversal in trend. Therefore bearish divergence is a sign confirming an uptrend. Similarly, bullish divergence is a sign confirming a downtrend.

Example of RSI Indicator Divergence

Reversals

Finally, Cardwell discovered the existence of positive and negative reversals in the RSI. Reversals are the opposite of divergence. For example, a positive reversal occurs when an uptrend price correction results in a higher low compared to the last price correction, while RSI results in a lower low compared to the prior correction. A negative reversal happens when a downtrend rally results in a lower high compared to the last downtrend rally, but RSI makes a higher high compared to the prior rally.

In other words, despite stronger momentum as seen by the higher high or lower low in the RSI, price could not make a higher high or lower low. This is evidence the main trend is about to resume. Cardwell noted that positive reversals only happen in uptrends while negative reversals only occur in downtrends, and therefore their existence confirms the trend.

Cutler's RSI

A variation called Cutler's RSI is based on a simple moving average of U and D,[4] instead of the exponential average above. Cutler had found that since Wilder used an exponential moving average to calculate RSI, the value of Wilder's RSI depended upon where in the data file his calculations started. Cutler termed this Data Length Dependency. Cutler's RSI is not data length dependent, and returns consistent results regardless of the length of, or the starting point within a data file.

$$RS = \frac{\mathrm{SMA}(U, n)}{\mathrm{SMA}(D, n)}$$

Cutler's RSI generally comes out slightly different from the normal Wilder RSI, but the two are similar, since SMA and EMA are also similar.

References

[1] J. Welles Wilder, *New Concepts in Technical Trading Systems*, ISBN 0-89459-027-8

[2] John J. Murphy (2009). *The Visual Investor: How to Spot Market Trends* (http://books.google.com/books?id=pdhp5CbXBJEC&pg=PA100) (2nd ed.). John Wiley and Sons. p. 100. ISBN 9780470382059. .

[3] Michael D. Sheimo (1998). *Cashing in on the Dow: using Dow theory to trade and determine trends in today's markets* (http://books.google.com/books?id=jDlZNrcs1RkC&pg=PA125). CRC Press. p. 125–127. ISBN 9780910944069. .

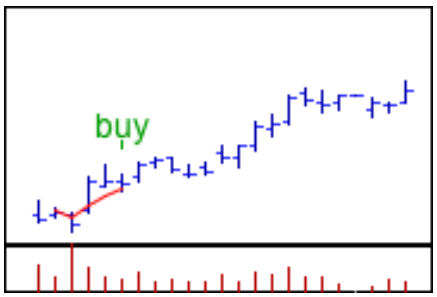
[4] Cutler's RSI page (http://www.aspenres.com/Documents/AspenGraphics4.0/CutlersRSI.htm) at Aspen Graphics Technical Analysis Software

External links

- Cardwell Techniques with the RSI (http://www.optionetics.com/market/articles/19819)

Rising moving average

The rising moving average is a technical indicator used in trading. Most commonly found visually, the pattern is spotted with a moving average overlay on a stock chart or price series. When the moving average has been rising consecutively for a number of days, this is used as a buy signal, to indicate a rising trend forming.



3-day Rising moving average on a 5-day close-price weighted moving average

While the rising moving average indicator is commonly used by investors without realising, there has been significant backtesting on historic stock data to calculate the performance of the rising moving average. Simulations have found that shorter rising averages, within the 3-10 day period, are more profitable overall than longer rising averages (e.g. 20 days). These have only been tested on US equity stocks however. [1]

Notes

[1] http://www.stockmarketfreedom.com/how-to-invest/Follow-the-Rising-Trend.html

Smart money index

Smart money index (SMI) of **smart money flow index** is a technical analysis indicator demonstrating investors' sentiment. The index was invented and popularized by money manager Don Hays.[1] The indicator is based on intra-day price patterns.[2]

The main idea is that the majority of traders (emotional, news-driven) overreact at the beginning of the trading day because of the overnight news and economic data. There is also a lot of buying on market orders and short covering at the opening. Smart, experienced investors start trading closer to the end of the day having the opportunity to evaluate market performance. Therefore, the basic strategy is to bet against the morning price trend and bet with the evening price trend. The SMI may be calculated for many markets and market indices (S&P 500, DJIA, etc.)

Basic formula

The today's SMI reading = yesterday's SMI − opening gain or loss + last hour change

For example, the SMI closed yesterday at 10000. During the first 30 minutes of today's trading, the DJIA has gained a total of 100 points. During the final hour, the DJIA has lost 80 points. So, today's SMI is 10000 − 100 − 80 = 9820.

Interpretation

The SMI sends no clear signal whether the market is bullish or bearish. There are also no fixed absolute or relative readings signaling about the trend. Traders need to look at the SMI dynamics relative to that of the market. If, for example, SMI rises sharply when the market falls, this fact would mean that smart money is buying, and the market is to revert to an uptrend soon. The opposite situation is also true. A rapidly falling SMI during a bullish market means that smart money is selling and that market is to revert to a downtrend soon. The SMI is, therefore, a trend-based indicator.

References

[1] SMART MONEY INDEX (SMI) (http://www.sentimentrader.com/subscriber/smart_money_index_description.htm)
[2] Hertler Market Signal (http://hertlermarketsignal.com/index2.html)

External links

- Smart Money Index: Accumulation During Decline (http://www.tradersnarrative.com/smart-money-index-accumulation-during-decline-1566.html)
- SMI charts (http://www.technicalanalysissite.com/indicators/smart_money_flow_index1.htm)
- Hertler Market Signal Update (http://www.hmsupdate.hertlermarketsignal.com/)

Stochastic oscillator

In technical analysis of securities trading, the **stochastic oscillator** is a momentum indicator that uses support and resistance levels. Dr. George Lane promoted this indicator in the 1950s. The term *stochastic* refers to the location of a current price in relation to its price range over a period of time.[1] This method attempts to predict price turning points by comparing the closing price of a security to its price range.

The indicator is defined as follows:

$$\%K = 100\frac{\text{closing price} - \text{L}}{\text{H} - \text{L}},$$

where H and L are respectively the highest and the lowest price over the last n periods, and

$$\%D = 3 \text{ period exponential moving average of } \%K.$$

In working with %D it is important to remember that there is only one valid signal—a divergence between %D and the analyzed security.[2]

Definition

The calculation above finds the range between an asset's high and low price during a given period of time. The current security's price is then expressed as a percentage of this range with 0% indicating the bottom of the range and 100% indicating the upper limits of the range over the time period covered. The idea behind this indicator is that prices tend to close near the extremes of the recent range before turning points. The Stochastic oscillator is calculated:

Where

$Price$ is the last closing price

$LOW_N(Price)$ is the lowest price over the last N periods

$HIGH_N(Price)$ is the highest price over the last N periods

$\%D$ is a 3-period exponential moving average of %K, $EMA_3(\%K)$.

$\%D - Slow$ is a 3-period exponential moving average of %D, $EMA_3(\%D)$.

A 3-line Stochastics will give an anticipatory signal in %K, a signal in the turnaround of %D at or before a bottom, and a confirmation of the turnaround in %D-Slow.[3] Typical values for *N* are 5, 9, or 14 periods. Smoothing the indicator over 3 periods is standard.

Dr. George Lane, a financial analyst, is one of the first to publish on the use of stochastic oscillators to forecast prices. According to Lane, the Stochastics indicator is to be used with cycles, Elliot Wave Theory and Fibonacci retracement for timing. In low margin, calendar futures spreads, one might use Wilders parabolic as a trailing stop after a stochastics entry. A centerpiece of his teaching is the divergence and convergence of trendlines drawn on stochastics, as diverging/converging to trendlines drawn on price cycles. Stochastics predicts tops and bottoms.

Interpretation

The signal to act is when there is a divergence-convergence, in an extreme area, with a crossover on the right hand side, of a cycle bottom.[2] As plain crossovers can occur frequently, one typically waits for crossovers occurring together with an extreme pullback, after a peak or trough in the %D line. If price volatility is high, an exponential moving average of the %D indicator may be taken, which tends to smooth out rapid fluctuations in price.

Stochastics attempts to predict turning points by comparing the closing price of a security to its price range. Prices tend to close near the extremes of the recent range just before turning points. In the case of an uptrend, prices tend to make higher highs, and the settlement price usually tends to be in the upper end of that time period's trading range. When the momentum starts to slow, the settlement prices will start to retreat from the upper boundaries of the range, causing the stochastic indicator to turn down at or before the final price high.[4]

An alert or set-up is present when the %D line is in an extreme area and diverging from the price action. The actual signal takes place when the faster % K line crosses the % D line.[5]

Divergence-convergence is an indication that the momentum in the market is waning and a reversal may be in the making. The chart below illustrates an example of where a divergence in stochastics relative to price forecasts a reversal in the price's direction.

Stochastic divergence.

An event known as "stochastic pop" occurs when prices break out and keep going. This is interpreted as a signal to increase the current position, or liquidate if the direction is against the current position.[6]

References

[1] Murphy, John J. (1999). " John Murphy's Ten Laws of Technical Trading (http://stockcharts.com/school/doku.php?id=chart_school:trading_strategies:john_murphy_s_ten_laws)".

[2] Lane, George M.D. (May/June 1984) "Lane's Stochastics," second issue of Technical Analysis of Stocks and Commodities magazine. pp 87-90.

[3] Lane, George C. & Caire (1998) "Getting Started With Stochastics" pg 3

[4] Person, John L (2004). *A Complete Guide to Technical Trading Tactics: How to Profit Using Pivot Points, Candlesticks & Other Indicators*. Hoboken, NJ: Wiley. pp. 144–145. ISBN 0-471-58455-X.

[5] Murphy, John J (1999). *Technical Analysis of the Financial Markets: A Comprehensive Guide to Trading Methods and Applications*. New York: New York Institute of Finance. p. 247. ISBN 0-7352-0066-1.

[6] Bernstein, Jake (1995). *The Complete Day Trader*. New York: McGraw Hill. ISBN 0-07-009251-6.

External links

- Stochastic Oscillator at Investopedia (http://www.investopedia.com/terms/s/stochasticoscillator.asp)

Trix (technical analysis)

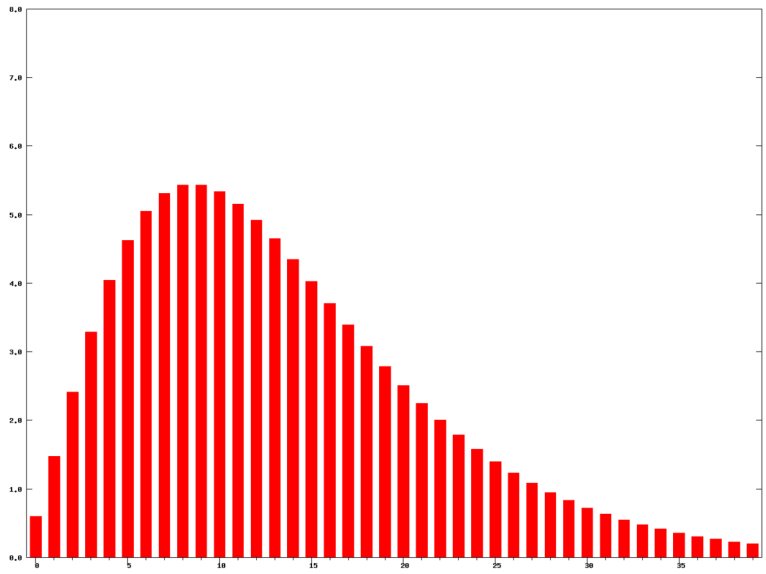
Trix (or **TRIX**) is a technical analysis oscillator developed in the 1980s by Jack Hutson, editor of Technical Analysis of Stocks and Commodities magazine. It shows the slope (i.e. derivative) of a triple-smoothed exponential moving average. The name Trix is from "**tri**ple e**x**ponential."

Trix is calculated with a given N-day period as follows:

- Smooth prices (often closing prices) using an N-day exponential moving average (EMA).
- Smooth that series using another N-day EMA.
- Smooth a third time, using a further N-day EMA.
- Calculate the percentage difference between today's and yesterday's value in that final smoothed series.

Like any moving average, the triple EMA is just a smoothing of price data and, therefore, is trend-following. A rising or falling line is an uptrend or downtrend and Trix shows the slope of that line, so it's positive for a steady uptrend, negative for a downtrend, and a crossing through zero is a trend-change, i.e. a peak or trough in the underlying average.

The triple-smoothed EMA is very different from a plain EMA. In a plain EMA the latest few days dominate and the EMA follows recent prices quite closely; however, applying it three times results in weightings spread much more broadly, and the weights for the latest few days are in fact smaller than those of days further past. The following graph shows the weightings for an N=10 triple EMA (most recent days at the left):



Triple exponential moving average weightings, N=10 (percentage versus days ago)

Note that the distribution's mode will lie with p_{N-2}'s weight, i.e. in the graph above p_8 carries the highest weighting. An N of 1 is invalid.

The easiest way to calculate the triple EMA based on successive values is just to apply the EMA three times, creating single-, then double-, then triple-smoothed series. The triple EMA can also be expressed directly in terms of the prices as below, with $p_0$today's close, $p_1$yesterday's, etc., and with $f = 1 - \frac{2}{N+1} = \frac{N-1}{N+1}$(as for a plain EMA):

$$TripleEMA_0 = (1-f)^3(p_0 + 3fp_1 + 6f^2p_2 + 10f^3p_3 + \ldots)$$

The coefficients are the triangle numbers, *n(n+1)/2*. In theory, the sum is infinite, using all past data, but as *f* is less than 1 the powers f^nbecome smaller as the series progresses, and they decrease faster than the coefficients increase, so beyond a certain point the terms are negligible.

References

- StockCharts.com article on TRIX [1], by Nicholas Fisher

References

[1] http://stockcharts.com/education/IndicatorAnalysis/indic_trix.htm

True strength index

The **true strength index** (**TSI**) was first published in 1991 by William Blau[1][2] as an indicator of both trend direction and overbought / oversold conditions, using moving averages of the underlying momentum of a financial instrument.[3][4] Momentum is considered a leading indicator of price movements, and a moving average characteristically lags behind price. The TSI combines these characteristics to create an indication of price and direction more in sync with market turns than either momentum or moving average.[5] The TSI is provided as part of the standard collection of indicators offered by various trading platforms.

Calculations

The TSI is a "double smoothed" indicator; meaning that a moving average applied to the data (daily momentum in this case) is smoothed again by a second moving average. The calculation for TSI uses exponential moving averages. The formula for the TSI is:

$$TSI(c_0, r, s) = 100\frac{EMA(EMA(m, r), s)}{EMA(EMA(|m|, r), s)}$$

where

c_0 = today's closing price

$m = c_0 - c_1$ = momentum (difference between today's and yesterday's close)

$EMA(m,n)$ = exponential moving average of m over n periods, that is,

$$EMA(m_0, n) = \frac{2}{n+1}\left[m_0 - EMA(m_1, n)\right] + EMA(m_1, n)$$

r = EMA smoothing period for momentum, typically 25

s = EMA smoothing period for smoothed momentum, typically 13

Interpretation

While the TSI output is bound between +100 and −100, most values fall between +25 and −25. Blau suggests interpreting these values as overbought and oversold levels, respectively, at which point a trader may anticipate a market turn. Trend direction is indicated by the slope of the TSI; a rising TSI suggests an up-trend in the market, and a falling TSI suggests a down-trend.[5][6]

References

[1] William Blau (Nov. 1991). "True Strength Index" (http://store.traders.com/-v11-c01-sidetru-pdf.html). *(Technical Analysis of) Stocks & Commodities* (traders.com) **11** (1): 438–446. .

[2] William Blau (Jan. 1993). "Stochastic Momentum" (http://store.traders.com/-v11-c01-sidetru-pdf.html). *(Technical Analysis of) Stocks & Commodities* (traders.com) **9** (11): 11–18. . "The true strength index (TSI) I introduce here is based on momentum as it occurs in nature, with no need to segregate up and down components of the momentum. ... The double-smoothed true strength index gives a choice to the trader; selection of the smaller smoothing time can be made to satisfy individual trading personalities. ... Calculating the true strength index requires an introduction to exponentially smoothed moving averages (EMA)"

[3] Blau, William (1995-03-06). *Momentum, Direction, and Divergence: Applying the Latest Momentum Indicators for Technical Analysis*. Wiley. ISBN 978-0-471-02729-4.

[4] Mark Etzkorn (1997). *Trading with oscillators: pinpointing market extremes—theory and practice* (http://books.google.com/books?id=MAcar1r3gEgC&pg=PA93). John Wiley and Sons. pp. 91–93. ISBN 978-0-471-15538-6. .

[5] Hartle, Tom (January 2002). "The True Strength Index" (https://www.tradestation.com/discussions/Data/2002125162918_AT-TSI JAN2002.pdf) (PDF). *Active Trader*. .

[6] Michael C. Thomsett (2009). *The options trading body of knowledge: the definitive source for information* (http://books.google.com/books?id=icIQyXSaLu8C&pg=PA268). FT Press. p. 268. ISBN 978-0-13-714293-4. .

Ulcer index

The **ulcer index** is a stock market risk measure or technical analysis indicator devised by Peter Martin in 1987,[1] and published by him and Byron McCann in their 1989 book *The Investors Guide to Fidelity Funds*. It's designed as a measure of volatility, but only volatility in the downward direction, i.e. the amount of drawdown or retracement occurring over a period.

Other volatility measures like standard deviation treat up and down movement equally, but a trader doesn't mind upward movement, it's the downside that causes stress and stomach ulcers that the index's name suggests. (The name pre-dates the discovery, described in the ulcer article, that most gastric ulcers are actually caused by a bacterium.)

The term ulcer index has also been used (later) by Steve Shellans, editor and publisher of MoniResearch Newsletter for a different calculation, also based on the ulcer causing potential of drawdowns.[2] Shellans index is not described in this article.

Calculation

The index is based on a given past period of N days. Working from oldest to newest a highest price (highest closing price) seen so-far is maintained, and any close below that is a retracement, expressed as a percentage

$$R_i = 100 \times \frac{price_i - maxprice}{maxprice}$$

For example if the high so far is $5.00 then a price of $4.50 is a retracement of −10%. The first R is always 0, there being no drawdown from a single price. The quadratic mean (or root mean square) of these values is taken, similar to a standard deviation calculation.

$$Ulcer = \sqrt{\frac{R_1^2 + R_2^2 + \cdots R_N^2}{N}}$$

The squares mean it doesn't matter if the R values are expressed as positives or negatives, both come out as a positive Ulcer Index.

The calculation is relatively immune to the sampling rate used. It gives similar results when calculated on weekly prices as it does on daily prices. Martin advises against sampling less often than weekly though, since for instance with quarterly prices a fall and recovery could take place entirely within such a period and thereby not appear in the index.

Usage

Martin recommends his index as a measure of risk in various contexts where usually the standard deviation (SD) is used for that purpose. For example the Sharpe ratio, which rates an investment's excess return (return above a safe cash rate) against risk, is

$$Sharpe\,ratio = \frac{Return - RiskFreeReturn}{standard\,deviation}$$

The ulcer index can replace the SD to make an ulcer performance index (UPI) or Martin ratio,

$$UPI = \frac{Return - RiskFreeReturn}{ulcer\,index}$$

In both cases annualized rates of return would be used (net of costs, inclusive of dividend reinvestment, etc.).

The index can also be charted over time and used as a kind of technical analysis indicator, to show stocks going into ulcer-forming territory (for one's chosen time-frame), or to compare volatility in different stocks.[3] As with the Sharpe Ratio, a higher value is better than a lower value (investors prefer more return for less risk).

References

[1] Peter Martin's Ulcer Index page (http://www.tangotools.com/ui/ui.htm)

[2] Pankin Managed Funds, client newsletter 3rd Quarter 1996, Questions and Answers (http://www.pankin.com/qa963.htm)

[3] Discovering the Absolute-Breadth Index and the Ulcer Index (http://www.investopedia.com/articles/technical/03/030403.asp) at Investopedia.com

Further reading

Related topics

- Hindenburg Omen
- The "Gilt Dragon" Omen (http://hireme.geek.nz/parity-condition-USD.html)

Books

- *The Investor's Guide to Fidelity Funds*, Peter Martin and Byron McCann, John Wiley & Sons, 1989. Now out of print, but offered for sale in electronic form by Martin at his web site (http://www.tangotools.com/ui/igff.htm).

External links

- Peter Martin's web site (http://www.tangotools.com/misc/index.html)

Ultimate oscillator

Larry Williams developed the **ultimate oscillator** as a way to account for the problems experienced in most oscillators when used over different lengths of time.[1]

The oscillator is a technical analysis oscillator developed by Larry Williams based on a notion of buying or selling "pressure" represented by where a day's closing price falls within the day's true range.

The calculation starts with "buying pressure", which is the amount by which the close is above the "true low" on a given day. The true low is the lesser of the given day's trading low and the previous close.

$$bp = close - \min(low, prev\,close)$$

The true range (the same as used in average true range) is the difference between the "true high" and the true low above. The true high is the greater of the given day's trading high and the previous close.

$$tr = \max(high, prev\,close) - \min(low, prev\,close)$$

The total buying pressure over the past 7 days is expressed as a fraction of the total true range over the same period. If $bp_1$is today, $bp_2$is yesterday, etc., then

$$avg_7 = \frac{bp_1 + bp_2 + \cdots + bp_7}{tr_1 + tr_2 + \cdots + tr_7}$$

The same is done for the past 14 days and past 28 days and the resulting three ratios combined in proportions 4:2:1, and scaled to make a percentage 0 to 100. The idea of the 7, 14 and 28 day periods is to combine short, intermediate and longer time frames.

$$UltOsc = 100 \times \frac{4 \times avg_7 + 2 \times avg_{14} + avg_{28}}{4 + 2 + 1}$$

Williams had specific criteria for a buy or sell signal. A buy signal occurs when,

- Bullish divergence between price and the oscillator is observed, meaning prices make new lows but the oscillator doesn't
- During the divergence the oscillator has fallen below 30.
- The oscillator then rises above its high during the divergence, i.e. the high in between the two lows. The buy trigger is the rise through that high.

The position is closed when the oscillator rises above 70 (considered overbought), or a rise above 50 but then a fallback through 45.

A sell signal is generated conversely on a bearish divergence above level 70, to be subsequently closed out below 30 (as oversold).

References

[1] AsiaPacFinance.com Trading Indicator Glossary (http://www.asiapacfinance.com/trading-strategies/technicalindicators/UltimateOscillator)

- Ultimate Oscillator (http://www.stockcharts.com/education/IndicatorAnalysis/indic_ultimate.html) at StockCharts.com
- Ultimate Oscillator (http://www.chartfilter.com/reports/c52.htm) at ChartFilter.com
- The Ultimate Buy Signal (http://www.fool.com/investing/small-cap/2007/04/05/the-ultimate-buy-signal.aspx), Motley Fool

Further reading

- *The Ultimate Oscillator*, by Larry Williams, Technical Analysis of Stocks and Commodities magazine V.3:4 (140–141) (introduction (http://store.traders.com/-v03-c04-ulti-pdf.html))

Volume spread analysis

Volume spread analysis (VSA) is a form of technical analysis which attempts to look into price action in relation to volume. VSA is the study of the market forces of supply and demand and the manipulation of those forces through examining the relationship between the quantity of volume on a price bar, the spread of the price or range of the bar, and the closing price on the height of that bar. Studying these variables allows users to decipher the state of balance between supply and demand for the underlying security, as well as the likely near term direction of the market.

History

VSA is the result of Tom Williams experience and discoveries applied to Richard Wyckoff's methods. Williams describes volume as "the major indicator" for institutional traders as it largely represents professional activity. Richard Wyckoff's Methods are based on his extensive experience and studies which rely heavily on Jesse Livermore's (*The Reminiscences of A Stock Operator*) discoveries. These three traders came to understand the underlying mechanics of the markets and developed methods to trade them and, later, to teach others how to trade effectively.

Jesse Livermore detailed how the markets are manipulated by professional traders and developed methods to profit from this dynamic. Richard Wyckoff codified Livermore's discoveries and methods and applied them to chart reading. Tom Williams added VSA to Wyckoff's work to reveal the specific tactics used to manipulate crowd behavior and expose the footprints of the "smart money" composite operators.

Analysis

VSA is a quantitative study of price action and uses three components on the chart to determine the balance of supply and demand as well as the probable near term direction of the market. These components are the amount of **volume** affecting the price, the **price spread** or range of the price (not the bid/ask spread), and where price closes. VSA analyses volume and price action relative to the preceding bars on a chart to identify accumulation or distribution phases, which suggest future direction of price.

References

Volume–price trend

Volume–price trend (VPT) (sometimes **price–volume trend**) is a technical analysis indicator intended to relate price and volume in the stock market. VPT is based on a running cumulative volume that adds or subtracts a multiple of the percentage change in share price trend and current volume, depending upon their upward or downward movements.[1]

Formula

$$VPT = VPT_{prev} + volume \times \frac{close_{today} - close_{prev}}{close_{prev}}$$

The starting point for the VPT total, i.e. the zero point, is arbitrary. Only the shape of the resulting indicator is used, not the actual level of the total.

VPT is similar to on-balance volume (OBV),[2] but where OBV takes volume just according to whether the close was higher or lower, VPT includes how much higher or lower it was.

VPT is interpreted in similar ways to OBV. Generally, the idea is that volume is higher on days with a price move in the dominant direction, for example in a strong uptrend more volume on up days than down days. So, when prices are going up, VPT should be going up too, and when prices make a new rally high, VPT should too. If VPT fails to go past its previous rally high then this is a negative divergence, suggesting a weak move.

Similar indicators

Other price × volume indicators:

- Money Flow Index
- On-balance volume
- Accumulation/distribution index

References

[1] Volume Price Trend Indicator – VPT (http://www.investopedia.com/terms/v/vptindicator.asp)

[2] Price/Volume Trend (http://www.gannalyst.com/Gannalyst_Professional/Gannalyst_Indicators_Price_Volume_Trend.shtml)

External links

- STEP3_Tutorial (https://www.psg-online.co.za/Documentation/Webdocs/Education/STEP3_Tutorial.ppt)
- OUTPERFORMANCE WITH TECHNICAL ANALYSIS? — AN INTRADAY STUDY ON THE SWISS STOCK MARKET (http://www.sbf.unisg.ch/org/sbf/web.nsf/SysWebRessources/AmmannRekatevonWyss04/$FILE/TechAnalysis.pdf)

Vortex indicator

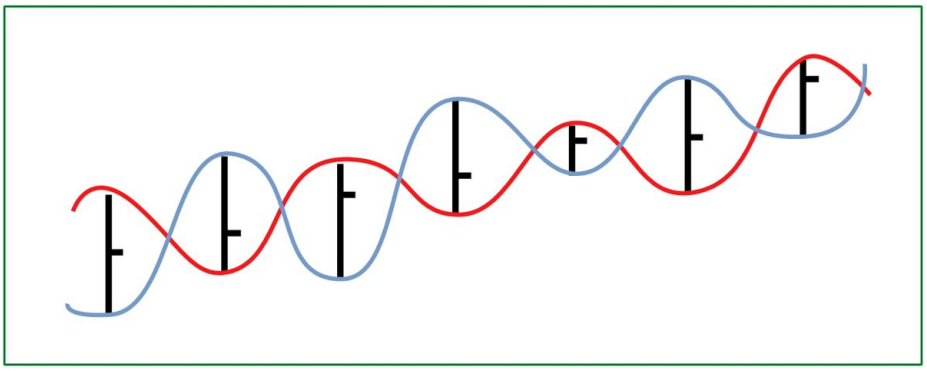
The Vortex Indicator is a technical indicator invented by Etienne Botes and Douglas Siepman to identify the start of a new trend or the continuation of an existing trend within financial markets. It was published in the January 2010 edition of *Technical Analysis of Stocks & Commodities*.[1]

Inspiration

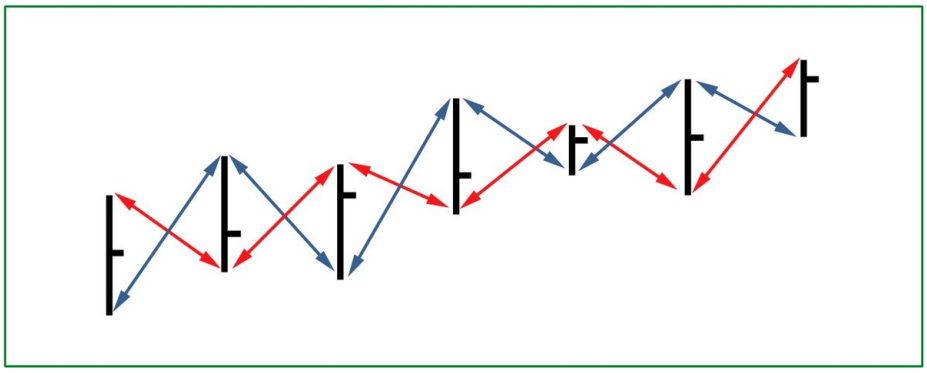
The Vortex Indicator was inspired by the work of an Austrian inventor, Viktor Schauberger, who studied the flow of water in rivers and turbines. Etienne Botes and Douglas Siepman developed the idea that movements and flows within financial markets are similar to the vortex motions found in water. The Vortex Indicator was also partly inspired by J. Welles Wilder's concept of directional movement, which assumes the relationship between price bars gives clues as to the direction of a market.[2]

Description

A vortex pattern may be observed in any market by connecting the lows of that market's price bars with the consecutive bars' highs, and then price bar highs with consecutive lows. The greater the distance between the low of a price bar and the subsequent bar's high, the greater the upward or positive Vortex movement (VM+). Similarly, the greater the distance between a price bar's high and the subsequent bar's low, the greater the downward or negative Vortex movement (VM-).



A Vortex Pattern in the Market: By connecting the lows of price bars with the consecutive bars' highs, and then price bar highs with consecutive lows, one can observe a vortex pattern in the market.



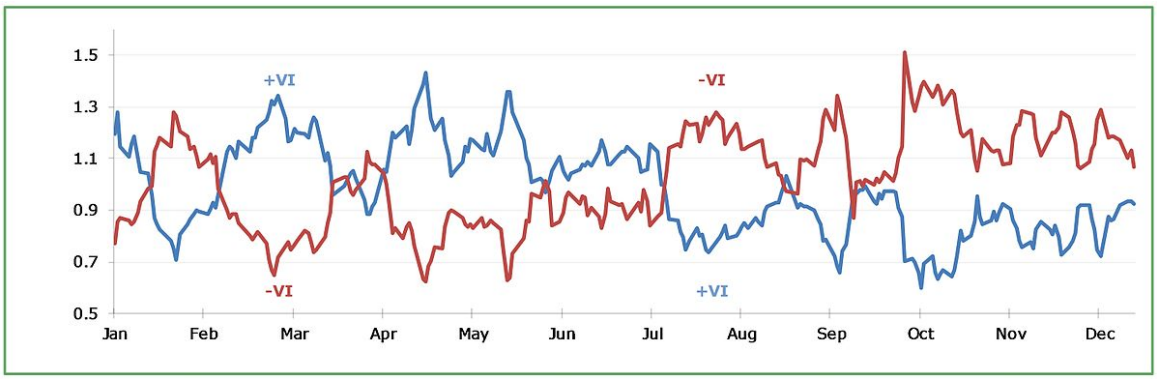
The Vortex Indicator: The greater the distance between the low of a price bar and the subsequent bar's high, the stronger the positive vortex movement (VM+). Similarly, the greater the distance between a price bar's high and the subsequent bar's low, the stronger the negative vortex movement (VM−).

Identifying a trend

On a chart, VI+ and VI– will be seen to intersect each other at a change of trend, and begin to diverge ever wider as the strength of a trend increases. When VI+ is larger and above VI–, the market is trending up. Conversely, when VI– is bigger and above VI+, the market is trending down.

A trader should focus on the key trend change points of the Vortex Indicator (i.e. the crossing points of VI+ and VI–). When VI+ crosses above VI–, a long (buy) position is indicated. A short or sell position is suggested when VI– crosses above VI+.

The published article also suggested further measures to ensure an effective trading strategy, for example, only entering a trade at the extreme high or low of the price bar that corresponds with a crossing of the Vortex Indicator.[3]



14-period daily vortex indicator: When VI+ is greater than VI–, it indicates that the market is trending up. The market is trending down when VI– is above VI+. The potential change of trend points are found where VI+ and VI– intersect one another.

Price chart: As the trend strengthens, notice how the VI+ and VI– lines increasingly diverge. As the trend weakens, you will observe the two lines converging again.

Calculation

The high, low and close values are required for any chosen stock, future, commodity or currency. These values may be 15-minute, hourly, daily, etc.

- **First, calculate the current true range:**
  - Current true range (TR) = Maximum absolute value of either (current high–current low), (current low–previous close), (current high–previous close)
- **Next, calculate the current upward (positive) and downward (negative) vortex movements:**
  - Current Vortex Movement Up (VM+) = absolute value of current high − previous low
  - Current Vortex Movement Down (VM–) = absolute value of current low − previous high

- **Decide on a parameter length (21 periods was used for this example). Now, sum the last 21 period's True Range, VM+ and VM-:**
  - Sum of the last 21 periods' True Range = SUM TR21
  - Sum of the last 21 periods' VM+ = SUM VM21+
  - Sum of the last 21 periods' VM- = SUM VM21−
- **Finally, divide SUM VM21+ and SUM VM21− respectively with the SUM TR21 to obtain the Vortex Indicator:**
  - SUM VM21+/SUM TR21 = VI21+
  - SUM VM21-/SUM TR21 = VI21−

If this process is repeated, the resulting VI21+ and VI21− can be drawn graphically to represent the two lines of the Vortex Indicator.

Practical application

The Vortex Indicator is simple to use as the only required inputs are the high, low and close of a price bar. Traders may use the Vortex Indicator on its own, in combination with other technical indicators to confirm a change of trend or as part of a larger trading system.

In addition, the Vortex Indicator may be used for any:

- **market** (such as stocks, futures or currencies)
- **time frame** (for example, 15 minute, hourly or weekly charts)
- **parameter** (such as 13, 21 or 34 periods)

The inventors of the Vortex Indicator recommend using longer time frames and parameters in order to filter out false signals. If a trader does opt to use a very short time frame, such as 5 minutes, this should be combined with a long parameter of 34 or 55 periods.

Because of its universal applicability, the Vortex Indicator is suitable for both short term traders as well as longer term fund managers who may wish to identify larger macro trends within a market.

Coding and strategies

The Vortex Indicator is available on most charting software.[4] Some of these companies have suggested additional trading strategies to use in conjunction with the Vortex Indicator, including the implementation of a trailing stop [5] and making use of supporting indicators in order to reduce the number of false signals.[6]

Comparative studies

To test the Vortex Indicator against Welles Wilder's directional movement indicator (DMI), a portfolio of 38 of the most actively traded, full sized, futures contracts was created. These 38 futures included a number of index and financial futures, currencies, metals, energy futures and commodities like grains, oils and foods. The test period was from 3 January 1978 to 6 November 2009, using a 14 day parameter for both indicators. Over the entire test period, and also during the last 10 years, the Vortex Indicator showed a better performance than the DMI.[7]

However, using a similar test based on 101 NASDAQ stocks, on a smaller sample (for the period 2 January 1992 to 14 August 2009), the DMI showed a better performance than the Vortex Indicator.[8]

Possible improvement and variation

An alternative version of the indicator was created by Jez Liberty.[9] This version enables the calculation to account for gap days which sometimes occur in non-continuous markets such as stocks.[10]

References

[1] Botes, Etienne & Siepman, Douglas (January 2010). "The Vortex Indicator". *Technical Analysis of Stocks & Commodities* 28(1), p. 21.

[2] Wilder, J. Welles (1978). *New Concepts In Technical Trading Systems*. Trend Research.

[3] Botes, Etienne & Siepman, Douglas (January 2010). "The Vortex Indicator". *Technical Analysis of Stocks & Commodities* 28(1), p. 25.

[4] "Traders' Tips" (http://www.traders.com/Documentation/FEEDbk_docs/2010/01/TradersTips.html). Technical Analysis of Stocks & Commodities. January 2010. . Retrieved 17 January 2010.

[5] Mills, Mark (January 2010). "TRADESTATION: VORTEX INDICATOR" (http://www.traders.com/Documentation/FEEDbk_docs/2010/01/TradersTips.html). TradeStation Securities, Inc. . Retrieved 17 January 2010.

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[7] Denning, Richard (January 2010). "January 2010 Stocks and Commodities Traders Tips" (http://www.tradersedgesystems.com/aiq/traderstips/traders-tips-january-2010.htm). Traders Edge Systems. . Retrieved 17 January 2010.

[8] Denning, Richard (January 2010). "January 2010 Stocks and Commodities Traders Tips" (http://www.tradersedgesystems.com/aiq/traderstips/traders-tips-january-2010.htm). Traders Edge Systems. . Retrieved 17 January 2010.

[9] Liberty, Jez (April 2010). "Vortex Indicator In Trading Blox". *Technical Analysis of Stocks & Commodities* 28(5), p. 76.

[10] Liberty, Jez (February 2010). "Vortex indicator" (http://www.automated-trading-system.com/free-code-improved-vortex/). Automated Trading System. . Retrieved 11 November 2010.

External links

- The Vortex Indicator (http://www.traders.com/Reprints/PDF_reprints/VFX_VORTEX.PDF)

Williams %R

Williams %R, or just **%R**, is a technical analysis oscillator showing the current closing price in relation to the high and low of the past N days (for a given N). It was developed by a publisher and promoter of trading materials, Larry Williams. Its purpose is to tell whether a stock or commodity market is trading near the high or the low, or somewhere in between, of its recent trading range.

$$\%R = \frac{high_{N days} - close_{today}}{high_{N days} - low_{N days}} \times 100$$ [1]

The oscillator is on a negative scale, from -100 (lowest) up to 0 (highest), considered unusual since it is the obverse of the more common 0 to 100 scale found in many Technical Analysis oscillators. Although sometimes altered (by simply adding 100), this scale needn't cause any confusion. A value of -100 is the close today at the lowest low of the past N days, and 0 is a close today at the highest high of the past N days.

Buy-/Sell-Signalling

Williams used a 10 trading day period and considered values below -80 as oversold and above -20 as overbought. But they were not to be traded directly, instead his rule to buy an oversold was

- %R reaches -100%.
- Five trading days pass since -100% was last reached
- %R rises above -95% or -85%.

or conversely to sell an overbought condition

- %R reaches 0%.
- Five trading days pass since 0% was last reached
- %R falls below -5% or -15%.

The timeframe can be changed for either more sensitive or smoother results. The more sensitive you make it, though, the more false signals you will get.

Notes

Due to the equivalence

$$(close_{today} - low_{Ndays}) - (close_{today} - high_{Ndays}) = high_{Ndays} - low_{Ndays}$$

the %R indicator is arithmetically exactly equivalent to the %K stochastic oscillator, mirrored at the 0%-line, when using the same time interval.

References

[1] http://stockcharts.com/school/doku.php?id=chart_school:technical_indicators:williams_r

Chart Patterns

Technical Analysis of Stocks & Commodities

Technical Analysis of Stocks & Commodities is an American, Seattle-based monthly magazine about commodity futures contracts, stocks, options, derivatives, and forex.

It was established in 1982 and today covers global industry trends, prominent people, trading technology, managed funds, and fundamental and technical analysis.[1]

The magazine is a respected source of information on the financial markets, with articles on industry issues, current market developments, trading techniques and strategies, and many other areas of interest to traders and risk managers. It contains feature articles, analysis and strategies for derivatives traders and money managers, and more.

The magazine is published by Technical Analysis, Inc. Its primary competitors are *Active Trader* and *Futures* among others.

References

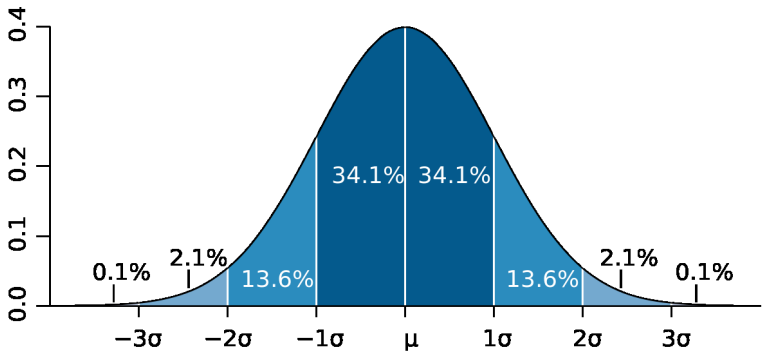
[1] About Technical Analysis of STOCKS & COMMODITIES (http://www.traders.com/S&C_Homepg.html) at *STOCKS & COMMODITIES* web site

External links

- Official web site (http://www.traders.com/S&C_Homepg.html)
- *Technical Analysis of Stocks & Commodities* (http://www.thinktrade.net/technical-analysis-stocks-and-commodities.php) ThinkTrade review

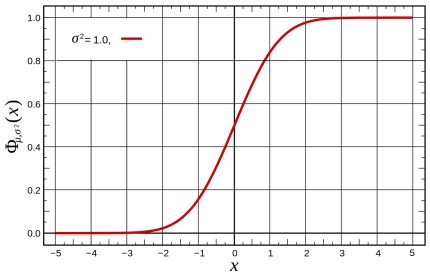
Standard deviation

In statistics and probability theory, **standard deviation** (represented by the symbol sigma, **σ**) shows how much variation or "dispersion" exists from the average (mean, or expected value). A low standard deviation indicates that the data points tend to be very close to the mean; high standard deviation indicates that the data points are spread out over a large range of values.



A plot of a normal distribution (or bell curve). Each band has a width of 1 standard deviation. See also: 68-95-99.7 rule

The standard deviation of a random variable, statistical population, data set, or probability distribution is the square root of its variance. It is algebraically simpler though practically less robust than the average absolute deviation.[1][2] A useful property of standard deviation is that, unlike variance, it is expressed in the same units as the data.



Cumulative probability of a normal distribution with expected value 0 and standard deviation 1

In addition to expressing the variability of a population, standard deviation is commonly used to measure confidence in statistical conclusions. For example, the margin of error in polling data is determined by calculating the expected standard deviation in the results if the same poll were to be conducted multiple times. The reported margin of error is typically about twice the standard deviation – the radius of a 95 percent confidence interval. In science, researchers commonly report the standard deviation of experimental data, and only effects that fall far outside the range of standard deviation are considered statistically significant – normal random error or variation in the measurements is in this way distinguished from causal variation. Standard deviation is also important in finance, where the standard deviation on the rate of return on an investment is a measure of the volatility of the investment.

When only a sample of data from a population is available, the population standard deviation can be estimated by a modified quantity called the sample standard deviation, explained below.

Basic examples

Consider a population consisting of the following eight values:

$$2,\ 4,\ 4,\ 4,\ 5,\ 5,\ 7,\ 9.$$

These eight data points have the mean (average) of 5:

$$\frac{2+4+4+4+5+5+7+9}{8} = 5.$$

To calculate the population standard deviation, first compute the difference of each data point from the mean, and square the result of each:

$$\begin{array}{ll} (2-5)^2 = (-3)^2 = 9 & (5-5)^2 = 0^2 = 0 \\ (4-5)^2 = (-1)^2 = 1 & (5-5)^2 = 0^2 = 0 \\ (4-5)^2 = (-1)^2 = 1 & (7-5)^2 = 2^2 = 4 \\ (4-5)^2 = (-1)^2 = 1 & (9-5)^2 = 4^2 = 16. \end{array}$$

Next, compute the average of these values, and take the square root:

$$\sqrt{\frac{(9+1+1+1+0+0+4+16)}{8}} = 2.$$

This quantity is the **population standard deviation**; it is equal to the square root of the variance. The formula is valid *only* if the eight values we began with form the *complete* population. If they instead were a random sample, drawn from some larger, "parent" population, then we should have used 7 (which is $n - 1$) instead of 8 (which is n) in the denominator of the last formula, and then the quantity thus obtained would have been called the **sample standard deviation**. See the section Estimation below for more details.

A slightly more complicated real life example, the average height for adult men in the United States is about 70", with a standard deviation of around 3". This means that most men (about 68%, assuming a normal distribution) have a height within 3" of the mean (67"–73") — one standard deviation — and almost all men (about 95%) have a height within 6" of the mean (64"–76") — two standard deviations. If the standard deviation were zero, then all men would be exactly 70" tall. If the standard deviation were 20", then men would have much more variable heights, with a typical range of about 50"–90". Three standard deviations account for 99.7% of the sample population being studied, assuming the distribution is normal (bell-shaped).

Definition of population values

Let X be a random variable with mean value μ:

$$\mathrm{E}[X] = \mu.$$

Here the operator E denotes the average or expected value of X. Then the **standard deviation** of X is the quantity

$$\sigma = \sqrt{\mathrm{E}[(X-\mu)^2]} = \sqrt{\mathrm{E}[X^2] - (\mathrm{E}[X])^2}.$$

That is, the standard deviation σ (sigma) is the square root of the variance of X, i.e., it is the square root of the average value of $(X-\mu)^2$.

The standard deviation of a (univariate) probability distribution is the same as that of a random variable having that distribution. Not all random variables have a standard deviation, since these expected values need not exist. For example, the standard deviation of a random variable that follows a Cauchy distribution is undefined because its expected value μ is undefined.

Discrete random variable

In the case where X takes random values from a finite data set x_1, x_2, …, x_N, with each value having the same probability, the standard deviation is

$$\sigma = \sqrt{\frac{1}{N}\left[(x_1-\mu)^2 + (x_2-\mu)^2 + \cdots + (x_N-\mu)^2\right]}, \quad \text{where} \quad \mu = \frac{1}{N}(x_1 + \cdots + x_N),$$

or, using summation notation,

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2}, \quad \text{where} \quad \mu = \frac{1}{N}\sum_{i=1}^{N} x_i.$$

If, instead of having equal probabilities, the values have different probabilities, let x_1 have probability p_1, x_2 have probability p_2, ..., x_N have probability p_N. In this case, the standard deviation will be

$$\sigma = \sqrt{\sum_{i=1}^{N} p_i(x_i-\mu)^2}, \quad \text{where} \quad \mu = \sum_{i=1}^{N} p_i x_i.$$

Continuous random variable

The standard deviation of a continuous real-valued random variable X with probability density function $p(x)$ is

$$\sigma = \sqrt{\int_{\mathbf{X}} (x-\mu)^2\, p(x)\, dx}, \quad \text{where} \quad \mu = \int_{\mathbf{X}} x\, p(x)\, dx,$$

and where the integrals are definite integrals taken for x ranging over the set of possible values of the random variable X.

In the case of a parametric family of distributions, the standard deviation can be expressed in terms of the parameters. For example, in the case of the log-normal distribution with parameters μ and σ^2, the standard deviation is $[(\exp(\sigma^2) - 1)\exp(2\mu + \sigma^2)]^{1/2}$.

Estimation

One can find the standard deviation of an entire population in cases (such as standardized testing) where every member of a population is sampled. In cases where that cannot be done, the standard deviation σ is estimated by examining a random sample taken from the population. Some estimators are given below:

Standard deviation of the sample

An estimator for σ sometimes used is the **standard deviation of the sample**, denoted by s_N and defined as follows:

$$s_N = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i-\overline{x})^2}.$$

where $\{x_1, x_2, ..., x_N\}$ are the observed values of the sample items and $\overline{x}$ is the mean value of these observations, while the denominator N stands for the size of the sample.

This estimator has a uniformly smaller mean squared error than the *sample standard deviation* (see below), and is the maximum-likelihood estimate when the population is normally distributed. But this estimator, when applied to a small or moderately sized sample, tends to be too low: it is a biased estimator.

The standard deviation of the sample is the same as the population standard deviation of a discrete random variable that can assume precisely the values from the data set, where the probability for each value is proportional to its multiplicity in the data set.

Sample standard deviation

The most commonly used estimator for σ is an adjusted version, the **sample standard deviation**, denoted by *s* and defined as follows:

$$s = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2},$$

Where $N - 1$ equals the number of degrees of freedom in the vector of residuals, $(x_1 - \overline{x}, \ldots, x_n - \overline{x})$. This correction (the use of $N - 1$ instead of N) is known as Bessel's correction. The reason for this correction is that s^2 is an unbiased estimator for the variance σ^2 of the underlying population, if that variance exists and the sample values are drawn independently with replacement. Additionally, if $N = 1$, then there is no indication of deviation from the mean, and standard deviation should therefore be undefined.

The term *standard deviation of the sample* is used for the uncorrected estimator (using *N*) while the term *sample standard deviation* is used for the corrected estimator (using $N - 1$).

Other estimators

Further information: Unbiased estimation of standard deviation and Bias of an estimator

Although an unbiased estimator for σ is known when the random variable is normally distributed, the formula is complicated and amounts to a minor correction. Moreover, unbiasedness (in this sense of the word) is not always desirable.

Confidence interval of a sampled standard deviation

The standard deviation we obtain by sampling a distribution is itself not absolutely accurate. This is especially true if the number of samples is very low. This effect can be described by the confidence interval or CI. For example, for N=2 the 95% CI of the SD is from 0.45*SD to 31.9*SD. In other words the standard deviation of the distribution in 95% of the cases can be up to a factor of 31 larger or up to a factor 2 smaller! For N=10 the interval is 0.69*SD to 1.83*SD, the actual SD can still be almost a factor 2 higher than the sampled SD. For N=100 this is down to 0.88*SD to 1.16*SD. So to be sure the sampled SD is close to the actual SD we need to sample a large number of points.

Identities and mathematical properties

The standard deviation is invariant under changes in location, and scales directly with the scale of the random variable. Thus, for a constant *c* and random variables *X* and *Y*:

$$\begin{aligned}
&\operatorname{stdev}(c) = 0 \\
&\operatorname{stdev}(X + c) = \operatorname{stdev}(X), \\
&\operatorname{stdev}(cX) = |c|\operatorname{stdev}(X).
\end{aligned}$$

The standard deviation of the sum of two random variables can be related to their individual standard deviations and the covariance between them:

$$\operatorname{stdev}(X + Y) = \sqrt{\operatorname{var}(X) + \operatorname{var}(Y) + 2\operatorname{cov}(X, Y)}.$$

where $\operatorname{var} = \operatorname{stdev}^2$ and cov stand for variance and covariance, respectively.

The calculation of the sum of squared deviations can be related to moments calculated directly from the data. The standard deviation of the sample can be computed as:

$$\operatorname{stdev}(X) = \sqrt{E[(X - E(X))^2]} = \sqrt{E[X^2] - (E[X])^2}.$$

The sample standard deviation can be computed as:

$$\operatorname{stdev}(X) = \sqrt{\frac{N}{N-1}}\sqrt{E[(X - E(X))^2]}.$$

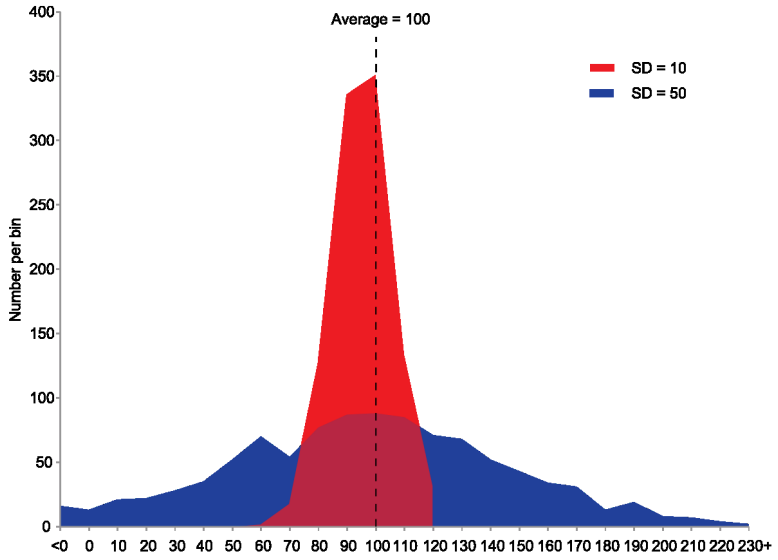
For a finite population with equal probabilities at all points, we have

$$\sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2} = \sqrt{\frac{1}{N}\left(\sum_{i=1}^{N} x_i^2\right) - \overline{x}^2} = \sqrt{\frac{1}{N}\sum_{i=1}^{N} x_i^2 - \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2}.$$

This means that the standard deviation is equal to the square root of (the average of the squares less the square of the average). *See computational formula for the variance for proof, and for an analogous result for the sample standard deviation.*

Interpretation and application

A large standard deviation indicates that the data points are far from the mean and a small standard deviation indicates that they are clustered closely around the mean.



Example of two sample populations with the same mean and different standard deviations. Red population has mean 100 and SD 10; blue population has mean 100 and SD 50.

For example, each of the three populations {0, 0, 14, 14}, {0, 6, 8, 14} and {6, 6, 8, 8} has a mean of 7. Their standard deviations are 7, 5, and 1, respectively. The third population has a much smaller standard deviation than the other two because its values are all close to 7. It will have the same units as the data points themselves. If, for instance, the data set {0, 6, 8, 14} represents the ages of a population of four siblings in years, the standard deviation is 5 years. As another example, the population {1000, 1006, 1008, 1014} may represent the distances traveled by four athletes, measured in meters. It has a mean of 1007 meters, and a standard deviation of 5 meters.

Standard deviation may serve as a measure of uncertainty. In physical science, for example, the reported standard deviation of a group of repeated measurements gives the precision of those measurements. When deciding whether measurements agree with a theoretical prediction, the standard deviation of those measurements is of crucial importance: if the mean of the measurements is too far away from the prediction (with the distance measured in standard deviations), then the theory being tested probably needs to be revised. This makes sense since they fall outside the range of values that could reasonably be expected to occur, if the prediction were correct and the standard deviation appropriately quantified. See prediction interval.

While the standard deviation does measure how far typical values tend to be from the mean, other measures are available. An example is the mean absolute deviation, which might be considered a more direct measure of average distance, compared to the root mean square distance inherent in the standard deviation.

Application examples

The practical value of understanding the standard deviation of a set of values is in appreciating how much variation there is from the "average" (mean).

Climate

As a simple example, consider the average daily maximum temperatures for two cities, one inland and one on the coast. It is helpful to understand that the range of daily maximum temperatures for cities near the coast is smaller than for cities inland. Thus, while these two cities may each have the same average maximum temperature, the standard deviation of the daily maximum temperature for the coastal city will be less than that of the inland city as, on any particular day, the actual maximum temperature is more likely to be farther from the average maximum temperature for the inland city than for the coastal one.

Particle physics

Particle physics uses a standard of "5 sigma" for the declaration of a discovery.[3] At five-sigma there is only one chance in nearly two million that a random fluctuation would yield the result. This level of certainty prompted the announcement that a particle consistent with the Higgs boson has been discovered in two independent experiments at CERN.[4]

Sports

Another way of seeing it is to consider sports teams. In any set of categories, there will be teams that rate highly at some things and poorly at others. Chances are, the teams that lead in the standings will not show such disparity but will perform well in most categories. The lower the standard deviation of their ratings in each category, the more balanced and consistent they will tend to be. Teams with a higher standard deviation, however, will be more unpredictable. For example, a team that is consistently bad in most categories will have a low standard deviation. A team that is consistently good in most categories will also have a low standard deviation. However, a team with a high standard deviation might be the type of team that scores a lot (strong offense) but also concedes a lot (weak defense), or, vice versa, that might have a poor offense but compensates by being difficult to score on.

Trying to predict which teams, on any given day, will win, may include looking at the standard deviations of the various team "stats" ratings, in which anomalies can match strengths vs. weaknesses to attempt to understand what factors may prevail as stronger indicators of eventual scoring outcomes.

In racing, a driver is timed on successive laps. A driver with a low standard deviation of lap times is more consistent than a driver with a higher standard deviation. This information can be used to help understand where opportunities might be found to reduce lap times.

Finance

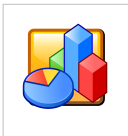
In finance, standard deviation is often used as a measure of the risk associated with price-fluctuations of a given asset (stocks, bonds, property, etc.), or the risk of a portfolio of assets [5] (actively managed mutual funds, index mutual funds, or ETFs). Risk is an important factor in determining how to efficiently manage a portfolio of investments because it determines the variation in returns on the asset and/or portfolio and gives investors a mathematical basis for investment decisions (known as mean-variance optimization). The fundamental concept of risk is that as it increases, the expected return on an investment should increase as well, an increase known as the "risk premium." In other words, investors should expect a higher return on an investment when that investment carries a higher level of risk or uncertainty. When evaluating investments, investors should estimate both the expected return and the uncertainty of future returns. Standard deviation provides a quantified estimate of the uncertainty of future returns.

For example, let's assume an investor had to choose between two stocks. Stock A over the past 20 years had an average return of 10 percent, with a standard deviation of 20 percentage points (pp) and Stock B, over the same period, had average returns of 12 percent but a higher standard deviation of 30 pp. On the basis of risk and return, an investor may decide that Stock A is the safer choice, because Stock B's additional two percentage points of return is not worth the additional 10 pp standard deviation (greater risk or uncertainty of the expected return). Stock B is likely to fall short of the initial investment (but also to exceed the initial investment) more often than Stock A under the same circumstances, and is estimated to return only two percent more on average. In this example, Stock A is expected to earn about 10 percent, plus or minus 20 pp (a range of 30 percent to -10 percent), about two-thirds of the future year returns. When considering more extreme possible returns or outcomes in future, an investor should expect results of as much as 10 percent plus or minus 60 pp, or a range from 70 percent to −50 percent, which includes outcomes for three standard deviations from the average return (about 99.7 percent of probable returns).

Calculating the average (or arithmetic mean) of the return of a security over a given period will generate the expected return of the asset. For each period, subtracting the expected return from the actual return results in the difference from the mean. Squaring the difference in each period and taking the average gives the overall variance of the return of the asset. The larger the variance, the greater risk the security carries. Finding the square root of this variance will give the standard deviation of the investment tool in question.

Population standard deviation is used to set the width of Bollinger Bands, a widely adopted technical analysis tool. For example, the upper Bollinger Band is given as $x + n\sigma_x$. The most commonly used value for n is 2; there is about a five percent chance of going outside, assuming a normal distribution of returns.

Geometric interpretation



It is requested that a **diagram** or **diagrams** be included in this article to improve its quality. Specific illustrations, plots or diagrams can be requested at the Graphic Lab.
For more information, refer to discussion on this page and/or the listing at Wikipedia:Requested images.

To gain some geometric insights and clarification, we will start with a population of three values, x_1, x_2, x_3. This defines a point $P = (x_1, x_2, x_3)$ in $\mathbf{R}^3$. Consider the line $L = \{(r, r, r) : r \in \mathbf{R}\}$. This is the "main diagonal" going through the origin. If our three given values were all equal, then the standard deviation would be zero and P would lie on L. So it is not unreasonable to assume that the standard deviation is related to the *distance* of P to L. And that is indeed the case. To move orthogonally from L to the point P, one begins at the point:

$$M = (\overline{x}, \overline{x}, \overline{x})$$

whose coordinates are the mean of the values we started out with. A little algebra shows that the distance between P and M (which is the same as the orthogonal distance between P and the line L) is equal to the standard deviation of the vector x_1, x_2, x_3, multiplied by the square root of the number of dimensions of the vector (3 in this case.)

Chebyshev's inequality

An observation is rarely more than a few standard deviations away from the mean. Chebyshev's inequality ensures that, for all distributions for which the standard deviation is defined, the amount of data within a number of standard deviations of the mean is at least as much as given in the following table.

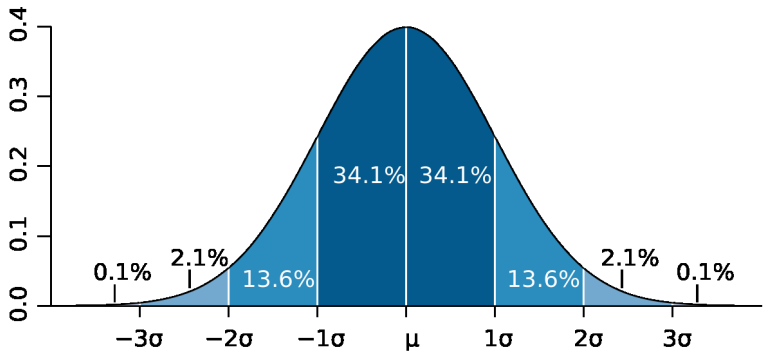
Minimum population	Distance from mean
50%	$\sqrt{2}$
75%	2
89%	3
94%	4
96%	5
97%	6
$1-\frac{1}{k^2}$[6]	k
l	$\frac{1}{\sqrt{1-l}}$

Rules for normally distributed data

The central limit theorem says that the distribution of an average of many independent, identically distributed random variables tends toward the famous bell-shaped normal distribution with a probability density function of:

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where μ is the expected value of the random variables, σ equals their distribution's standard deviation divided by $n^{1/2}$, and n is the number of random variables. The standard deviation therefore is simply a scaling variable that adjusts how broad the curve will be, though it also appears in the normalizing constant.



Dark blue is less than one standard deviation from the mean. For the normal distribution, this accounts for 68.27 percent of the set; while two standard deviations from the mean (medium and dark blue) account for 95.45 percent; three standard deviations (light, medium, and dark blue) account for 99.73 percent; and four standard deviations account for 99.994 percent. The two points of the curve that are one standard deviation from the mean are also the inflection points.

If a data distribution is approximately normal, then the proportion of data values within z standard deviations of the mean is defined by:

$$\text{Proportion} = \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right)$$

where erf is the error function. If a data distribution is approximately normal then about 68 percent of the data values are within one standard deviation of the mean (mathematically, μ ± σ, where μ is the arithmetic mean), about 95 percent are within two standard deviations (μ ± 2σ), and about 99.7 percent lie within three standard deviations (μ ± 3σ). This is known as the *68-95-99.7 rule*, or *the empirical rule*.

For various values of z, the percentage of values expected to lie in and outside the symmetric interval, CI = $(-z\sigma, z\sigma)$, are as follows:

zσ	Percentage within CI	Percentage outside CI	Fraction outside CI
0674490σ	50%	50%	1 / 2
0994458σ	68%	32%	1 / 3.125
1σ	682689492%	317310508%	1 / 31514872
1281552σ	80%	20%	1 / 5
1644854σ	90%	10%	1 / 10
1959964σ	95%	5%	1 / 20
2σ	954499736%	45500264%	1 / 21977895
2575829σ	99%	1%	1 / 100
3σ	997300204%	02699796%	1 / 370.398
3290527σ	99.9%	0.1%	1 / 1,000
3890592σ	99.99%	0.01%	1 / 10,000
4σ	99993666%	0006334%	1 / 15,787
4417173σ	99.999%	0.001%	1 / 100,000
4891638σ	999999%	00001%	1 / 1,000,000
5σ	999999426697%	00000573303%	1 / 1,744,278
5326724σ	9999999%	000001%	1 / 10,000,000
5730729σ	99999999%	0000001%	1 / 100,000,000
6σ	999999998027%	00000001973%	1 / 506,797,346
6109410σ	999999999%	00000001%	1 / 1,000,000,000
6466951σ	9999999999%	000000001%	1 / 10,000,000,000
6806502σ	99999999999%	0000000001%	1 / 100,000,000,000
7σ	99.9999999997440%	0000000000256%	1 / 390,682,215,445

Relationship between standard deviation and mean

The mean and the standard deviation of a set of data are usually reported together. In a certain sense, the standard deviation is a "natural" measure of statistical dispersion if the center of the data is measured about the mean. This is because the standard deviation from the mean is smaller than from any other point. The precise statement is the following: suppose $x_1, ..., x_n$ are real numbers and define the function:

$$\sigma(r) = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - r)^2}.$$

Using calculus or by completing the square, it is possible to show that σ(*r*) has a unique minimum at the mean:

$$r = \overline{x}.$$

Variability can also be measured by the coefficient of variation, which is the ratio of the standard deviation to the mean. It is a dimensionless number.

Often, we want some information about the precision of the mean we obtained. We can obtain this by determining the standard deviation of the sampled mean. The standard deviation of the mean is related to the standard deviation of the distribution by:

$$\sigma_{\text{mean}} = \frac{1}{\sqrt{N}}\sigma$$

where N is the number of observations in the sample used to estimate the mean. This can easily be proven with:

$$\begin{aligned}\operatorname{var}(X) &\equiv \sigma_X^2 \\ \operatorname{var}(X_1 + X_2) &\equiv \operatorname{var}(X_1) + \operatorname{var}(X_2) \\ \operatorname{var}(cX_1) &\equiv c^2 \operatorname{var}(X_1)\end{aligned}$$

hence

$$\begin{aligned}\operatorname{var}(\text{mean}) &= \operatorname{var}\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) = \frac{1}{N^2}\operatorname{var}\left(\sum_{i=1}^{N} X_i\right) \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\operatorname{var}(X_i) = \frac{N}{N^2}\operatorname{var}(X) = \frac{1}{N}\operatorname{var}(X).\end{aligned}$$

Resulting in:

$$\sigma_{\text{mean}} = \frac{\sigma}{\sqrt{N}}.$$

Rapid calculation methods

The following two formulas can represent a running (continuous) standard deviation. A set of three power sums s_0, s_1, s_2 are each computed over a set of N values of x, denoted as x_1, ..., x_N:

$$s_j = \sum_{k=1}^{N} x_k^j.$$

Note that s_0 raises x to the zero power, and since x^0 is always 1, s_0 evaluates to N.

Given the results of these three running summations, the values s_0, s_1, s_2 can be used at any time to compute the *current* value of the running standard deviation:

$$\sigma = \frac{\sqrt{s_0 s_2 - s_1^2}}{s_0}$$

Similarly for sample standard deviation,

$$s = \sqrt{\frac{s_0 s_2 - s_1^2}{s_0(s_0 - 1)}}.$$

In a computer implementation, as the three s_j sums become large, we need to consider round-off error, arithmetic overflow, and arithmetic underflow. The method below calculates the running sums method with reduced rounding errors.[7] This is a "one pass" algorithm for calculating variance of n samples without the need to store prior data during the calculation. Applying this method to a time series will result in successive values of standard deviation corresponding to n data points as n grows larger with each new sample, rather than a constant-width sliding window calculation.

For $k = 0, ..., n$:

$$\begin{aligned}A_0 &= 0 \\ A_k &= A_{k-1} + \frac{x_k - A_{k-1}}{k}\end{aligned}$$

where A is the mean value.

$$\begin{aligned}Q_0 &= 0 \\ Q_k &= Q_{k-1} + \frac{k-1}{k}(x_k - A_{k-1})^2 = Q_{k-1} + (x_k - A_{k-1})(x_k - A_k)\end{aligned}$$

Sample variance:

$$s_n^2 = \frac{Q_n}{n-1}$$

Standard variance:

$$\sigma_n^2 = \frac{Q_n}{n}$$

Weighted calculation

When the values x_i are weighted with unequal weights w_i, the power sums s_0, s_1, s_2 are each computed as:

$$s_j = \sum_{k=1}^{N} w_k x_k^j.$$

And the standard deviation equations remain unchanged. Note that s_0 is now the sum of the weights and not the number of samples *N*.

The incremental method with reduced rounding errors can also be applied, with some additional complexity.

A running sum of weights must be computed for each *k* from 1 to *n*:

$$W_0 = 0$$
$$W_k = W_{k-1} + w_k$$

and places where 1/*n* is used above must be replaced by w_i/W_n:

$$A_0 = 0$$
$$A_k = A_{k-1} + \frac{w_k}{W_k}(x_k - A_{k-1})$$
$$Q_0 = 0$$
$$Q_k = Q_{k-1} + \frac{w_k W_{k-1}}{W_k}(x_k - A_{k-1})^2 = Q_{k-1} + w_k(x_k - A_{k-1})(x_k - A_k)$$

In the final division,

$$\sigma_n^2 = \frac{Q_n}{W_n}$$

and

$$s_n^2 = \frac{n'}{n'-1}\sigma_n^2$$

where n is the total number of elements, and n' is the number of elements with non-zero weights. The above formulas become equal to the simpler formulas given above if weights are taken as equal to one.

Combining standard deviations

Population-based statistics

The populations of sets, which may overlap, can be calculated simply as follows:

$$\begin{aligned} & N_{X\cup Y} = N_X + N_Y - N_{X\cap Y} \\ X \cap Y = \varnothing \Rightarrow \quad & N_{X\cap Y} = 0 \\ \Rightarrow \quad & N_{X\cup Y} = N_X + N_Y \end{aligned}$$

Standard deviations of non-overlapping ($X \cap Y = \varnothing$) sub-populations can be aggregated as follows if the size (actual or relative to one another) and means of each are known:

$$\mu_{X\cup Y} = \frac{N_X\mu_X + N_Y\mu_Y}{N_X + N_Y}$$
$$\sigma_{X\cup Y} = \sqrt{\frac{N_X\sigma_X^2 + N_Y\sigma_Y^2}{N_X + N_Y} + \frac{N_X N_Y}{(N_X + N_Y)^2}(\mu_X - \mu_Y)^2}$$

For example, suppose it is known that the average American man has a mean height of 70 inches with a standard deviation of three inches and that the average American woman has a mean height of 65 inches with a standard deviation of two inches. Also assume that the number of men, *N*, is equal to the number of women. Then the mean and standard deviation of heights of American adults could be calculated as:

$$\mu = \frac{N\cdot 70 + N\cdot 65}{N+N} = \frac{70+65}{2} = 67.5$$
$$\sigma = \sqrt{\frac{3^2+2^2}{2} + \frac{(70-65)^2}{2^2}} = \sqrt{12.75} \approx 3.57$$

For the more general case of *M* non-overlapping populations, X_1 through X_M, and the aggregate population $X = \bigcup_i X_i$:

$$\mu_X = \frac{\sum_i N_{X_i}\mu_{X_i}}{\sum_i N_{X_i}}$$
$$\sigma_X = \sqrt{\frac{\sum_i N_{X_i}(\sigma_{X_i}^2 + \mu_{X_i}^2)}{\sum_i N_{X_i}} - \mu_X^2} = \sqrt{\frac{\sum_i N_{X_i}\sigma_{X_i}^2}{\sum_i N_{X_i}} + \frac{\sum_{i<j} N_{X_i}N_{X_j}(\mu_{X_i} - \mu_{X_j})^2}{\left(\sum_i N_{X_i}\right)^2}}$$

where

$$X_i \cap X_j = \varnothing, \quad \forall\, i < j.$$

If the size (actual or relative to one another), mean, and standard deviation of two overlapping populations are known for the populations as well as their intersection, then the standard deviation of the overall population can still be calculated as follows:

$$\mu_{X\cup Y} = \frac{1}{N_{X\cup Y}}\left(N_X\mu_X + N_Y\mu_Y - N_{X\cap Y}\mu_{X\cap Y}\right)$$
$$\sigma_{X\cup Y} = \sqrt{\frac{1}{N_{X\cup Y}}\left(N_X[\sigma_X^2 + \mu_X^2] + N_Y[\sigma_Y^2 + \mu_Y^2] - N_{X\cap Y}[\sigma_{X\cap Y}^2 + \mu_{X\cap Y}^2]\right) - \mu_{X\cup Y}^2}$$

If two or more sets of data are being added together datapoint by datapoint, the standard deviation of the result can be calculated if the standard deviation of each data set and the covariance between each pair of data sets is known:

$$\sigma_X = \sqrt{\sum_i \sigma_{X_i}^2 + \sum_{i,j} \operatorname{cov}(X_i, X_j)}$$

For the special case where no correlation exists between any pair of data sets, then the relation reduces to the root-mean-square:

$$\operatorname{cov}(X_i, X_j) = 0, \quad \forall i < j$$
$$\Rightarrow \sigma_X = \sqrt{\sum_i \sigma_{X_i}^2}.$$

Sample-based statistics

Standard deviations of non-overlapping ($X \cap Y = \varnothing$) sub-samples can be aggregated as follows if the actual size and means of each are known:

$$\mu_{X\cup Y} = \frac{1}{N_{X\cup Y}}\left(N_X\mu_X + N_Y\mu_Y\right)$$

$$\sigma_{X\cup Y} = \sqrt{\frac{1}{N_{X\cup Y}-1}\left([N_X-1]\sigma_X^2 + N_X\mu_X^2 + [N_Y-1]\sigma_Y^2 + N_Y\mu_Y^2 - [N_X+N_Y]\mu_{X\cup Y}^2\right)}$$

For the more general case of M non-overlapping data sets, X_1 through X_M, and the aggregate data set $X = \bigcup_i X_i$:

$$\mu_X = \frac{1}{\sum_i N_{X_i}}\left(\sum_i N_{X_i}\mu_{X_i}\right)$$

$$\sigma_X = \sqrt{\frac{1}{\sum_i N_{X_i}-1}\left(\sum_i\left[(N_{X_i}-1)\sigma_{X_i}^2 + N_{X_i}\mu_{X_i}^2\right] - \left[\sum_i N_{X_i}\right]\mu_X^2\right)}$$

where:

$$X_i \cap X_j = \varnothing, \quad \forall i < j.$$

If the size, mean, and standard deviation of two overlapping samples are known for the samples as well as their intersection, then the standard deviation of the aggregated sample can still be calculated. In general:

$$\mu_{X\cup Y} = \frac{1}{N_{X\cup Y}}\left(N_X\mu_X + N_Y\mu_Y - N_{X\cap Y}\mu_{X\cap Y}\right)$$

$$\sigma_{X\cup Y} = \sqrt{\frac{1}{N_{X\cup Y}-1}\left([N_X-1]\sigma_X^2 + N_X\mu_X^2 + [N_Y-1]\sigma_Y^2 + N_Y\mu_Y^2 - [N_{X\cap Y}-1]\sigma_{X\cap Y}^2 - N_{X\cap Y}\mu_{X\cap Y}^2 - [N_X+N_Y-N_{X\cap Y}]\mu_{X\cup Y}^2\right)}$$

History

The term *standard deviation* was first used[8] in writing by Karl Pearson[9] in 1894, following his use of it in lectures. This was as a replacement for earlier alternative names for the same idea: for example, Gauss used *mean error*.[10]

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- Standard Deviation, a simpler explanation for writers and journalists (http://www.robertniles.com/stats/stdev.shtml)
- The concept of Standard Deviation is shown in this 8-foot-tall (2.4 m) Probability Machine (named Sir Francis) comparing stock market returns to the randomness of the beans dropping through the quincunx pattern. (http://www.youtube.com/watch?v=AUSKTk9ENzg) from Index Funds Advisors IFA.com (http://www.ifa.com)

Technical analysis

In finance, **technical analysis** is a security analysis discipline used for forecasting the direction of prices through the study of past market data, primarily price and volume.[1] Behavioral economics and quantitative analysis use many of the same tools of technical analysis [2] [3][4] [5], which, being an aspect of active management, stands in contradiction to much of modern portfolio theory. The efficacy of both technical and fundamental analysis is disputed by efficient-market hypothesis which states that stock market prices are essentially unpredictable.[6]

History

The principles of technical analysis are derived from hundreds of years of financial markets data.[7] Some aspects of technical analysis began to appear in Joseph de la Vega's accounts of the Dutch markets in the 17th century. In Asia, technical analysis is said to be a method developed by Homma Munehisa during early 18th century which evolved into the use of candlestick techniques, and is today a technical analysis charting tool.[8][9] In the 1920s and 1930s Richard W. Schabacker published several books which continued the work of Charles Dow and William Peter Hamilton in their books *Stock Market Theory and Practice* and *Technical Market Analysis*. In 1948 Robert D. Edwards and John Magee published *Technical Analysis of Stock Trends* which is widely considered to be one of the seminal works of the discipline. It is exclusively concerned with trend analysis and chart patterns and remains in use to the present. As is obvious, early technical analysis was almost exclusively the analysis of charts, because the processing power of computers was not available for statistical analysis. Charles Dow reportedly originated a form of point and figure chart analysis.

Dow Theory is based on the collected writings of Dow Jones co-founder and editor Charles Dow, and inspired the use and development of modern technical analysis at the end of the 19th century. Other pioneers of analysis techniques include Ralph Nelson Elliott, William Delbert Gann and Richard Wyckoff who developed their respective techniques in the early 20th century. More technical tools and theories have been developed and enhanced in recent decades, with an increasing emphasis on computer-assisted techniques using specially designed computer software.

General description

While fundamental analysts examine earnings, dividends, new products, research and the like, technical analysts examine what investors fear or think about those developments and whether or not investors have the wherewithal to back up their opinions; these two concepts are called psych (psychology) and supply/demand. Technicians employ many techniques, one of which is the use of charts. Using charts, technical analysts seek to identify price patterns and market trends in financial markets and attempt to exploit those patterns.[10] Technicians use various methods and tools, the study of price charts is but one.

Technicians using charts search for archetypal price chart patterns, such as the well-known head and shoulders or double top/bottom reversal patterns, study technical indicators, moving averages, and look for forms such as lines of support, resistance, channels, and more obscure formations such as flags, pennants, balance days and cup and handle patterns.[11]

Technical analysts also widely use market indicators of many sorts, some of which are mathematical transformations of price, often including up and down volume, advance/decline data and other inputs. These indicators are used to help assess whether an asset is trending, and if it is, the probability of its direction and of continuation. Technicians also look for relationships between price/volume indices and market indicators. Examples include the relative strength index, and MACD. Other avenues of study include correlations between changes in Options (implied volatility) and put/call ratios with price. Also important are sentiment indicators such as Put/Call ratios, bull/bear ratios, short interest, Implied Volatility, etc.

There are many techniques in technical analysis. Adherents of different techniques (for example, candlestick charting, Dow Theory, and Elliott wave theory) may ignore the other approaches, yet many traders combine elements from more than one technique. Some technical analysts use subjective judgment to decide which pattern(s) a particular instrument reflects at a given time and what the interpretation of that pattern should be. Others employ a strictly mechanical or systematic approach to pattern identification and interpretation.

Technical analysis is frequently contrasted with *fundamental analysis*, the study of economic factors that influence the way investors price financial markets. Technical analysis holds that prices already reflect all such trends before investors are aware of them. Uncovering those trends is what technical indicators are designed to do, imperfect as they may be. Fundamental indicators are subject to the same limitations, naturally. Some traders use technical or fundamental analysis exclusively, while others use both types to make trading decisions.[12]

Characteristics

Technical analysis employs models and trading rules based on price and volume transformations, such as the relative strength index, moving averages, regressions, inter-market and intra-market price correlations, business cycles, stock market cycles or, classically, through recognition of chart patterns.

Technical analysis stands in contrast to the fundamental analysis approach to security and stock analysis. Technical analysis analyzes price, volume and other market information, whereas fundamental analysis looks at the facts of the company, market, currency or commodity. Most large brokerage, trading group, or financial institutions will typically have both a technical analysis and fundamental analysis team.

Technical analysis is widely used among traders and financial professionals and is very often used by active day traders, market makers and pit traders. In the 1960s and 1970s it was widely dismissed by academics. In a recent review, Irwin and Park[13] reported that 56 of 95 modern studies found that it produces positive results but noted that many of the positive results were rendered dubious by issues such as data snooping, so that the evidence in support of technical analysis was inconclusive; it is still considered by many academics to be pseudoscience.[14] Academics such as Eugene Fama say the evidence for technical analysis is sparse and is inconsistent with the *weak form* of the efficient-market hypothesis.[15][16] Users hold that even if technical analysis cannot predict the future, it helps to identify trading opportunities.[17]

In the foreign exchange markets, its use may be more widespread than fundamental analysis.[18][19] This does not mean technical analysis is more applicable to foreign markets, but that technical analysis is more recognized as to its efficacy there than elsewhere. While some isolated studies have indicated that technical trading rules might lead to consistent returns in the period prior to 1987,[20][21][22][23] most academic work has focused on the nature of the anomalous position of the foreign exchange market.[24] It is speculated that this anomaly is due to central bank intervention, which obviously technical analysis is not designed to predict.[25] Recent research suggests that combining various trading signals into a Combined Signal Approach may be able to increase profitability and reduce dependence on any single rule.[26]

Principles

A fundamental principle of technical analysis is that a market's price reflects all relevant information, so their analysis looks at the history of a security's trading pattern rather than external drivers such as economic, fundamental and news events. Therefore, price action would also tend to repeat itself due many investors collectively tend toward patterned behavior – hence technicians' focus on identifiable trends and conditions.[27][28]

Stock chart showing levels of support (4,5,6, 7, and 8) and resistance (1, 2, and 3); levels of resistance tend to become levels of support and vice versa.

Market action discounts everything

Based on the premise that all relevant information is already reflected by prices, technical analysts believe it is important to understand what investors think of that information, known and perceived.

Prices move in trends

Technical analysts believe that prices trend directionally, i.e., up, down, or sideways (flat) or some combination. The basic definition of a price trend was originally put forward by Dow Theory.[10]Murphy, John J. *Technical Analysis of the Financial Markets*. New York Institute of Finance, 1999, pp. 1-5, 24-31. ISBN 0-7352-0066-1</ref>

An example of a security that had an apparent trend is AOL from November 2001 through August 2002. A technical analyst or trend follower recognizing this trend would look for opportunities to sell this security. AOL consistently moves downward in price. Each time the stock rose, sellers would enter the market and sell the stock; hence the "zig-zag" movement in the price. The series of "lower highs" and "lower lows" is a tell tale sign of a stock in a down trend.[29] In other words, each time the stock moved lower, it fell below its previous relative low price. Each time the stock moved higher, it could not reach the level of its previous relative high price.

Note that the sequence of lower lows and lower highs did not begin until August. Then AOL makes a low price that does not pierce the relative low set earlier in the month. Later in the same month, the stock makes a relative high equal to the most recent relative high. In this a technician sees strong indications that the down trend is at least pausing and possibly ending, and would likely stop actively selling the stock at that point.

History tends to repeat itself

Technical analysts believe that investors collectively repeat the behavior of the investors that preceded them. To a technician, the emotions in the market may be irrational, but they exist. Because investor behavior repeats itself so often, technicians believe that recognizable (and predictable) price patterns will develop on a chart.[10]

Technical analysis is not limited to charting, but it always considers price trends.[1] For example, many technicians monitor surveys of investor sentiment. These surveys gauge the attitude of market participants, specifically whether they are bearish or bullish. Technicians use these surveys to help determine whether a trend will continue or if a reversal could develop; they are most likely to anticipate a change when the surveys report extreme investor sentiment[30] Surveys that show overwhelming bullishness, for example, are evidence that an uptrend may reverse; the premise being that if most investors are bullish they have already bought the market (anticipating higher prices). And because most investors *are* bullish and invested, one assumes that few buyers remain. This leaves more potential sellers than buyers, despite the bullish sentiment. This suggests that prices will trend down, and is an example of contrarian trading.[31]

Recently, Kim Man Lui, Lun Hu, and Keith C.C. Chan have suggested that there is statistical evidence of association relationships between some of the index composite stocks whereas there is no evidence for such a relationship between some index composite others. They show that the price behavior of these Hang Seng index composite stocks is easier to understand than that of the index.[32]

Industry

The industry is globally represented by the International Federation of Technical Analysts (IFTA), which is a Federation of regional and national organizations. In the United States, the industry is represented by both the Market Technicians Association (MTA) and the American Association of Professional Technical Analysts (AAPTA). The United States is also represented by the Technical Security Analysts Association of San Francisco (TSAASF). In the United Kingdom, the industry is represented by the Society of Technical Analysts (STA). In Canada the industry is represented by the Canadian Society of Technical Analysts.[33] In Australia, the industry is represented by the Australian Professional Technical Analysts (APTA) Inc [34] and the Australian Technical Analysts Association (ATAA).

Professional technical analysis societies have worked on creating a body of knowledge that describes the field of Technical Analysis. A body of knowledge is central to the field as a way of defining how and why technical analysis may work. It can then be used by academia, as well as regulatory bodies, in developing proper research and standards for the field.[35] The Market Technicians Association (MTA) has published a body of knowledge, which is the structure for the MTA's Chartered Market Technician (CMT) exam.[36]

Systematic trading

Neural networks

Since the early 1990s when the first practically usable types emerged, artificial neural networks (ANNs) have rapidly grown in popularity. They are artificial intelligence adaptive software systems that have been inspired by how biological neural networks work. They are used because they can learn to detect complex patterns in data. In mathematical terms, they are universal function approximators,[37][38] meaning that given the right data and configured correctly, they can capture and model any input-output relationships. This not only removes the need for human interpretation of charts or the series of rules for generating entry/exit signals, but also provides a bridge to fundamental analysis, as the variables used in fundamental analysis can be used as input.

As ANNs are essentially non-linear statistical models, their accuracy and prediction capabilities can be both mathematically and empirically tested. In various studies, authors have claimed that neural networks used for

generating trading signals given various technical and fundamental inputs have significantly outperformed buy-hold strategies as well as traditional linear technical analysis methods when combined with rule-based expert systems.[39][40][41]

While the advanced mathematical nature of such adaptive systems has kept neural networks for financial analysis mostly within academic research circles, in recent years more user friendly neural network software has made the technology more accessible to traders. However, large-scale application is problematic because of the problem of matching the correct neural topology to the market being studied.

Backtesting

Systematic trading is most often employed after testing an investment strategy on historic data. This is known as backtesting. Backtesting is most often performed for technical indicators, but can be applied to most investment strategies (e.g. fundamental analysis). While traditional backtesting was done by hand, this was usually only performed on human-selected stocks, and was thus prone to prior knowledge in stock selection. With the advent of computers, backtesting can be performed on entire exchanges over decades of historic data in very short amounts of time.

The use of computers does have its drawbacks, being limited to algorithms that a computer can perform. Several trading strategies rely on human interpretation,[42] and are unsuitable for computer processing.[43] Only technical indicators which are entirely algorithmic can be programmed for computerised automated backtesting.

Combination with other market forecast methods

John Murphy states that the principal sources of information available to technicians are price, volume and open interest.[10] Other data, such as indicators and sentiment analysis, are considered secondary.

However, many technical analysts reach outside pure technical analysis, combining other market forecast methods with their technical work. One advocate for this approach is John Bollinger, who coined the term *rational analysis* in the middle 1980s for the intersection of technical analysis and fundamental analysis.[44] Another such approach, fusion analysis,[45] overlays fundamental analysis with technical, in an attempt to improve portfolio manager performance.

Technical analysis is also often combined with quantitative analysis and economics. For example, neural networks may be used to help identify intermarket relationships.[46] A few market forecasters combine financial astrology with technical analysis. Chris Carolan's article "Autumn Panics and Calendar Phenomenon", which won the Market Technicians Association Dow Award for best technical analysis paper in 1998, demonstrates how technical analysis and lunar cycles can be combined.[47] Calendar phenomena, such as the January effect in the stock market, are generally believed to be caused by tax and accounting related transactions, and are not related to the subject of financial astrology.

Investor and newsletter polls, and magazine cover sentiment indicators, are also used by technical analysts.[48]

Empirical evidence

Whether technical analysis actually works is a matter of controversy. Methods vary greatly, and different technical analysts can sometimes make contradictory predictions from the same data. Many investors claim that they experience positive returns, but academic appraisals often find that it has little predictive power.[49] Of 95 modern studies, 56 concluded that technical analysis had positive results, although data-snooping bias and other problems make the analysis difficult.[13] Nonlinear prediction using neural networks occasionally produces statistically significant prediction results.[50] A Federal Reserve working paper[21] regarding support and resistance levels in short-term foreign exchange rates "offers strong evidence that the levels help to predict intraday trend interruptions," although the "predictive power" of those levels was "found to vary across the exchange rates and firms examined".

Technical trading strategies were found to be effective in the Chinese marketplace by a recent study that states, "Finally, we find significant positive returns on buy trades generated by the contrarian version of the moving-average crossover rule, the channel breakout rule, and the Bollinger band trading rule, after accounting for transaction costs of 0.50 percent."[51]

An influential 1992 study by Brock et al. which appeared to find support for technical trading rules was tested for data snooping and other problems in 1999;[52] the sample covered by Brock et al. was robust to data snooping.

Subsequently, a comprehensive study of the question by Amsterdam economist Gerwin Griffioen concludes that: "for the U.S., Japanese and most Western European stock market indices the recursive out-of-sample forecasting procedure does not show to be profitable, after implementing little transaction costs. Moreover, for sufficiently high transaction costs it is found, by estimating CAPMs, that technical trading shows no statistically significant risk-corrected out-of-sample forecasting power for almost all of the stock market indices."[16] Transaction costs are particularly applicable to "momentum strategies"; a comprehensive 1996 review of the data and studies concluded that even small transaction costs would lead to an inability to capture any excess from such strategies.[53]

In a paper published in the Journal of Finance, Dr. Andrew W. Lo, director MIT Laboratory for Financial Engineering, working with Harry Mamaysky and Jiang Wang found that "

> Technical analysis, also known as "charting," has been a part of financial practice for many decades, but this discipline has not received the same level of academic scrutiny and acceptance as more traditional approaches such as fundamental analysis. One of the main obstacles is the highly subjective nature of technical analysis – the presence of geometric shapes in historical price charts is often in the eyes of the beholder. In this paper, we propose a systematic and automatic approach to technical pattern recognition using nonparametric kernel regression, and apply this method to a large number of U.S. stocks from 1962 to 1996 to evaluate the effectiveness of technical analysis. By comparing the unconditional empirical distribution of daily stock returns to the conditional distribution – conditioned on specific technical indicators such as head-and-shoulders or double-bottoms – we find that over the 31-year sample period, several technical indicators do provide incremental information and may have some practical value.[54]

In that same paper Dr. Lo wrote that "several academic studies suggest that ... technical analysis may well be an effective means for extracting useful information from market prices."[55] Some techniques such as Drummond Geometry attempt to overcome the past data bias by projecting support and resistance levels from differing time frames into the near-term future and combining that with reversion to the mean techniques.[56]

Efficient market hypothesis

The efficient-market hypothesis (EMH) contradicts the basic tenets of technical analysis by stating that past prices cannot be used to profitably predict future prices. Thus it holds that technical analysis cannot be effective. Economist Eugene Fama published the seminal paper on the EMH in the *Journal of Finance* in 1970, and said "In short, the evidence in support of the efficient markets model is extensive, and (somewhat uniquely in economics) contradictory evidence is sparse."[57]

Technicians say that EMH ignores the way markets work, in that many investors base their expectations on past earnings or track record, for example. Because future stock prices can be strongly influenced by investor expectations, technicians claim it only follows that past prices influence future prices.[58] They also point to research in the field of behavioral finance, specifically that people are not the rational participants EMH makes them out to be. Technicians have long said that irrational human behavior influences stock prices, and that this behavior leads to predictable outcomes.[59] Author David Aronson says that the theory of behavioral finance blends with the practice of technical analysis:

> By considering the impact of emotions, cognitive errors, irrational preferences, and the dynamics of group behavior, behavioral finance offers succinct explanations of excess market volatility as well as the excess returns earned by stale information strategies.... cognitive errors may also explain the existence

> of market inefficiencies that spawn the systematic price movements that allow objective TA [technical analysis] methods to work.[58]

EMH advocates reply that while individual market participants do not always act rationally (or have complete information), their aggregate decisions balance each other, resulting in a rational outcome (optimists who buy stock and bid the price higher are countered by pessimists who sell their stock, which keeps the price in equilibrium).[60] Likewise, complete information is reflected in the price because all market participants bring their own individual, but incomplete, knowledge together in the market.[60]

Random walk hypothesis

The random walk hypothesis may be derived from the weak-form efficient markets hypothesis, which is based on the assumption that market participants take full account of any information contained in past price movements (but not necessarily other public information). In his book *A Random Walk Down Wall Street*, Princeton economist Burton Malkiel said that technical forecasting tools such as pattern analysis must ultimately be self-defeating: "The problem is that once such a regularity is known to market participants, people will act in such a way that prevents it from happening in the future."[61] Malkiel has stated that while momentum may explain some stock price movements, there is not enough momentum to make excess profits. Malkiel has compared technical analysis to "astrology".[62]

In the late 1980s, professors Andrew Lo and Craig McKinlay published a paper which cast doubt on the random walk hypothesis. In a 1999 response to Malkiel, Lo and McKinlay collected empirical papers that questioned the hypothesis' applicability[63] that suggested a non-random and possibly predictive component to stock price movement, though they were careful to point out that rejecting random walk does not necessarily invalidate EMH, which is an entirely separate concept from RWH. In a 2000 paper, Andrew Lo back-analyzed data from U.S. from 1962 to 1996 and found that "several technical indicators do provide incremental information and may have some practical value".[55] Burton Malkiel dismissed the irregularities mentioned by Lo and McKinlay as being too small to profit from.[62]

Technicians say that the EMH and random walk theories both ignore the realities of markets, in that participants are not completely rational and that current price moves are not independent of previous moves.[29][64] Some signal processing researchers negate the random walk hypothesis that stock market prices resemble Wiener processes, because the statistical moments of such processes and real stock data vary significantly with respect window size and similarity measure.[65] They argue that feature transformations used for the description of audio and biosignals can also be used to predict stock market prices successfully which would contradict the random walk hypothesis.

The random walk index (RWI) is a technical indicator that attempts to determine if a stock's price movement is random in nature or a result of a statistically significant trend. The random walk index attempts to determine when the market is in a strong uptrend or downtrend by measuring price ranges over N and how it differs from what would be expected by a random walk (randomly going up or down). The greater the range suggests a stronger trend.[66]

Ticker tape reading

See the main article: Ticker tape

In recent decades with the popularity of PCs and later the internet, and through them, the electronic trading, the chart analysis became the main and most popular branch of technical analysis. But it is not the only one branch of this type of analysis.

One very popular form of technical analysis until the mid-1960s was the **"tape reading"**. It was consisted in reading the market informations as price, volume, orders size, speed, conditions, bids for buying and selling, etc; printed in a paper strip which ran through a machine called a stock ticker. It was sent to the brokerage houses and to the homes and offices of most active speculators. Such a system fell into disuse with the advent in the late 60's, of the electronic panels.

Quotation board

Another form of technical analysis used so far was via interpretation of stock market data contained in quotation boards, that in the times before electronic screens, were huge chalkboards located into the stock exchanges, with data of the main financial assets listed on exchanges for analysis of their movements.[67] It was manually updated with chalk, with the updates regarding some of these data being transmitted to environments outside of exchanges (such as brokerage houses, bucket shops, etc.) via the aforementioned tape, telegraph, telephone and later telex.[68]

This analysis tool was used both, on the spot, mainly by market professionals for day trading and scalping, as well as by general public through the printed versions in newspapers showing the data of the negotiations of the previous day, for swing and position trades.[69]

Despite to continue appearing in print in newspapers, as well as computerized versions in some websites, analysis via quotation board is another form of technical analysis that has fallen into disuse by the majority.

Charting terms and indicators

Concepts

- Resistance – a price level that may prompt a net increase of selling activity
- Support – a price level that may prompt a net increase of buying activity
- Breakout – the concept whereby prices forcefully penetrate an area of prior support or resistance, usually, but not always, accompanied by an increase in volume.
- Trending – the phenomenon by which price movement tends to persist in one direction for an extended period of time
- Average true range – averaged daily trading range, adjusted for price gaps
- Chart pattern – distinctive pattern created by the movement of security prices on a chart
- Dead cat bounce – the phenomenon whereby a spectacular decline in the price of a stock is immediately followed by a moderate and temporary rise before resuming its downward movement
- Elliott wave principle and the golden ratio to calculate successive price movements and retracements
- Fibonacci ratios – used as a guide to determine support and resistance
- Momentum – the rate of price change
- Point and figure analysis – A priced-based analytical approach employing numerical filters which may incorporate time references, though ignores time entirely in its construction.
- Cycles – time targets for potential change in price action (price only moves up, down, or sideways)

Types of charts

- Open-high-low-close chart – OHLC charts, also known as bar charts, plot the span between the high and low prices of a trading period as a vertical line segment at the trading time, and the open and close prices with horizontal tick marks on the range line, usually a tick to the left for the open price and a tick to the right for the closing price.
- Candlestick chart – Of Japanese origin and similar to OHLC, candlesticks widen and fill the interval between the open and close prices to emphasize the open/close relationship. In the West, often black or red candle bodies represent a close lower than the open, while white, green or blue candles represent a close higher than the open price.
- Line chart – Connects the closing price values with line segments.
- Point and figure chart – a chart type employing numerical filters with only passing references to time, and which ignores time entirely in its construction.

Overlays

Overlays are generally superimposed over the main price chart.

- Resistance – a price level that may act as a ceiling above price
- Support – a price level that may act as a floor below price
- Trend line – a sloping line described by at least two peaks or two troughs
- Channel – a pair of parallel trend lines
- Moving average – the last n-bars of price divided by "n" -- where "n" is the number of bars specified by the length of the average. A moving average can be thought of as a kind of dynamic trend-line.
- Bollinger bands – a range of price volatility
- Parabolic SAR – Wilder's trailing stop based on prices tending to stay within a parabolic curve during a strong trend
- Pivot point – derived by calculating the numerical average of a particular currency's or stock's high, low and closing prices
- Ichimoku kinko hyo – a moving average-based system that factors in time and the average point between a candle's high and low

Price-based indicators

These indicators are generally shown below or above the main price chart.

- Average directional index – a widely used indicator of trend strength
- Commodity Channel Index – identifies cyclical trends
- MACD – moving average convergence/divergence
- Momentum – the rate of price change
- Relative strength index (RSI) – oscillator showing price strength
- Stochastic oscillator – close position within recent trading range
- Trix – an oscillator showing the slope of a triple-smoothed exponential moving average
- %C – denotes current market environment as range expansion or a range contraction, it also forecast when extremes in trend or choppiness are being reached, so the trader can expect change.

Breadth Indicators

These indicators are based on statistics derived from the broad market

- Advance–decline line – a popular indicator of market breadth
- McClellan Oscillator - a popular closed-form indicator of breadth
- McClellan Summation Index - a popular open-form indicator of breadth

Volume-based indicators

- Accumulation/distribution index – based on the close within the day's range
- Money Flow – the amount of stock traded on days the price went up
- On-balance volume – the momentum of buying and selling stocks

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External links

International and national organizations

- International Federation of Technical Analysts (http://www.ifta.org)
- New Zealand: Society of Technical Analysts of New Zealand (http://www.stanz.co.nz)
- Singapore: Technical Analysts Society (Singapore) (http://www.tass.org.sg)

Volatility (finance)

In finance, **volatility** is a measure for variation of price of a financial instrument over time. Historic volatility is derived from time series of past market prices. An implied volatility is derived from the market price of a market traded derivative (in particular an option).

Volatility terminology

Volatility as described here refers to the **actual current** volatility of a financial instrument for a specified period (for example 30 days or 90 days). It is the volatility of a financial instrument based on historical prices over the specified period with the last observation the most recent price. This phrase is used particularly when it is wished to distinguish between the actual current volatility of an instrument and

- **actual historical** volatility which refers to the volatility of a financial instrument over a specified period but with the last observation on a date in the past
- **actual future** volatility which refers to the volatility of a financial instrument over a specified period starting at the current time and ending at a future date (normally the expiry date of an option)
- **historical implied volatility** which refers to the implied volatility observed from historical prices of the financial instrument (normally options)
- **current implied volatility** which refers to the implied volatility observed from current prices of the financial instrument
- **future implied volatility** which refers to the implied volatility observed from future prices of the financial instrument

For a financial instrument whose price follows a Gaussian random walk, or Wiener process, the width of the distribution increases as time increases. This is because there is an increasing probability that the instrument's price will be farther away from the initial price as time increases. However, rather than increase linearly, the volatility increases with the square-root of time as time increases, because some fluctuations are expected to cancel each other out, so the most likely deviation after twice the time will not be twice the distance from zero.

Since observed price changes do not follow Gaussian distributions, others such as the Lévy distribution are often used.[1] These can capture attributes such as "fat tails".

Volatility and Liquidity

Much research has been devoted to modeling and forecasting the volatility of financial returns, and yet few theoretical models explain how volatility comes to exist in the first place.

Roll (1984) shows that volatility is affected by market microstructure.[2] Glosten and Milgrom (1985) shows that at least one source of volatility can be explained by the liquidity provision process. When market makers infer the possibility of adverse selection, they adjust their trading ranges, which in turn increases the band of price oscillation.[3]

Volatility for investors

Investors care about volatility for five reasons. 1) The wider the swings in an investment's price the harder emotionally it is to not worry. 2) When certain cash flows from selling a security are needed at a specific future date, higher volatility means a greater chance of a shortfall. 3) Higher volatility of returns while saving for retirement results in a wider distribution of possible final portfolio values. 4) Higher volatility of return when retired gives withdrawals a larger permanent impact on the portfolio's value. 5) Price volatility presents opportunities to buy assets cheaply and sell when overpriced.[4]

In today's markets, it is also possible to trade volatility directly, through the use of derivative securities such as options and variance swaps. See Volatility arbitrage.

Volatility versus direction

Volatility does not measure the direction of price changes, merely their dispersion. This is because when calculating standard deviation (or variance), all differences are squared, so that negative and positive differences are combined into one quantity. Two instruments with different volatilities may have the same expected return, but the instrument with higher volatility will have larger swings in values over a given period of time.

For example, a lower volatility stock may have an expected (average) return of 7%, with annual volatility of 5%. This would indicate returns from approximately negative 3% to positive 17% most of the time (19 times out of 20, or 95% via a two standard deviation rule). A higher volatility stock, with the same expected return of 7% but with annual volatility of 20%, would indicate returns from approximately negative 33% to positive 47% most of the time (19 times out of 20, or 95%). These estimates assume a normal distribution; in reality stocks are found to be leptokurtotic.

Volatility over time

Although the Black Scholes equation assumes predictable constant volatility, this is not observed in real markets, and amongst the models are Bruno Dupire's Local Volatility, Poisson Process where volatility jumps to new levels with a predictable frequency, and the increasingly popular Heston model of Stochastic Volatility.[5]

It is common knowledge that types of assets experience periods of high and low volatility. That is, during some periods prices go up and down quickly, while during other times they barely move at all.

Periods when prices fall quickly (a crash) are often followed by prices going down even more, or going up by an unusual amount. Also, a time when prices rise quickly (a possible bubble) may often be followed by prices going up even more, or going down by an unusual amount.

The converse behavior, 'doldrums', can last for a long time as well.

Most typically, extreme movements do not appear 'out of nowhere'; they are presaged by larger movements than usual. This is termed autoregressive conditional heteroskedasticity. Of course, whether such large movements have the same direction, or the opposite, is more difficult to say. And an increase in volatility does not always presage a further increase—the volatility may simply go back down again.

Mathematical definition

The annualized volatility σ is the standard deviation of the instrument's yearly logarithmic returns.[6]

The generalized volatility σ_T for time horizon T in years is expressed as:

$$\sigma_T = \sigma\sqrt{T}.$$

Therefore, if the daily logarithmic returns of a stock have a standard deviation of σ_{SD} and the time period of returns is P, the annualized volatility is

$$\sigma = \frac{\sigma_{SD}}{\sqrt{P}}.$$

A common assumption is that P = 1/252 (there are 252 trading days in any given year). Then, if σ_{SD} = 0.01 the annualized volatility is

$$\sigma_{\text{annual}} = \frac{0.01}{\sqrt{\frac{1}{252}}} = 0.01\sqrt{252} = 0.1587.$$

The monthly volatility (i.e., T = 1/12 of a year) would be

$$\sigma_{\text{monthly}} = 0.1587\sqrt{\tfrac{1}{12}} = 0.0458.$$

The formula used above to convert returns or volatility measures from one time period to another assume a particular underlying model or process. These formulas are accurate extrapolations of a random walk, or Wiener process, whose steps have finite variance. However, more generally, for natural stochastic processes, the precise relationship between volatility measures for different time periods is more complicated. Some use the Lévy stability exponent α to extrapolate natural processes:

$$\sigma_T = T^{1/\alpha}\sigma.$$

If α = 2 you get the Wiener process scaling relation, but some people believe α < 2 for financial activities such as stocks, indexes and so on. This was discovered by Benoît Mandelbrot, who looked at cotton prices and found that they followed a Lévy alpha-stable distribution with α = 1.7. (See New Scientist, 19 April 1997.)

Crude volatility estimation

Using a simplification of the formulae above it is possible to estimate annualized volatility based solely on approximate observations. Suppose you notice that a market price index, which has a current value near 10,000, has moved about 100 points a day, on average, for many days. This would constitute a 1% daily movement, up or down.

To annualize this, you can use the "rule of 16", that is, multiply by 16 to get 16% as the annual volatility. The rationale for this is that 16 is the square root of 256, which is approximately the number of trading days in a year (252). This also uses the fact that the standard deviation of the sum of *n* independent variables (with equal standard deviations) is √n times the standard deviation of the individual variables.

Of course, the average magnitude of the observations is merely an approximation of the standard deviation of the market index. Assuming that the market index daily changes are normally distributed with mean zero and standard deviation σ, the expected value of the magnitude of the observations is $\sqrt{(2/\pi)}\sigma = 0.798\sigma$. The net effect is that this crude approach underestimates the true volatility by about 20%.

Estimate of compound annual growth rate (CAGR)

Consider the Taylor series:

$$\log(1+y) = y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \ldots$$

Taking only the first two terms one has:

$$CAGR \approx AR - \tfrac{1}{2}\sigma^2$$

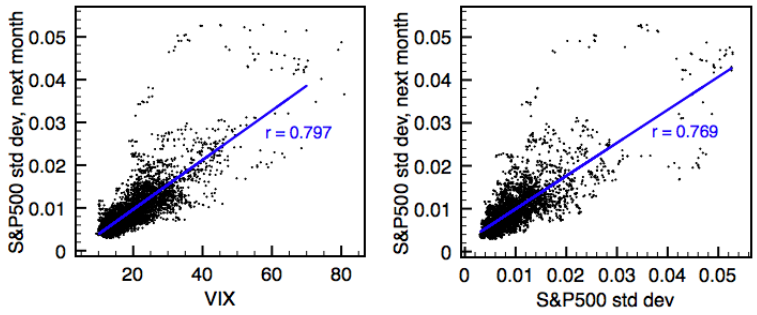
Realistically, most financial assets have negative skewness and leptokurtosis, so this formula tends to be over-optimistic. Some people use the formula:

$$CAGR \approx AR - \tfrac{1}{2}k\sigma^2$$

for a rough estimate, where *k* is an empirical factor (typically five to ten).

Criticisms

Despite their sophisticated composition, critics claim the predictive power of most volatility forecasting models is similar to that of plain-vanilla measures, such as simple past volatility.[7][8] However, other works have countered that these critiques failed to correctly implement the more complicated models.[9] Some practitioners and portfolio managers seem to completely ignore or dismiss volatility forecasting models. For example, Nassim Taleb famously titled one of his *Journal of Portfolio Management* papers "We Don't Quite Know What We are Talking About When We Talk About Volatility".[10] In a similar note, Emanuel Derman expressed his disillusion with the enormous supply of empirical models unsupported by theory.[11] He argues that, while *"theories are attempts to uncover the hidden principles underpinning the world around us, as Albert Einstein did with his theory of relativity"*, we should remember that *"models are metaphors -- analogies that describe one thing relative to another"*.



Performance of VIX (left) compared to past volatility (right) as 30-day volatility predictors, for the period of Jan 1990-Sep 2009. Volatility is measured as the standard deviation of S&P500 one-day returns over a month's period. The blue lines indicate linear regressions, resulting in the correlation coefficients *r* shown. Note that VIX has virtually the same predictive power as past volatility, insofar as the shown correlation coefficients are nearly identical.

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External links

- Graphical Comparison of Implied and Historical Volatility (http://training.thomsonreuters.com/video/v.php?v=273), video
- An introduction to volatility and how it can be calculated in excel, by Dr A. A. Kotzé (http://quantonline.co.za/Articles/article_volatility.htm)
- Interactive Java Applet " What is Historic Volatility? (http://www.frog-numerics.com/ifs/ifs_LevelA/HistVolaBasic.html)"
- Diebold, Francis X.; Hickman, Andrew; Inoue, Atsushi & Schuermannm, Til (1996) "Converting 1-Day Volatility to h-Day Volatility: Scaling by sqrt(h) is Worse than You Think" (http://citeseer.ist.psu.edu/244698.html)
- A short introduction to alternative mathematical concepts of volatility (http://staff.science.uva.nl/~marvisse/volatility.html)
- Volatility estimation from predicted return density (http://www.macroaxis.com/invest/market/GOOG--symbolVolatility) Example based on Google daily return distribution using standard density function
- Research paper including excerpt from report entitled Identifying Rich and Cheap Volatility (http://www.iijournals.com/doi/abs/10.3905/JOT.2010.5.2.035) Excerpt from Enhanced Call Overwriting, a report by Ryan Renicker and Devapriya Mallick at Lehman Brothers (2005).

Moving average

In statistics, a **moving average**, also called **rolling average**, **rolling mean** or **running average**, is a type of finite impulse response filter used to analyze a set of datum points by creating a series of averages of different subsets of the full data set.

Given a series of numbers and a fixed subset size, the first element of the moving average is obtained by taking the average of the initial fixed subset of the number series. Then the subset is modified by "shifting forward", that is excluding the first number of the series and including the next number following the original subset in the series. This creates a new subset of numbers, which is averaged. This process is repeated over the entire data series. The plot line connecting all the (fixed) averages is the moving average. A moving average is a set of numbers, each of which is the average of the corresponding subset of a larger set of datum points. A moving average may also use unequal weights for each datum value in the subset to emphasize particular values in the subset.

A moving average is commonly used with time series data to smooth out short-term fluctuations and highlight longer-term trends or cycles. The threshold between short-term and long-term depends on the application, and the parameters of the moving average will be set accordingly. For example, it is often used in technical analysis of financial data, like stock prices, returns or trading volumes. It is also used in economics to examine gross domestic product, employment or other macroeconomic time series. Mathematically, a moving average is a type of convolution and so it can be viewed as an example of a low-pass filter used in signal processing. When used with non-time series data, a moving average filters higher frequency components without any specific connection to time, although typically some kind of ordering is implied. Viewed simplistically it can be regarded as smoothing the data.

Simple moving average

In financial applications a **simple moving average** (SMA) is the unweighted mean of the previous *n* datum points. However, in science and engineering the mean is normally taken from an equal number of data either side of a central value. This ensures that variations in the mean are aligned with the variations in the data rather than being shifted in time. An example of a simple unweighted running mean for a n-day sample of closing price is the mean of the previous n days' closing prices. If those prices are $p_M, p_{M-1}, \cdots, p_{M-(n-1)}$ then the formula is

$$SMA = \frac{p_M + p_{M-1} + \cdots + p_{M-(n-1)}}{n}$$

When calculating successive values, a new value comes into the sum and an old value drops out, meaning a full summation each time is unnecessary for this simple case,

$$SMA_{\text{today}} = SMA_{\text{yesterday}} - \frac{p_{M-n}}{n} + \frac{p_M}{n}$$

The period selected depends on the type of movement of interest, such as short, intermediate, or long term. In financial terms moving average levels can be interpreted as support in a rising market, or resistance in a falling market.

If the data used are not centred around the mean, a simple moving average lags behind the latest datum point by half the sample width. An SMA can also be disproportionately influenced by old datum points dropping out or new data coming in. One characteristic of the SMA is that if the data have a periodic fluctuation, then applying an SMA of that period will eliminate that variation (the average always containing one complete cycle). But a perfectly regular cycle is rarely encountered.[1]

For a number of applications it is advantageous to avoid the shifting induced by using only 'past' data. Hence a **central moving average** can be computed, using data equally spaced either side of the point in the series where the mean is calculated. This requires using an odd number of datum points in the sample window.

Cumulative moving average

In a **cumulative moving average**, the data arrive in an ordered datum stream and the statistician would like to get the average of all of the data up until the current datum point. For example, an investor may want the average price of all of the stock transactions for a particular stock up until the current time. As each new transaction occurs, the average price at the time of the transaction can be calculated for all of the transactions up to that point using the cumulative average, typically an unweighted average of the sequence of *i* values x_1, ..., x_i up to the current time:

$$CA_i = \frac{x_1 + \cdots + x_i}{i}.$$

The brute-force method to calculate this would be to store all of the data and calculate the sum and divide by the number of datum points every time a new datum point arrived. However, it is possible to simply update cumulative average as a new value x_{i+1} becomes available, using the formula:

$$CA_{i+1} = \frac{x_{i+1} + iCA_i}{i+1},$$

where CA_0 can be taken to be equal to 0.

Thus the current cumulative average for a new datum point is equal to the previous cumulative average plus the difference between the latest datum point and the previous average divided by the number of points received so far. When all of the datum points arrive ($i = N$), the cumulative average will equal the final average.

The derivation of the cumulative average formula is straightforward. Using

$$x_1 + \cdots + \cdots + x_i = iCA_i,$$

and similarly for $i + 1$, it is seen that

$$x_{i+1} = (x_1 + \cdots + x_{i+1}) - (x_1 + \cdots + x_i) = (i+1)CA_{i+1} - iCA_i.$$

Solving this equation for CA_{i+1} results in:

$$CA_{i+1} = \frac{(x_{i+1} + iCA_i)}{i+1} = CA_i + \frac{x_{i+1} - CA_i}{i+1}.$$

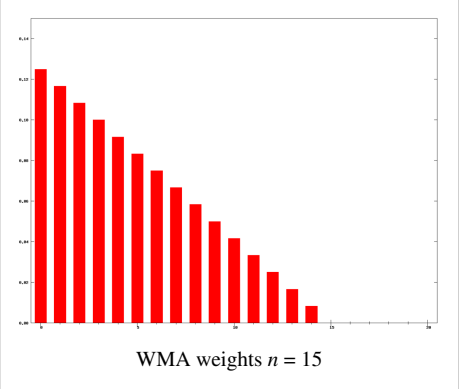
Weighted moving average

A weighted average is any average that has multiplying factors to give different weights to data at different positions in the sample window. Mathematically, the moving average is the convolution of the datum points with a fixed weighting function. One application is removing pixelisation from a digital graphical image.

In technical analysis of financial data, a **weighted moving average** (WMA) has the specific meaning of weights that decrease in arithmetical progression.[2] In an *n*-day WMA the latest day has weight *n*, the second latest *n* – 1, etc., down to one.

$$\mathrm{WMA}_M = \frac{np_M + (n-1)p_{M-1} + \cdots + 2p_{(M-n+2)} + p_{(M-n+1)}}{n + (n-1) + \cdots + 2 + 1}$$

The denominator is a triangle number equal to $\frac{n(n+1)}{2}$. In the more general case the denominator will always be the sum of the individual weights.



WMA weights *n* = 15

When calculating the WMA across successive values, the difference between the numerators of WMA_{M+1} and WMA_M is $np_{M+1} - p_M - \ldots - p_{M-n+1}$. If we denote the sum $p_M + \ldots + p_{M-n+1}$ by Total_M, then

$$\mathrm{Total}_{M+1} = Total_M + p_{M+1} - p_{M-n+1}$$

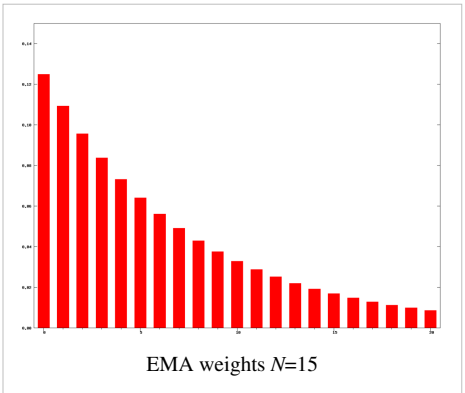
$$\mathrm{Numerator}_{M+1} = \mathrm{Numerator}_M + np_{M+1} - Total_M$$

$$\mathrm{WMA}_{M+1} = \frac{\mathrm{Numerator}_{M+1}}{n + (n-1) + \cdots + 2 + 1}$$

The graph at the right shows how the weights decrease, from highest weight for the most recent datum points, down to zero. It can be compared to the weights in the exponential moving average which follows.

Exponential moving average

An **exponential moving average** (EMA), also known as an **exponentially weighted moving average** (EWMA),[3] is a type of infinite impulse response filter that applies weighting factors which decrease exponentially. The weighting for each older datum point decreases exponentially, never reaching zero. The graph at right shows an example of the weight decrease.



EMA weights *N*=15

The EMA for a series *Y* may be calculated recursively:

$$S_1 = Y_1$$

$$\text{for } t > 1, \quad S_t = \alpha \cdot Y_t + (1-\alpha) \cdot S_{t-1}$$

Where:

- The coefficient α represents the degree of weighting decrease, a constant smoothing factor between 0 and 1. A higher α discounts older observations faster. Alternatively, α may be expressed in terms of *N* time periods, where $\alpha = 2/(N+1)$. For example, $N = 19$ is equivalent to $\alpha = 0.1$. The half-life of the weights (the interval over which the weights decrease by a factor of two) is approximately $N/2.8854$ (within 1% if $N > 5$).

- Y_t is the value at a time period t.
- S_t is the value of the EMA at any time period t.

S_1 is undefined. S_1 may be initialized in a number of different ways, most commonly by setting S_1 to Y_1, though other techniques exist, such as setting S_1 to an average of the first 4 or 5 observations. The prominence of the S_1 initialization's effect on the resultant moving average depends on α; smaller α values make the choice of S_1 relatively more important than larger α values, since a higher α discounts older observations faster.

This formulation is according to Hunter (1986).[4] By repeated application of this formula for different times, we can eventually write S_t as a weighted sum of the datum points Y_t, as:

$$S_t = \alpha \times (Y_{t-1} + (1-\alpha) \times Y_{t-2} + (1-\alpha)^2 \times Y_{t-3} + \ldots + (1-\alpha)^k \times Y_{t-(k+1)}) + (1-\alpha)^{k+1} \times S_{t-(k+1)}$$

for any suitable k = 0, 1, 2, ... The weight of the general datum point Y_{t-i} is $\alpha(1-\alpha)^{i-1}$.

An alternate approach by Roberts (1959) uses Y_t in lieu of Y_{t-1}[5]:

$$S_{t,\,\text{alternate}} = \alpha \times Y_t + (1-\alpha) \times S_{t-1}$$

This formula can also be expressed in technical analysis terms as follows, showing how the EMA steps towards the latest datum point, but only by a proportion of the difference (each time):

$$\text{EMA}_{\text{today}} = \text{EMA}_{\text{yesterday}} + \alpha \times (\text{price}_{\text{today}} - \text{EMA}_{\text{yesterday}})$$

Expanding out $\text{EMA}_{\text{yesterday}}$ each time results in the following power series, showing how the weighting factor on each datum point p_1, p_2, etc., decreases exponentially:

$$\text{EMA}_{\text{today}} = \alpha \times (p_1 + (1-\alpha)p_2 + (1-\alpha)^2 p_3 + (1-\alpha)^3 p_4 + \cdots)$$

where

- p_1 is $\text{price}_{\text{today}}$
- p_2 is $\text{price}_{\text{yesterday}}$
- and so on

$$\text{EMA}_{\text{today}} = \frac{p_1 + (1-\alpha)p_2 + (1-\alpha)^2 p_3 + (1-\alpha)^3 p_4 + \cdots}{1 + (1-\alpha) + (1-\alpha)^2 + (1-\alpha)^3 + \cdots},$$

since $1/\alpha = 1 + (1-\alpha) + (1-\alpha)^2 + \cdots$.

This is an infinite sum with decreasing terms.

The N periods in an N-day EMA only specify the α factor. N is not a stopping point for the calculation in the way it is in an SMA or WMA. For sufficiently large N, The first N datum points in an EMA represent about 86% of the total weight in the calculation[6]:

$$\frac{\alpha \times \left(1 + (1-\alpha) + (1-\alpha)^2 + \cdots + (1-\alpha)^N\right)}{\alpha \times (1 + (1-\alpha) + (1-\alpha)^2 + \cdots + (1-\alpha)^\infty)} = 1 - \left(1 - \frac{2}{N+1}\right)^{N+1}$$

i.e. $\lim_{N\to\infty} \left[1 - \left(1 - \frac{2}{N+1}\right)^{N+1}\right]$ simplified,[7] tends to $1 - \mathrm{e}^{-2} \approx 0.8647$.

The power formula above gives a starting value for a particular day, after which the successive days formula shown first can be applied. The question of how far back to go for an initial value depends, in the worst case, on the data. Large price values in old data will affect on the total even if their weighting is very small. If prices have small variations then just the weighting can be considered. The weight omitted by stopping after k terms is

$$\alpha \times \left((1-\alpha)^k + (1-\alpha)^{k+1} + (1-\alpha)^{k+2} + \cdots\right),$$

which is

$$\alpha \times (1-\alpha)^k \times \left(1 + (1-\alpha) + (1-\alpha)^2 + \cdots\right),$$

i.e. a fraction

$$\frac{\text{weight omitted by stopping after k terms}}{\text{total weight}} = \frac{\alpha \times \left[(1-\alpha)^k + (1-\alpha)^{k+1} + (1-\alpha)^{k+2} + \cdots\right]}{\alpha \times \left[1 + (1-\alpha) + (1-\alpha)^2 + \cdots\right]}$$
$$= \frac{\alpha(1-\alpha)^k \times \frac{1}{1-(1-\alpha)}}{\frac{\alpha}{1-(1-\alpha)}}$$
$$= (1-\alpha)^k$$

out of the total weight.

For example, to have 99.9% of the weight, set above ratio equal to 0.1% and solve for *k*:

$$k = \frac{\log(0.001)}{\log(1-\alpha)}$$

terms should be used. Since $\log(1-\alpha)$ approaches $\frac{-2}{N+1}$ as N increases,[8] this simplifies to approximately[9]

$$k = 3.45(N+1)$$

for this example (99.9% weight).

Modified moving average

A **modified moving average** (MMA), **running moving average** (RMA), or **smoothed moving average** is defined as:

$$\text{MMA}_{\text{today}} = \frac{(N-1) \times \text{MMA}_{\text{yesterday}} + \text{price}}{N}$$

In short, this is exponential moving average, with $\alpha = 1/N$.

Application to measuring computer performance

Some computer performance metrics, e.g. the average process queue length, or the average CPU utilization, use a form of exponential moving average.

$$S_n = \alpha(t_n - t_{n-1}) \times Y_n + (1 - \alpha(t_n - t_{n-1})) \times S_{n-1}.$$

Here α is defined as a function of time between two readings. An example of a coefficient giving bigger weight to the current reading, and smaller weight to the older readings is

$$\alpha(t_n - t_{n-1}) = 1 - \exp\left(-\frac{t_n - t_{n-1}}{W \times 60}\right)$$

where time for readings t_n is expressed in seconds, and W is the period of time in minutes over which the reading is said to be averaged (the mean lifetime of each reading in the average). Given the above definition of α, the moving average can be expressed as

$$S_n = \left(1 - \exp\left(-\frac{t_n - t_{n-1}}{W \times 60}\right)\right) \times Y_n + e^{-\frac{tn - t_{n-1}}{W \times 60}} \times S_{n-1}$$

For example, a 15-minute average *L* of a process queue length *Q*, measured every 5 seconds (time difference is 5 seconds), is computed as

$$L_n = \left(1 - \exp\left(-\frac{5}{15 \times 60}\right)\right) \times Q_n + e^{-\frac{5}{15 \times 60}} \times L_{n-1} = \left(1 - \exp\left(-\frac{1}{180}\right)\right) \times Q_n + e^{-1/180} \times L_{n-1} = Q_n + e^{-1/180} \times (L_{n-1} - Q_n)$$

Other weightings

Other weighting systems are used occasionally – for example, in share trading a **volume weighting** will weight each time period in proportion to its trading volume.

A further weighting, used by actuaries, is Spencer's 15-Point Moving Average[10] (a central moving average). The symmetric weight coefficients are -3, -6, -5, 3, 21, 46, 67, 74, 67, 46, 21, 3, -5, -6, -3.

Outside the world of finance, weighted running means have many forms and applications. Each weighting function or "kernel" has its own characteristics. In engineering and science the frequency and phase response of the filter is often of primary importance in understanding the desired and undesired distortions that a particular filter will apply to the data.

A mean does not just "smooth" the data. A mean is a form of low-pass filter. The effects of the particular filter used should be understood in order to make an appropriate choice.

Moving median

From a statistical point of view, the moving average, when used to estimate the underlying trend in a time series, is susceptible to rare events such as rapid shocks or other anomalies. A more robust estimate of the trend is the **simple moving median** over *n* time points:

$$SMM = \text{Median}(p_M, p_{M-1}, \ldots, p_{M-n+1})$$

where the median is found by, for example, sorting the values inside the brackets and finding the value in the middle.

Statistically, the moving average is optimal for recovering the underlying trend of the time series when the fluctuations about the trend are normally distributed. However, the normal distribution does not place high probability on very large deviations from the trend which explains why such deviations will have a disproportionately large effect on the trend estimate. It can be shown that if the fluctuations are instead assumed to be Laplace distributed, then the moving median is statistically optimal.[11] For a given variance, the Laplace distribution places higher probability on rare events than does the normal, which explains why the moving median tolerates shocks better than the moving mean.

When the simple moving median above is central, the smoothing is identical to the median filter which has applications in, for example, image signal processing.

Notes and references

[1] *Statistical Analysis*, Ya-lun Chou, Holt International, 1975, ISBN 0-03-089422-0, section 17.9.

[2] "Weighted Moving Averages: The Basics" (http://www.investopedia.com/articles/technical/060401.asp). Investopedia. .

[3] http://lorien.ncl.ac.uk/ming/filter/filewma.htm

[4] NIST/SEMATECH e-Handbook of Statistical Methods: Single Exponential Smoothing (http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc431.htm) at the National Institute of Standards and Technology

[5] NIST/SEMATECH e-Handbook of Statistical Methods: EWMA Control Charts (http://www.itl.nist.gov/div898/handbook/pmc/section3/pmc324.htm) at the National Institute of Standards and Technology

[6] The denominator on the left-hand side should be unity, and the numerator will become the right-hand side (geometric series), $\alpha\left(\frac{1-(1-\alpha)^{N+1}}{1-(1-\alpha)}\right)$.

[7] Because $(1+x/n)^n$ becomes e^x for large *n*.

[8] It means α -> 0, and the Taylor series of $\log(1-\alpha) = -\alpha - \alpha^2/2 - \cdots$ tends to $-\alpha$.

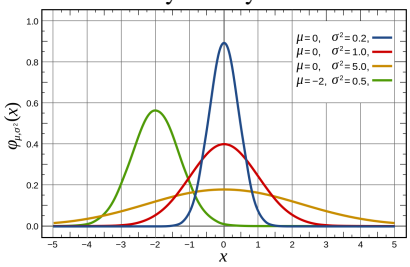
[9] $\log_e(0.001) / 2 = -3.45$

[10] Spencer's 15-Point Moving Average — from Wolfram MathWorld (http://mathworld.wolfram.com/Spencers15-PointMovingAverage.html)

[11] G.R. Arce, "Nonlinear Signal Processing: A Statistical Approach", Wiley:New Jersey, USA, 2005.

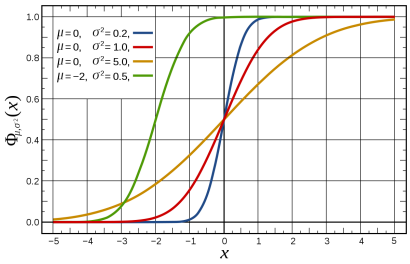
Normal distribution

Probability density function



The red curve is the *standard normal distribution*

Cumulative distribution function



Notation	$\mathcal{N}(\mu, \sigma^2)$
Parameters	$\mu \in \mathbf{R}$ — mean (location) $\sigma^2 > 0$ — variance (squared scale)
Support	$x \in \mathbf{R}$
PDF	$\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
CDF	$\frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sqrt{2\sigma^2}}\right)\right]$
Mean	μ
Median	μ
Mode	μ
Variance	σ^2
Skewness	0
Ex. kurtosis	0
Entropy	$\frac{1}{2}\ln(2\pi e\,\sigma^2)$
MGF	$\exp\{\mu t + \frac{1}{2}\sigma^2 t^2\}$
CF	$\exp\{i\mu t - \frac{1}{2}\sigma^2 t^2\}$
Fisher information	$\begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/(2\sigma^4) \end{pmatrix}$

In probability theory, the **normal** (or **Gaussian**) **distribution** is a continuous probability distribution, defined on the entire real line, that has a bell-shaped probability density function, known as the **Gaussian function** or informally as the **bell curve**:[1]

$$f(x;\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The parameter μ is the *mean* or *expectation* (location of the peak) and σ^2 is the variance. σ is known as the standard deviation. The distribution with $\mu = 0$ and $\sigma^2 = 1$ is called the **standard normal distribution** or the **unit normal distribution**. A normal distribution is often used as a first approximation to describe real-valued random variables that cluster around a single mean value.

The normal distribution is considered the most prominent probability distribution in statistics. There are several reasons for this:[2] First, the normal distribution arises from the central limit theorem, which states that under mild conditions, the mean of a large number of random variables independently drawn from the same distribution is distributed approximately normally, irrespective of the form of the original distribution. This gives it exceptionally wide application in, for example, sampling. Secondly, the normal distribution is very tractable analytically, that is, a large number of results involving this distribution can be derived in explicit form.

For these reasons, the normal distribution is commonly encountered in practice, and is used throughout statistics, the natural sciences, and the social sciences[3] as a simple model for complex phenomena. For example, the observational error in an experiment is usually assumed to follow a normal distribution, and the propagation of uncertainty is computed using this assumption. Note that a normally distributed variable has a symmetric distribution about its mean. Quantities that grow exponentially, such as prices, incomes or populations, are often skewed to the right, and hence may be better described by other distributions, such as the log-normal distribution or the Pareto distribution. In addition, the probability of seeing a normally distributed value that is far (i.e. more than a few standard deviations) from the mean drops off extremely rapidly. As a result, statistical inference using a normal distribution is not robust to the presence of outliers (data that are unexpectedly far from the mean, due to exceptional circumstances, observational error, etc.). When outliers are expected, data may be better described using a heavy-tailed distribution such as the Student's t-distribution.

From a technical perspective, alternative characterizations are possible, for example:

- The normal distribution is the only absolutely continuous distribution all of whose cumulants beyond the first two (i.e. other than the mean and variance) are zero.
- For a given mean and variance, the corresponding normal distribution is the continuous distribution with the maximum entropy.[4][5]

The normal distributions are a subclass of the elliptical distributions.

Definition

The simplest case of a normal distribution is known as the 'standard normal distribution*, described by this probability density function :*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

The factor in this expression ensures that the total area under the curve $\phi(x)$ is equal to one[proof], and 12 in the exponent makes the "width" of the curve (measured as half the distance between the inflection points) also equal to one. It is traditional in statistics to denote this function with the Greek letter ϕ (phi), whereas density functions for all other distributions are usually denoted with letters *f* or *p*.[6] The alternative glyph φ is also used quite often, however within this article "φ" is reserved to denote characteristic functions.

Every normal distribution is the result of exponentiating a quadratic function (just as an exponential distribution results from exponentiating a linear function):

$$f(x) = e^{ax^2+bx+c}.$$

This yields the classic "bell curve" shape, provided that $a < 0$ so that the quadratic function is concave for x close to 0. $f(x) > 0$ everywhere. One can adjust a to control the "width" of the bell, then adjust b to move the central peak of

the bell along the x-axis, and finally one must choose c such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (which is only possible when $a < 0$).

Rather than using a, b, and c, it is far more common to describe a normal distribution by its mean $\mu = -b2a$ and variance $\sigma^2 = -12a$. Changing to these new parameters allows one to rewrite the probability density function in a convenient standard form:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right).$$

For a standard normal distribution, $\mu = 0$ and $\sigma^2 = 1$. The last part of the equation above shows that any other normal distribution can be regarded as a version of the standard normal distribution that has been stretched horizontally by a factor σ and then translated rightward by a distance μ. Thus, μ specifies the position of the bell curve's central peak, and σ specifies the "width" of the bell curve.

The parameter μ is at the same time the mean, the median and the mode of the normal distribution. The parameter σ^2 is called the *variance*; as for any random variable, it describes how concentrated the distribution is around its mean. The square root of σ^2 is called the *standard deviation* and is the width of the density function.

The normal distribution is usually denoted by $N(\mu, \sigma^2)$.[7] Thus when a random variable X is distributed normally with mean μ and variance σ^2, we write

$$X \sim \mathcal{N}(\mu, \sigma^2).$$

Alternative formulations

Some authors advocate using the precision instead of the variance. The precision is normally defined as the reciprocal of the variance ($\tau = \sigma^{-2}$), although it is occasionally defined as the reciprocal of the standard deviation ($\tau = \sigma^{-1}$).[8] This parametrization has an advantage in numerical applications where σ^2 is very close to zero and is more convenient to work with in analysis as τ is a natural parameter of the normal distribution. This parametrization is common in Bayesian statistics, as it simplifies the Bayesian analysis of the normal distribution. Another advantage of using this parametrization is in the study of conditional distributions in the multivariate normal case. The form of the normal distribution with the more common definition $\tau = \sigma^{-2}$ is as follows:

$$f(x;\, \mu, \tau) = \sqrt{\frac{\tau}{2\pi}}\, e^{\frac{-\tau(x-\mu)^2}{2}}.$$

The question of which normal distribution should be called the "*standard*" one is also answered differently by various authors. Starting from the works of Gauss the standard normal was considered to be the one with variance $\sigma^2 = 12$:

$$f(x) = \frac{1}{\sqrt{\pi}}\, e^{-x^2}$$

Stigler (1982) goes even further and insists the standard normal to be with the variance $\sigma^2 = 12\pi$:

$$f(x) = e^{-\pi x^2}$$

According to the author, this formulation is advantageous because of a much simpler and easier-to-remember formula, the fact that the pdf has unit height at zero, and simple approximate formulas for the quantiles of the distribution.

Characterization

In the previous section the normal distribution was defined by specifying its probability density function. However there are other ways to characterize a probability distribution. They include: the cumulative distribution function, the moments, the cumulants, the characteristic function, the moment-generating function, etc.

Probability density function

The probability density function (pdf) of a random variable describes the relative frequencies of different values for that random variable. The pdf of the normal distribution is given by the formula explained in detail in the previous section:

$$f(x;\,\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-(x-\mu)^2/(2\sigma^2)} = \frac{1}{\sigma}\,\phi\left(\frac{x-\mu}{\sigma}\right), \qquad x \in \mathbb{R}.$$

This is a proper function only when the variance σ^2 is not equal to zero. In that case this is a continuous smooth function, defined on the entire real line, and which is called the "Gaussian function".

Properties:

- Function $f(x)$ is unimodal and symmetric around the point $x = \mu$, which is at the same time the mode, the median and the mean of the distribution.[9]
- The inflection points of the curve occur one standard deviation away from the mean (i.e., at $x = \mu - \sigma$ and $x = \mu + \sigma$).[9]
- Function $f(x)$ is log-concave.[9]
- The standard normal density $\phi(x)$ is an eigenfunction of the Fourier transform in that if f is a normalized Gaussian function with variance σ^2, centered at zero, then its Fourier transform is a Gaussian function with variance $1/\sigma^2$.
- The function is supersmooth of order 2, implying that it is infinitely differentiable.[10]
- The first derivative of $\phi(x)$ is $\phi'(x) = -x\cdot\phi(x)$; the second derivative is $\phi''(x) = (x^2 - 1)\phi(x)$. More generally, the n-th derivative is given by $\phi^{(n)}(x) = (-1)^n H_n(x)\phi(x)$, where H_n is the Hermite polynomial of order n.[11]

When $\sigma^2 = 0$, the density function doesn't exist. However a generalized function that defines a measure on the real line, and it can be used to calculate, for example, expected value is

$$f(x;\,\mu,0) = \delta(x-\mu).$$

where $\delta(x)$ is the Dirac delta function which is equal to infinity at $x = \mu$ and is zero elsewhere.

Cumulative distribution function

The cumulative distribution function (CDF) describes probability of a random variable falling in the interval $(-\infty, x]$.

The CDF of the standard normal distribution is denoted with the capital Greek letter Φ (phi), and can be computed as an integral of the probability density function:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\,dt = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right], \quad x \in \mathbb{R}.$$

This integral cannot be expressed in terms of elementary functions, so is simply called a transformation of the error function, or **erf**, a special function. Numerical methods for calculation of the standard normal CDF are discussed below. For a generic normal random variable with mean μ and variance $\sigma^2 > 0$ the CDF will be equal to

$$F(x;\,\mu,\sigma^2) = \Phi\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right)\right], \quad x \in \mathbb{R}.$$

The complement of the standard normal CDF, $Q(x) = 1 - \Phi(x)$, is referred to as the Q-function, especially in engineering texts.[12][13] This represents the upper tail probability of the Gaussian distribution: that is, the probability that a standard normal random variable X is greater than the number x. Other definitions of the Q-function, all of which are simple transformations of Φ, are also used occasionally.[14]

Properties:

- The standard normal CDF is 2-fold rotationally symmetric around point (0, ½): $\Phi(-x) = 1 - \Phi(x)$.
- The derivative of $\Phi(x)$ is equal to the standard normal pdf $\phi(x)$: $\Phi'(x) = \phi(x)$.
- The antiderivative of $\Phi(x)$ is: $\int \Phi(x)\,dx = x\,\Phi(x) + \phi(x)$.

For a normal distribution with zero variance, the CDF is the Heaviside step function (with $H(0) = 1$ convention):

$$F(x;\,\mu, 0) = \mathbf{1}\{x \geq \mu\}\,.$$

Quantile function

The quantile function of a distribution is the inverse of the CDF. The quantile function of the standard normal distribution is called the probit function, and can be expressed in terms of the inverse error function:

$$\Phi^{-1}(p) \equiv z_p = \sqrt{2}\,\operatorname{erf}^{-1}(2p-1), \quad p \in (0,1).$$

Quantiles of the standard normal distribution are commonly denoted as z_p. The quantile z_p represents such a value that a standard normal random variable X has the probability of exactly p to fall inside the $(-\infty, z_p]$ interval. The quantiles are used in hypothesis testing, construction of confidence intervals and Q-Q plots. The most "famous" normal quantile is $1.96 = z_{0.975}$. A standard normal random variable is greater than 1.96 in absolute value in 5% of cases.

For a normal random variable with mean μ and variance σ^2, the quantile function is

$$F^{-1}(p;\,\mu, \sigma^2) = \mu + \sigma\Phi^{-1}(p) = \mu + \sigma\sqrt{2}\,\operatorname{erf}^{-1}(2p-1), \quad p \in (0,1).$$

Characteristic function and moment generating function

The characteristic function $\varphi_X(t)$ of a random variable X is defined as the expected value of e^{itX}, where i is the imaginary unit, and $t \in \mathbf{R}$ is the argument of the characteristic function. Thus the characteristic function is the Fourier transform of the density $\phi(x)$. For a normally distributed X with mean μ and variance σ^2, the characteristic function is[15]

$$\varphi(t;\,\mu, \sigma^2) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2}\,dx = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}.$$

The characteristic function can be analytically extended to the entire complex plane: one defines $\varphi(z) = e^{i\mu z - 12\sigma} 2z^2$ for all $z \in \mathbf{C}$.[16]

The moment generating function is defined as the expected value of e^{tX}. For a normal distribution, the moment generating function exists and is equal to

$$M(t;\,\mu, \sigma^2) = \mathrm{E}[e^{tX}] = \varphi(-it;\,\mu, \sigma^2) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}.$$

The cumulant generating function is the logarithm of the moment generating function:

$$g(t;\,\mu, \sigma^2) = \ln M(t;\,\mu, \sigma^2) = \mu t + \frac{1}{2}\sigma^2 t^2.$$

Since this is a quadratic polynomial in t, only the first two cumulants are nonzero.

Moments

The normal distribution has moments of all orders. That is, for a normally distributed X with mean μ and variance σ^2, the expectation $\mathrm{E}[|X|^p]$ exists and is finite for all p such that $\mathrm{Re}[p] > -1$. Usually we are interested only in moments of integer orders: $p = 1, 2, 3, \ldots$.

- *Central moments* are the moments of X around its mean μ. Thus, a central moment of order p is the expected value of $(X - \mu)^p$. Using standardization of normal random variables, this expectation will be equal to $\sigma^p \cdot \mathrm{E}[Z^p]$, where Z is standard normal.

$$\mathrm{E}\left[(X-\mu)^p\right] = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ \sigma^p\,(p-1)!! & \text{if } p \text{ is even.} \end{cases}$$

  Here $n!!$ denotes the double factorial, that is the product of every odd number from n to 1.
- *Central absolute moments* are the moments of $|X - \mu|$. They coincide with regular moments for all even orders, but are nonzero for all odd p's.

$$\mathrm{E}\left[|X-\mu|^p\right] = \sigma^p (p-1)!! \cdot \left\{ \begin{matrix} \sqrt{2/\pi} & \text{if } p \text{ is odd,} \\ 1 & \text{if } p \text{ is even,} \end{matrix} \right\} = \sigma^p \cdot \frac{2^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right)}{\sqrt{\pi}}$$

  The last formula is true for any non-integer $p > -1$.
- *Raw moments* and *raw absolute moments* are the moments of X and $|X|$ respectively. The formulas for these moments are much more complicated, and are given in terms of confluent hypergeometric functions ${}_1F_1$ and U.

$$\mathrm{E}\left[X^p\right] = \sigma^p \cdot (-i\sqrt{2}\,\mathrm{sgn}\,\mu)^p\; U\left(-\frac{1}{2}p,\, \frac{1}{2},\, -\frac{1}{2}(\mu/\sigma)^2\right),$$

$$\mathrm{E}\left[|X|^p\right] = \sigma^p \cdot 2^{\frac{p}{2}} \frac{\Gamma\left(\frac{1+p}{2}\right)}{\sqrt{\pi}}\; {}_1F_1\left(-\frac{1}{2}p,\, \frac{1}{2},\, -\frac{1}{2}(\mu/\sigma)^2\right).$$

  These expressions remain valid even if p is not integer. See also generalized Hermite polynomials.
- First two *cumulants* are equal to μ and σ^2 respectively, whereas all higher-order cumulants are equal to zero.

Order	Raw moment	Central moment	Cumulant
1	μ	0	μ
2	$\mu^2 + \sigma^2$	σ^2	σ^2
3	$\mu^3 + 3\mu\sigma^2$	0	0
4	$\mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$	$3\sigma^4$	0
5	$\mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4$	0	0
6	$\mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6$	$15\sigma^6$	0
7	$\mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6$	0	0
8	$\mu^8 + 28\mu^6\sigma^2 + 210\mu^4\sigma^4 + 420\mu^2\sigma^6 + 105\sigma^8$	$105\sigma^8$	0

Properties

Standardizing normal random variables

Because the normal distribution is a location-scale family, it is possible to relate all normal random variables to the standard normal. For example if X is normal with mean μ and variance σ^2, then

$$Z = \frac{X - \mu}{\sigma}$$

has mean zero and unit variance, that is Z has the standard normal distribution. Conversely, having a standard normal random variable Z we can always construct another normal random variable with specific mean μ and variance σ^2:

$$X = \sigma Z + \mu.$$

This "standardizing" transformation is convenient as it allows one to compute the PDF and especially the CDF of a normal distribution having the table of PDF and CDF values for the standard normal. They will be related via

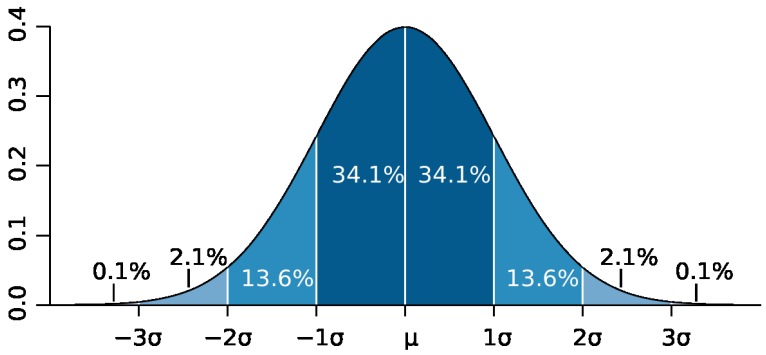
$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right), \quad f_X(x) = \frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right).$$

Standard deviation and tolerance intervals

About 68% of values drawn from a normal distribution are within one standard deviation σ away from the mean; about 95% of the values lie within two standard deviations; and about 99.7% are within three standard deviations. This fact is known as the *68-95-99.7 rule*, or the *empirical rule*, or the *3-sigma rule*. To be more precise, the area under the bell curve between $\mu - n\sigma$ and $\mu + n\sigma$ is given by

$$F(\mu + n\sigma;\, \mu, \sigma^2) - F(\mu - n\sigma;\, \mu, \sigma^2) = \Phi(n) - \Phi(-n) = \operatorname{erf}\left(\frac{n}{\sqrt{2}}\right),$$

where erf is the error function. To 12 decimal places, the values for the 1-, 2-, up to 6-sigma points are:[17]



Dark blue is less than one standard deviation away from the mean. For the normal distribution, this accounts for about 68% of the set, while two standard deviations from the mean (medium and dark blue) account for about 95%, and three standard deviations (light, medium, and dark blue) account for about 99.7%.

n	$\operatorname{erf}\left(\frac{n}{\sqrt{2}}\right)$	i.e. 1 minus ...	or 1 in ...
1	0682689492137	0317310507863	315148718753
2	0954499736104	0045500263896	219778945080
3	0997300203937	0002699796063	370398347345
4	0999936657516	0000063342484	15,7871927673
5	0999999426697	0000000573303	1.74428e+0689362
6	0999999998027	0000000001973	5.06797e+08897

The next table gives the reverse relation of sigma multiples corresponding to a few often used values for the area under the bell curve. These values are useful to determine (asymptotic) tolerance intervals of the specified levels based on normally distributed (or asymptotically normal) estimators:[18]

$\operatorname{erf}\left(\frac{n}{\sqrt{2}}\right)$	n	$\operatorname{erf}\left(\frac{n}{\sqrt{2}}\right)$	n
0.80	1281551565545	0.999	3290526731492
0.90	1644853626951	0.9999	3890591886413
0.95	1959963984540	0.99999	4417173413469
0.98	2326347874041	0.999999	4891638475699
0.99	2575829303549	0.9999999	5326723886384
0.995	2807033768344	0.99999999	5730728868236
0.998	3090232306168	0.999999999	6109410204869

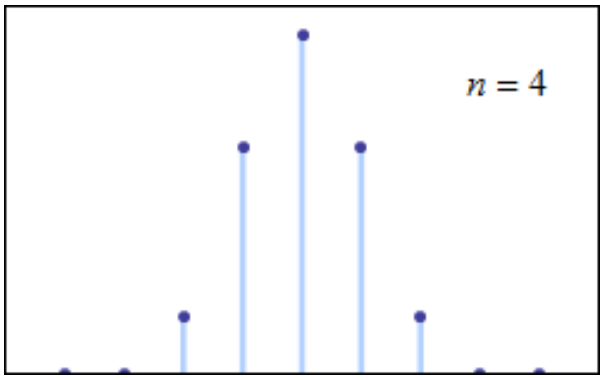
where the value on the left of the table is the proportion of values that will fall within a given interval and n is a multiple of the standard deviation that specifies the width of the interval.

Central limit theorem

The theorem states that under certain (fairly common) conditions, the sum of a large number of random variables will have an approximately normal distribution. For example if $(x_1, \ldots, x_n)$ is a sequence of iid random variables, each having mean μ and variance σ^2, then the central limit theorem states that

$$\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} x_i - \mu\right) \xrightarrow{d} \mathcal{N}(0,\, \sigma^2).$$

The theorem will hold even if the summands x_i are not iid, although some constraints on the degree of dependence and the growth rate of moments still have to be imposed.



As the number of discrete events increases, the function begins to resemble a normal distribution

The importance of the central limit theorem cannot be overemphasized. A great number of test statistics, scores, and estimators encountered in practice contain sums of certain random variables in them, even more estimators can be represented as sums of random variables through the use of influence functions – all of these quantities are governed by the central limit theorem and will have asymptotically normal distribution as a result.

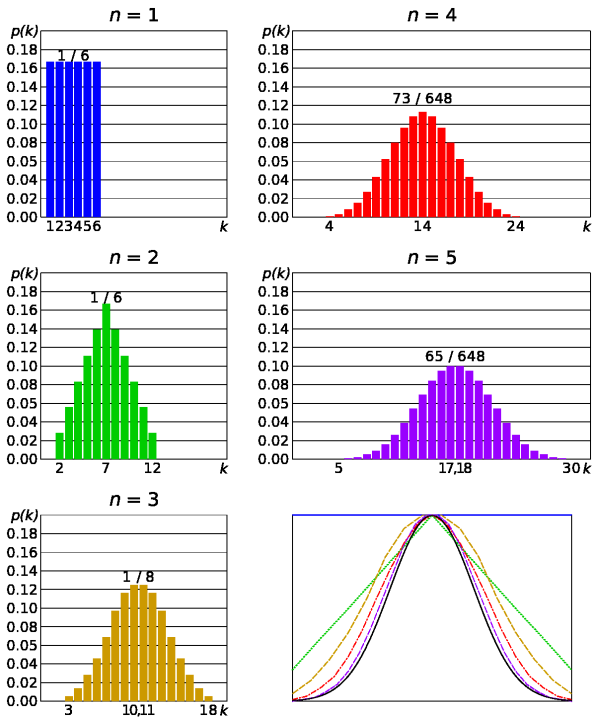
Another practical consequence of the central limit theorem is that certain other distributions can be approximated by the normal distribution, for example:

- The binomial distribution $B(n, p)$ is approximately normal $N(np, np(1 - p))$ for large n and for p not too close to zero or one.
- The Poisson(λ) distribution is approximately normal $N(\lambda, \lambda)$ for large values of λ.

- The chi-squared distribution $\chi^2(k)$ is approximately normal $N(k, 2k)$ for large k.
- The Student's t-distribution $t(\nu)$ is approximately normal $N(0, 1)$ when ν is large.

Whether these approximations are sufficiently accurate depends on the purpose for which they are needed, and the rate of convergence to the normal distribution. It is typically the case that such approximations are less accurate in the tails of the distribution.

A general upper bound for the approximation error in the central limit theorem is given by the Berry–Esseen theorem, improvements of the approximation are given by the Edgeworth expansions.



Comparison of probability density functions, $p(k)$ for the sum of n fair 6-sided dice to show their convergence to a normal distribution with increasing n, in accordance to the central limit theorem. In the bottom-right graph, smoothed profiles of the previous graphs are rescaled, superimposed and compared with a normal distribution (black curve).

Miscellaneous

1. The family of normal distributions is closed under linear transformations. That is, if X is normally distributed with mean μ and variance σ^2, then a linear transform $aX + b$ (for some real numbers a and b) is also normally distributed:

$$aX + b \ \sim \ \mathcal{N}(a\mu + b,\, a^2\sigma^2).$$

   Also if X_1, X_2 are two independent normal random variables, with means μ_1, μ_2 and standard deviations σ_1, σ_2, then their linear combination will also be normally distributed: [proof]

$$aX_1 + bX_2 \ \sim \ \mathcal{N}(a\mu_1 + b\mu_2,\, a^2\sigma_1^2 + b^2\sigma_2^2)$$

2. The converse of (1) is also true: if X_1 and X_2 are independent and their sum $X_1 + X_2$ is distributed normally, then both X_1 and X_2 must also be normal.[19] This is known as **Cramér's decomposition theorem**. The interpretation of this property is that a normal distribution is only divisible by other normal distributions. Another application of this property is in connection with the central limit theorem: although the CLT asserts that the distribution of a sum of arbitrary non-normal iid random variables is approximately normal, the Cramér's theorem shows that it can never become exactly normal.[20]
3. If the characteristic function φ_X of some random variable X is of the form $\varphi_X(t) = e^{Q(t)}$, where $Q(t)$ is a polynomial, then the **Marcinkiewicz theorem** (named after Józef Marcinkiewicz) asserts that Q can be at most a quadratic polynomial, and therefore X a normal random variable.[20] The consequence of this result is that the normal distribution is the only distribution with a finite number (two) of non-zero cumulants.
4. If X and Y are jointly normal and uncorrelated, then they are independent. The requirement that X and Y should be *jointly* normal is essential, without it the property does not hold.[proof] For non-normal random variables

uncorrelatedness does not imply independence.

5. If X and Y are independent $N(\mu, \sigma^2)$ random variables, then $X + Y$ and $X - Y$ are also independent and identically distributed (this follows from the polarization identity).[21] This property uniquely characterizes normal distribution, as can be seen from the **Bernstein's theorem**: if X and Y are independent and such that $X + Y$ and $X - Y$ are also independent, then both X and Y must necessarily have normal distributions.
   More generally, if $X_1, ..., X_n$ are independent random variables, then two linear combinations $\sum a_k X_k$ and $\sum b_k X_k$ will be independent if and only if all X_k's are normal and $\sum a_k b_k \sigma^2_k = 0$, where σ^2_k denotes the variance of X_k.[22]
6. The normal distribution is infinitely divisible:[23] for a normally distributed X with mean μ and variance σ^2 we can find n independent random variables $\{X_1, \ldots, X_n\}$ each distributed normally with means μ/n and variances σ^2/n such that
$$X_1 + X_2 + \cdots + X_n \sim \mathcal{N}(\mu, \sigma^2)$$
7. The normal distribution is stable (with exponent $\alpha = 2$): if X_1, X_2 are two independent $N(\mu, \sigma^2)$ random variables and a, b are arbitrary real numbers, then
$$aX_1 + bX_2 \sim \sqrt{a^2 + b^2} \cdot X_3 + (a + b - \sqrt{a^2 + b^2})\mu,$$
   where X_3 is also $N(\mu, \sigma^2)$. This relationship directly follows from property (1).
8. The Kullback–Leibler divergence between two normal distributions $X_1 \sim N(\mu_1, \sigma^2_1)$ and $X_2 \sim N(\mu_2, \sigma^2_2)$ is given by:[24]
$$D_{\mathrm{KL}}(X_1 \,\|\, X_2) = \frac{(\mu_1 - \mu_2)^2}{2\sigma_2^2} + \frac{1}{2}\left(\frac{\sigma_1^2}{\sigma_2^2} - 1 - \ln\frac{\sigma_1^2}{\sigma_2^2}\right).$$
   The Hellinger distance between the same distributions is equal to
$$H^2(X_1, X_2) = 1 - \sqrt{\frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}}\, e^{-\frac{1}{4}\frac{(\mu_1 - \mu_2)^2}{\sigma_1^2 + \sigma_2^2}}.$$
9. The Fisher information matrix for a normal distribution is diagonal and takes the form
$$\mathcal{I} = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$$
10. Normal distributions belongs to an exponential family with natural parameters $\theta_1 = \frac{\mu}{\sigma^2}$ and $\theta_2 = \frac{-1}{2\sigma^2}$, and natural statistics x and x^2. The dual, expectation parameters for normal distribution are $\eta_1 = \mu$ and $\eta_2 = \mu^2 + \sigma^2$.
11. The conjugate prior of the mean of a normal distribution is another normal distribution.[25] Specifically, if $x_1, \ldots, x_n$ are iid $N(\mu, \sigma^2)$ and the prior is $\mu \sim N(\mu_0, \sigma^2_0)$, then the posterior distribution for the estimator of μ will be
$$\mu | x_1, \ldots, x_n \sim \mathcal{N}\left(\frac{\frac{\sigma^2}{n}\mu_0 + \sigma_0^2\bar{x}}{\frac{\sigma^2}{n} + \sigma_0^2}, \left(\frac{n}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1}\right)$$
12. Of all probability distributions over the reals with mean μ and variance σ^2, the normal distribution $N(\mu, \sigma^2)$ is the one with the maximum entropy.[26]
13. The family of normal distributions forms a manifold with constant curvature -1. The same family is flat with respect to the (± 1)-connections $\nabla^{(e)}$ and $\nabla^{(m)}$.[27]

Related distributions

Operations on a single random variable

If X is distributed normally with mean μ and variance σ^2, then

- The exponential of X is distributed log-normally: $e^X \sim \ln N(\mu, \sigma^2)$.
- The absolute value of X has folded normal distribution: $|X| \sim N_f(\mu, \sigma^2)$. If $\mu = 0$ this is known as the half-normal distribution.
- The square of X/σ has the noncentral chi-squared distribution with one degree of freedom: $X^2/\sigma^2 \sim \chi^2{}_1(\mu^2/\sigma^2)$. If $\mu = 0$, the distribution is called simply chi-squared.
- The distribution of the variable X restricted to an interval $[a, b]$ is called the truncated normal distribution.
- $(X - \mu)^{-2}$ has a Lévy distribution with location 0 and scale σ^{-2}.

Combination of two independent random variables

If X_1 and X_2 are two independent standard normal random variables with mean 0 and variance 1, then

- Their sum and difference is distributed normally with mean zero and variance two: $X_1 \pm X_2 \sim N(0, 2)$.
- Their product $Z = X_1 \cdot X_2$ follows the "product-normal" distribution[28] with density function $f_Z(z) = \pi^{-1} K_0(|z|)$, where K_0 is the modified Bessel function of the second kind. This distribution is symmetric around zero, unbounded at $z = 0$, and has the characteristic function $\varphi_Z(t) = (1 + t^2)^{-1/2}$.
- Their ratio follows the standard Cauchy distribution: $X_1 \div X_2 \sim \text{Cauchy}(0, 1)$.
- Their Euclidean norm $\sqrt{X_1^2 + X_2^2}$ has the Rayleigh distribution, also known as the chi distribution with 2 degrees of freedom.

Combination of two or more independent random variables

- If $X_1, X_2, \ldots, X_n$ are independent standard normal random variables, then the sum of their squares has the chi-squared distribution with n degrees of freedom

$$X_1^2 + \cdots + X_n^2 \sim \chi_n^2.$$

- If $X_1, X_2, \ldots, X_n$ are independent normally distributed random variables with means μ and variances σ^2, then their sample mean is independent from the sample standard deviation, which can be demonstrated using Basu's theorem or Cochran's theorem. The ratio of these two quantities will have the Student's t-distribution with $n - 1$ degrees of freedom:

$$t = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{\frac{1}{n}(X_1 + \cdots + X_n) - \mu}{\sqrt{\frac{1}{n(n-1)}\left[(X_1 - \overline{X})^2 + \cdots + (X_n - \overline{X})^2\right]}} \sim t_{n-1}.$$

- If $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are independent standard normal random variables, then the ratio of their normalized sums of squares will have the F-distribution with (n, m) degrees of freedom :

$$F = \frac{(X_1^2 + X_2^2 + \cdots + X_n^2)/n}{(Y_1^2 + Y_2^2 + \cdots + Y_m^2)/m} \sim F_{n,m}.$$

Operations on the density function

The split normal distribution is most directly defined in terms of joining scaled sections of the density functions of different normal distributions and rescaling the density to integrate to one. The truncated normal distribution results from rescaling a section of a single density function.

Extensions

The notion of normal distribution, being one of the most important distributions in probability theory, has been extended far beyond the standard framework of the univariate (that is one-dimensional) case (Case 1). All these extensions are also called *normal* or *Gaussian* laws, so a certain ambiguity in names exists.

- Multivariate normal distribution describes the Gaussian law in the k-dimensional Euclidean space. A vector $X \in \mathbf{R}^k$ is multivariate-normally distributed if any linear combination of its components $\sum_{j=1}^{k} a_j X_j$ has a (univariate) normal distribution. The variance of X is a $k \times k$ symmetric positive-definite matrix V.
- Rectified Gaussian distribution a rectified version of normal distribution with all the negative elements reset to 0
- Complex normal distribution deals with the complex normal vectors. A complex vector $X \in \mathbf{C}^k$ is said to be normal if both its real and imaginary components jointly possess a $2k$-dimensional multivariate normal distribution. The variance-covariance structure of X is described by two matrices: the *variance* matrix Γ, and the *relation* matrix C.
- Matrix normal distribution describes the case of normally distributed matrices.
- Gaussian processes are the normally distributed stochastic processes. These can be viewed as elements of some infinite-dimensional Hilbert space H, and thus are the analogues of multivariate normal vectors for the case $k = \infty$. A random element $h \in H$ is said to be normal if for any constant $a \in H$ the scalar product (a, h) has a (univariate) normal distribution. The variance structure of such Gaussian random element can be described in terms of the linear *covariance operator* K: $H \to H$. Several Gaussian processes became popular enough to have their own names:
  - Brownian motion,
  - Brownian bridge,
  - Ornstein–Uhlenbeck process.
- Gaussian q-distribution is an abstract mathematical construction which represents a "q-analogue" of the normal distribution.
- the q-Gaussian is an analogue of the Gaussian distribution, in the sense that it maximises the Tsallis entropy, and is one type of Tsallis distribution. Note that this distribution is different from the Gaussian q-distribution above.

One of the main practical uses of the Gaussian law is to model the empirical distributions of many different random variables encountered in practice. In such case a possible extension would be a richer family of distributions, having more than two parameters and therefore being able to fit the empirical distribution more accurately. The examples of such extensions are:

- Pearson distribution — a four-parametric family of probability distributions that extend the normal law to include different skewness and kurtosis values.

Normality tests

Normality tests assess the likelihood that the given data set $\{x_1, \ldots, x_n\}$ comes from a normal distribution. Typically the null hypothesis H_0 is that the observations are distributed normally with unspecified mean μ and variance σ^2, versus the alternative H_a that the distribution is arbitrary. A great number of tests (over 40) have been devised for this problem, the more prominent of them are outlined below:

- **"Visual" tests** are more intuitively appealing but subjective at the same time, as they rely on informal human judgement to accept or reject the null hypothesis.
  - Q-Q plot — is a plot of the sorted values from the data set against the expected values of the corresponding quantiles from the standard normal distribution. That is, it's a plot of point of the form $(\Phi^{-1}(p_k), x_{(k)})$, where plotting points p_k are equal to $p_k = (k - \alpha)/(n + 1 - 2\alpha)$ and α is an adjustment constant which can be anything between 0 and 1. If the null hypothesis is true, the plotted points should approximately lie on a straight line.
  - P-P plot — similar to the Q-Q plot, but used much less frequently. This method consists of plotting the points $(\Phi(z_{(k)}), p_k)$, where $z_{(k)} = (x_{(k)} - \hat{\mu})/\hat{\sigma}$. For normally distributed data this plot should lie on a 45° line between (0, 0) and (1, 1).
  - Wilk–Shapiro test employs the fact that the line in the Q-Q plot has the slope of σ. The test compares the least squares estimate of that slope with the value of the sample variance, and rejects the null hypothesis if these two quantities differ significantly.
  - Normal probability plot (rankit plot)
- **Moment tests**:
  - D'Agostino's K-squared test
  - Jarque–Bera test
- **Empirical distribution function tests**:
  - Lilliefors test (an adaptation of the Kolmogorov–Smirnov test)
  - Anderson–Darling test

Estimation of parameters

It is often the case that we don't know the parameters of the normal distribution, but instead want to estimate them. That is, having a sample $(x_1, \ldots, x_n)$ from a normal $N(\mu, \sigma^2)$ population we would like to learn the approximate values of parameters μ and σ^2. The standard approach to this problem is the maximum likelihood method, which requires maximization of the *log-likelihood function*:

$$\ln \mathcal{L}(\mu, \sigma^2) = \sum_{i=1}^{n} \ln f(x_i;\, \mu, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2.$$

Taking derivatives with respect to μ and σ^2 and solving the resulting system of first order conditions yields the *maximum likelihood estimates*:

$$\hat{\mu} = \overline{x} \equiv \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

Estimator $\hat{\mu}$ is called the *sample mean*, since it is the arithmetic mean of all observations. The statistic $\overline{x}$ is complete and sufficient for μ, and therefore by the Lehmann–Scheffé theorem, $\hat{\mu}$ is the uniformly minimum variance unbiased (UMVU) estimator.[29] In finite samples it is distributed normally:

$$\hat{\mu} \sim \mathcal{N}(\mu,\ \sigma^2/n).$$

The variance of this estimator is equal to the $\mu\mu$-element of the inverse Fisher information matrix $\mathcal{I}^{-1}$. This implies that the estimator is finite-sample efficient. Of practical importance is the fact that the standard error of $\hat{\mu}$ is proportional to $1/\sqrt{n}$, that is, if one wishes to decrease the standard error by a factor of 10, one must increase the number of points in the sample by a factor of 100. This fact is widely used in determining sample sizes for opinion

polls and the number of trials in Monte Carlo simulations.

From the standpoint of the asymptotic theory, $\hat{\mu}$ is consistent, that is, it converges in probability to μ as $n \to \infty$. The estimator is also asymptotically normal, which is a simple corollary of the fact that it is normal in finite samples:

$$\sqrt{n}(\hat{\mu} - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

The estimator $\hat{\sigma}^2$ is called the *sample variance*, since it is the variance of the sample (x_1, …, x_n). In practice, another estimator is often used instead of the $\hat{\sigma}^2$. This other estimator is denoted s^2, and is also called the *sample variance*, which represents a certain ambiguity in terminology; its square root s is called the *sample standard deviation*. The estimator s^2 differs from $\hat{\sigma}^2$ by having $(n - 1)$ instead of n in the denominator (the so called Bessel's correction):

$$s^2 = \frac{n}{n-1}\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{x})^2.$$

The difference between s^2 and $\hat{\sigma}^2$ becomes negligibly small for large n's. In finite samples however, the motivation behind the use of s^2 is that it is an unbiased estimator of the underlying parameter σ^2, whereas $\hat{\sigma}^2$ is biased. Also, by the Lehmann–Scheffé theorem the estimator s^2 is uniformly minimum variance unbiased (UMVU),[29] which makes it the "best" estimator among all unbiased ones. However it can be shown that the biased estimator $\hat{\sigma}^2$ is "better" than the s^2 in terms of the mean squared error (MSE) criterion. In finite samples both s^2 and $\hat{\sigma}^2$ have scaled chi-squared distribution with $(n - 1)$ degrees of freedom:

$$s^2 \sim \frac{\sigma^2}{n-1}\cdot\chi^2_{n-1}, \qquad \hat{\sigma}^2 \sim \frac{\sigma^2}{n}\cdot\chi^2_{n-1}\ .$$

The first of these expressions shows that the variance of s^2 is equal to $2\sigma^4/(n-1)$, which is slightly greater than the $\sigma\sigma$-element of the inverse Fisher information matrix $\mathcal{I}^{-1}$. Thus, s^2 is not an efficient estimator for σ^2, and moreover, since s^2 is UMVU, we can conclude that the finite-sample efficient estimator for σ^2 does not exist.

Applying the asymptotic theory, both estimators s^2 and $\hat{\sigma}^2$ are consistent, that is they converge in probability to σ^2 as the sample size $n \to \infty$. The two estimators are also both asymptotically normal:

$$\sqrt{n}(\hat{\sigma}^2 - \sigma^2) \simeq \sqrt{n}(s^2 - \sigma^2) \xrightarrow{d} \mathcal{N}(0, 2\sigma^4).$$

In particular, both estimators are asymptotically efficient for σ^2.

By Cochran's theorem, for normal distribution the sample mean $\hat{\mu}$ and the sample variance s^2 are independent, which means there can be no gain in considering their joint distribution. There is also a reverse theorem: if in a sample the sample mean and sample variance are independent, then the sample must have come from the normal distribution. The independence between $\hat{\mu}$ and s can be employed to construct the so-called *t-statistic*:

$$t = \frac{\hat{\mu} - \mu}{s/\sqrt{n}} = \frac{\overline{x} - \mu}{\sqrt{\frac{1}{n(n-1)}\sum(x_i - \overline{x})^2}} \sim t_{n-1}$$

This quantity t has the Student's t-distribution with $(n - 1)$ degrees of freedom, and it is an ancillary statistic (independent of the value of the parameters). Inverting the distribution of this t-statistics will allow us to construct the confidence interval for μ;[30] similarly, inverting the χ^2 distribution of the statistic s^2 will give us the confidence interval for σ^2:[31]

$$\mu \in \left[\hat{\mu} + t_{n-1,\alpha/2}\frac{1}{\sqrt{n}}s,\ \ \hat{\mu} + t_{n-1,1-\alpha/2}\frac{1}{\sqrt{n}}s\right] \approx \left[\hat{\mu} - |z_{\alpha/2}|\frac{1}{\sqrt{n}}s,\ \ \hat{\mu} + |z_{\alpha/2}|\frac{1}{\sqrt{n}}s\right],$$

$$\sigma^2 \in \left[\frac{(n-1)s^2}{\chi^2_{n-1,1-\alpha/2}},\ \ \frac{(n-1)s^2}{\chi^2_{n-1,\alpha/2}}\right] \approx \left[s^2 - |z_{\alpha/2}|\frac{\sqrt{2}}{\sqrt{n}}s^2,\ \ s^2 + |z_{\alpha/2}|\frac{\sqrt{2}}{\sqrt{n}}s^2\right],$$

where $t_{k,p}$ and $\chi^2_{k,p}$ are the p^{th} quantiles of the t- and χ^2-distributions respectively. These confidence intervals are of the *level* $1 - \alpha$, meaning that the true values μ and σ^2 fall outside of these intervals with probability α. In practice people usually take $\alpha = 5\%$, resulting in the 95% confidence intervals. The approximate formulas in the display above were derived from

the asymptotic distributions of $\hat{\mu}$ and s^2. The approximate formulas become valid for large values of *n*, and are more convenient for the manual calculation since the standard normal quantiles $z_{\alpha/2}$ do not depend on *n*. In particular, the most popular value of $\alpha = 5\%$, results in $|z_{0.025}| = 1.96$.

Bayesian analysis of the normal distribution

Bayesian analysis of normally distributed data is complicated by the many different possibilities that may be considered:

- Either the mean, or the variance, or neither, may be considered a fixed quantity.
- When the variance is unknown, analysis may be done directly in terms of the variance, or in terms of the precision, the reciprocal of the variance. The reason for expressing the formulas in terms of precision is that the analysis of most cases is simplified.
- Both univariate and multivariate cases need to be considered.
- Either conjugate or improper prior distributions may be placed on the unknown variables.
- An additional set of cases occurs in Bayesian linear regression, where in the basic model the data is assumed to be normally distributed, and normal priors are placed on the regression coefficients. The resulting analysis is similar to the basic cases of independent identically distributed data, but more complex.

The formulas for the non-linear-regression cases are summarized in the conjugate prior article.

The sum of two quadratics

Scalar form

The following auxiliary formula is useful for simplifying the posterior update equations, which otherwise become fairly tedious.

$$a(y-x)^2 + b(x-z)^2 = (a+b)\left(x - \frac{ay+bz}{a+b}\right)^2 + \frac{ab}{a+b}(y-z)^2$$

This equation rewrites the sum of two quadratics in *x* by expanding the squares, grouping the terms in *x*, and completing the square. Note the following about the complex constant factors attached to some of the terms:

1. The factor $\frac{ay+bz}{a+b}$ has the form of a weighted average of *y* and *z*.
2. $\frac{ab}{a+b} = \frac{1}{\frac{1}{a}+\frac{1}{b}} = (a^{-1}+b^{-1})^{-1}$. This shows that this factor can be thought of as resulting from a situation where the reciprocals of quantities *a* and *b* add directly, so to combine *a* and *b* themselves, it's necessary to reciprocate, add, and reciprocate the result again to get back into the original units. This is exactly the sort of operation performed by the harmonic mean, so it is not surprising that $\frac{ab}{a+b}$ is one-half the harmonic mean of *a* and *b*.

Vector form

A similar formula can be written for the sum of two vector quadratics: If $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are vectors of length k, and $\mathbf{A}$ and $\mathbf{B}$ are symmetric, invertible matrices of size $k \times k$, then

$$(\mathbf{y}-\mathbf{x})'\mathbf{A}(\mathbf{y}-\mathbf{x})+(\mathbf{x}-\mathbf{z})'\mathbf{B}(\mathbf{x}-\mathbf{z}) = (\mathbf{x}-\mathbf{c})'(\mathbf{A}+\mathbf{B})(\mathbf{x}-\mathbf{c})+(\mathbf{y}-\mathbf{z})'(\mathbf{A}^{-1}+\mathbf{B}^{-1})^{-1}(\mathbf{y}-\mathbf{z})$$

where

$$\mathbf{c} = (\mathbf{A}+\mathbf{B})^{-1}(\mathbf{A}\mathbf{y}+\mathbf{B}\mathbf{z})$$

Note that the form $\mathbf{x}'\mathbf{A}\mathbf{x}$ is called a quadratic form and is a scalar:

$$\mathbf{x}'\mathbf{A}\mathbf{x} = \sum_{i,j} a_{ij}x_i x_j$$

In other words, it sums up all possible combinations of products of pairs of elements from $\mathbf{x}$, with a separate coefficient for each. In addition, since $x_i x_j = x_j x_i$, only the sum $a_{ij} + a_{ji}$ matters for any off-diagonal elements of $\mathbf{A}$, and there is no loss of generality in assuming that $\mathbf{A}$ is symmetric. Furthermore, if $\mathbf{A}$ is symmetric, then the form $\mathbf{x}'\mathbf{A}\mathbf{y} = \mathbf{y}'\mathbf{A}\mathbf{x}$.

The sum of differences from the mean

Another useful formula is as follows:

$$\sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.

With known variance

For a set of i.i.d. normally distributed data points **X** of size *n* where each individual point *x* follows $x \sim \mathcal{N}(\mu, \sigma^2)$ with known variance σ^2, the conjugate prior distribution is also normally distributed.

This can be shown more easily by rewriting the variance as the precision, i.e. using $\tau = 1/\sigma^2$. Then if $x \sim \mathcal{N}(\mu, \tau)$ and $\mu \sim \mathcal{N}(\mu_0, \tau_0)$, we proceed as follows.

First, the likelihood function is (using the formula above for the sum of differences from the mean):

$$\begin{aligned} p(\mathbf{X}|\mu,\tau) &= \prod_{i=1}^{n}\sqrt{\frac{\tau}{2\pi}}\exp\left(-\frac{1}{2}\tau(x_i-\mu)^2\right) \\ &= \left(\frac{\tau}{2\pi}\right)^{n/2}\exp\left(-\frac{1}{2}\tau\sum_{i=1}^{n}(x_i-\mu)^2\right) \\ &= \left(\frac{\tau}{2\pi}\right)^{n/2}\exp\left[-\frac{1}{2}\tau\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right)\right] \end{aligned}$$

Then, we proceed as follows:

$$\begin{aligned}
p(\mu|\mathbf{X}) \propto p(\mathbf{X}|\mu)p(\mu) &= \left(\frac{\tau}{2\pi}\right)^{n/2} \exp\left[-\frac{1}{2}\tau\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right)\right]\sqrt{\frac{\tau_0}{2\pi}}\exp\left(-\frac{1}{2}\tau_0(\mu-\mu_0)^2\right) \\
&\propto \exp\left(-\frac{1}{2}\left(\tau\left(\sum_{i=1}^{n}(x_i-\bar{x})^2 + n(\bar{x}-\mu)^2\right) + \tau_0(\mu-\mu_0)^2\right)\right) \\
&\propto \exp\left(-\frac{1}{2}(n\tau(\bar{x}-\mu)^2 + \tau_0(\mu-\mu_0)^2)\right) \\
&= \exp\left(-\frac{1}{2}(n\tau+\tau_0)\left(\mu - \frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}\right)^2 + \frac{n\tau\tau_0}{n\tau+\tau_0}(\bar{x}-\mu_0)^2\right) \\
&\propto \exp\left(-\frac{1}{2}(n\tau+\tau_0)\left(\mu - \frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}\right)^2\right)
\end{aligned}$$

In the above derivation, we used the formula above for the sum of two quadratics and eliminated all constant factors not involving μ. The result is the kernel of a normal distribution, with mean $\frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}$ and precision $n\tau+\tau_0$, i.e.

$$p(\mu|\mathbf{X}) \sim \mathcal{N}\left(\frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0}, n\tau+\tau_0\right)$$

This can be written as a set of Bayesian update equations for the posterior parameters in terms of the prior parameters:

$$\begin{aligned}
\tau_0' &= \tau_0 + n\tau \\
\mu_0' &= \frac{n\tau\bar{x}+\tau_0\mu_0}{n\tau+\tau_0} \\
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i
\end{aligned}$$

That is, to combine n data points with total precision of $n\tau$ (or equivalently, total variance of n/σ^2) and mean of values $\bar{x}$, derive a new total precision simply by adding the total precision of the data to the prior total precision, and form a new mean through a *precision-weighted average*, i.e. a weighted average of the data mean and the prior mean, each weighted by the associated total precision. This makes logical sense if the precision is thought of as indicating the certainty of the observations: In the distribution of the posterior mean, each of the input components is weighted by its certainty, and the certainty of this distribution is the sum of the individual certainties. (For the intuition of this, compare the expression "the whole is (or is not) greater than the sum of its parts". In addition, consider that the knowledge of the posterior comes from a combination of the knowledge of the prior and likelihood, so it makes sense that we are more certain of it than of either of its components.)

The above formula reveals why it is more convenient to do Bayesian analysis of conjugate priors for the normal distribution in terms of the precision. The posterior precision is simply the sum of the prior and likelihood precisions, and the posterior mean is computed through a precision-weighted average, as described above. The same formulas can be written in terms of variance by reciprocating all the precisions, yielding the more ugly formulas

$$\begin{aligned}
\sigma_0^{2\prime} &= \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}} \\
\mu_0' &= \frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\sigma_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\sigma_0^2}} \\
\bar{x} &= \frac{1}{n}\sum_{i=1}^{n} x_i
\end{aligned}$$

With known mean

For a set of i.i.d. normally distributed data points **X** of size *n* where each individual point *x* follows $x \sim \mathcal{N}(\mu, \sigma^2)$ with known mean μ, the conjugate prior of the variance has an inverse gamma distribution or a scaled inverse chi-squared distribution. The two are equivalent except for having different parameterizations. The use of the inverse gamma is more common, but the scaled inverse chi-squared is more convenient, so we use it in the following derivation. The prior for σ^2 is as follows:

$$p(\sigma^2|\nu_0,\sigma_0^2) = \frac{(\sigma_0^2\nu_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)}\,\frac{\exp\left[\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right]}{(\sigma^2)^{1+\nu_0/2}} \propto \frac{\exp\left[\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right]}{(\sigma^2)^{1+\nu_0/2}}$$

The likelihood function from above, written in terms of the variance, is:

$$\begin{aligned} p(\mathbf{X}|\mu,\sigma^2) &= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right] \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{S}{2\sigma^2}\right] \end{aligned}$$

where $S = \sum_{i=1}^{n}(x_i-\mu)^2$.

Then:

$$\begin{aligned} p(\sigma^2|\mathbf{X}) \propto p(\mathbf{X}|\sigma^2)p(\sigma^2) &= \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} \exp\left[-\frac{S}{2\sigma^2}\right] \frac{(\sigma_0^2\nu_0/2)^{\nu_0/2}}{\Gamma(\nu_0/2)}\,\frac{\exp\left[\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right]}{(\sigma^2)^{1+\nu_0/2}} \\ &\propto \left(\frac{1}{\sigma^2}\right)^{n/2} \frac{1}{(\sigma^2)^{1+\nu_0/2}} \exp\left[-\frac{S}{2\sigma^2} + \frac{-\nu_0\sigma_0^2}{2\sigma^2}\right] \\ &= \frac{1}{(\sigma^2)^{1+(\nu_0+n)/2}} \exp\left[-\frac{\nu_0\sigma_0^2 + S}{2\sigma^2}\right] \end{aligned}$$

This is also a scaled inverse chi-squared distribution, where

$$\begin{aligned} \nu_0' &= \nu_0 + n \\ \nu_0'\sigma_0^{2\prime} &= \nu_0\sigma_0^2 + \sum_{i=1}^{n}(x_i-\mu)^2 \end{aligned}$$

or equivalently

$$\begin{aligned} \nu_0' &= \nu_0 + n \\ \sigma_0^{2\prime} &= \frac{\nu_0\sigma_0^2 + \sum_{i=1}^{n}(x_i-\mu)^2}{\nu_0+n} \end{aligned}$$

Reparameterizing in terms of an inverse gamma distribution, the result is:

$$\begin{aligned} \alpha' &= \alpha + \frac{n}{2} \\ \beta' &= \beta + \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2} \end{aligned}$$

With unknown mean and variance

For a set of i.i.d. normally distributed data points **X** of size *n* where each individual point *x* follows $x \sim \mathcal{N}(\mu, \sigma^2)$ with unknown mean μ and variance σ^2, the a combined (multivariate) conjugate prior is placed over the mean and variance, consisting of a normal-inverse-gamma distribution. Logically, this originates as follows:

1. From the analysis of the case with unknown mean but known variance, we see that the update equations involve sufficient statistics computed from the data consisting of the mean of the data points and the total variance of the data points, computed in turn from the known variance divided by the number of data points.
2. From the analysis of the case with unknown variance but known mean, we see that the update equations involve sufficient statistics over the data consisting of the number of data points and sum of squared deviations.
3. Keep in mind that the posterior update values serve as the prior distribution when further data is handled. Thus, we should logically think of our priors in terms of the sufficient statistics just described, with the same semantics kept in mind as much as possible.
4. To handle the case where both mean and variance are unknown, we could place independent priors over the mean and variance, with fixed estimates of the average mean, total variance, number of data points used to compute the variance prior, and sum of squared deviations. Note however that in reality, the total variance of the mean depends on the unknown variance, and the sum of squared deviations that goes into the variance prior (appears to) depend on the unknown mean. In practice, the latter dependence is relatively unimportant: Shifting the actual mean shifts the generated points by an equal amount, and on average the squared deviations will remain the same. This is not the case, however, with the total variance of the mean: As the unknown variance increases, the total variance of the mean will increase proportionately, and we would like to capture this dependence.
5. This suggests that we create a *conditional prior* of the mean on the unknown variance, with a hyperparameter specifying the mean of the pseudo-observations associated with the prior, and another parameter specifying the number of pseudo-observations. This number serves as a scaling parameter on the variance, making it possible to control the overall variance of the mean relative to the actual variance parameter. The prior for the variance also has two hyperparameters, one specifying the sum of squared deviations of the pseudo-observations associated with the prior, and another specifying once again the number of pseudo-observations. Note that each of the priors has a hyperparameter specifying the number of pseudo-observations, and in each case this controls the relative variance of that prior. These are given as two separate hyperparameters so that the variance (aka the confidence) of the two priors can be controlled separately.
6. This leads immediately to the normal-inverse-gamma distribution, which is defined as the product of the two distributions just defined, with conjugate priors used (an inverse gamma distribution over the variance, and a normal distribution over the mean, *conditional* on the variance) and with the same four parameters just defined.

The priors are normally defined as follows:

$$p(\mu|\sigma^2; \mu_0, n_0) \sim \mathcal{N}(\mu_0, \sigma_0^2/n_0)$$
$$p(\sigma^2; \nu_0, \sigma_0^2) \sim I\chi^2(\nu_0, \sigma_0^2) = IG(\nu_0/2, \nu_0\sigma_0^2/2)$$

The update equations can be derived, and look as follows:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$\mu_0' = \frac{n_0\mu_0 + n\bar{x}}{n_0 + n}$$
$$n_0' = n_0 + n$$
$$\nu_0' = \nu_0 + n$$
$$\nu_0'\sigma_0^{2\prime} = \nu_0\sigma_0^2 + \sum_{i=1}^{n}(x_i - \bar{x})^2 + \frac{n_0 n}{n_0 + n}(\mu_0 - \bar{x})^2$$

The respective numbers of pseudo-observations just add the number of actual observations to them. The new mean hyperparameter is once again a weighted average, this time weighted by the relative numbers of observations. Finally, the update for $\nu_0'\sigma_0^{2\prime}$ is similar to the case with known mean, but in this case the sum of squared deviations is taken with respect to the observed data mean rather than the true mean, and as a result a new "interaction term" needs to be added to take care of the additional error source stemming from the deviation between prior and data mean.

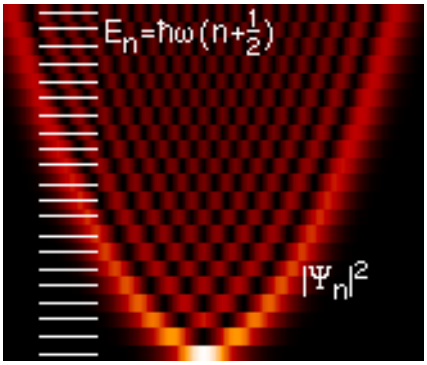
Proof is as follows.

Occurrence

The occurrence of normal distribution in practical problems can be loosely classified into three categories:

1. Exactly normal distributions;
2. Approximately normal laws, for example when such approximation is justified by the central limit theorem; and
3. Distributions modeled as normal – the normal distribution being the distribution with maximum entropy for a given mean and variance.

Exact normality



The ground state of a quantum harmonic oscillator has the Gaussian distribution.

Certain quantities in physics are distributed normally, as was first demonstrated by James Clerk Maxwell. Examples of such quantities are:

- Velocities of the molecules in the ideal gas. More generally, velocities of the particles in any system in thermodynamic equilibrium will have normal distribution, due to the maximum entropy principle.
- Probability density function of a ground state in a quantum harmonic oscillator.
- The position of a particle which experiences diffusion. If initially the particle is located at a specific point (that is its probability distribution is the dirac delta function), then after time t its location is described by a normal distribution with variance t, which satisfies the diffusion equation $\frac{\partial}{\partial t} f(x,t) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f(x,t)$. If the initial location is given by a certain density function $g(x)$, then the density at time t is the convolution of g and the normal PDF.

Approximate normality

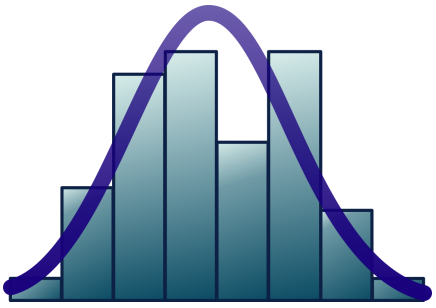
Approximately normal distributions occur in many situations, as explained by the central limit theorem. When the outcome is produced by a large number of small effects acting *additively and independently*, its distribution will be close to normal. The normal approximation will not be valid if the effects act multiplicatively (instead of additively), or if there is a single external influence which has a considerably larger magnitude than the rest of the effects.

- In counting problems, where the central limit theorem includes a discrete-to-continuum approximation and where infinitely divisible and decomposable distributions are involved, such as
  - Binomial random variables, associated with binary response variables;
  - Poisson random variables, associated with rare events;
- Thermal light has a Bose–Einstein distribution on very short time scales, and a normal distribution on longer timescales due to the central limit theorem.

Assumed normality

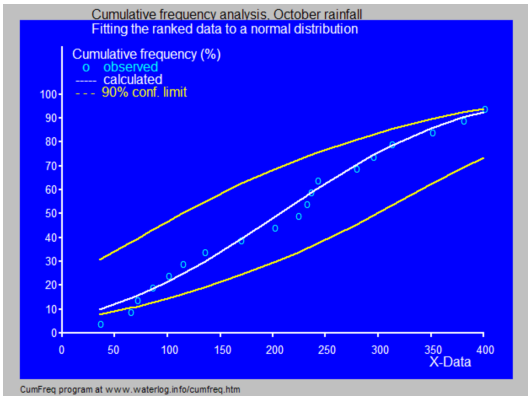
> I can only recognize the occurrence of the normal curve – the Laplacian curve of errors – as a very abnormal phenomenon. It is roughly approximated to in certain distributions; for this reason, and on account for its beautiful simplicity, we may, perhaps, use it as a first approximation, particularly in theoretical investigations.
>
> —Pearson (1901)

There are statistical methods to empirically test that assumption, see the above Normality tests section.



Histogram of sepal widths for *Iris versicolor* from Fisher's Iris flower data set, with superimposed best-fitting normal distribution.

- In biology, the *logarithm* of various variables tend to have a normal distribution, that is, they tend to have a log-normal distribution (after separation on male/female subpopulations), with examples including:
  - Measures of size of living tissue (length, height, skin area, weight);[32]
  - The *length* of *inert* appendages (hair, claws, nails, teeth) of biological specimens, *in the direction of growth*; presumably the thickness of tree bark also falls under this category;
  - Certain physiological measurements, such as blood pressure of adult humans.
- In finance, in particular the Black–Scholes model, changes in the *logarithm* of exchange rates, price indices, and stock market indices are assumed normal (these variables behave like compound interest, not like simple interest, and so are multiplicative). Some mathematicians such as Benoît Mandelbrot have argued that log-Levy distributions which possesses heavy tails would be a more appropriate model, in particular for the analysis for stock market crashes.
- Measurement errors in physical experiments are often modeled by a normal distribution. This use of a normal distribution does not imply that one is assuming the measurement errors are normally distributed, rather using the normal distribution produces the most conservative predictions possible given only knowledge about the mean and variance of the errors.[33]
- In standardized testing, results can be made to have a normal distribution. This is done by either selecting the number and difficulty of questions (as in the IQ test), or by transforming the raw test scores into "output" scores by fitting them to the normal distribution. For example, the SAT's traditional range of 200–800 is based on a normal distribution with a mean of 500 and a standard deviation of 100.



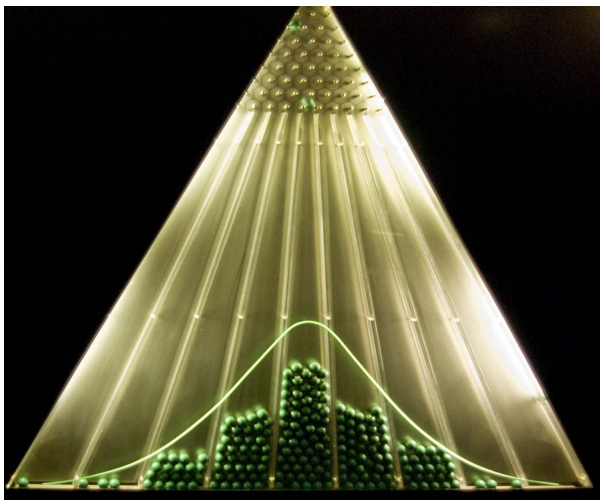
Fitted cumulative normal distribution to October rainfalls, see distribution fitting

- Many scores are derived from the normal distribution, including percentile ranks ("percentiles" or "quantiles"), normal curve equivalents, stanines, z-scores, and T-scores. Additionally, a number of behavioral statistical procedures are based on the assumption that scores are normally distributed; for example, t-tests and ANOVAs. Bell curve grading assigns relative grades based on a normal distribution of scores.
- In hydrology the distribution of long duration river discharge or rainfall, e.g. monthly and yearly totals, is often thought to be practically normal according to the central limit theorem.[34] The blue picture illustrates an example of fitting the normal distribution to ranked October rainfalls showing the 90% confidence belt based on the binomial distribution. The rainfall data are represented by plotting positions as part of the cumulative frequency

analysis.

Generating values from normal distribution

In computer simulations, especially in applications of the Monte-Carlo method, it is often desirable to generate values that are normally distributed. The algorithms listed below all generate the standard normal deviates, since a $N(\mu, \sigma^2)$ can be generated as $X = \mu + \sigma Z$, where Z is standard normal. All these algorithms rely on the availability of a random number generator U capable of producing uniform random variates.



The bean machine, a device invented by Francis Galton, can be called the first generator of normal random variables. This machine consists of a vertical board with interleaved rows of pins. Small balls are dropped from the top and then bounce randomly left or right as they hit the pins. The balls are collected into bins at the bottom and settle down into a pattern resembling the Gaussian curve.

- The most straightforward method is based on the probability integral transform property: if U is distributed uniformly on (0,1), then $\Phi^{-1}(U)$ will have the standard normal distribution. The drawback of this method is that it relies on calculation of the probit function Φ^{-1}, which cannot be done analytically. Some approximate methods are described in Hart (1968) and in the erf article. Wichura[35] gives a fast algorithm for computing this function to 16 decimal places, which is used by R to compute random variates of the normal distribution.
- An easy to program approximate approach, that relies on the central limit theorem, is as follows: generate 12 uniform $U(0,1)$ deviates, add them all up, and subtract 6 – the resulting random variable will have approximately standard normal distribution. In truth, the distribution will be Irwin–Hall, which is a 12-section eleventh-order polynomial approximation to the normal distribution. This random deviate will have a limited range of (−6, 6).[36]
- The Box–Muller method uses two independent random numbers U and V distributed uniformly on (0,1). Then the two random variables X and Y

$$X = \sqrt{-2\ln U}\,\cos(2\pi V),$$
$$Y = \sqrt{-2\ln U}\,\sin(2\pi V).$$

  will both have the standard normal distribution, and will be independent. This formulation arises because for a bivariate normal random vector $(X\ Y)$ the squared norm $X^2 + Y^2$ will have the chi-squared distribution with two degrees of freedom, which is an easily generated exponential random variable corresponding to the quantity $-2\ln(U)$ in these equations; and the angle is distributed uniformly around the circle, chosen by the random variable V.
- Marsaglia polar method is a modification of the Box–Muller method algorithm, which does not require computation of functions `sin()` and `cos()`. In this method U and V are drawn from the uniform (−1,1) distribution, and then $S = U^2 + V^2$ is computed. If S is greater or equal to one then the method starts over, otherwise two quantities

$$X = U\sqrt{\frac{-2\ln S}{S}}, \qquad Y = V\sqrt{\frac{-2\ln S}{S}}$$

  are returned. Again, X and Y will be independent and standard normally distributed.
- The Ratio method[37] is a rejection method. The algorithm proceeds as follows:
  - Generate two independent uniform deviates U and V;

- Compute $X = \sqrt{8}/e\ (V - 0.5)/U$;
- If $X^2 \le 5 - 4e^{1/4}U$ then accept X and terminate algorithm;
- If $X^2 \ge 4e^{-1.35}/U + 1.4$ then reject X and start over from step 1;
- If $X^2 \le -4 / \ln U$ then accept X, otherwise start over the algorithm.

- The ziggurat algorithm Marsaglia & Tsang (2000) is faster than the Box–Muller transform and still exact. In about 97% of all cases it uses only two random numbers, one random integer and one random uniform, one multiplication and an if-test. Only in 3% of the cases where the combination of those two falls outside the "core of the ziggurat" a kind of rejection sampling using logarithms, exponentials and more uniform random numbers has to be employed.
- There is also some investigation into the connection between the fast Hadamard transform and the normal distribution, since the transform employs just addition and subtraction and by the central limit theorem random numbers from almost any distribution will be transformed into the normal distribution. In this regard a series of Hadamard transforms can be combined with random permutations to turn arbitrary data sets into a normally distributed data.

Numerical approximations for the normal CDF

The standard normal CDF is widely used in scientific and statistical computing. The values $\Phi(x)$ may be approximated very accurately by a variety of methods, such as numerical integration, Taylor series, asymptotic series and continued fractions. Different approximations are used depending on the desired level of accuracy.

- Zelen & Severo (1964) give the approximation for $\Phi(x)$ for $x > 0$ with the absolute error $|\varepsilon(x)| < 7.5 \cdot 10^{-8}$ (algorithm 26.2.17 [38]):

$$\Phi(x) = 1 - \phi(x)\left(b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + b_5 t^5\right) + \varepsilon(x), \qquad t = \frac{1}{1 + b_0 x},$$

  where $\phi(x)$ is the standard normal PDF, and $b_0 = 0.2316419$, $b_1 = 0.319381530$, $b_2 = -0.356563782$, $b_3 = 1.781477937$, $b_4 = -1.821255978$, $b_5 = 1.330274429$.
- Hart (1968) lists almost a hundred of rational function approximations for the `erfc()` function. His algorithms vary in the degree of complexity and the resulting precision, with maximum absolute precision of 24 digits. An algorithm by West (2009) combines Hart's algorithm 5666 with a continued fraction approximation in the tail to provide a fast computation algorithm with a 16-digit precision.
- Cody (1969) after recalling Hart68 solution is not suited for *erf*, gives a solution for both *erf* and *erfc*, with maximal relative error bound, via Rational Chebyshev Approximation.
- Marsaglia (2004) suggested a simple algorithm[39] based on the Taylor series expansion

$$\Phi(x) = \frac{1}{2} + \phi(x)\left(x + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \frac{x^9}{3 \cdot 5 \cdot 7 \cdot 9} + \cdots\right)$$

  for calculating $\Phi(x)$ with arbitrary precision. The drawback of this algorithm is comparatively slow calculation time (for example it takes over 300 iterations to calculate the function with 16 digits of precision when $x = 10$).
- The GNU Scientific Library calculates values of the standard normal CDF using Hart's algorithms and approximations with Chebyshev polynomials.

History

Development

Some authors[40][41] attribute the credit for the discovery of the normal distribution to de Moivre, who in 1738[42] published in the second edition of his "*The Doctrine of Chances*" the study of the coefficients in the binomial expansion of $(a + b)^n$. De Moivre proved that the middle term in this expansion has the approximate magnitude of $2/\sqrt{2\pi n}$, and that "If m or $\frac{1}{2}n$ be a Quantity infinitely great, then the Logarithm of the Ratio, which a Term distant from the middle by the Interval ℓ, has to the middle Term, is $-\frac{2\ell\ell}{n}$."[43] Although this theorem can be interpreted as the first obscure expression for the normal probability law, Stigler points out that de Moivre himself did not interpret his results as anything more than the approximate rule for the binomial coefficients, and in particular de Moivre lacked the concept of the probability density function.[44]

Carl Friedrich Gauss discovered the normal distribution in 1809 as a way to rationalize the method of least squares.

In 1809 Gauss published his monograph "*Theoria motus corporum coelestium in sectionibus conicis solem ambientium*" where among other things he introduces several important statistical concepts, such as the method of least squares, the method of maximum likelihood, and the *normal distribution*. Gauss used M, M', M'', … to denote the measurements of some unknown quantity V, and sought the "most probable" estimator: the one which maximizes the probability $\varphi(M-V) \cdot \varphi(M'-V) \cdot \varphi(M''-V) \cdot \ldots$ of obtaining the observed experimental results. In his notation $\varphi\Delta$ is the probability law of the measurement errors of magnitude Δ. Not knowing what the function φ is, Gauss requires that his method should reduce to the well-known answer: the arithmetic mean of the measured values.[45] Starting from these principles, Gauss demonstrates that the only law which rationalizes the choice of arithmetic mean as an estimator of the location parameter, is the normal law of errors:[46]

$$\varphi\Delta = \frac{h}{\sqrt{\pi}}\, e^{-\mathrm{hh}\Delta\Delta},$$

where h is "the measure of the precision of the observations". Using this normal law as a generic model for errors in the experiments, Gauss formulates what is now known as the non-linear weighted least squares (NWLS) method.[47]

Although Gauss was the first to suggest the normal distribution law, Laplace made significant contributions.[48] It was Laplace who first posed the problem of aggregating several observations in 1774,[49] although his own solution led to the Laplacian distribution. It was Laplace who first calculated the value of the integral $\int e^{-t^2} dt = \sqrt{\pi}$ in 1782, providing the normalization constant for the normal distribution.[50] Finally, it was Laplace who in 1810 proved and presented to the Academy the fundamental central limit theorem, which emphasized the theoretical importance of the normal distribution.[51]

Marquis de Laplace proved the central limit theorem in 1810, consolidating the importance of the normal distribution in statistics.

It is of interest to note that in 1809 an American mathematician Adrain published two derivations of the normal probability law, simultaneously and independently from Gauss.[52] His works remained largely unnoticed by the scientific community, until in 1871 they were "rediscovered" by Abbe.[53]

In the middle of the 19th century Maxwell demonstrated that the normal distribution is not just a convenient mathematical tool, but may also occur in natural phenomena:[54] "The number of particles whose velocity, resolved in a certain direction, lies between x and $x + dx$ is

$$\mathrm{N}\,\frac{1}{\alpha\sqrt{\pi}}\,e^{-\frac{x^2}{\alpha^2}}dx$$

Naming

Since its introduction, the normal distribution has been known by many different names: the law of error, the law of facility of errors, Laplace's second law, Gaussian law, etc. Gauss himself apparently coined the term with reference to the "normal equations" involved in its applications, with normal having its technical meaning of orthogonal rather than "usual".[55] However, by the end of the 19th century some authors[56] had started using the name *normal distribution*, where the word "normal" was used as an adjective – the term now being seen as a reflection of the fact that this distribution was seen as typical, common – and thus "normal". Peirce (one of those authors) once defined "normal" thus: "...the 'normal' is not the average (or any other kind of mean) of what actually occurs, but of what *would*, in the long run, occur under certain circumstances."[57] Around the turn of the 20th century Pearson popularized the term *normal* as a designation for this distribution.[58]

> Many years ago I called the Laplace–Gaussian curve the *normal* curve, which name, while it avoids an international question of priority, has the disadvantage of leading people to believe that all other distributions of frequency are in one sense or another 'abnormal'.
>
> —Pearson (1920)

Also, it was Pearson who first wrote the distribution in terms of the standard deviation σ as in modern notation. Soon after this, in year 1915, Fisher added the location parameter to the formula for normal distribution, expressing it in the way it is written nowadays:

$$df = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} dx$$

The term "standard normal" which denotes the normal distribution with zero mean and unit variance came into general use around 1950s, appearing in the popular textbooks by P.G. Hoel (1947) "*Introduction to mathematical statistics*" and A.M. Mood (1950) "*Introduction to the theory of statistics*".[59]

When the name is used, the "Gaussian distribution" was named after Carl Friedrich Gauss, who introduced the distribution in 1809 as a way of rationalizing the method of least squares as outlined above. The related work of Laplace, also outlined above has led to the normal distribution being sometimes called Laplacian, especially in French-speaking countries. Among English speakers, both "normal distribution" and "Gaussian distribution" are in common use, with different terms preferred by different communities.

Notes

[1] The designation "bell curve" is ambiguous: many other distributions are "bell"-shaped: the Cauchy distribution, Student's t-distribution, the generalized normal, the logistic, etc.

[2] Casella & Berger (2001, p. 102)

[3] *Normal Distribution* (http://findarticles.com/p/articles/mi_g2699/is_0002/ai_2699000241), Gale Encyclopedia of Psychology

[4] Cover, Thomas M.; Thomas, Joy A. (2006). *Elements of Information Theory*. John Wiley and Sons. p. 254.

[5] Park, Sung Y.; Bera, Anil K. (2009). "Maximum Entropy Autoregressive Conditional Heteroskedasticity Model" (http://www.wise.xmu.edu.cn/Master/Download/..\..\UploadFiles\paper-masterdownload\20095199323270554751157 76.pdf). *Journal of Econometrics* (Elsevier): 219–230. . Retrieved 2011-06-02.

[6] Halperin, Hartley & Hoel (1965, item 7)

[7] McPherson (1990, p. 110)

[8] Bernardo & Smith (2000, p. 121)

[9] Patel & Read (1996, [2.1.4])

[10] Fan (1991, p. 1258)

[11] Patel & Read (1996, [2.1.8])

[12] Scott, Clayton; Nowak, Robert (August 7, 2003). "The Q-function" (http://cnx.org/content/m11537/1.2/). *Connexions*. .

[13] Barak, Ohad (April 6, 2006). "Q Function and Error Function" (http://www.eng.tau.ac.il/~jo/academic/Q.pdf). Tel Aviv University. .

[14] Weisstein, Eric W., " Normal Distribution Function (http://mathworld.wolfram.com/NormalDistributionFunction.html)" from MathWorld.

[15] Bryc (1995, p. 23)

[16] Bryc (1995, p. 24)

[17] WolframAlpha.com (http://www.wolframalpha.com/input/?i=Table[{N(Erf(n/Sqrt(2)),+12),+N(1-Erf(n/Sqrt(2)),+12),+N(1/(1-Erf(n/Sqrt(2))),+12)},+{n,1,6}])

[18] part 1 (http://www.wolframalpha.com/input/?i=Table[Sqrt(2)*InverseErf(x),+{x,+N({8/10,+9/10,+19/20,+49/50,+99/100,+995/1000,+998/1000},+13)}]), part 2 (http://www.wolframalpha.com/input/?i=Table[{N(1-10^(-x),9),N(Sqrt(2)*InverseErf(1-10^(-x)),13)},{x,3,9}])

[19] Galambos & Simonelli (2004, Theorem 3.5)

[20] Bryc (1995, p. 35)

[21] Bryc (1995, p. 27)

[22] Lukacs & King (1954)

[23] Patel & Read (1996, [2.3.6])

[24] http://www.allisons.org/ll/MML/KL/Normal/

[25] Jordan, Michael I. (February 8th, 2010). "Stat260: Bayesian Modeling and Inference: The Conjugate Prior for the Normal Distribution" (http://www.cs.berkeley.edu/~jordan/courses/260-spring10/lectures/lecture5.pdf). .

[26] Cover & Thomas (2006, p. 254)

[27] Amari & Nagaoka (2000)

[28] *Normal Product Distribution* (http://mathworld.wolfram.com/NormalProductDistribution.html), Mathworld

[29] Krishnamoorthy (2006, p. 127)

[30] Krishnamoorthy (2006, p. 130)

[31] Krishnamoorthy (2006, p. 133)

[32] Huxley (1932)

[33] Jaynes, Edwin T. (2003). *Probability Theory: The Logic of Science*. Cambridge University Press. pp. 592–593.

[34] Oosterbaan, Roland J. (1994). "Chapter 6: Frequency and Regression Analysis of Hydrologic Data" (http://www.waterlog.info/pdf/freqtxt.pdf). In Ritzema, Henk P.. *Drainage Principles and Applications, Publication 16* (second revised ed.). Wageningen, The Netherlands: International Institute for Land Reclamation and Improvement (ILRI). pp. 175–224. ISBN 90-70754-33-9. .

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[36] Johnson, Kotz & Balakrishnan (1995, Equation (26.48))

[37] Kinderman & Monahan (1977)

[38] http://www.math.sfu.ca/~cbm/aands/page_932.htm

[39] For example, this algorithm is given in the article Bc programming language.

[40] Johnson, Kotz & Balakrishnan (1994, p. 85)
[41] Le Cam & Lo Yang (2000, p. 74)
[42] De Moivre first published his findings in 1733, in a pamphlet "Approximatio ad Summam Terminorum Binomii $(a + b)^n$ in Seriem Expansi" that was designated for private circulation only. But it was not until the year 1738 that he made his results publicly available. The original pamphlet was reprinted several times, see for example Walker (1985).
[43] De Moivre, Abraham (1733), Corollary I – see Walker (1985, p. 77)
[44] Stigler (1986, p. 76)
[45] "It has been customary certainly to regard as an axiom the hypothesis that if any quantity has been determined by several direct observations, made under the same circumstances and with equal care, the arithmetical mean of the observed values affords the most probable value, if not rigorously, yet very nearly at least, so that it is always most safe to adhere to it." — Gauss (1809, section 177)
[46] Gauss (1809, section 177)
[47] Gauss (1809, section 179)
[48] "My custom of terming the curve the Gauss–Laplacian or *normal* curve saves us from proportioning the merit of discovery between the two great astronomer mathematicians." quote from Pearson (1905, p. 189)
[49] Laplace (1774, Problem III)
[50] Pearson (1905, p. 189)
[51] Stigler (1986, p. 144)
[52] Stigler (1978, p. 243)
[53] Stigler (1978, p. 244)
[54] Maxwell (1860, p. 23)
[55] Jaynes, Edwin J.; *Probability Theory: The Logic of Science*, Ch 7 (http://www-biba.inrialpes.fr/Jaynes/cc07s.pdf)
[56] Besides those specifically referenced here, such use is encountered in the works of Peirce, Galton (Galton (1889, chapter V)) and Lexis (Lexis (1878), Rohrbasser & Véron (2003)) approximately around 1875.
[57] Peirce, Charles S. (c. 1909 MS), *Collected Papers* v. 6, paragraph 327
[58] Kruskal & Stigler (1997)
[59] "Earliest uses… (entry `STANDARD NORMAL CURVE`)" (http://jeff560.tripod.com/s.html). .

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External links

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- Normal Distribution Video Tutorial Part 1-2 (http://www.youtube.com/watch?v=kB_kYUbS_ig)
- An 8-foot-tall (2.4 m) Probability Machine (named Sir Francis) comparing stock market returns to the randomness of the beans dropping through the quincunx pattern. (http://www.youtube.com/watch?v=AUSKTk9ENzg) YouTube link originating from Index Funds Advisors (http://www.ifa.com)
- An interactive Normal (Gaussian) distribution plot (http://peter.freeshell.org/gaussian/)

Implied volatility

In financial mathematics, the **implied volatility** of an option contract is the volatility of the price of the underlying security that is implied by the market price of the option based on an option pricing model. In other words, it is the volatility that, when used in a given pricing model (such as Black-Scholes), yields a theoretical value for the option equal to the current market price of that option. Non-option financial instruments that have *embedded* optionality, such as an interest rate cap, can also have an implied volatility. Implied volatility, a forward-looking measure, differs from historical volatility because the latter is calculated from known past returns of a security.

Motivation

An option pricing model, such as Black–Scholes, uses a variety of inputs to derive a theoretical value for an option. Inputs to pricing models vary depending on the type of option being priced and the pricing model used. However, in general, the value of an option depends on an estimate of the future realized price volatility, σ, of the underlying. Or, mathematically:

$$C = f(\sigma, \cdot)$$

where C is the theoretical value of an option, and f is a pricing model that depends on σ, along with other inputs.

The function f is monotonically increasing in σ, meaning that a higher value for volatility results in a higher theoretical value of the option. Conversely, by the inverse function theorem, there can be at most one value for σ that, when applied as an input to $f(\sigma, \cdot)$, will result in a particular value for C.

Put in other terms, assume that there is some inverse function $g = f^{-1}$, such that

$$\sigma_{\bar{C}} = g(\bar{C}, \cdot)$$

where $\bar{C}$ is the market price for an option. The value $\sigma_{\bar{C}}$ is the volatility **implied** by the market price $\bar{C}$, or the **implied volatility**.

In general, it is not possible to give a closed form formula for implied volatility in terms of call price. However, in some cases (large strike, low strike, short expiry, large expiry) it is possible to give an asymptotic expansion of implied volatility in terms of call price.[1]

Example

A European call option, C_{XYZ}, on 100 shares of non-dividend-paying XYZ Corp. The option is struck at $50 and expires in 32 days. The risk-free interest rate is 5%. XYZ stock is currently trading at $51.25 and the current market price of C_{XYZ} is $2.00. Using a standard Black–Scholes pricing model, the volatility implied by the market price C_{XYZ} is 18.7%, or:

$$\sigma_{\bar{C}} = g(\bar{C}, \cdot) = 18.7\%$$

To verify, we apply the implied volatility back into the pricing model, f and we generate a theoretical value of $2.0004:

$$C_{theo} = f(\sigma_{\bar{C}}, \cdot) = \$2.0004$$

which confirms our computation of the market implied volatility.

Solving the inverse pricing model function

In general, a pricing model function, *f*, does not have a closed-form solution for its inverse, *g*. Instead, a root finding technique is used to solve the equation:

$$f(\sigma_{\bar{C}}, \cdot) - \bar{C} = 0$$

While there are many techniques for finding roots, two of the most commonly used are Newton's method and Brent's method. Because options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities.

Newton's method provides rapid convergence; however, it requires the first partial derivative of the option's theoretical value with respect to volatility; i.e., $\frac{\partial C}{\partial \sigma}$, which is also known as *vega* (see The Greeks). If the pricing model function yields a closed-form solution for *vega*, which is the case for Black–Scholes model, then Newton's method can be more efficient. However, for most practical pricing models, such as a binomial model, this is not the case and *vega* must be derived numerically. When forced to solve for *vega* numerically, it usually turns out that Brent's method is more efficient as a root-finding technique.

Implied volatility as measure of relative value

Often, the implied volatility of an option is a more useful measure of the option's relative value than its price. The reason is that the price of an option depends most directly on the price of its underlying asset. If an option is held as part of a delta neutral portfolio (that is, a portfolio that is hedged against small moves in the underlying's price), then the next most important factor in determining the value of the option will be its implied volatility.

Implied volatility is so important that options are often quoted in terms of volatility rather than price, particularly between professional traders.

Example

A call option is trading at $1.50 with the underlying trading at $42.05. The implied volatility of the option is determined to be 18.0%. A short time later, the option is trading at $2.10 with the underlying at $43.34, yielding an implied volatility of 17.2%. Even though the option's price is higher at the second measurement, it is still considered cheaper based on volatility. The reason is that the underlying needed to hedge the call option can be sold for a higher price.

Implied volatility as a price

Another way to look at implied volatility is to think of it as a price, not as a measure of future stock moves. In this view it simply is a more convenient way to communicate option prices than currency. Prices are different in nature from statistical quantities: one can estimate volatility of future underlying returns using any of a large number of estimation methods; however, the number one gets is not a price. A price requires two counterparties, a buyer and a seller. Prices are determined by supply and demand. Statistical estimates depend on the time-series and the mathematical structure of the model used. It is a mistake to confuse a price, which implies a transaction, with the result of a statistical estimation, which is merely what comes out of a calculation. Implied volatilities are prices: they have been derived from actual transactions. Seen in this light, it should not be surprising that implied volatilities might not conform to what a particular statistical model would predict.

Non-constant implied volatility

In general, options based on the same underlying but with different strike values and expiration times will yield different implied volatilities. This is generally viewed as evidence that an underlying's volatility is not constant but instead depends on factors such as the price level of the underlying, the underlying's recent price variance, and the passage of time. See stochastic volatility and volatility smile for more information.

Volatility instruments

Volatility instruments are financial instruments that track the value of implied volatility of other derivative securities. For instance, the CBOE Volatility Index (VIX) is calculated from a weighted average of implied volatilities of various options on the S&P 500 Index. There are also other commonly referenced volatility indices such as the VXN index (Nasdaq 100 index futures volatility measure), the QQV (QQQ volatility measure), IVX - Implied Volatility Index (an expected stock volatility over a future period for any of US securities and exchange traded instruments), as well as options and futures derivatives based directly on these volatility indices themselves.

Notes

[1] Asymptotic Expansions of the Lognormal Implied Volatility (http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1965977), Grunspan, C. (2011)

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External links

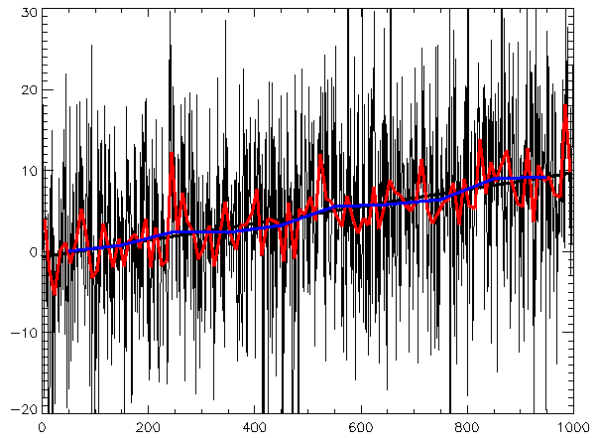
- Interactive Java Applet " Implied Volatility vs. Historic Volatility (http://www.frog-numerics.com/ifs/ifs_LevelA/HistVolaVsVDAX.html)"

Computer implementations

- Real-time calculator of implied volatilities when the underlying follows a Mean-Reverting Geometric Brownian Motion (http://www.cba.ua.edu/~rpascala/impliedvol/BSOPMSForm.php), by Razvan Pascalau, Univ. of Alabama
- Test online implied volatility calculation (http://pricing-option.com/impliedvolatility.aspx) by Christophe Rougeaux, ESILV

Time series

In statistics, signal processing, pattern recognition, econometrics, mathematical finance, Weather forecasting, Earthquake prediction, Electroencephalography, Control engineering and Communications engineering a **time series** is a sequence of data points, measured typically at successive time instants spaced at uniform time intervals. Examples of time series are the daily closing value of the Dow Jones index or the annual flow volume of the Nile River at Aswan. **Time series *analysis*** comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data. **Time series *forecasting*** is the use of a model to predict future values based on previously observed values. Time series are very frequently plotted via line charts.



Time series: random data plus trend, with best-fit line and different smoothings

Time series data have a natural temporal ordering. This makes time series analysis distinct from other common data analysis problems, in which there is no natural ordering of the observations (e.g. explaining people's wages by reference to their respective education levels, where the individuals' data could be entered in any order). Time series analysis is also distinct from spatial data analysis where the observations typically relate to geographical locations (e.g. accounting for house prices by the location as well as the intrinsic characteristics of the houses). A stochastic model for a time series will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values (see time reversibility.)

Methods for time series analyses may be divided into two classes: frequency-domain methods and time-domain methods. The former include spectral analysis and recently wavelet analysis; the latter include auto-correlation and cross-correlation analysis.

Additionally time series analysis techniques may be divided into parametric and non-parametric methods. The parametric approaches assume that the underlying stationary Stochastic process has a certain structure which can be described using a small number of parameters (for example, using an autoregressive or moving average model). In these approaches, the task is to estimate the parameters of the model that describes the stochastic process. By contrast, non-parametric approaches explicitly estimate the covariance or the spectrum of the process without assuming that the process has any particular structure.

Additionally methods of time series analysis may be divided into linear and non-linear, univariate and multivariate.

Time series analysis can be applied to:

- real-valued, continuous data
- discrete numeric data
- discrete symbolic data (i.e. sequences of characters, such as letters and words in English language[1]).

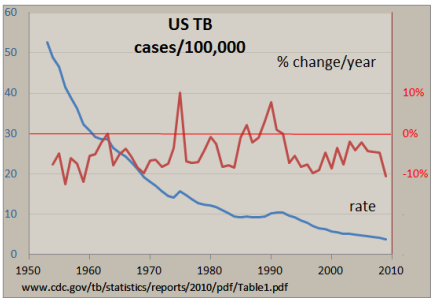
Analysis

There are several types of data analysis available for time series which are appropriate for different purposes.

In the context of statistics, econometrics, quantitative finance, seismology,meteorology, geophysics the primary goal of time series analysis is forecasting, in the context of signal processing, control engineering and communication engineering it is used for signal detection and estimation while in the context of data mining, pattern recognition and machine learning time series analysis can be used for clustering, classification, query by content, anomaly detection as well as forecasting.

Exploratory analysis

The clearest way to examine a regular time series manually is with a line chart such as the one shown for tuberculosis in the United States, made with a spreadsheet program. The number of cases was standardized to a rate per 100,000 and the percent change per year in this rate was calculated. The nearly steadily dropping line shows that the TB incidence was decreasing in most years, but the percent change in this rate varied by as much as +/- 10%, with 'surges' in 1975 and around the early 1990s. The use of both vertical axes allows the comparison of two time series in one graphic. Other techniques include:



Tuberculosis incidence US 1953-2009

- Autocorrelation analysis to examine serial dependence
- Spectral analysis to examine cyclic behaviour which need not be related to seasonality. For example, sun spot activity varies over 11 year cycles.[2][3] Other common examples include celestial phenomena, weather patterns, neural activity, commodity prices, and economic activity.
- Separation into components representing trend, seasonality, slow and fast variation, cyclical irregular: see decomposition of time series
- Simple properties of marginal distributions

Prediction and forecasting

- Fully formed statistical models for stochastic simulation purposes, so as to generate alternative versions of the time series, representing what might happen over non-specific time-periods in the future
- Simple or fully formed statistical models to describe the likely outcome of the time series in the immediate future, given knowledge of the most recent outcomes (forecasting).
- Forecasting on time series is usually done using automated statistical software packages and programming languages, such as R (programming language),S (programming language), SAS (software), SPSS, Minitab and many others.

Classification

- Assigning time series pattern to a specific category, for example identify a word based on series of hand movements in Sign language

See main article: Statistical classification

Regression analysis

- Estimating future value of a signal based on its previous behavior, e.g. predict the price of AAPL stock based on its previous price movements for that hour, day or month, or predict position of Apollo 11 spacecraft at a certain future moment based on its current trajectory (i.e. time series of its previous locations).[4]
- Regression analysis is usually based on statistical interpretation of time series properties in time domain, pioneered by statisticians George Box and Gwilym Jenkins in the 50s: see Box–Jenkins

See main article: Regression analysis

Signal Estimation

- This approach is based on Harmonic analysis and filtering of signals in Frequency domain using Fourier transform, and Spectral density estimation, the development of which was significantly accelerated during World War II by mathematician Norbert Weiner, electrical engineers Rudolf E. Kálmán, Dennis Gabor and others for filtering signal from noise and predicting signal value at a certain point in time, see Kalman Filter, Estimation theory and Digital Signal Processing

Models

Models for time series data can have many forms and represent different stochastic processes. When modeling variations in the level of a process, three broad classes of practical importance are the *autoregressive* (AR) models, the *integrated* (I) models, and the *moving average* (MA) models. These three classes depend linearly[5] on previous data points. Combinations of these ideas produce autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models. The autoregressive fractionally integrated moving average (ARFIMA) model generalizes the former three. Extensions of these classes to deal with vector-valued data are available under the heading of multivariate time-series models and sometimes the preceding acronyms are extended by including an initial "V" for "vector". An additional set of extensions of these models is available for use where the observed time-series is driven by some "forcing" time-series (which may not have a causal effect on the observed series): the distinction from the multivariate case is that the forcing series may be deterministic or under the experimenter's control. For these models, the acronyms are extended with a final "X" for "exogenous".

Non-linear dependence of the level of a series on previous data points is of interest, partly because of the possibility of producing a chaotic time series. However, more importantly, empirical investigations can indicate the advantage of using predictions derived from non-linear models, over those from linear models, as for example in nonlinear autoregressive exogenous models.

Among other types of non-linear time series models, there are models to represent the changes of variance along time (heteroskedasticity). These models represent autoregressive conditional heteroskedasticity (ARCH) and the collection comprises a wide variety of representation (GARCH, TARCH, EGARCH, FIGARCH, CGARCH, etc.). Here changes in variability are related to, or predicted by, recent past values of the observed series. This is in contrast to other possible representations of locally varying variability, where the variability might be modelled as being driven by a separate time-varying process, as in a doubly stochastic model.

In recent work on model-free analyses, wavelet transform based methods (for example locally stationary wavelets and wavelet decomposed neural networks) have gained favor. Multiscale (often referred to as multiresolution) techniques decompose a given time series, attempting to illustrate time dependence at multiple scales. See also

Markov switching multifractal (MSMF) techniques for modeling volatility evolution.

Notation

A number of different notations are in use for time-series analysis. A common notation specifying a time series X that is indexed by the natural numbers is written

$$X = \{X_1, X_2, ...\}.$$

Another common notation is

$$Y = \{Y_t : t \in T\},$$

where T is the index set.

Conditions

There are two sets of conditions under which much of the theory is built:

- Stationary process
- Ergodic process

However, ideas of stationarity must be expanded to consider two important ideas: strict stationarity and second-order stationarity. Both models and applications can be developed under each of these conditions, although the models in the latter case might be considered as only partly specified.

In addition, time-series analysis can be applied where the series are seasonally stationary or non-stationary. Situations where the amplitudes of frequency components change with time can be dealt with in time-frequency analysis which makes use of a time–frequency representation of a time-series or signal.[6]

Models

The general representation of an autoregressive model, well known as AR(p), is

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \cdots + \alpha_p Y_{t-p} + \varepsilon_t$$

where the term ε_t is the source of randomness and is called white noise. It is assumed to have the following characteristics:

- $E[\varepsilon_t] = 0$,
- $E[\varepsilon_t^2] = \sigma^2$,
- $E[\varepsilon_t \varepsilon_s] = 0 \quad \text{for all } t \neq s$.

With these assumptions, the process is specified up to second-order moments and, subject to conditions on the coefficients, may be second-order stationary.

If the noise also has a normal distribution, it is called normal or Gaussian white noise. In this case, the AR process may be strictly stationary, again subject to conditions on the coefficients.

Tools for investigating time-series data include:

- Consideration of the autocorrelation function and the spectral density function (also cross-correlation functions and cross-spectral density functions)
- Scaled cross- and auto-correlation functions [7]
- Performing a Fourier transform to investigate the series in the frequency domain
- Use of a filter to remove unwanted noise
- Principal components analysis (or empirical orthogonal function analysis)
- Singular spectrum analysis
- "Structural" models:

  - General State Space Models
  - Unobserved Components Models
- Machine Learning
  - Artificial neural networks
  - Support Vector Machine
  - Fuzzy Logic
- Hidden Markov model
- Control chart
  - Shewhart individuals control chart
  - CUSUM chart
  - EWMA chart
  - Real-time contrasts chart
- Detrended fluctuation analysis
- Dynamic time warping
- Dynamic Bayesian network
- Time-frequency analysis techniques:
  - Fast Fourier Transform
  - Continuous wavelet transform
  - Short-time Fourier transform
  - Chirplet transform
  - Fractional Fourier transform
- Chaotic analysis
  - Correlation dimension
  - Recurrence plots
  - Recurrence quantification analysis
  - Lyapunov exponents
  - Entropy encoding

Measures

Time series metrics or features that can be used for time series classification or regression analysis [8]:

- **Univariate linear measures**
  - Moment (mathematics)
  - Spectral band power
  - Spectral edge frequency
  - Accumulated Energy (signal processing)
  - Characteristics of the autocorrelation function
  - Hjorth parameters
  - FFT parameters
  - Autoregressive model parameters
- **Univariate non-linear measures**
  - Measures based on the correlation sum
  - Correlation dimension
  - Correlation integral
  - Correlation density

- Correlation entropy
- Approximate Entropy [9]
- Sample Entropy
- Fourier entropy
- Wavelet entropy
- Rényi entropy
- Higher-order methods
- Marginal predictability
- Dynamical similarity index
- State space dissimilarity measures
- Lyapunov exponent
- Permutation methods
- Local flow

- **Other univariate measures**
  - Algorithmic complexity
  - Kolmogorov complexity estimates
  - Hidden Markov Model states
  - Surrogate time series and surrogate correction
  - Loss of recurrence (degree of non-stationarity)
- **Bivariate linear measures**
  - Maximum linear cross-correlation
  - Linear Coherence (signal processing)
- **Bivariate non-linear measures**
  - Non-linear interdependence
  - Dynamical Entrainment (physics)
  - Measures for Phase synchronization
- **Similarity measures** [10]:
  - Dynamic Time Warping
  - Hidden Markov Models
  - Edit distance
  - Total correlation
  - Newey–West estimator
  - Prais-Winsten transformation
  - Data as Vectors in a Metrizable Space
    - Minkowski distance
    - Mahalanobis distance
  - Data as Time Series with Envelopes
    - Global Standard Deviation
    - Local Standard Deviation
    - Windowed Standard Deviation
  - Data Interpreted as Stochastic Series
    - Pearson product-moment correlation coefficient
    - Spearman's rank correlation coefficient
  - Data Interpreted as a Probability Distribution Function
    - Kolmogorov-Smirnov test

- Cramér-von Mises criterion

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External links

- A First Course on Time Series Analysis (http://statistik.mathematik.uni-wuerzburg.de/timeseries/) - an open source book on time series analysis with SAS
- Introduction to Time series Analysis (Engineering Statistics Handbook) (http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc4.htm) - a practical guide to Time series analysis
- [HTTP://www.jstatsoft.org/v33/i05/paper MATLAB Toolkit for Computation of Multiple Measures on Time Series Data Bases]

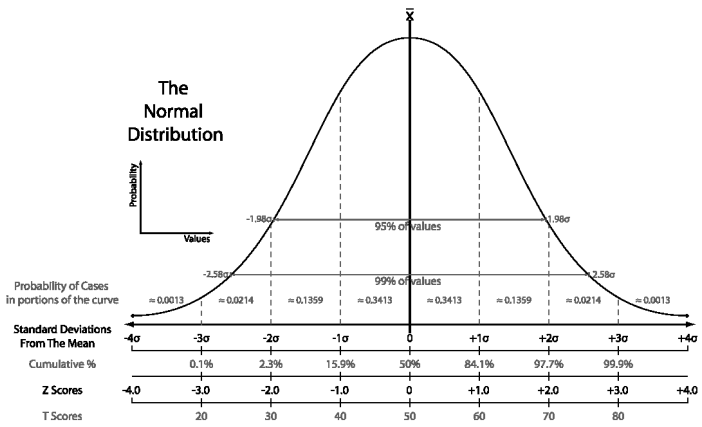
Statistics

Statistics is the study of the collection, organization, analysis, interpretation, and presentation of data.[1][2] It deals with all aspects of this, including the planning of data collection in terms of the design of surveys and experiments.[1]

A statistician is someone who is particularly well versed in the ways of thinking necessary for the successful application of statistical analysis. Such people have often gained this experience through working in any of a wide number of fields. There is also a discipline called *mathematical statistics* that studies statistics mathematically.

The word *statistics*, when referring to the scientific discipline, is singular, as in "Statistics is an art."[3] This should not be confused with the word *statistic*, referring to a quantity (such as mean or median) calculated from a set of data,[4] whose plural is *statistics* ("this statistic seems wrong" or "these statistics are misleading").

Scope



More probability density will be found the closer one gets to the expected (mean) value in a normal distribution. Statistics used in standardized testing assessment are shown. The scales include *standard deviations, cumulative percentages, percentile equivalents, Z-scores, T-scores, standard nines,* and *percentages in standard nines.*

Some consider statistics to be a mathematical body of science pertaining to the collection, analysis, interpretation or explanation, and presentation of data,[5] while others consider it a branch of mathematics[6] concerned with collecting and interpreting data. Because of its empirical roots and its focus on applications, statistics is usually considered to be a distinct mathematical science rather than a branch of mathematics.[7][8] Much of statistics is non-mathematical: ensuring that data collection is undertaken in a way that allows valid conclusions to be drawn; coding and archiving of data so that information is retained and made useful for international comparisons of official statistics; reporting of results and summarised data (tables and graphs) in ways that are comprehensible to those who need to make use of them; implementing procedures that ensure the privacy of census information.

Statisticians improve the quality of data with the design of experiments and survey sampling. Statistics also provides tools for prediction and forecasting using data and statistical models. Statistics is applicable to a wide variety of academic disciplines, including natural and social sciences, government, and business. Statistical consultants are

available to provide help for organizations and companies without direct access to expertise relevant to their particular problems.

Statistical methods can be used for summarizing or describing a collection of data; this is called *descriptive statistics*. This is useful in research, when communicating the results of experiments. In addition, patterns in the data may be modeled in a way that accounts for randomness and uncertainty in the observations, and are then used for drawing inferences about the process or population being studied; this is called *inferential statistics*. Inference is a vital element of scientific advance, since it provides a means for drawing conclusions from data that are subject to random variation. To prove the propositions being investigated further, the conclusions are tested as well, as part of the scientific method. Descriptive statistics and analysis of the new data tend to provide more information as to the truth of the proposition.

Descriptive statistics and the application of inferential statistics (a.k.a., predictive statistics) together comprise *applied statistics*.[9] *Theoretical statistics* concerns both the logical arguments underlying justification of approaches to statistical inference, as well encompassing *mathematical statistics*. Mathematical statistics includes not only the manipulation of probability distributions necessary for deriving results related to methods of estimation and inference, but also various aspects of computational statistics and the design of experiments.

Statistics is closely related to probability theory, with which it is often grouped; the difference is roughly that in probability theory, one starts from the given parameters of a total population to deduce probabilities pertaining to samples, but statistical inference moves in the opposite direction, inductive inference from samples to the parameters of a larger or total population.

History

The use of statistical methods dates back at least to the 5th century BC. The earliest writing on statistics was found in a 9th century book entitled: "Manuscript on Deciphering Cryptographic Messages", written by Al-Kindi. In his book, he gave a detailed description of how to use statistics and frequency analysis to decipher encrypted messages, this was the birth of both statistics and cryptanalysis, according to the Saudi engineer Ibrahim Al-Kadi.[10][11]

The *Nuova Cronica*, a 14th century history of Florence by the Florentine banker and official Giovanni Villani, includes much statistical information on population, ordinances, commerce and trade, education, and religious facilities and has been described as the first introduction of statistics as a positive element in history.[12]

Some scholars pinpoint the origin of statistics to 1663, with the publication of *Natural and Political Observations upon the Bills of Mortality* by John Graunt.[13] Early applications of statistical thinking revolved around the needs of states to base policy on demographic and economic data, hence its *stat-* etymology. The scope of the discipline of statistics broadened in the early 19th century to include the collection and analysis of data in general. Today, statistics is widely employed in government, business, and the natural and social sciences.

Its mathematical foundations were laid in the 17th century with the development of probability theory by Blaise Pascal and Pierre de Fermat. Probability theory arose from the study of games of chance. The method of least squares was first described by Carl Friedrich Gauss around 1794. The use of modern computers has expedited large-scale statistical computation, and has also made possible new methods that are impractical to perform manually.

Overview

In applying statistics to a scientific, industrial, or societal problem, it is necessary to begin with a population or process to be studied. Populations can be diverse topics such as "all persons living in a country" or "every atom composing a crystal". A population can also be composed of observations of a process at various times, with the data from each observation serving as a different member of the overall group. Data collected about this kind of "population" constitutes what is called a time series.

For practical reasons, a chosen subset of the population called a sample is studied — as opposed to compiling data about the entire group (an operation called census). Once a sample that is representative of the population is determined, data are collected for the sample members in an observational or experimental setting. This data can then be subjected to statistical analysis, serving two related purposes: description and inference.

- Descriptive statistics summarize the population data by describing what was observed in the sample numerically or graphically. Numerical descriptors include mean and standard deviation for continuous data types (like heights or weights), while frequency and percentage are more useful in terms of describing categorical data (like race).
- Inferential statistics uses patterns in the sample data to draw inferences about the population represented, accounting for randomness. These inferences may take the form of: answering yes/no questions about the data (hypothesis testing), estimating numerical characteristics of the data (estimation), describing associations within the data (correlation) and modeling relationships within the data (for example, using regression analysis). Inference can extend to forecasting, prediction and estimation of unobserved values either in or associated with the population being studied; it can include extrapolation and interpolation of time series or spatial data, and can also include data mining.[14]

"... it is only the manipulation of uncertainty that interests us. We are not concerned with the matter that is uncertain. Thus we do not study the mechanism of rain; only whether it will rain."

Dennis Lindley, 2000[15]

The concept of correlation is particularly noteworthy for the potential confusion it can cause. Statistical analysis of a data set often reveals that two variables (properties) of the population under consideration tend to vary together, as if they were connected. For example, a study of annual income that also looks at age of death might find that poor people tend to have shorter lives than affluent people. The two variables are said to be correlated; however, they may or may not be the cause of one another. The correlation phenomena could be caused by a third, previously unconsidered phenomenon, called a lurking variable or confounding variable. For this reason, there is no way to immediately infer the existence of a causal relationship between the two variables. (See Correlation does not imply causation.)

For a sample to be used as a guide to an entire population, it is important that it is truly a representative of that overall population. Representative sampling assures that the inferences and conclusions can be safely extended from the sample to the population as a whole. A major problem lies in determining the extent to which the sample chosen is actually representative. Statistics offers methods to estimate and correct for any random trending within the sample and data collection procedures. There are also methods of experimental design for experiments that can lessen these issues at the outset of a study, strengthening its capability to discern truths about the population.

Randomness is studied using the mathematical discipline of probability theory. Probability is used in "mathematical statistics" (alternatively, "statistical theory") to study the sampling distributions of sample statistics and, more generally, the properties of statistical procedures. The use of any statistical method is valid when the system or population under consideration satisfies the assumptions of the method.

Misuse of statistics can produce subtle, but serious errors in description and interpretation — subtle in the sense that even experienced professionals make such errors, and serious in the sense that they can lead to devastating decision errors. For instance, social policy, medical practice, and the reliability of structures like bridges all rely on the proper use of statistics. See below for further discussion.

Even when statistical techniques are correctly applied, the results can be difficult to interpret for those lacking expertise. The statistical significance of a trend in the data — which measures the extent to which a trend could be caused by random variation in the sample — may or may not agree with an intuitive sense of its significance. The set of basic statistical skills (and skepticism) that people need to deal with information in their everyday lives properly is referred to as statistical literacy.

Statistical methods

Experimental and observational studies

A common goal for a statistical research project is to investigate causality, and in particular to draw a conclusion on the effect of changes in the values of predictors or independent variables on dependent variables or response. There are two major types of causal statistical studies: experimental studies and observational studies. In both types of studies, the effect of differences of an independent variable (or variables) on the behavior of the dependent variable are observed. The difference between the two types lies in how the study is actually conducted. Each can be very effective. An experimental study involves taking measurements of the system under study, manipulating the system, and then taking additional measurements using the same procedure to determine if the manipulation has modified the values of the measurements. In contrast, an observational study does not involve experimental manipulation. Instead, data are gathered and correlations between predictors and response are investigated.

Experiments

The basic steps of a statistical experiment are:

1. Planning the research, including finding the number of replicates of the study, using the following information: preliminary estimates regarding the size of treatment effects, alternative hypotheses, and the estimated experimental variability. Consideration of the selection of experimental subjects and the ethics of research is necessary. Statisticians recommend that experiments compare (at least) one new treatment with a standard treatment or control, to allow an unbiased estimate of the difference in treatment effects.
2. Design of experiments, using blocking to reduce the influence of confounding variables, and randomized assignment of treatments to subjects to allow unbiased estimates of treatment effects and experimental error. At this stage, the experimenters and statisticians write the *experimental protocol* that shall guide the performance of the experiment and that specifies the *primary analysis* of the experimental data.
3. Performing the experiment following the experimental protocol and analyzing the data following the experimental protocol.
4. Further examining the data set in secondary analyses, to suggest new hypotheses for future study.
5. Documenting and presenting the results of the study.

Experiments on human behavior have special concerns. The famous Hawthorne study examined changes to the working environment at the Hawthorne plant of the Western Electric Company. The researchers were interested in determining whether increased illumination would increase the productivity of the assembly line workers. The researchers first measured the productivity in the plant, then modified the illumination in an area of the plant and checked if the changes in illumination affected productivity. It turned out that productivity indeed improved (under the experimental conditions). However, the study is heavily criticized today for errors in experimental procedures, specifically for the lack of a control group and blindness. The Hawthorne effect refers to finding that an outcome (in this case, worker productivity) changed due to observation itself. Those in the Hawthorne study became more productive not because the lighting was changed but because they were being observed.

Observational study

An example of an observational study is one that explores the correlation between smoking and lung cancer. This type of study typically uses a survey to collect observations about the area of interest and then performs statistical analysis. In this case, the researchers would collect observations of both smokers and non-smokers, perhaps through a case-control study, and then look for the number of cases of lung cancer in each group.

Levels of measurement

There are four main levels of measurement used in statistics: nominal, ordinal, interval, and ratio.[16] Each of these have different degrees of usefulness in statistical research. Ratio measurements have both a meaningful zero value and the distances between different measurements defined; they provide the greatest flexibility in statistical methods that can be used for analyzing the data. Interval measurements have meaningful distances between measurements defined, but the zero value is arbitrary (as in the case with longitude and temperature measurements in Celsius or Fahrenheit). Ordinal measurements have imprecise differences between consecutive values, but have a meaningful order to those values. Nominal measurements have no meaningful rank order among values.

Because variables conforming only to nominal or ordinal measurements cannot be reasonably measured numerically, sometimes they are grouped together as categorical variables, whereas ratio and interval measurements are grouped together as quantitative variables, which can be either discrete or continuous, due to their numerical nature.

Key terms used in statistics

Null hypothesis

Interpretation of statistical information can often involve the development of a null hypothesis in that the assumption is that whatever is proposed as a cause has no effect on the variable being measured.

The best illustration for a novice is the predicament encountered by a jury trial. The null hypothesis, H_0, asserts that the defendant is innocent, whereas the alternative hypothesis, H_1, asserts that the defendant is guilty. The indictment comes because of suspicion of the guilt. The H_0 (status quo) stands in opposition to H_1 and is maintained unless H_1 is supported by evidence"beyond a reasonable doubt". However,"failure to reject H_0" in this case does not imply innocence, but merely that the evidence was insufficient to convict. So the jury does not necessarily *accept* H_0 but *fails to reject* H_0. While one can not "prove" a null hypothesis one can test how close it is to being true with a power test, which tests for type II errors.

Error

Working from a null hypothesis two basic forms of error are recognized:

- Type I errors where the null hypothesis is falsely rejected giving a "false positive".
- Type II errors where the null hypothesis fails to be rejected and an actual difference between populations is missed giving a "false negative".

Error also refers to the extent to which individual observations in a sample differ from a central value, such as the sample or population mean. Many statistical methods seek to minimize the mean-squared error, and these are called "methods of least squares."

Measurement processes that generate statistical data are also subject to error. Many of these errors are classified as random (noise) or systematic (bias), but other important types of errors (e.g., blunder, such as when an analyst reports incorrect units) can also be important.

Interval estimation

Most studies will only sample part of a population and so the results are not fully representative of the whole population. Any estimates obtained from the sample only approximate the population value. Confidence intervals allow statisticians to express how closely the sample estimate matches the true value in the whole population. Often they are expressed as 95% confidence intervals. Formally, a 95% confidence interval for a value is a range where, if the sampling and analysis were repeated under the same conditions (yielding a different dataset), the interval would include the true (population) value 95% of the time. This does *not* imply that the probability that the true value is in the confidence interval is 95%. From the frequentist perspective, such a claim does not even make sense, as the true value is not a random variable. Either the true value is or is not within the given interval. However, it is true that, before any data are sampled and given a plan for how the confidence interval will be constructed, the probability is 95% that the yet-to-be-calculated interval will cover the true value: at this point, the limits of the interval are yet-to-be-observed random variables. One approach that does yield an interval that can be interpreted as having a given probability of containing the true value is to use a credible interval from Bayesian statistics: this approach depends on a different way of interpreting what is meant by "probability", that is as a Bayesian probability.

Significance

Statistics rarely give a simple Yes/No type answer to the question asked of them. Interpretation often comes down to the level of statistical significance applied to the numbers and often refers to the probability of a value accurately rejecting the null hypothesis (sometimes referred to as the p-value).

Referring to statistical significance does not necessarily mean that the overall result is significant in real world terms. For example, in a large study of a drug it may be shown that the drug has a statistically significant but very small beneficial effect, such that the drug will be unlikely to help the patient in a noticeable way.

Criticisms arise because the hypothesis testing approach forces one hypothesis (the null hypothesis) to be "favored," and can also seem to exaggerate the importance of minor differences in large studies. A difference that is highly statistically significant can still be of no practical significance, but it is possible to properly formulate tests in account for this. (See also criticism of hypothesis testing.)

One response involves going beyond reporting only the significance level to include the *p*-value when reporting whether a hypothesis is rejected or accepted. The *p*-value, however, does not indicate the size of the effect. A better and increasingly common approach is to report confidence intervals. Although these are produced from the same calculations as those of hypothesis tests or *p*-values, they describe both the size of the effect and the uncertainty surrounding it.

Examples

Some well-known statistical tests and procedures are:

- Analysis of variance (ANOVA)
- Chi-squared test
- Correlation
- Factor analysis
- Mann–Whitney U
- Mean square weighted deviation (MSWD)
- Pearson product-moment correlation coefficient
- Regression analysis
- Spearman's rank correlation coefficient
- Student's t-test
- Time series analysis

Specialized disciplines

Statistical techniques are used in a wide range of types of scientific and social research, including: biostatistics, computational biology, computational sociology, network biology, social science, sociology and social research. Some fields of inquiry use applied statistics so extensively that they have specialized terminology. These disciplines include:

- Actuarial science
- Applied information economics
- Biostatistics
- Business statistics
- Chemometrics (for analysis of data from chemistry)
- Data mining (applying statistics and pattern recognition to discover knowledge from data)
- Demography
- Econometrics
- Energy statistics
- Engineering statistics
- Epidemiology
- Geography and Geographic Information Systems, specifically in Spatial analysis
- Image processing
- Psychological statistics
- Reliability engineering
- Social statistics

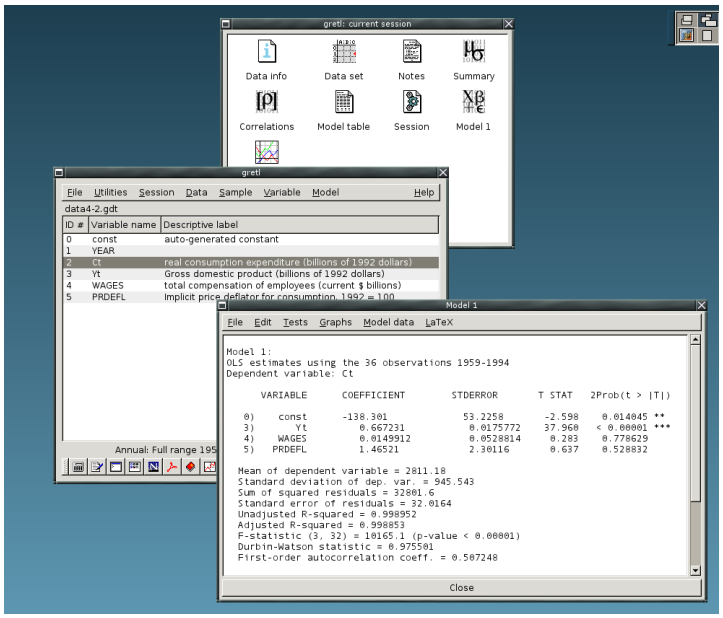
In addition, there are particular types of statistical analysis that have also developed their own specialised terminology and methodology:

- Bootstrap & Jackknife Resampling
- Multivariate statistics
- Statistical classification
- Statistical surveys
- Structured data analysis (statistics)
- Structural equation modelling
- Survival analysis
- Statistics in various sports, particularly baseball and cricket

Statistics form a key basis tool in business and manufacturing as well. It is used to understand measurement systems variability, control processes (as in statistical process control or SPC), for summarizing data, and to make data-driven decisions. In these roles, it is a key tool, and perhaps the only reliable tool.

Statistical computing

The rapid and sustained increases in computing power starting from the second half of the 20th century have had a substantial impact on the practice of statistical science. Early statistical models were almost always from the class of linear models, but powerful computers, coupled with suitable numerical algorithms, caused an increased interest in nonlinear models (such as neural networks) as well as the creation of new types, such as generalized linear models and multilevel models.



gretl, an example of an open source statistical package

Increased computing power has also led to the growing popularity of computationally intensive methods based on resampling, such as permutation tests and the bootstrap, while techniques such as Gibbs sampling have made use of Bayesian models more feasible. The computer revolution has implications for the future of statistics with new emphasis on "experimental" and "empirical" statistics. A large number of both general and special purpose statistical software are now available.

Misuse

There is a general perception that statistical knowledge is all-too-frequently intentionally misused by finding ways to interpret only the data that are favorable to the presenter.[17] A mistrust and misunderstanding of statistics is associated with the quotation, "There are three kinds of lies: lies, damned lies, and statistics". Misuse of statistics can be both inadvertent and intentional, and the book *How to Lie With Statistics*[17] outlines a range of considerations. In an attempt to shed light on the use and misuse of statistics, reviews of statistical techniques used in particular fields are conducted (e.g. Warne, Lazo, Ramos, and Ritter (2012)).[18]

Ways to avoid misuse of statistics include using proper diagrams and avoiding bias.[19] "The misuse occurs when such conclusions are held to be representative of the universe by those who either deliberately or unconsciously overlook the sampling bias.[20] Bar graphs are arguably the easiest diagrams to use and understand, and they can be made either with simple computer programs or hand drawn.[19] Unfortunately, most people do not look for bias or errors, so they do not see them. Thus, we believe something to be truth that is not well-represented.[20] In order to make data gathered from statistics believable and accurate, the sample taken must be representative of the whole.[21] As Huff's book states,"The dependability of a sample can be destroyed by [bias]… allow yourself some degree of skepticism."[22]

Statistics applied to mathematics or the arts

Traditionally, statistics was concerned with drawing inferences using a semi-standardized methodology that was "required learning" in most sciences. This has changed with use of statistics in non-inferential contexts. What was once considered a dry subject, taken in many fields as a degree-requirement, is now viewed enthusiastically. Initially derided by some mathematical purists, it is now considered essential methodology in certain areas.

- In number theory, scatter plots of data generated by a distribution function may be transformed with familiar tools used in statistics to reveal underlying patterns, which may then lead to hypotheses.
- Methods of statistics including predictive methods in forecasting, are combined with chaos theory and fractal geometry to create video works that are considered to have great beauty.
- The process art of Jackson Pollock relied on artistic experiments whereby underlying distributions in nature were artistically revealed. With the advent of computers, methods of statistics were applied to formalize such distribution driven natural processes, in order to make and analyze moving video art.
- Methods of statistics may be used predicatively in performance art, as in a card trick based on a Markov process that only works some of the time, the occasion of which can be predicted using statistical methodology.
- Statistics can be used to predicatively create art, as in the statistical or stochastic music invented by Iannis Xenakis, where the music is performance-specific. Though this type of artistry does not always come out as expected, it does behave in ways that are predictable and tunable using statistics.

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Variance

In probability theory and statistics, the **variance** is a measure of how far a set of numbers is spread out. It is one of several descriptors of a probability distribution, describing how far the numbers lie from the mean (expected value). In particular, the variance is one of the moments of a distribution. In that context, it forms part of a systematic approach to distinguishing between probability distributions. While other such approaches have been developed, those based on moments are advantageous in terms of mathematical and computational simplicity.

The variance is a parameter describing in part either the actual probability distribution of an observed population of numbers, or the theoretical probability distribution of a sample (a not-fully-observed population) of numbers. In the latter case a sample of data from such a distribution can be used to construct an estimate of its variance: in the simplest cases this estimate can be the **sample variance**, defined below.

Basic discussion

Examples

The **variance** of a random variable or distribution is the expectation, or mean, of the squared deviation of that variable from its expected value or mean. Thus the variance is a measure of the amount of variation of the values of that variable, taking account of all possible values and their probabilities or weightings (not just the extremes which give the range).

For example, a perfect six-sided die, when thrown, has expected value of

$$\frac{1}{6}(1+2+3+4+5+6) = 3.5.$$

Its expected absolute deviation—the mean of the equally likely absolute deviations from the mean—is

$$\frac{1}{6}(|1-3.5|+|2-3.5|+|3-3.5|+|4-3.5|+|5-3.5|+|6-3.5|) = \frac{1}{6}(2.5+1.5+0.5+0.5+1.5+2.5) = 1.5.$$

But its expected *squared* deviation—its variance (the mean of the equally likely squared deviations)—is

$$\frac{1}{6}(2.5^2 + 1.5^2 + 0.5^2 + 0.5^2 + 1.5^2 + 2.5^2) = 17.5/6 \approx 2.9.$$

As another example, if a coin is tossed twice, the number of heads is: 0 with probability 0.25, 1 with probability 0.5 and 2 with probability 0.25. Thus the expected value of the number of heads is:

$$0.25 \times 0 + 0.5 \times 1 + 0.25 \times 2 = 1,$$

and the variance is:

$$0.25 \times (0-1)^2 + 0.5 \times (1-1)^2 + 0.25 \times (2-1)^2 = 0.25 + 0 + 0.25 = 0.5.$$

Units of measurement

Unlike expected absolute deviation, the variance of a variable has units that are the square of the units of the variable itself. For example, a variable measured in inches will have a variance measured in square inches. For this reason, describing data sets via their standard deviation or root mean square deviation is often preferred over using the variance. In the dice example the standard deviation is $\sqrt{2.9} \approx 1.7$, slightly larger than the expected absolute deviation of 1.5.

The standard deviation and the expected absolute deviation can both be used as an indicator of the "spread" of a distribution. The standard deviation is more amenable to algebraic manipulation than the expected absolute deviation, and, together with variance and its generalization covariance, is used frequently in theoretical statistics; however the expected absolute deviation tends to be more robust as it is less sensitive to outliers arising from measurement anomalies or an unduly heavy-tailed distribution.

Estimating the variance

Real-world distributions such as the distribution of yesterday's rain throughout the day are typically not fully known, unlike the behavior of perfect dice or an ideal distribution such as the normal distribution, because it is impractical to account for every raindrop. Instead one estimates the mean and variance of the whole distribution as the computed mean and variance of a sample of n observations drawn suitably randomly from the whole sample space, in this example the set of all measurements of yesterday's rainfall in all available rain gauges.

This method of estimation is close to optimal, with the caveat that it underestimates the variance by a factor of $(n - 1) / n$. (For example, when $n = 1$ the variance of a single observation is obviously zero regardless of the true variance). This gives a bias which should be corrected for when n is small by multiplying by $n / (n - 1)$. If the mean is determined in some other way than from the same samples used to estimate the variance then this bias does not arise and the variance can safely be estimated as that of the samples.

To illustrate the relation between the population variance and the sample variance, suppose that in the (not entirely observed) population of numerical values, the value 1 occurs 1/3 of the time, the value 2 occurs 1/3 of the time, and the value 4 occurs 1/3 of the time. The population mean is $(1/3)[1 + 2 + 4] = 7/3$. The equally likely deviations from the population mean are $1 - 7/3$, $2 - 7/3$, and $4 - 7/3$. The population variance — the expected squared deviation from the mean 7/3 — is $(1/3)[(-4/3)^2 + (-1/3)^2 + (5/3)^2] = 14/9$. Now suppose for the sake of a simple example that we take a very small sample of $n = 2$ observations, and consider the nine equally likely possibilities for the set of numbers within that sample: (1, 1), (1, 2), (1,4), (2, 1), (2,2), (2, 4), (4,1), (4, 2), and (4, 4). For these nine possible samples, the sample variance of the two numbers is respectively 0, 1/4, 9/4, 1/4, 0, 4/4, 9/4, 4/4, and 0. With our plan to observe two values, we could end up computing any of these sample variances (and indeed if we hypothetically could observe a pair of numbers many times, we would compute each of these sample variances 1/9 of the time). So the expected value, over all possible samples that might be drawn from the population, of the computed sample variance is $(1/9)[0 + 1/4 + 9/4 + 1/4 + 0 + 4/4 + 9/4 + 4/4 + 0] = 7/9$. This value of 7/9 for the expected value of our sample variance computation is a substantial underestimate of the true population variance, which we computed as 14/9, because our sample size of just two observations was so small. But if we adjust for this downward bias by multiplying our computed sample variance, whichever it may be, by $n/(n - 1) = 2/(2 - 1) = 2$, then our estimate of the population variance would be any one of 0, 1/2, 9/2, 1/2, 0, 4/2, 9/2, 4/2, and 0. The average of these is indeed the correct population variance of 14/9, so on average over all possible samples we would have the correct estimate of the population variance.

The variance of a real-valued random variable is its second central moment, and it also happens to be its second cumulant. Just as some distributions do not have a mean, some do not have a variance. The mean exists whenever the variance exists, but the converse is not necessarily true.

Definition

If a random variable X has the expected value (mean) $\mu = \mathrm{E}[X]$, then the variance of X is the covariance of X with itself, given by:

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{Cov}(X, X) \\ &= \mathrm{E}\left[(X - \mu)(X - \mu)\right] \\ &= \mathrm{E}\left[(X - \mu)^2\right]\end{aligned}$$

That is, the variance is the expected value of the squared difference between the variable's realization and the variable's mean. This definition encompasses random variables that are discrete, continuous, neither, or mixed. From the corresponding expression for Covariance, it can be expanded:

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{Cov}(X, X) \\ &= \mathrm{E}\left[XX\right] - \mathrm{E}[X]\mathrm{E}[X] \\ &= \mathrm{E}\left[X^2\right] - (\mathrm{E}[X])^2\end{aligned}$$

A mnemonic for the above expression is "mean of square minus square of mean". The variance of random variable X is typically designated as $\operatorname{Var}(X)$, σ_X^2, or simply σ^2 (pronounced "sigma squared").

Continuous random variable

If the random variable X is continuous with probability density function $f(x)$, then the variance equals the second central moment, given by

$$\operatorname{Var}(X) = \int (x - \mu)^2 f(x)\, dx\,,$$

where μ is the expected value,

$$\mu = \int x\, f(x)\, dx\,,$$

and where the integrals are definite integrals taken for x ranging over the range of X.

If a continuous distribution does not have an expected value, as is the case for the Cauchy distribution, it does not have a variance either. Many other distributions for which the expected value does exist also do not have a finite variance because the integral in the variance definition diverges. An example is a Pareto distribution whose index k satisfies $1 < k \leq 2$.

Discrete random variable

If the random variable X is discrete with probability mass function $x_1 \mapsto p_1, \ldots, x_n \mapsto p_n$, then

$$\operatorname{Var}(X) = \sum_{i=1}^{n} p_i \cdot (x_i - \mu)^2$$

where μ is the expected value, i.e.

$$\mu = \sum_{i=1}^{n} p_i \cdot x_i\,.$$

(When such a discrete weighted variance is specified by weights whose sum is not 1, then one divides by the sum of the weights.) That is, it is the expected value of the square of the deviation of X from its own mean. In plain language, it can be expressed as "The mean of the squares of the deviations of the data points from the average". It is thus the *mean squared deviation*.

Examples

Exponential distribution

The exponential distribution with parameter λ is a continuous distribution whose support is the semi-infinite interval [0,∞). Its probability density function is given by:

$$f(x) = \lambda e^{-\lambda x},$$

and it has expected value $\mu = \lambda^{-1}$. Therefore the variance is equal to:

$$\int_0^\infty f(x)(x-\mu)^2\,dx = \int_0^\infty \lambda e^{-\lambda x}(x-\lambda^{-1})^2\,dx = \lambda^{-2}.$$

So for an exponentially distributed random variable $\sigma^2 = \mu^2$.

Fair die

A six-sided fair die can be modelled with a discrete random variable with outcomes 1 through 6, each with equal probability $\frac{1}{6}$. The expected value is (1 + 2 + 3 + 4 + 5 + 6)/6 = 3.5. Therefore the variance can be computed to be:

$$\begin{aligned}\sum_{i=1}^{6} \tfrac{1}{6}(i-3.5)^2 = \tfrac{1}{6}\sum_{i=1}^{6}(i-3.5)^2 &= \tfrac{1}{6}\left((-2.5)^2+(-1.5)^2+(-0.5)^2+0.5^2+1.5^2+2.5^2\right)\\ &= \tfrac{1}{6}\cdot 17.50 = \tfrac{35}{12} \approx 2.92.\end{aligned}$$

The general formula for the variance of the outcome *X* of a die of n sides is:

$$\begin{aligned}\sigma^2 = E(X^2) - (E(X))^2 &= \frac{1}{n}\sum_{i=1}^{n} i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} i\right)^2\\ &= \tfrac{1}{6}(n+1)(2n+1) - \tfrac{1}{4}(n+1)^2\\ &= \frac{n^2-1}{12}.\end{aligned}$$

Properties

Variance is non-negative because the squares are positive or zero.

$$\operatorname{Var}(X) \geq 0.$$

The variance of a constant random variable is zero, and if the variance of a variable in a data set is 0, then all the entries have the same value.

$$P(X = a) = 1 \Leftrightarrow \operatorname{Var}(X) = 0.$$

Variance is invariant with respect to changes in a location parameter. That is, if a constant is added to all values of the variable, the variance is unchanged.

$$\operatorname{Var}(X + a) = \operatorname{Var}(X).$$

If all values are scaled by a constant, the variance is scaled by the square of that constant.

$$\operatorname{Var}(aX) = a^2\operatorname{Var}(X).$$

The variance of a sum of two random variables is given by:

$$\operatorname{Var}(aX + bY) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 2ab\operatorname{Cov}(X,Y),$$
$$\operatorname{Var}(X - Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) - 2\operatorname{Cov}(X,Y),$$

In general we have for the sum of N random variables:

$$\operatorname{Var}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i,j=1}^{N} \operatorname{Cov}(X_i, X_j) = \sum_{i=1}^{N} \operatorname{Var}(X_i) + \sum_{i \neq j} \operatorname{Cov}(X_i, X_j).$$

The variance of a finite sum of *uncorrelated* random variables is equal to the sum of their variances. This stems from the above identity and the fact that for uncorrelated variables the covariance is zero.

$$\operatorname{Cov}(X_i, X_j) = 0 \ (i \neq j) \Rightarrow \operatorname{Var}\left(\sum_{i=1}^{N} X_i\right) = \sum_{i=1}^{N} \operatorname{Var}(X_i).$$

These results lead to the variance of a linear combination as:

$$\begin{aligned}
\operatorname{Var}\left(\sum_{i=1}^{N} a_i X_i\right) &= \sum_{i=1}^{N} \sum_{j=1}^{N} a_i a_j \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{N} a_i^2 \operatorname{Var}(X_i) + \sum_{i \neq j} a_i a_j \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{N} a_i^2 \operatorname{Var}(X_i) + 2 \sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j).
\end{aligned}$$

Suppose that the observations can be partitioned into equal-sized **subgroups** according to some second variable. Then the variance of the total group is equal to the mean of the variances of the subgroups plus the variance of the means of the subgroups. This property is known as variance decomposition or the law of total variance and plays an important role in the analysis of variance. For example, suppose that a group consists of a subgroup of men and an equally large subgroup of women. Suppose that the men have a mean height of 180 and that the variance of their heights is 100. Suppose that the women have a mean height of 160 and that the variance of their heights is 50. Then the mean of the variances is (100 + 50) / 2 = 75; the variance of the means is the variance of 180, 160 which is 100. Then, for the total group of men and women combined, the variance of the height will be 75 + 100 = 175. Note that this uses N for the denominator instead of N − 1.

In a more general case, if the subgroups have unequal sizes, then they must be weighted proportionally to their size in the computations of the means and variances. The formula is also valid with more than two groups, and even if the grouping variable is continuous.

This formula implies that the variance of the total group cannot be smaller than the mean of the variances of the subgroups. Note, however, that the total variance is not necessarily larger than the variances of the subgroups. In the above example, when the subgroups are analyzed separately, the variance is influenced only by the man-man differences and the woman-woman differences. If the two groups are combined, however, then the men-women differences enter into the variance also.

Many computational formulas for the variance are based on this equality: **The variance is equal to the mean of the square minus the square of the mean:**

$$\operatorname{Var}(X) = \mathrm{E}[X^2] - \mathrm{E}[X]^2.$$

For example, if we consider the numbers 1, 2, 3, 4 then the mean of the squares is (1 × 1 + 2 × 2 + 3 × 3 + 4 × 4) / 4 = 7.5. The regular mean of all four numbers is 2.5, so the square of the mean is 6.25. Therefore the variance is 7.5 − 6.25 = 1.25, which is indeed the same result obtained earlier with the definition formulas. Many pocket calculators use an algorithm that is based on this formula and that allows them to compute the variance while the data are entered, without storing all values in memory. The algorithm is to adjust only three variables when a new data value is entered: The number of data entered so far (*n*), the sum of the values so far (*S*), and the sum of the squared values so far (*SS*). For example, if the data are 1, 2, 3, 4, then after entering the first value, the algorithm would have *n* = 1, *S* = 1 and *SS* = 1. After entering the second value (2), it would have *n* = 2, *S* = 3 and *SS* = 5. When all data are entered, it would have *n* = 4, *S* = 10 and *SS* = 30. Next, the mean is computed as *M* = *S* / *n*, and finally the

variance is computed as *SS* / *n* – *M* × *M*. In this example the outcome would be 30 / 4 – 2.5 × 2.5 = 7.5 – 6.25 = 1.25. If the unbiased sample estimate is to be computed, the outcome will be multiplied by 1 / (*n* – 1), which yields 1.667 in this example.

Properties, formal

Sum of uncorrelated variables (Bienaymé formula)

One reason for the use of the variance in preference to other measures of dispersion is that the variance of the sum (or the difference) of uncorrelated random variables is the sum of their variances:

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i).$$

This statement is called the Bienaymé formula.[1] and was discovered in 1853. It is often made with the stronger condition that the variables are independent, but uncorrelatedness suffices. So if all the variables have the same variance σ^2, then, since division by *n* is a linear transformation, this formula immediately implies that the variance of their mean is

$$\operatorname{Var}\left(\overline{X}\right) = \operatorname{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \operatorname{Var}\left(X_i\right) = \frac{\sigma^2}{n}.$$

That is, the variance of the mean decreases when *n* increases. This formula for the variance of the mean is used in the definition of the standard error of the sample mean, which is used in the central limit theorem.

Product of independent variables

If two variables X and Y are independent, the variance of their product is given by[2][3]

$$\operatorname{Var}(XY) = [E(X)]^2 \operatorname{Var}(Y) + [E(Y)]^2 \operatorname{Var}(X) + \operatorname{Var}(X)\operatorname{Var}(Y).$$

Sum of correlated variables

In general, if the variables are correlated, then the variance of their sum is the sum of their covariances:

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} \operatorname{Cov}(X_i, X_j).$$

(Note: This by definition includes the variance of each variable, since $\operatorname{Cov}(X_i,X_i) = \operatorname{Var}(X_i)$.)

Here Cov is the covariance, which is zero for independent random variables (if it exists). The formula states that the variance of a sum is equal to the sum of all elements in the covariance matrix of the components. This formula is used in the theory of Cronbach's alpha in classical test theory.

So if the variables have equal variance σ^2 and the average correlation of distinct variables is ρ, then the variance of their mean is

$$\operatorname{Var}(\overline{X}) = \frac{\sigma^2}{n} + \frac{n-1}{n}\rho\sigma^2.$$

This implies that the variance of the mean increases with the average of the correlations. Moreover, if the variables have unit variance, for example if they are standardized, then this simplifies to

$$\operatorname{Var}(\overline{X}) = \frac{1}{n} + \frac{n-1}{n}\rho.$$

This formula is used in the Spearman–Brown prediction formula of classical test theory. This converges to ρ if *n* goes to infinity, provided that the average correlation remains constant or converges too. So for the variance of the mean of standardized variables with equal correlations or converging average correlation we have

$$\lim_{n\to\infty} \operatorname{Var}(\overline{X}) = \rho.$$

Therefore, the variance of the mean of a large number of standardized variables is approximately equal to their average correlation. This makes clear that the sample mean of correlated variables does generally not converge to the population mean, even though the Law of large numbers states that the sample mean will converge for independent variables.

Weighted sum of variables

The scaling property and the Bienaymé formula, along with this property from the covariance page: Cov(*aX*, *bY*) = *ab* Cov(*X*, *Y*) jointly imply that

$$\operatorname{Var}(aX + bY) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 2ab\operatorname{Cov}(X, Y).$$

This implies that in a weighted sum of variables, the variable with the largest weight will have a disproportionally large weight in the variance of the total. For example, if *X* and *Y* are uncorrelated and the weight of *X* is two times the weight of *Y*, then the weight of the variance of *X* will be four times the weight of the variance of *Y*.

The expression above can be extended to a weighted sum of multiple variables:

$$\operatorname{Var}\left(\sum_i a_i X_i\right) = \sum_i a_i^2 \operatorname{Var}(X_i) + 2\sum_i \sum_{j>i} a_i a_j \operatorname{Cov}(X_i, X_j)$$

Decomposition

The general formula for variance decomposition or the law of total variance is: If X and Y are two random variables and the variance of X exists, then

$$\operatorname{Var}(X) = \operatorname{Var}(\operatorname{E}(X|Y)) + \operatorname{E}(\operatorname{Var}(X|Y)).$$

Here, $\operatorname{E}(X|Y)$ is the conditional expectation of X given Y, and $\operatorname{Var}(X|Y)$ is the conditional variance of X given Y. (A more intuitive explanation is that given a particular value of Y, then X follows a distribution with mean $\operatorname{E}(X|Y)$ and variance $\operatorname{Var}(X|Y)$. The above formula tells how to find $\operatorname{Var}(X)$ based on the distributions of these two quantities when Y is allowed to vary.) This formula is often applied in analysis of variance, where the corresponding formula is

$$MS_{\text{Total}} = MS_{\text{Between}} + MS_{\text{Within}};$$

here MS refers to the Mean of the Squares. It is also used in linear regression analysis, where the corresponding formula is

$$MS_{\text{Total}} = MS_{\text{Regression}} + MS_{\text{Residual}}.$$

This can also be derived from the additivity of variances, since the total (observed) score is the sum of the predicted score and the error score, where the latter two are uncorrelated.

Similar decompositions are possible for the sum of squared deviations (sum of squares, SS):

$$SS_{\text{Total}} = SS_{\text{Between}} + SS_{\text{Within}},$$
$$SS_{\text{Total}} = SS_{\text{Regression}} + SS_{\text{Residual}}.$$

Computational formula

The **computational formula for the variance** follows in a straightforward manner from the linearity of expected values and the above definition:

$$\begin{aligned}\operatorname{Var}(X) &= \operatorname{E}(X^2 - 2\,X\,\operatorname{E}(X) + (\operatorname{E}(X))^2) \\ &= \operatorname{E}(X^2) - 2(\operatorname{E}(X))^2 + (\operatorname{E}(X))^2 \\ &= \operatorname{E}(X^2) - (\operatorname{E}(X))^2.\end{aligned}$$

This is often used to calculate the variance in practice, although it suffers from catastrophic cancellation if the two components of the equation are similar in magnitude.

Characteristic property

The second moment of a random variable attains the minimum value when taken around the first moment (i.e., mean) of the random variable, i.e. $\operatorname{argmin}_m \operatorname{E}((X-m)^2) = \operatorname{E}(X)$. Conversely, if a continuous function φ satisfies $\operatorname{argmin}_m \operatorname{E}(\varphi(X-m)) = \operatorname{E}(X)$for all random variables *X*, then it is necessarily of the form $\varphi(x) = ax^2 + b$, where *a* > 0. This also holds in the multidimensional case.[4]

Calculation from the CDF

The population variance for a non-negative random variable can be expressed in terms of the cumulative distribution function *F* using

$$2\int_0^\infty uH(u)\,du - \left(\int_0^\infty H(u)\,du\right)^2.$$

where $H(u) = 1 - F(u)$ is the right tail function. This expression can be used to calculate the variance in situations where the CDF, but not the density, can be conveniently expressed.

Matrix notation for the variance of a linear combination

Let's define X as a column vector of n random variables $X_1, \ldots, X_n$, and c as a column vector of N scalars $c_1, \ldots, c_n$. Therefore $c^T X$ is a linear combination of these random variables, where c^T denotes the transpose of vector c. Let also be $\Sigma$the variance-covariance matrix of the vector X. The variance of $c^T X$ is given by[5]:

$$\operatorname{Var}(c^T X) = c^T \Sigma c\,.$$

Approximating the variance of a function

The delta method uses second-order Taylor expansions to approximate the variance of a function of one or more random variables: see Taylor expansions for the moments of functions of random variables. For example, the approximate variance of a function of one variable is given by

$$\operatorname{Var}\left[f(X)\right] \approx \left(f'(\operatorname{E}\left[X\right])\right)^2 \operatorname{Var}\left[X\right]$$

provided that *f* is twice differentiable and that the mean and variance of *X* are finite.

Population variance and sample variance

In general, the ***population variance*** of a *finite* population of size *N* is given by

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i-\mu)^2 = \frac{1}{N}\sum_{i=1}^{N}x_i^2 - \mu^2$$

where

$$\mu = \frac{1}{N}\sum_{i=1}^{N}x_i$$

is the population mean, and

$$\begin{aligned}\sum_{i=1}^{N}(x_i-\mu)^2 &= \sum_{i=1}^{N}\left(x_i^2 - 2x_i\mu + \mu^2\right)\\ &= \sum_{i=1}^{N}\left(x_i^2 + \mu^2\right) - 2\mu\sum_{i=1}^{N}x_i\\ &= \sum_{i=1}^{N}\left(x_i^2 + \mu^2\right) - 2N\mu^2\\ &= \sum_{i=1}^{N}\left(x_i^2 + \mu^2 - 2\mu^2\right)\\ &= \sum_{i=1}^{N}\left(x_i^2 - \mu^2\right)\end{aligned}$$

In many practical situations, the true variance of a population is not known *a priori* and must be computed somehow. When dealing with extremely large populations, it is not possible to count every object in the population.

A common task is to estimate the variance of a population from a sample.[6] We take a sample with replacement of *n* values y_1, ..., y_n from the population, where $n < N$, and estimate the variance on the basis of this sample. There are several good estimators. Two of them are well known:[7]

$$s_n^2 = \frac{1}{n}\sum_{i=1}^{n}(y_i-\overline{y})^2 = \left(\frac{1}{n}\sum_{i=1}^{n}y_i^2\right) - \overline{y}^2, \text{and}$$

$$\begin{aligned}s^2 &= \frac{1}{n-1}\sum_{i=1}^{n}(y_i-\overline{y})^2\\ &= \frac{1}{n-1}\sum_{i=1}^{n}\left(y_i^2-\overline{y}^2\right)\\ &= \frac{1}{n-1}\sum_{i=1}^{n}y_i^2 - \frac{n}{n-1}\overline{y}^2\end{aligned}$$

The first estimator, also known as the second central moment, is called the ***biased sample variance***. The second estimator is called the ***unbiased sample variance***. Either estimator may be simply referred to as the ***sample variance*** when the version can be determined by context. Here, $\overline{y}$ denotes the sample mean:

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n}y_i.$$

The two estimators only differ slightly as can be seen, and for larger values of the sample size *n* the difference is negligible. While the first one may be seen as the variance of the sample considered as a population, the second one is the unbiased estimator of the population variance, meaning that its expected value E[s^2] is equal to the true

variance of the sampled random variable; the use of the term $n - 1$ is called Bessel's correction. In particular,

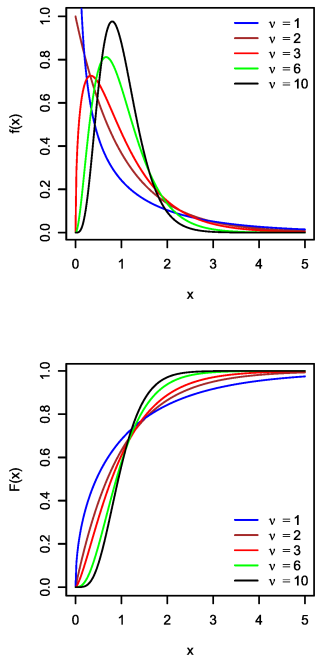
$$\mathrm{E}[s^2] = \sigma^2,$$

while, in contrast,

$$\mathrm{E}[s_n^2] = \frac{n-1}{n}\sigma^2.$$

The unbiased sample variance is a U-statistic for the function $f(x_1, x_2) = (x_1 - x_2)^2/2$, meaning that it is obtained by averaging a 2-sample statistic over 2-element subsets of the population.

Distribution of the sample variance



Distribution and cumulative distribution of s^2/σ^2, for various values of ν = n-1, when the y_i are independent normally distributed.

Being a function of random variables, the sample variance is itself a random variable, and it is natural to study its distribution. In the case that y_i are independent observations from a normal distribution, Cochran's theorem shows that s^2 follows a scaled chi-squared distribution (see Knight 2000, proposition 2.11 [8]):

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}.$$

As a direct consequence, it follows that $\mathrm{E}(s^2) = \sigma^2$.

If the y_i are independent and identically distributed, but not necessarily normally distributed, then

$$\mathrm{E}[s^2] = \sigma^2, \quad \mathrm{Var}[s^2] = \sigma^4\left(\frac{2}{n-1} + \frac{\kappa}{n}\right),$$

where κ is the excess kurtosis of the distribution.

If the y_i are furthermore normally distributed, the variance reduces then to[9]:

$$\mathrm{Var}[s^2] = \frac{2\sigma^4}{n-1},$$

If the conditions of the law of large numbers hold, s^2 is a consistent estimator of σ^2.. One can see indeed that the variance of the estimator tends asymptotically to zero.

Samuelson's inequality

Samuelson's inequality is a result that states, given that the sample mean and variance have been calculated from a particular sample, bounds on the values that individual values in the sample can take.[10] Values must lie within the limits $m \pm s\,(n-1)^{1/2}$.

Relations with the harmonic and arithmetic means

It has been shown[11] that for a sample of real numbers that

$$\mathrm{Var} \leq 2M(A-H)$$

where *M* is the maximum of the sample, *A* is the arithmetic mean, *H* is the harmonic mean of the sample and *Var* is the variance of the sample.

This bound has been improved on and it is known that variance is bounded by

$$\mathrm{Var} \leq \frac{M(A-H)(M-A)}{M-H}$$

$$\mathrm{Var} \geq \frac{m(A-H)(A-m)}{H-m}$$

where *m* is the minimum of the sample.[12]

Generalizations

If X is a vector-valued random variable, with values in $\mathbb{R}^n$, and thought of as a column vector, then the natural generalization of variance is $\mathrm{E}((X-\mu)(X-\mu)^{\mathrm{T}})$, where $\mu = \mathrm{E}(X)$ and X^{T} is the transpose of X, and so is a row vector. This variance is a positive semi-definite square matrix, commonly referred to as the covariance matrix.

If X is a complex-valued random variable, with values in $\mathbb{C}$, then its variance is $\mathrm{E}((X-\mu)(X-\mu)^{\dagger})$, where $X^{\dagger}$ is the conjugate transpose of X. This variance is also a positive semi-definite square matrix.

History

The term *variance* was first introduced by Ronald Fisher in his 1918 paper *The Correlation Between Relatives on the Supposition of Mendelian Inheritance*:[13]

> The great body of available statistics show us that the deviations of a human measurement from its mean follow very closely the Normal Law of Errors, and, therefore, that the variability may be uniformly measured by the standard deviation corresponding to the square root of the mean square error. When there are two independent causes of variability capable of producing in an otherwise uniform population distributions with standard deviations θ_1 and θ_2, it is found that the distribution, when both causes act together, has a standard deviation $\sqrt{\theta_1^2 + \theta_2^2}$. It is therefore desirable in analysing the causes of variability to deal with the square of the standard deviation as the measure of variability. We shall term this quantity the Variance...

Moment of Inertia

The variance of a probability distribution is analogous to the moment of inertia in classical mechanics of a corresponding mass distribution along a line, with respect to rotation about its center of mass. It is because of this analogy that such things as the variance are called *moments* of probability distributions. The covariance matrix is related to the moment of inertia tensor for multivariate distributions. The moment of inertia of a cloud of *n* points with a covariance matrix of Σ is given by

$$I = n(\mathbf{1}_{3\times 3}\operatorname{tr}(\Sigma) - \Sigma).$$

This difference between moment of inertia in physics and in statistics is clear for points that are gathered along a line. Suppose many points are close to the *x* axis and distributed along it. The covariance matrix might look like

$$\Sigma = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

That is, there is the most variance in the *x* direction. However, physicists would consider this to have a low moment *about* the *x* axis so the moment-of-inertia tensor is

$$I = n\begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 10.1 & 0 \\ 0 & 0 & 10.1 \end{bmatrix}.$$

Notes

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Absolute deviation

In statistics, the **absolute deviation** of an element of a data set is the absolute difference between that element and a given point. Typically the point from which the deviation is measured is a measure of central tendency, most often the median or sometimes the mean of the data set.

$$D_i = |x_i - m(X)|$$

where

D_i is the absolute deviation,

x_i is the data element

and $m(X)$ is the chosen measure of central tendency of the data set—sometimes the mean ($\overline{x}$), but most often the median.

Measures of dispersion

Several measures of statistical dispersion are defined in terms of the absolute deviation.

Average absolute deviation

The **average absolute deviation,** or simply **average deviation** of a data set is the average of the absolute deviations and is a summary statistic of statistical dispersion or variability. It is also called the **mean absolute deviation,** but this is easily confused with the median absolute deviation.

The average absolute deviation of a set $\{x_1, x_2, ..., x_n\}$ is

$$\frac{1}{n}\sum_{i=1}^{n}|x_i - m(X)|.$$

The choice of measure of central tendency, $m(X)$, has a marked effect on the value of the average deviation. For example, for the data set {2, 2, 3, 4, 14}:

Measure of central tendency $m(X)$	Average absolute deviation
Mean = 5	$\frac{\lvert2-5\rvert+\lvert2-5\rvert+\lvert3-5\rvert+\lvert4-5\rvert+\lvert14-5\rvert}{5} = 3.6$
Median = 3	$\frac{\lvert2-3\rvert+\lvert2-3\rvert+\lvert3-3\rvert+\lvert4-3\rvert+\lvert14-3\rvert}{5} = 2.8$
Mode = 2	$\frac{\lvert2-2\rvert+\lvert2-2\rvert+\lvert3-2\rvert+\lvert4-2\rvert+\lvert14-2\rvert}{5} = 3.0$

The average absolute deviation from the median is less than or equal to the average absolute deviation from the mean. In fact, the average absolute deviation from the median is always less than or equal to the average absolute deviation from any other fixed number.

The average absolute deviation from the mean is less than or equal to the standard deviation; one way of proving this relies on Jensen's inequality.

For the normal or "Gaussian" distribution, the ratio of mean absolute deviation to standard deviation is $\sqrt{2/\pi}$=0.79788456.... Thus if X is a normally distributed random variable with expected value 0 then

$$\frac{E|X|}{\sqrt{E(X^2)}} = \sqrt{\frac{2}{\pi}}.$$

In other words, for a Gaussian, mean absolute deviation is about 0.8 times the standard deviation.

Mean absolute deviation

The mean absolute deviation (MAD), also referred to as the mean deviation or **average absolute deviation**, is the mean of the absolute deviations of a set of data about the data's mean. In other words, it is the average distance of the data set from its mean.

The equation for MAD is as follows:

$MAD = 1/n \sum(|e_i|)$, where $e_i = F_i - D_i$

This method forecast accuracy is very closely related to the mean squared error (MSE) method which is just the average squared error of the forecasts. Although these methods are very closely related MAD is more commonly used because it does not require squaring.

The equation for MSE is as follows:

$MSE = 1/n \Sigma(e_i^2)$, where $e_i = F_i - D_i$

Median absolute deviation (MAD)

The median absolute deviation is the *median* of the absolute deviation from the *median*. It is a robust estimator of dispersion.

For the example {2, 2, 3, 4, 14}: 3 is the median, so the absolute deviations from the median are {1, 1, 0, 1, 11} (reordered as {0, 1, 1, 1, 11}) with a median of 1, in this case unaffected by the value of the outlier 14, so the median absolute deviation (also called MAD) is 1.

Maximum absolute deviation

The **maximum absolute deviation** about a point is the maximum of the absolute deviations of a sample from that point. It is realized by the sample maximum or sample minimum and cannot be less than half the range.

Minimization

The measures of statistical dispersion derived from absolute deviation characterize various measures of central tendency as *minimizing* dispersion: The median is the measure of central tendency most associated with the absolute deviation, in that

L^2 norm statistics

: just as the mean minimizes the standard deviation,

L^1 norm statistics

: the median minimizes *average* absolute deviation,

L^∞ norm statistics

: the mid-range minimizes the *maximum* absolute deviation, and

trimmed L^∞ norm statistics

: for example, the midhinge (average of first and third quartiles) which minimizes the *median* absolute deviation of the whole distribution, also minimizes the *maximum* absolute deviation of the distribution after the top and bottom 25% have been trimmed off.

Estimation

The mean absolute deviation of a sample is a biased estimator of the mean absolute deviation of the population. In order for the absolute deviation to be an unbiased estimator, the expected value (average) of all the sample absolute deviations must equal the population absolute deviation. However, it does not. For the population 1,2,3 the population absolute deviation is 2/3. The average of all the sample standard deviations of size 3 that can be drawn from the population is 40/81. Therefore the absolute deviation is a biased estimator.

External links

- Advantages of the mean absolute deviation [1]

References

[1] http://www.leeds.ac.uk/educol/documents/00003759.htm

Correlation and dependence

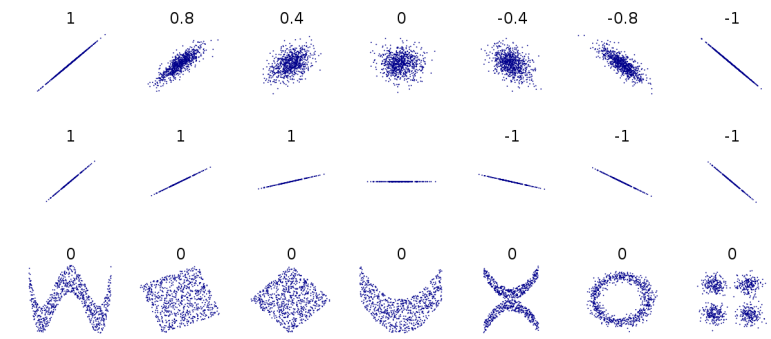
In statistics, **dependence** refers to any statistical relationship between two random variables or two sets of data. **Correlation** refers to any of a broad class of statistical relationships involving dependence.

Familiar examples of dependent phenomena include the correlation between the physical statures of parents and their offspring, and the correlation between the demand for a product and its price. Correlations are useful because they can indicate a predictive relationship that can be exploited in practice. For example, an electrical utility may produce less power on a mild day based on the correlation between electricity demand and weather. In this example there is a causal relationship, because extreme weather causes people to use more electricity for heating or cooling; however, statistical dependence is not sufficient to demonstrate the presence of such a causal relationship (i.e., Correlation does not imply causation).

Formally, *dependence* refers to any situation in which random variables do not satisfy a mathematical condition of probabilistic independence. In loose usage, *correlation* can refer to any departure of two or more random variables from independence, but technically it refers to any of several more specialized types of relationship between mean values. There are several **correlation coefficients**, often denoted ϱ or r, measuring the degree of correlation. The most common of these is the Pearson correlation coefficient, which is sensitive only to a linear relationship between two variables (which may exist even if one is a nonlinear function of the other). Other correlation coefficients have been developed to be more robust than the Pearson correlation – that is, more sensitive to nonlinear relationships.[1][2][3]

Pearson's product-moment coefficient

The most familiar measure of dependence between two quantities is the Pearson product-moment correlation coefficient, or "Pearson's correlation." It is obtained by dividing the covariance of the two variables by the product of their standard deviations. Karl Pearson developed the coefficient from a similar but slightly different idea by Francis Galton.[4]



Several sets of (*x*, *y*) points, with the Pearson correlation coefficient of *x* and *y* for each set. Note that the correlation reflects the noisiness and direction of a linear relationship (top row), but not the slope of that relationship (middle), nor many aspects of nonlinear relationships (bottom). N.B.: the figure in the center has a slope of 0 but in that case the correlation coefficient is undefined because the variance of *Y* is zero.

The population correlation coefficient $\rho_{X,Y}$ between two random variables *X* and *Y* with expected values μ_X and μ_Y and standard deviations σ_X and σ_Y is defined as:

$$\rho_{X,Y} = \operatorname{corr}(X,Y) = \frac{\operatorname{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y},$$

where *E* is the expected value operator, *cov* means covariance, and, *corr* a widely used alternative notation for Pearson's correlation.

The Pearson correlation is defined only if both of the standard deviations are finite and both of them are nonzero. It is a corollary of the Cauchy–Schwarz inequality that the correlation cannot exceed 1 in absolute value. The correlation coefficient is symmetric: corr(*X*,*Y*) = corr(*Y*,*X*).

The Pearson correlation is +1 in the case of a perfect positive (increasing) linear relationship (correlation), −1 in the case of a perfect decreasing (negative) linear relationship (**anticorrelation**),[5] and some value between −1 and 1 in all other cases, indicating the degree of linear dependence between the variables. As it approaches zero there is less of a relationship (closer to uncorrelated). The closer the coefficient is to either −1 or 1, the stronger the correlation between the variables.

If the variables are independent, Pearson's correlation coefficient is 0, but the converse is not true because the correlation coefficient detects only linear dependencies between two variables. For example, suppose the random variable *X* is symmetrically distributed about zero, and $Y = X^2$. Then *Y* is completely determined by *X*, so that *X* and *Y* are perfectly dependent, but their correlation is zero; they are uncorrelated. However, in the special case when *X* and *Y* are jointly normal, uncorrelatedness is equivalent to independence.

If we have a series of *n* measurements of *X* and *Y* written as x_i and y_i where $i = 1, 2, \ldots, n$, then the *sample correlation coefficient* can be used to estimate the population Pearson correlation *r* between *X* and *Y*. The sample correlation coefficient is written

$$r_{xy} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{(n-1)s_x s_y} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2 \sum_{i=1}^{n}(y_i-\bar{y})^2}},$$

where x and y are the sample means of *X* and *Y*, and s_x and s_y are the sample standard deviations of *X* and *Y*.

This can also be written as:

$$r_{xy} = \frac{\sum x_i y_i - n\bar{x}\bar{y}}{(n-1)s_x s_y} = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{\sqrt{n\sum x_i^2 - (\sum x_i)^2}\,\sqrt{n\sum y_i^2 - (\sum y_i)^2}}.$$

If x and y are results of measurements that contain measurement error, the realistic limits on the correlation coefficient are not −1 to +1 but a smaller range.[6]

Rank correlation coefficients

Rank correlation coefficients, such as Spearman's rank correlation coefficient and Kendall's rank correlation coefficient (τ) measure the extent to which, as one variable increases, the other variable tends to increase, without requiring that increase to be represented by a linear relationship. If, as the one variable increases, the other *decreases*, the rank correlation coefficients will be negative. It is common to regard these rank correlation coefficients as alternatives to Pearson's coefficient, used either to reduce the amount of calculation or to make the coefficient less sensitive to non-normality in distributions. However, this view has little mathematical basis, as rank correlation coefficients measure a different type of relationship than the Pearson product-moment correlation coefficient, and are best seen as measures of a different type of association, rather than as alternative measure of the population correlation coefficient.[7][8]

To illustrate the nature of rank correlation, and its difference from linear correlation, consider the following four pairs of numbers (x, y):

(0, 1), (10, 100), (101, 500), (102, 2000).

As we go from each pair to the next pair x increases, and so does y. This relationship is perfect, in the sense that an increase in x is *always* accompanied by an increase in y. This means that we have a perfect rank correlation, and both Spearman's and Kendall's correlation coefficients are 1, whereas in this example Pearson product-moment correlation coefficient is 0.7544, indicating that the points are far from lying on a straight line. In the same way if y always *decreases* when x *increases*, the rank correlation coefficients will be −1, while the Pearson product-moment correlation coefficient may or may not be close to −1, depending on how close the points are to a straight line. Although in the extreme cases of perfect rank correlation the two coefficients are both equal (being both +1 or both −1) this is not in general so, and values of the two coefficients cannot meaningfully be compared.[7] For example, for the three pairs (1, 1) (2, 3) (3, 2) Spearman's coefficient is 1/2, while Kendall's coefficient is 1/3.

Other measures of dependence among random variables

The information given by a correlation coefficient is not enough to define the dependence structure between random variables. The correlation coefficient completely defines the dependence structure only in very particular cases, for example when the distribution is a multivariate normal distribution. (See diagram above.) In the case of elliptical distributions it characterizes the (hyper-)ellipses of equal density, however, it does not completely characterize the dependence structure (for example, a multivariate t-distribution's degrees of freedom determine the level of tail dependence).

Distance correlation and Brownian covariance / Brownian correlation [9][10] were introduced to address the deficiency of Pearson's correlation that it can be zero for dependent random variables; zero distance correlation and zero Brownian correlation imply independence.

The correlation ratio is able to detect almost any functional dependency, and the entropy-based mutual information, total correlation and dual total correlation are capable of detecting even more general dependencies. These are sometimes referred to as multi-moment correlation measures, in comparison to those that consider only second moment (pairwise or quadratic) dependence.

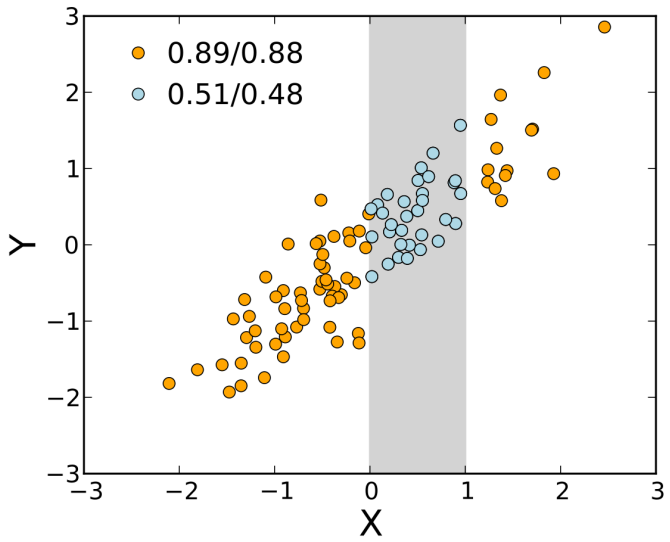
The polychoric correlation is another correlation applied to ordinal data that aims to estimate the correlation between theorised latent variables.

One way to capture a more complete view of dependence structure is to consider a copula between them.

Sensitivity to the data distribution

The degree of dependence between variables X and Y does not depend on the scale on which the variables are expressed. That is, if we are analyzing the relationship between X and Y, most correlation measures are unaffected by transforming X to $a + bX$ and Y to $c + dY$, where a, b, c, and d are constants. This is true of some correlation statistics as well as their population analogues. Some correlation statistics, such as the rank correlation coefficient, are also invariant to monotone transformations of the marginal distributions of X and/or Y.

Most correlation measures are sensitive to the manner in which X and Y are sampled. Dependencies tend to be stronger if viewed over a wider range of values. Thus, if we consider the correlation coefficient between the heights of fathers and their sons over all adult males, and compare it to the same correlation coefficient calculated when the fathers are selected to be between 165 cm and 170 cm in height, the correlation will be weaker in the latter case. Several techniques have been developed that attempt to correct for range restriction in one or both variables, and are commonly used in meta-analysis; the most common are Thorndike's case II and case III equations.[11]



Pearson/Spearman correlation coefficients between X and Y are shown when the two variables' ranges are unrestricted, and when the range of X is restricted to the interval (0,1).

Various correlation measures in use may be undefined for certain joint distributions of X and Y. For example, the Pearson correlation coefficient is defined in terms of moments, and hence will be undefined if the moments are undefined. Measures of dependence based on quantiles are always defined. Sample-based statistics intended to estimate population measures of dependence may or may not have desirable statistical properties such as being unbiased, or asymptotically consistent, based on the spatial structure of the population from which the data were sampled.

Sensitivity to the data distribution can be used to an advantage. For example, scaled correlation is designed to use the sensitivity to the range in order to pick out correlations between fast components of time series.[12] By reducing the range of values in a controlled manner, the correlations on long time scale are filtered out and only the correlations on short time scales are revealed.

Correlation matrices

The correlation matrix of n random variables X_1, ..., X_n is the $n \times n$ matrix whose i,j entry is $\operatorname{corr}(X_i, X_j)$. If the measures of correlation used are product-moment coefficients, the correlation matrix is the same as the covariance matrix of the standardized random variables $X_i / \sigma(X_i)$ for $i = 1, ..., n$. This applies to both the matrix of population correlations (in which case "σ" is the population standard deviation), and to the matrix of sample correlations (in which case "σ" denotes the sample standard deviation). Consequently, each is necessarily a positive-semidefinite matrix.

The correlation matrix is symmetric because the correlation between X_i and X_j is the same as the correlation between X_j and X_i.

Common misconceptions

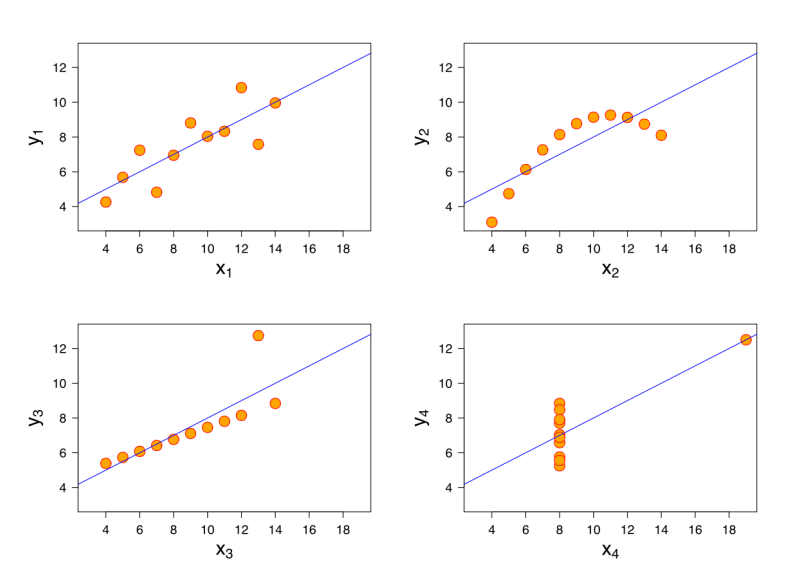
Correlation and causality

The conventional dictum that "correlation does not imply causation" means that correlation cannot be used to infer a causal relationship between the variables.[13] This dictum should not be taken to mean that correlations cannot indicate the potential existence of causal relations. However, the causes underlying the correlation, if any, may be indirect and unknown, and high correlations also overlap with identity relations (tautologies), where no causal process exists. Consequently, establishing a correlation between two variables is not a sufficient condition to establish a causal relationship (in either direction). For example, one may observe a correlation between an ordinary alarm clock ringing and daybreak, though there is no direct causal relationship between these events.

A correlation between age and height in children is fairly causally transparent, but a correlation between mood and health in people is less so. Does improved mood lead to improved health, or does good health lead to good mood, or both? Or does some other factor underlie both? In other words, a correlation can be taken as evidence for a possible causal relationship, but cannot indicate what the causal relationship, if any, might be.

Correlation and linearity

The Pearson correlation coefficient indicates the strength of a linear relationship between two variables, but its value generally does not completely characterize their relationship. In particular, if the conditional mean of Y given X, denoted $\mathrm{E}(Y|X)$, is not linear in X, the correlation coefficient will not fully determine the form of $\mathrm{E}(Y|X)$.



Four sets of data with the same correlation of 0.816

The image on the right shows scatterplots of Anscombe's quartet, a set of four different pairs of variables created by Francis Anscombe.[14] The four y variables have the same mean (7.5), standard deviation (4.12), correlation (0.816) and regression line ($y = 3 + 0.5x$). However, as can be seen on the plots, the distribution of the variables is very different. The first one (top left) seems to be distributed normally, and corresponds to what one would expect when considering two variables correlated and following the assumption of normality. The second one (top right) is not distributed normally; while an obvious relationship between the two variables can be observed, it is not linear. In this case the Pearson correlation coefficient does not indicate that there is an exact functional relationship: only the extent to which that relationship can be approximated by a linear relationship. In the third case (bottom left), the linear relationship is perfect, except for one outlier which exerts enough influence to lower the correlation coefficient from 1 to 0.816. Finally, the fourth example (bottom right) shows another example when one outlier is enough to produce a high correlation coefficient, even though the relationship between the two variables is not linear.

These examples indicate that the correlation coefficient, as a summary statistic, cannot replace visual examination of the data. Note that the examples are sometimes said to demonstrate that the Pearson correlation assumes that the data follow a normal distribution, but this is not correct.[4]

The coefficient of determination generalizes the correlation coefficient for relationships beyond simple linear regression.

Bivariate normal distribution

If a pair (*X*, *Y*) of random variables follows a bivariate normal distribution, the conditional mean E(*X*|*Y*) is a linear function of *Y*, and the conditional mean E(*Y*|*X*) is a linear function of *X*. The correlation coefficient *r* between *X* and *Y*, along with the marginal means and variances of *X* and *Y*, determines this linear relationship:

$$E(Y|X) = E(Y) + r\sigma_y \frac{X - E(X)}{\sigma_x},$$

where *E(X)* and *E(Y)* are the expected values of *X* and *Y*, respectively, and σ_x and σ_y are the standard deviations of *X* and *Y*, respectively.

Partial correlation

If a population or data-set is characterized by more than two variables, a partial correlation coefficient measures the strength of dependence between a pair of variables that is not accounted for by the way in which they both change in response to variations in a selected subset of the other variables.

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[4] J. L. Rodgers and W. A. Nicewander. Thirteen ways to look at the correlation coefficient (http://www.jstor.org/stable/2685263). The American Statistician, 42(1):59–66, February 1988.

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[10] Székely, G. J. and Rizzo, M. L. (2009). "Brownian distance covariance", *Annals of Applied Statistics*, 3/4, 1233–1303. doi: 10.1214/09-AOAS312 Reprint (http://personal.bgsu.edu/~mrizzo/energy/AOAS312.pdf)

[11] Thorndike, Robert Ladd (1947). *Research problems and techniques (Report No. 3)*. Washington DC: US Govt. print. off..

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[14] Anscombe, Francis J. (1973). "Graphs in statistical analysis". *The American Statistician* **27**: 17–21. doi:10.2307/2682899. JSTOR 2682899.

Further reading

- Cohen, J., Cohen P., West, S.G., & Aiken, L.S. (2002). *Applied multiple regression/correlation analysis for the behavioral sciences (3rd ed.)*. Psychology Press. ISBN 0-8058-2223-2.

External links

- Hazewinkel, Michiel, ed. (2001), "Correlation (in statistics)" (http://www.encyclopediaofmath.org/index.php?title=p/c026560), *Encyclopedia of Mathematics*, Springer, ISBN 978-1-55608-010-4
- Earliest Uses: Correlation (http://jeff560.tripod.com/c.html) – gives basic history and references.
- Understanding Correlation (http://www.hawaii.edu/powerkills/UC.HTM) – Introductory material by a U. of Hawaii Prof.
- Statsoft Electronic Textbook (http://www.statsoft.com/textbook/stathome.html?stbasic.html&1)
- Pearson's Correlation Coefficient (http://www.vias.org/tmdatanaleng/cc_corr_coeff.html) – How to calculate it quickly
- Learning by Simulations (http://www.vias.org/simulations/simusoft_rdistri.html) – The distribution of the correlation coefficient
- Correlation measures the strength of a *linear* relationship between two variables. (http://www.statisticalengineering.com/correlation.htm)
- MathWorld page on (cross-) correlation coefficient(s) of a sample. (http://mathworld.wolfram.com/CorrelationCoefficient.html)
- Compute Significance between two correlations (http://peaks.informatik.uni-erlangen.de/cgi-bin/usignificance.cgi) – A useful website if one wants to compare two correlation values.
- A MATLAB Toolbox for computing Weighted Correlation Coefficients (http://www.mathworks.com/matlabcentral/fileexchange/20846)
- Proof that the Sample Bivariate Correlation Coefficient has Limits ±1 (http://www.docstoc.com/docs/3530180/Proof-that-the-Sample-Bivariate-Correlation-Coefficient-has-Limits-(Plus-or-Minus)-1)

Chart pattern

A **Price pattern** is a pattern that is formed within a chart when prices are graphed. In stock and commodity markets trading, chart pattern studies play a large role during technical analysis. When data is plotted there is usually a pattern which naturally occurs and repeats over a period. Chart patterns are used as either reversal or continuation signals.

Some people claim that by recognizing chart patterns they are able to predict future stock prices and profit by this prediction; other people respond by quoting "past performance is no guarantee of future results" and argue that chart patterns are merely illusions created by people's subconscious. Certain theories of economics hold that if there were a way to predict future stock prices and profit by it then when enough people used these techniques they would become ineffective and cease to be profitable. On the other hand, predicting what others will predict the market will do, would be valuable information.

Patterns

Examples of "classical" chart patterns as popularized by Edwards and Magee and used widely by traders and investors include:

- Head and shoulders
- Trend lines
- Cup and handle
- Double top and double bottom
- Triple top and triple bottom
- Broadening top
- Price channels
- Wedge pattern
- Triangle (technical analysis)
- Flag and pennant patterns

Head and shoulders (chart pattern)

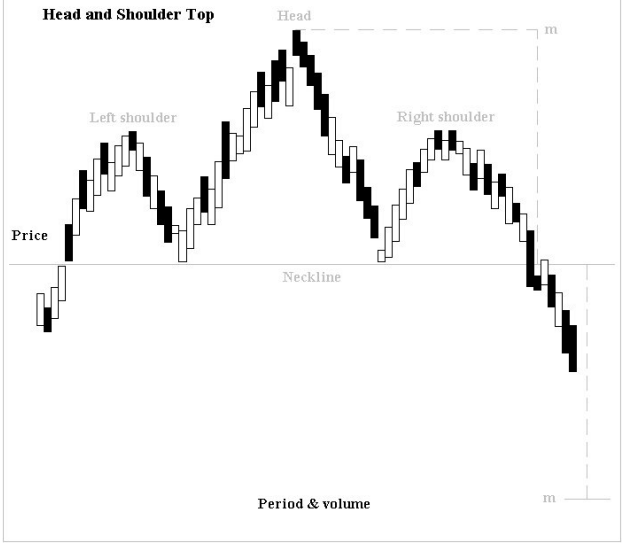
The **Head and Shoulders** formation is one of the most well known reversal patterns.

On the technical analysis chart, when a price trend is in the process of reversal either from a bullish or bearish trend, a characteristic pattern takes shape and is recognized as reversal formation.

Formations

Head and Shoulders Top and Head and Shoulders Bottom is discussed below.

Head and shoulders top

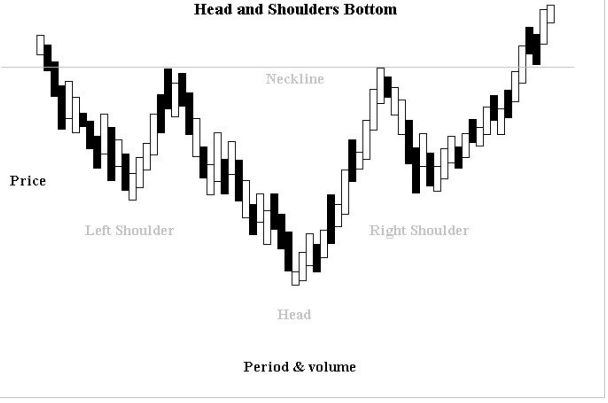


Head and Shoulders Top

Head and Shoulders formation consists of a left shoulder, a head, and a right shoulder and a line drawn as the neckline. The left shoulder is formed at the end of an extensive move during which volume is noticeably high. After the peak of the left shoulder is formed, there is a subsequent reaction and prices slide down to a certain extent which generally occurs on low volume. The prices rally up to form the head with normal or heavy volume and subsequent reaction downward is accompanied with lesser volume. The right shoulder is formed when prices move up again but remain below the central peak called the Head and fall down nearly equal to the first valley between the left shoulder and the head or at least below the peak of the left shoulder. Volume is lesser in the right shoulder formation compared to the left shoulder and the head formation. A neckline is drawn across the bottoms of the left shoulder, the head and the right shoulder. When prices break through this neckline and keep on falling after forming the right shoulder, it is the ultimate confirmation of the completion of the Head and Shoulders Top formation. It is quite possible that prices pull back to touch the neckline before continuing their declining trend.

Head and shoulders bottom

This formation is simply the inverse of a Head and Shoulders Top and often indicates a change in the trend and the sentiment. The formation is upside down in which volume pattern is different than a Head and Shoulder Top. Prices move up from first low with increase volume up to a level to complete the left shoulder formation and then falls down to a new low. It follows by a recovery move that is marked by somewhat more volume than seen before to complete the head formation. A corrective reaction on low volume occurs to start formation of the right shoulder and then a sharp move up that must be on quite heavy volume breaks though the neckline.



Head and Shoulders Bottom

Another difference between the Head and Shoulders Top and Bottom is that the Top Formations are completed in a few weeks, whereas a Major Bottom (Left, right shoulder or the head) usually takes a longer, and as observed, may prolong for a period of several months or sometimes more than a year.

Importance of neckline

The drawn neckline of the pattern represents a support level, and assumption cannot be taken that the Head and Shoulder formation is completed unless it is broken and such breakthrough may happen to be on more volume or may not be. The breakthrough should not be observed carelessly. A serious situation can occur if such a break is more than three to four percent.

When a stock drifts through the neckline on small volume, there may be a *wave up*, although it is not certain, but it is observed, the rally normally does not cross the general level of the Neckline and before selling pressure increases, the steep decline occurs and prices tumble with greater volume.

Characteristics

- Most of the time Head and Shoulders are not perfectly shaped. This formation is slightly tilted upward or downward.
- One shoulder may appear to droop.
- On many chart patterns, any one of the two shoulders may appear broader than the other which is caused by the time involved in the formation of the valleys.
- The neckline may not be perfectly horizontal; it may be ascending or descending.
- If the neckline is ascending then the only qualification of the formation lies in the fact that the lowest point of the right shoulder must be noticeably lower than the peak of the left shoulder.

Usage as a tool

Head and Shoulders is an extremely useful tool after its confirmation to estimate and measure the minimum probable extent of the subsequent move from the neckline. To find the distance of subsequent move, measure the vertical distance from the peak of the head to the neckline. Then measure this same distance down from the neckline beginning at the point where prices penetrate the neckline after the completion of the right shoulder. This gives the minimum objective of how far prices can decline after the completion of this top formation.

If the price advance preceding the Head and Shoulders top is not long, the subsequent price fall after its completion may be small as well.

Complex head and shoulders

Further information: Double top and double bottom and Triple top and triple bottom

This type of Head and Shoulders pattern has more than one left and/or right shoulders and/or head. It is also known as *Multiple Head and Shoulders* pattern.

External links

- Head and Shoulders Video of pattern, breakout, average expected decline from sell signal; all information has cited sources. [1]
- Head and shoulders [2] at onlinetradingconcepts.com
- Head and Shoulders Reversal Pattern [3] at stockresearch.co.in
- *Analyzing Chart Patterns: Head And Shoulders* [4] at investopedia.com

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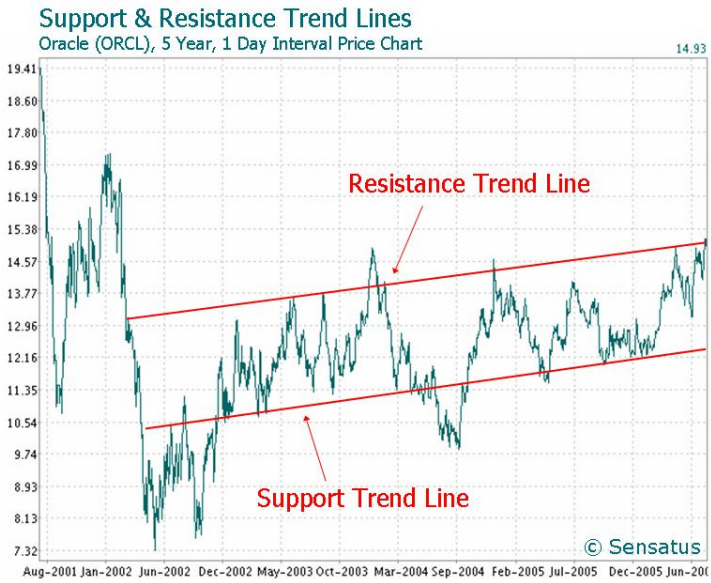
[1] http://finvids.com/Chart-Pattern/Head-And-Shoulders
[2] http://www.onlinetradingconcepts.com/TechnicalAnalysis/ClassicCharting/HeadShoulders.html
[3] http://www.stockresearch.co.in/head-and-shoulders-reversal-pattern.php
[4] http://www.investopedia.com/university/charts/charts2.asp

Trend line (technical analysis)

A **trend line** is formed when a diagonal line can be drawn between two or more price pivot points. They are commonly used to judge entry and exit investment timing when trading securities.[1] It can also be referred to as a **dutch line** as it was first used in Holland.

A trend line is a bounding line for the price movement of a security. A **support trend line** is formed when a securities price decreases and then rebounds at a pivot point that aligns with at least two previous support pivot points. Similarly a **resistance trend line** is formed when a securities price increases and then rebounds at a pivot point that aligns with at least two previous resistance pivot points.

Trend lines are a simple and widely used technical analysis approach to judging entry and exit investment timing. To establish a trend line historical data, typically presented in the format of a chart such as the above price chart, is required. Historically, trend lines have been drawn by hand on paper charts, but it is now more common to use charting software that enables trend lines to be drawn on computer based charts. There are some charting software that will automatically generate trend lines, however most traders prefer to draw their own trend lines.



Trend lines on a price chart.

When establishing trend lines it is important to choose a chart based on a price interval period that aligns with your trading strategy. Short term traders tend to use charts based on interval periods, such as 1 minute (i.e. the price of the security is plotted on the chart every 1 minute), with longer term traders using price charts based on hourly, daily, weekly and monthly interval periods.

However, time periods can also be viewed in terms of years. For example, below is a chart of the S&P 500 since the earliest data point until April 2008. While the Oracle example above uses a linear scale of price changes, long term data is more often viewed as logarithmic: e.g. the changes are really an attempt to approximate percentage changes than pure numerical value.

Trend lines are typically used with price charts, however they can also be used with a range of technical analysis charts such as MACD and RSI. Trend lines can be used to identify positive and negative trending charts, whereby a positive trending chart forms an upsloping line when the support and the resistance pivots points are aligned, and a negative trending chart forms a downsloping line when the support and resistance pivot points are aligned.

Previous chart from 1950 to about 1990, showing how linear scale obscures details by compressing the data.

Trend lines are used in many ways by traders. If a stock price is moving between support and resistance trend lines, then a basic investment strategy commonly used by traders, is to buy a stock at support and sell at resistance, then short at resistance and cover the short at support. The logic behind this, is that when the price returns to an existing principal trend line it may be an opportunity to open new positions in the direction of the trend, in the belief that the trend line will hold and the trend will continue further. A second way is that when price action breaks through the principal trend line of an existing trend, it is evidence that the trend may be going to fail, and a trader may consider trading in the opposite direction to the existing trend, or exiting positions in the direction of the trend.

References

[1] Edwards, Robert D.; Magee, John (1948). "14". *Technical Analysis of Stock Trends*. Springfield, MA, USA: Stock Trend Service. pp. 505. ISBN 1-880408-00-7.

Cup and handle

The **cup and handle** formation (also called the **cup with handle** formation) is a bullish chart pattern that is defined by a chart where a stock drops in value, then rises back up to the original value, then drops a small amount in value, and then rises a small amount in value. The "cup and handle" formation was discovered by William O'Neil, Founder of Investor's Business Daily, and explained in his top selling book, "How to Make Money in Stocks."[1]

Important characteristics

- **Trend:** A cup and handle formation should follow an increase trend, ideally one that is only a few months old. The older the increase trend, the less likely it is that the cup and handle will be an accurate indicator.
- **Shape:** In a cup and handle formation, the cup must always precede the handle. The cup should form a rounded bowl shape, with an obvious bottom. A V-shaped bowl is said to be avoided. The cup should be fairly shallow, and ideally should retrace about 30% to 50% of the previous increase.[2] The perfect pattern would have equal highs on either side of the cup, but this is not always the case.
- **Duration:** The cup should last 1 to 6 months, while the handle should only last for 1 to 4 weeks.[2] These are only approximate values, however; a cup may last anywhere from a few weeks to a few years.
- **Volume:** The volume of the stock should decrease along with the price during the cup and should increase rapidly near the end of the handle when the price begins to rise.[2]

Significance for traders

A cup and handle formation is considered to be a bullish signal, and is usually followed by a sharp rise in value. A rather accurate estimation of the expected price rise is found by measuring the price rise from the bottom of the cup to the right side.[2] The reason for a price rise following a cup and handle formation is largely unknown. Likely because many traders know about this signal and buy when they see it forming thus artificially creating the bullish uptrend.

References

[1] http://stockcharts.com/school/doku.php?id=chart_school:chart_analysis:chart_patterns:cup_with_handle_cont

[2] Cup and handle at stockcharts.com (http://stockcharts.com/school/doku.php?id=chart_school:chart_analysis:chart_patterns:cup_with_handle)

External links

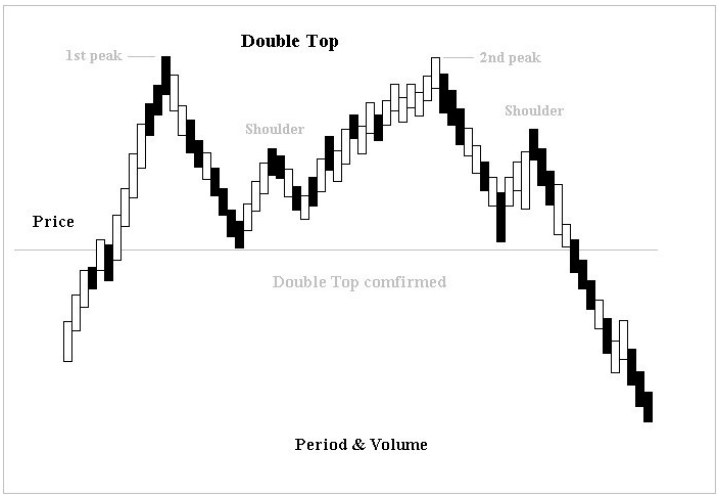
- Video explaining Cup & Handle Pattern (http://finvids.com/Chart-Pattern/Cup-And-Handle)
- *Analyzing Chart Patterns: Cup And Handle* (http://www.investopedia.com/university/charts/charts3.asp) at investopedia.com

Double top and double bottom

Double top and double bottom are reversal chart patterns observed in the technical analysis of financial trading markets of stocks, commodities, currencies, and other assets.

Double top

The double top is a frequent price formation at the end of a bull market. It appears as two consecutive peaks of approximately the same price on a price-versus-time chart of a market. The two peaks are separated by a minimum in price, a *valley*. The price level of this minimum is called the neck line of the formation. The formation is completed and confirmed when the price falls below the neck line, indicating that further price decline is imminent or highly likely.



Double top confirmation

The double top pattern shows that demand is outpacing supply (buyers predominate) up to the first top, causing prices to rise. The supply-demand balance then reverses; supply outpaces demand (sellers predominate), causing prices to fall. After a price valley, buyers again predominate and prices rise. If traders see that prices are not pushing past their level at the first top, sellers may again prevail, lowering prices and causing a double top to form. It is generally regarded as a bearish signal if prices drop below the neck line.

The time between the two peaks is also a determining factor for the existence of a double top pattern. If the tops appear at the same level but are very close in time, then the probability is high that they are part of the consolidation and the trend will resume.

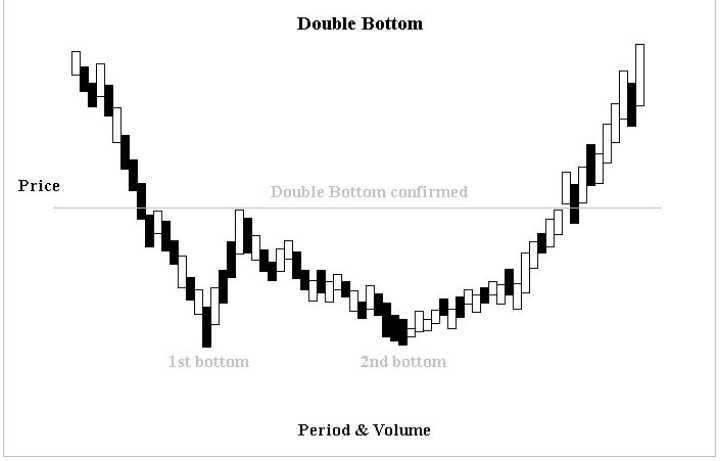
Volume is another indicator for interpreting this formation. Price reaches the first peak on increased volume then falls down the valley with low volume. Another attempt on the rally up to the second peak should be on a lower volume.

Double bottom

A double bottom is the end formation in a declining market. It is identical to the double top, except for the inverse relationship in price. The pattern is formed by two price minima separated by local peak defining the neck line. The formation is completed and confirmed when the price rises above the neck line, indicating that further price rise is imminent or highly likely.

Most of the rules that are associated with double top formation also apply to the double bottom pattern.

Volume should show a marked increase on the rally up while prices are flat at the second bottom.



Double bottom confirmation

External links

- Double Bottom video describing the time and price movement traits of the double bottom with the four double bottom patterns each with chart examples: Eve & Eve, Adam & Adam, Eve & Adam, and Adam & Adam double bottom; all information has cited sources [1]
- Double Top video with chart examples of the: Eve & Eve, Adam & Adam, Eve & Adam, and Adam & Adam double top; all information has cited sources [2]
- Trading the Double Bottom Pattern [3] at StockChartPatterns.org
- *Analyzing Chart Patterns: Double Top And Double Bottom* [4] at investopedia.com

References

[1] http://finvids.com/Chart-Pattern/Double-Bottom
[2] http://finvids.com/Chart-Pattern/Double-Top
[3] http://stockchartpatterns.org/how-to-trade-double-bottom/
[4] http://www.investopedia.com/university/charts/charts4.asp

Triple top and triple bottom

Triple top and triple bottom are reversal chart patterns used in the technical analysis of stocks, commodites, currencies, and other assets.

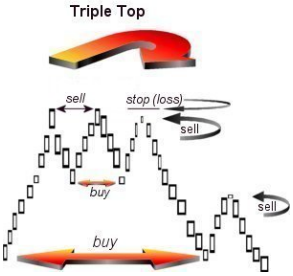
Triple top

Formation

Triple top confirmation

The formation of triple tops is rarer than that of double tops in the rising market trend. The volume is usually low during the second rally up and lesser during the formation of the *third top.* The peaks may not necessarily be spaced evenly like those which constitute a Double top. The intervening valleys may not bottom out at exactly the same level, i.e. either the first or second may be lower. The *triple top* is confirmed when the price decline from the *third top* falls below the *bottom* of the lowest valley between the three peaks.

Selling strategy

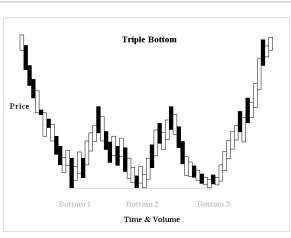


Opportunity

There are several different trading strategies that can be employed to take advantage of this formation. Of course, first and second peaks are perfect point to place sell orders. After the double top has been confirmed and if prices are moving up again with low volume, it is an opportune point to sell. One can sell short with a stop (calculated loss) above the highest peak of the Double top. The next opportune point to sell would be after a Triple top has formed and a fourth top is being formed at the lower level.

Notes Observation shows that it is rare to see four tops or bottoms at equal levels. In case prices continue to rally up to the level of the three previous tops, there is a good chance that they will rally up higher. If they come down to the same level a fourth time, they usually decline.

Triple bottom



Triple bottom

Most of the rules that are applied in the formation of the triple top can be reversed in the formation of *triple bottom.* As far as volume is concerned, the third low bottom should be on low volume and the rally up from that bottom should show a marked increase in activity.

The formation of Triple bottom occurs during the period of accumulation.

External links

- Triple Bottom Video with Statistics and Price Targets [1]
- *Analyzing Chart Patterns: Triple Tops And Bottoms* [2] at investopedia.com
- *How to Trade the Double Bottom Pattern* [3] at StockChartPatterns.org

References

[1] http://finvids.com/Chart-Pattern/Triple-Bottom
[2] http://www.investopedia.com/university/charts/charts9.asp

Broadening top

Broadening top is technical analysis chart pattern describing trends of stocks, commodities, currencies, and other assets.

Point of formation

Broadening Top formation appears much more frequently at tops than at bottoms. It is a difficult formation to trade in. Its formation usually has *bearish* implications.

Role of big players

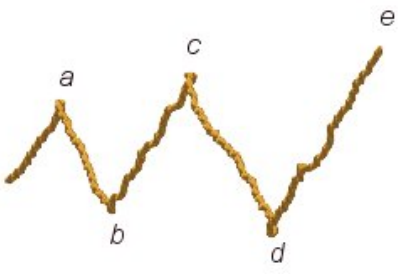
It is a common saying that *smart money* is out of market in such formation and market is out of control. In its formation, most of the selling is completed in the early stage by big players and the participation is from general public in the later stage.

Price and volume

Price keeps on swinging unpredictably and one can't be sure where the next swing will end. Regarding the *shares volume*, it is very irregular and leaves no clue to the direction of the next move.

How broadening top is formed

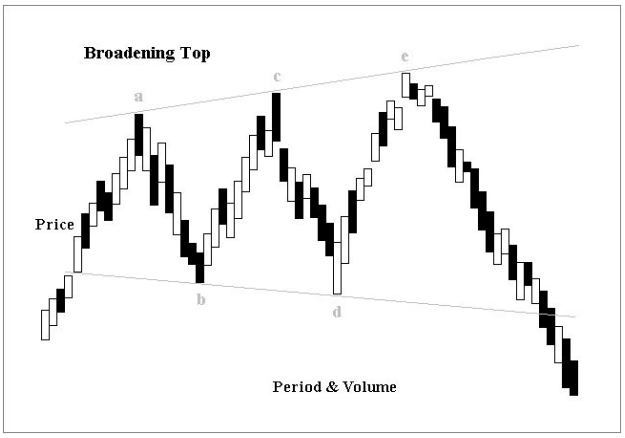
In the broadening top formation *five minor reversals* are followed by a substantial decline.



Five minor reversals a-b-c-d-e

In the figure above, price of the share reverses five times, reversal point *d* is made at a lower point than reversal point *b* and reversal point *c* and *e* occur successively higher than reversal point *a*.

One can't be sure of the trend unless price breaks down the lower of the two points *(b & d)* and keeps on falling. In the figure below, *Broading Top is confirmed*.



Broadening Top confirmation

Other chart patterns

- Candlestick pattern
- Double top and double bottom
- Gap (chart pattern)
- Head and shoulders top and bottom
- Island reversal
- Triple top and triple bottom
- Wedge pattern

Price channels

A **price channel** is a pair of parallel trend lines that form a chart pattern for a stock or commodity.[1] Channels may be horizontal, ascending or descending. When prices pass through and stay through a trendline representing support or resistance, the trend is said to be broken and there is a "breakout".[2]

Trend channel

References

[1] Murphy, pages 80-85
[2] Murphy, pages 400-401

- John J. Murphy, *Technical Analysis of the Financial Markets*, New York Institute of Finance, 1999, ISBN 0-7352-0066-1

Wedge pattern

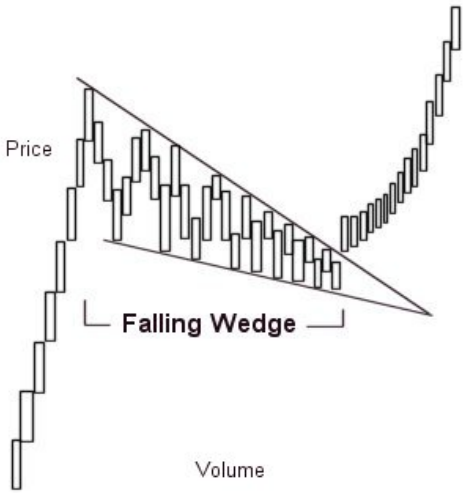
The **wedge pattern** is a commonly found pattern in the price charts of financially traded assets (stocks, bonds, futures, etc.). The pattern is characterized by a contracting range in prices coupled with an upward trend in prices (known as a rising wedge) or a downward trend in prices (known as a falling wedge).

A wedge pattern is considered to be a temporary halt of primary trend. It is a type of formation in which trading activities are confined within converging straight lines which form a pattern. It should take about 3 to 4 weeks to complete the wedge. This pattern has a rising or falling slant pointing in the same direction. It differs from the triangle in the sense that both boundary lines either slope up or down. Price breaking out point creates another difference from the triangle. Falling and rising wedges are a small part of intermediate or major trend. As they are reserved for minor trends, they are not considered to be major patterns. Once that basic or primary trend resumes itself, the wedge pattern loses its effectiveness as a technical indicator.

Falling wedge

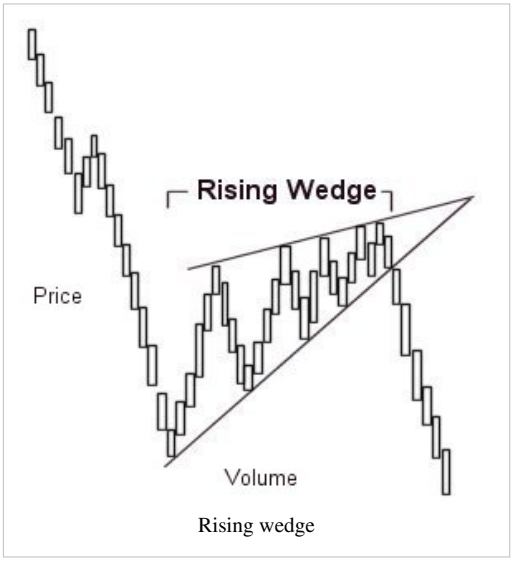
The falling wedge pattern is characterized by a chart pattern which forms when the market makes lower lows and lower highs with a contracting range. When this pattern is found in a downward trend, it is considered a reversal pattern, as the contraction of the range indicates the downtrend is losing steam. When this pattern is found in an uptrend, it is considered a bullish pattern, as the market range becomes narrower into the correction, indicating that the downward trend is losing strength and the resumption of the uptrend is in the making.

In a falling wedge, both boundary lines slant down from left to right. The upper descends at a steeper angle than the lower line. Volume keeps on diminishing and trading activity slows down due to narrowing prices. There comes the breaking point, and trading activity after the breakout differs. Once prices move out of the specific boundary lines of a falling wedge, they are more likely to move sideways and saucer-out before they resume the basic trend.



Falling wedge

Rising wedge



Rising wedge

The rising wedge pattern is characterized by a chart pattern which forms when the market makes higher highs and higher lows with a contracting range. When this pattern is found in an uptrend, it is considered a reversal pattern, as the contraction of the range indicates that the uptrend is losing strength. When this pattern is found in a downtrend, it is considered a bearish pattern, as the market range becomes narrower into the correction, indicating that the correction is losing strength, and that the resumption of the downtrend is in the making.

In a rising wedge, both boundary lines slant up from left to right. Although both lines point in the same direction, the lower line rises at a steeper angle then the upper one. Prices usually decline after breaking through the lower boundary line. As far as volumes are concerned, they keep on declining with each new price advance or wave up, indicating that the demand is weakening at the higher price level. A rising wedge is more reliable when found in a bearish market. In a bullish trend what seems to be a Rising Wedge may actually be a Flag or a Pennant (stepbrother of a wedge) requiring about 4 weeks to complete.

External links

- *How To Trade Wedges* [1] at investopedia.com
- *Analyzing Chart Patterns: The Wedge* [2] at investopedia.com

References

[1] http://stockchartpatterns.org/how-to-trade-wedges/
[2] http://www.investopedia.com/university/charts/charts7.asp

Triangle (chart pattern)

Triangles are a commonly found in the price charts of financially traded assets (stocks, bonds, futures, etc.). The pattern derives its name from the fact that it is characterized by a contraction in price range and converging trendlines, thus giving it a triangular shape.

Triangle Patterns can be broken down into three categories: the ascending triangle, the descending triangle, and the symmetrical triangle. While the shape of the triangle is significant, of more importance is the direction that the market moves when it breaks out of the triangle. Lastly, while triangles can sometimes be reversal patterns—meaning a reversal of the prior trend—they are normally seen as continuation patterns (meaning a continuation of the prior trend).

The ascending triangle

The ascending triangle is formed when the market makes higher lows and the same level highs. These patterns are normally seen in an uptrend and viewed as a continuation pattern as buying demand gain more and more control, running up to the top resistance line of the pattern. While you normally will see this pattern form in an uptrend, if you do see it in a downtrend it should be paid attention to as it can act as a powerful reversal signal.

The chart below offers an example of an ascending triangle.

The descending triangle

The descending triangle is formed when the market makes lower highs and the same level lows. These patterns are normally seen in a downtrend and viewed as a continuation pattern as the bears gain more and more control running down to the bottom support line of the pattern. While you normally will see this pattern form in a downtrend, if you do see it in an uptrend it should be paid attention to as it can act as a powerful reversal signal.

The image below illustrates.

The symmetrical triangle

The symmetrical triangle is formed when the market makes lower highs and higher lows and is commonly associated with directionless markets as the contraction of the market range indicates that neither the bulls nor the bears are in control. If this pattern forms in an uptrend then it is considered a continuation pattern if the market breaks out to the upside and a reversal pattern if the market breaks to the downside. Similarly if the pattern forms in a downtrend it is considered a continuation pattern if the market breaks out to the downside and a reversal pattern if the market breaks to the upside.

The image below illustrates.

External links

- Video of Triangle Pattern with breakout direction and gain/decline statistics all fully sourced content [1]
- How To Trade the Triangle Chart Pattern [2] at StockChartPatterns.org

References

[1] http://finvids.com/Chart-Pattern/Triangles
[2] http://stockchartpatterns.org/how-to-trade-the-triangle-chart-pattern/

Flag and pennant patterns

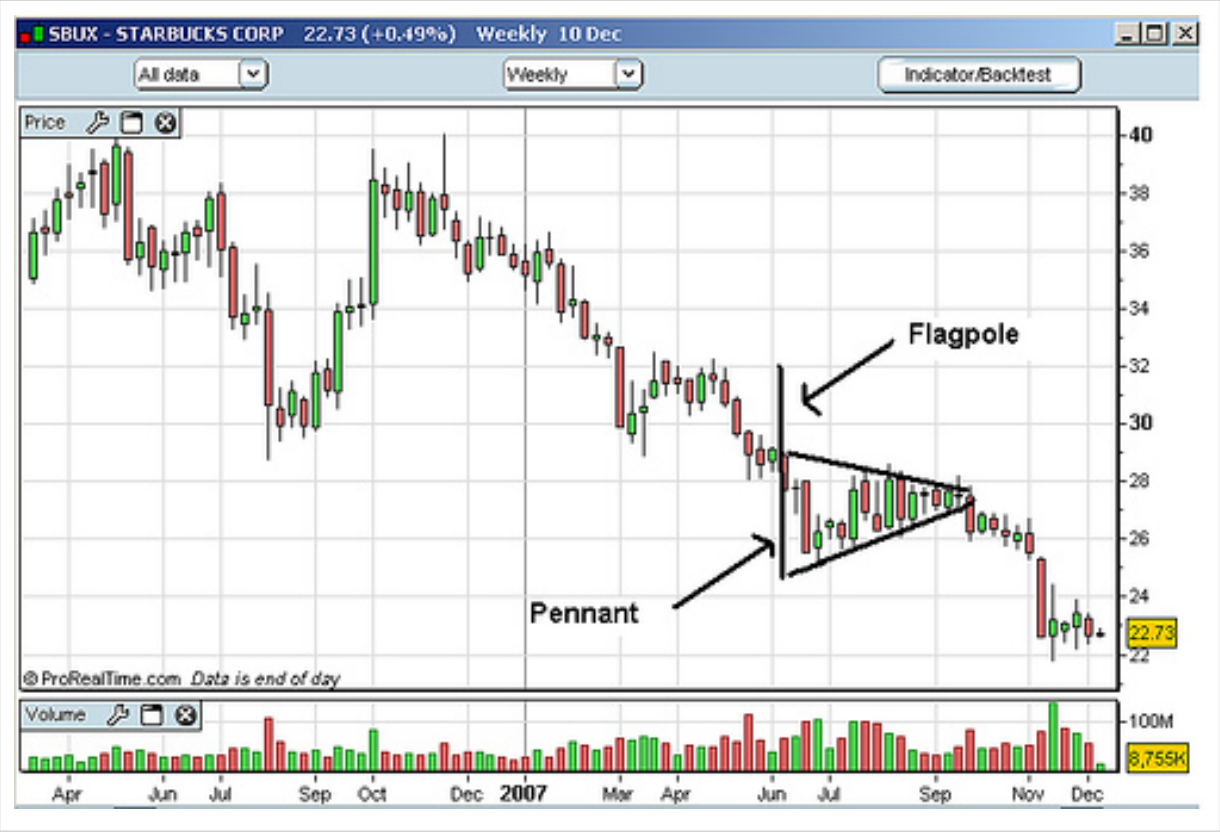
The **flag and pennant patterns** are commonly found patterns in the price charts of financially traded assets (stocks, bonds, futures, etc.). The patterns are characterized by a clear direction of the price trend, followed by a consolidation and rangebound movement, which is then followed by a resumption of the trend.

Flag pattern

The flag pattern is encompassed by two parallel lines. These lines can be either flat or pointed in the opposite direction of the primary market trend. The pole is then formed by a line which represents the primary trend in the market. The pattern is seen as the market potentially just taking a “breather” after a big move before continuing its primary trend. The chart below illustrates.

Pennant pattern

The pennant pattern is identical to the flag pattern in its setup and implications; the only difference is that the consolidation phase of a pennant pattern is characterized by converging trendlines rather than parallel trendlines. The image below illustrates.



External links

- Video of Flag Pattern and High & Tight Flag; referenced sources [1]
- Video of Pennant Pattern with cited sources [2]
- *Analyzing Chart Patterns: Flags And Pennants* [3] at investopedia.com

References

[1] http://finvids.com/Chart-Pattern/Flag
[2] http://finvids.com/Chart-Pattern/Pennants
[3] http://www.investopedia.com/university/charts/charts6.asp

Fibonacci number

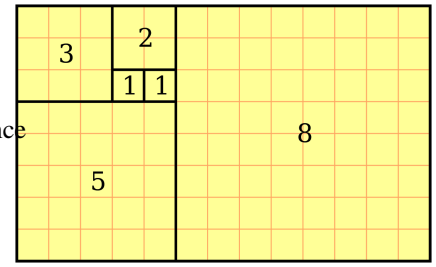
In mathematics, the **Fibonacci numbers** or **Fibonacci series** or **Fibonacci sequence** are the numbers in the following integer sequence:

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$ (sequence A000045 in OEIS)

or, alternatively,[1]

$$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots.$$

By definition, the first two numbers in the Fibonacci sequence are 0 and 1 (alternatively, 1 and 1), and each subsequent number is the sum of the previous two.



A tiling with squares whose sides are successive Fibonacci numbers in length

In mathematical terms, the sequence F_n of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

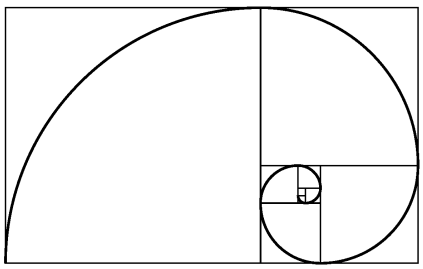
with seed values[2]

$$F_0 = 0,\ F_1 = 1$$

in the first form, or

$$F_1 = 1,\ F_2 = 1$$

in the second form.


A Fibonacci spiral created by drawing circular arcs connecting the opposite corners of squares in the Fibonacci tiling; this one uses squares of sizes 1, 1, 2, 3, 5, 8, 13, 21, and 34. See golden spiral.

The Fibonacci sequence is named after Leonardo of Pisa, who was known as Fibonacci. Fibonacci's 1202 book *Liber Abaci* introduced the sequence to Western European mathematics,[3] although the sequence had been described earlier in Indian mathematics.[4][5][6] (By modern convention, the sequence begins either with $F_0 = 0$ or with $F_1 = 1$. The *Liber Abaci* began the sequence with $F_1 = 1$, without an initial 0.)

Fibonacci numbers are closely related to Lucas numbers in that they are a complementary pair of Lucas sequences. They are intimately connected with the golden ratio, for example the closest rational approximations to the ratio are 2/1, 3/2, 5/3, 8/5, ... Applications include computer algorithms such as the Fibonacci search technique and the Fibonacci heap data structure, and graphs called Fibonacci cubes used for interconnecting parallel and distributed systems. They also appear in biological settings,[7] such as branching in trees, phyllotaxis (the arrangement of leaves on a stem), the fruit sprouts of a pineapple,[8] the flowering of artichoke, an uncurling fern and the arrangement of a pine cone.[9]

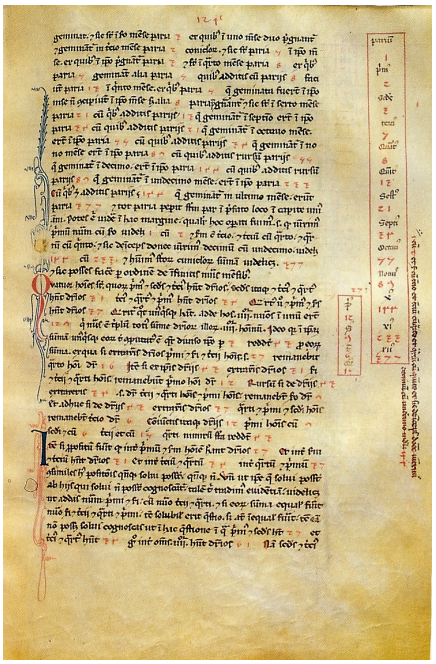
Origins

The Fibonacci sequence appears in Indian mathematics, in connection with Sanskrit prosody.[5][10] In the Sanskrit oral tradition, there was much emphasis on how long (L) syllables mix with the short (S), and counting the different patterns of L and S within a given fixed length results in the Fibonacci numbers; the number of patterns that are *m* short syllables long is the Fibonacci number F_{m+1}.[6]

Susantha Goonatilake writes that the development of the Fibonacci sequence "is attributed in part to Pingala (200 BC), later being associated with Virahanka (c. 700 AD), Gopāla (c. 1135), and Hemachandra (c. 1150)".[4] Parmanand Singh cites Pingala's cryptic formula *misrau cha* ("the two are mixed") and cites scholars who interpret it in context as saying that the cases for *m* beats (F_{m+1}) is obtained by adding a [S] to F_m cases and [L] to the F_{m-1} cases. He dates Pingala before 450 BCE.[11]

However, the clearest exposition of the series arises in the work of Virahanka (c. 700 AD), whose own work is lost, but is available in a quotation by Gopala (c. 1135):

> Variations of two earlier meters [is the variation]... For example, for [a meter of length] four, variations of meters of two [and] three being mixed, five happens. [works out examples 8, 13, 21]... In this way, the process should be followed in all *mātrā-vṛttas* [prosodic combinations].[12]



A page of Fibonacci's *Liber Abaci* from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Roman numerals and the value in Hindu-Arabic numerals.

The series is also discussed by Gopala (before 1135 AD) and by the Jain scholar Hemachandra (c. 1150).

In the West, the Fibonacci sequence first appears in the book *Liber Abaci* (1202) by Leonardo of Pisa, known as Fibonacci.[3] Fibonacci considers the growth of an idealized (biologically unrealistic) rabbit population, assuming that: a newly born pair of rabbits, one male, one female, are put in a field; rabbits are able to mate at the age of one month so that at the end of its second month a female can produce another pair of rabbits; rabbits never die and a mating pair always produces one new pair (one male, one female) every month from the second month on. The puzzle that Fibonacci posed was: how many pairs will there be in one year?

- At the end of the first month, they mate, but there is still only 1 pair.
- At the end of the second month the female produces a new pair, so now there are 2 pairs of rabbits in the field.
- At the end of the third month, the original female produces a second pair, making 3 pairs in all in the field.
- At the end of the fourth month, the original female has produced yet another new pair, the female born two months ago produces her first pair also, making 5 pairs.

At the end of the *n*th month, the number of pairs of rabbits is equal to the number of new pairs (which is the number of pairs in month $n - 2$) plus the number of pairs alive last month ($n - 1$). This is the *n*th Fibonacci number.[13]

The name "Fibonacci sequence" was first used by the 19th-century number theorist Édouard Lucas.[14]

List of Fibonacci numbers

The first 21 Fibonacci numbers F_n for $n = 0, 1, 2, ..., 20$ are:[15]

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}	F_{15}	F_{16}	F_{17}	F_{18}	F_{19}	F_{20}
0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987	1597	2584	4181	6765

The sequence can also be extended to negative index n using the re-arranged recurrence relation

$$F_{n-2} = F_n - F_{n-1},$$

which yields the sequence of "negafibonacci" numbers[16] satisfying

$$F_{-n} = (-1)^{n+1} F_n.$$

Thus the bidirectional sequence is

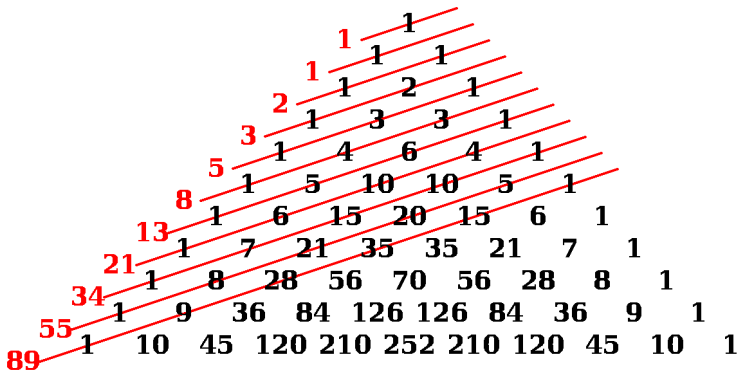
F_{-8}	F_{-7}	F_{-6}	F_{-5}	F_{-4}	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
−21	13	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21

Occurrences in mathematics

The Fibonacci numbers occur in the sums of "shallow" diagonals in Pascal's triangle (see Binomial coefficient).[17]

$$F_n = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-k-1}{k}.$$

The Fibonacci numbers can be found in different ways in the sequence of binary strings.



The Fibonacci numbers are the sums of the "shallow" diagonals (shown in red) of Pascal's triangle.

- The number of binary strings of length n without consecutive 1s is the Fibonacci number F_{n+2}. For example, out of the 16 binary strings of length 4, there are $F_6 = 8$ without consecutive 1s – they are 0000, 0100, 0010, 0001, 0101, 1000, 1010 and 1001. By symmetry, the number of strings of length n without consecutive 0s is also F_{n+2}.
- The number of binary strings of length n without an odd number of consecutive 1s is the Fibonacci number F_{n+1}. For example, out of the 16 binary strings of length 4, there are $F_5 = 5$ without an odd number of consecutive 1s – they are 0000, 0011, 0110, 1100, 1111.
- The number of binary strings of length n without an even number of consecutive 0s or 1s is $2F_n$. For example, out of the 16 binary strings of length 4, there are $2F_4 = 6$ without an even number of consecutive 0s or 1s – they are 0001, 1000, 1110, 0111, 0101, 1010.

Relation to the golden ratio

Closed-form expression

Like every sequence defined by a linear recurrence with constant coefficients, the Fibonacci numbers have a closed-form solution. It has become known as Binet's formula, even though it was already known by Abraham de Moivre:[18]

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

where

$$\varphi = \frac{1+\sqrt{5}}{2} \approx 1.61803\,39887\cdots$$

is the golden ratio (sequence A001622 in OEIS), and

$$\psi = \frac{1-\sqrt{5}}{2} = 1 - \varphi = -\frac{1}{\varphi} \approx -0.61803\,39887\cdots$$ [19]

To see this,[20] note that φ and ψ are both solutions of the equations

$$x^2 = x + 1,\ x^n = x^{n-1} + x^{n-2},$$

so the powers of φ and ψ satisfy the Fibonacci recursion. In other words

$$\varphi^n = \varphi^{n-1} + \varphi^{n-2}$$

and

$$\psi^n = \psi^{n-1} + \psi^{n-2}\,.$$

It follows that for any values *a* and *b*, the sequence defined by

$$U_n = a\varphi^n + b\psi^n$$

satisfies the same recurrence

$$U_n = a\varphi^{n-1} + b\psi^{n-1} + a\varphi^{n-2} + b\psi^{n-2} = U_{n-1} + U_{n-2}.$$

If *a* and *b* are chosen so that $U_0 = 0$ and $U_1 = 1$ then the resulting sequence U_n must be the Fibonacci sequence. This is the same as requiring *a* and *b* satisfy the system of equations:

$$\begin{cases} a + b = 0 \\ \varphi a + \psi b = 1 \end{cases}$$

which has solution

$$a = \frac{1}{\varphi - \psi} = \frac{1}{\sqrt{5}},\ b = -a$$

producing the required formula.

Computation by rounding

Since

$$\frac{|\psi|^n}{\sqrt{5}} < \frac{1}{2}$$

for all $n \geq 0$, the number F_n is the closest integer to

$$\frac{\varphi^n}{\sqrt{5}}\,.$$

Therefore it can be found by rounding, or in terms of the floor function:

$$F_n = \left\lfloor \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2} \right\rfloor, \ n \geq 0.$$

Or the nearest integer function:

$$F_n = \left[\frac{\varphi^n}{\sqrt{5}} \right], \ n \geq 0.$$

Similarly, if we already know that the number $F > 1$ is a Fibonacci number, we can determine its index within the sequence by

$$n(F) = \left\lfloor \log_\varphi \left(F \cdot \sqrt{5} + \frac{1}{2} \right) \right\rfloor$$

Limit of consecutive quotients

Johannes Kepler observed that the ratio of consecutive Fibonacci numbers converges. He wrote that "as 5 is to 8 so is 8 to 13, practically, and as 8 is to 13, so is 13 to 21 almost", and concluded that the limit approaches the golden ratio φ.[21]

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi$$

This convergence does not depend on the starting values chosen, excluding 0, 0. For example, the initial values 19 and 31 generate the sequence 19, 31, 50, 81, 131, 212, 343, 555 ... etc. The ratio of consecutive terms in this sequence shows the same convergence towards the golden ratio.

In fact this holds for any sequence which satisfies the Fibonacci recurrence other than a sequence of 0's. This can be derived from Binet's formula.

Another consequence is that the limit of the ratio of two Fibonacci numbers offset by a particular finite deviation in index corresponds to the golden ratio raised by that deviation. Or, in other words:

$$\lim_{n\to\infty} \frac{F_{n+\alpha}}{F_n} = \varphi^\alpha$$

Decomposition of powers of the golden ratio

Since the golden ratio satisfies the equation

$$\varphi^2 = \varphi + 1,$$

this expression can be used to decompose higher powers φ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of φ and 1. The resulting recurrence relationships yield Fibonacci numbers as the linear coefficients:

$$\varphi^n = F_n\varphi + F_{n-1}.$$

This equation can be proved by induction on n.

This expression is also true for $n < 1$ if the Fibonacci sequence F_n is extended to negative integers using the Fibonacci rule $F_n = F_{n-1} + F_{n-2}$.

Matrix form

A 2-dimensional system of linear difference equations that describes the Fibonacci sequence is

$$\begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$$
$$\vec{F}_{k+1} = A\vec{F}_k$$

The eigenvalues of the matrix A are $\varphi = \frac{1}{2}(1+\sqrt{5})$ and $1-\varphi = \frac{1}{2}(1-\sqrt{5})$, and the elements of the eigenvectors of A, $\binom{\varphi}{1}$ and $\binom{1-\varphi}{1}$, are in the ratios φ and $1-\varphi$. Using these facts, and the properties of eigenvalues, we can derive a direct formula for the nth element in the Fibonacci series as an analytic function of n:

$$F_n = \frac{1}{\sqrt{5}} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \cdot \left(\frac{1-\sqrt{5}}{2} \right)^n$$

The matrix has a determinant of −1, and thus it is a 2×2 unimodular matrix. This property can be understood in terms of the continued fraction representation for the golden ratio:

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\ddots}}}$$

The Fibonacci numbers occur as the ratio of successive convergents of the continued fraction for φ, and the matrix formed from successive convergents of any continued fraction has a determinant of +1 or −1.

The matrix representation gives the following closed expression for the Fibonacci numbers:

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

Taking the determinant of both sides of this equation yields Cassini's identity

$$(-1)^n = F_{n+1}F_{n-1} - F_n^2 .$$

Additionally, since $A^n A^m = A^{m+n}$ for any square matrix *A*, the following identities can be derived:

$$F_m F_n + F_{m-1} F_{n-1} = F_{m+n-1}$$
$$F_{n+1} F_m + F_n F_{m-1} = F_{m+n}$$

In particular, with $m = n$,

$$F_{2n-1} = F_n^2 + F_{n-1}^2$$
$$F_{2n} = (F_{n-1} + F_{n+1}) F_n$$
$$= (2F_{n-1} + F_n) F_n$$

Recognizing Fibonacci numbers

The question may arise whether a positive integer *z* is a Fibonacci number. Since $F(n)$ is the closest integer to $\varphi^n/\sqrt{5}$, the most straightforward, brute-force test is the identity

$$F\left(\left\lfloor \log_\varphi \left(z \cdot \sqrt{5} + \frac{1}{2} \right) \right\rfloor \right) = z,$$

which is true if and only if *z* is a Fibonacci number. In this formula, $F(n)$ can be computed rapidly using any of the previously discussed closed-form expressions.

One implication of the above expression is this: if it is known that a number *z* is a Fibonacci number, we may determine an *n* such that *F(n) = z* by the following:

$$\left\lfloor \log_\varphi \left(z \cdot \sqrt{5} + \frac{1}{2} \right) \right\rfloor = n$$

Alternatively, a positive integer z is a Fibonacci number if and only if one of $5z^2 + 4$ or $5z^2 - 4$ is a perfect square.[22]

A slightly more sophisticated test uses the fact that the convergents of the continued fraction representation of φ are ratios of successive Fibonacci numbers. That is, the inequality

$$\left| \varphi - \frac{p}{q} \right| < \frac{1}{q^2}$$

(with coprime positive integers p, q) is true if and only if p and q are successive Fibonacci numbers. From this one derives the criterion that z is a Fibonacci number if and only if the closed interval

$$\left[\varphi z - \frac{1}{z}, \varphi z + \frac{1}{z} \right]$$

contains a positive integer.[23] For $z \geq 2$, it is easy to show that this interval contains at most one integer, and in the event that z is a Fibonacci number, the contained integer is equal to the next successive Fibonacci number after z. Somewhat remarkably, this result still holds for the case $z = 1$, but it must be stated carefully since 1 appears twice in the Fibonacci sequence, and thus has two distinct successors.

Combinatorial identities

Most identities involving Fibonacci numbers can be proven using combinatorial arguments using the fact that F_n can be interpreted as the number of sequences of 1s and 2s that sum to $n - 1$. This can be taken as the definition of F_n, with the convention that $F_0 = 0$, meaning no sum will add up to −1, and that $F_1 = 1$, meaning the empty sum will "add up" to 0. Here the order of the summand matters. For example, 1 + 2 and 2 + 1 are considered two different sums.

For example, the recurrence relation

$$F_n = F_{n-1} + F_{n-2},$$

or in words, the nth Fibonacci number is the sum of the previous two Fibonacci numbers, may be shown by dividing the $F(n)$ sums of 1s and 2s that add to $n-1$ into two non-overlapping groups. One group contains those sums whose first term is 1 and the other those sums whose first term is 2. In the first group the remaining terms add to $n-2$, so it has $F(n-1)$ sums, and in the second group the remaining terms add to $n-3$, so there are $F(n-2)$ sums. So there are a total of $F(n-1)+F(n-2)$ sums altogether, showing this is equal to $F(n)$.

Similarly, it may be shown that the sum of the first Fibonacci numbers up to the nth is equal to the n+2nd Fibonacci number minus 1.[24] In symbols:

$$\sum_{i=1}^{n} F_i = F_{n+2} - 1$$

This is done by dividing the sums adding to n+1 in a different way, this time by the location of the first 2. Specifically, the first group consists of those sums that start with 2, the second group those that start 1+2, the third 1+1+2, and so on, until the last group which consists of the single sum where only 1's are used. The number of sums in the first group is $F(n)$, $F(n\text{-}1)$ in the second group, and so on, with 1 sum in the last group. So the total number of sums is $F(n) + F(n-1) + \ldots + F(1)+1$ and therefore this quantity is equal to $F(n+2)$

A similar argument, grouping the sums by the position of the first 1 rather than the first 2, gives two more identities:

$$\sum_{i=0}^{n-1} F_{2i+1} = F_{2n}$$

and

$$\sum_{i=1}^{n} F_{2i} = F_{2n+1} - 1.$$

In words, the sum of the first Fibonacci numbers with odd index up to F_{2n-1} is the (2n)th Fibonacci number, and the sum of the first Fibonacci numbers with even index up to F_{2n} is the (2n+1)th Fibonacci number minus 1.[25]

A different trick may be used to prove

$$\sum_{i=1}^{n} {F_i}^2 = F_n F_{n+1},$$

or in words, the sum of the squares of the first Fibonacci numbers up to F_n is the product of the nth and $(n + 1)$th Fibonacci numbers. In this case note that Fibonacci rectangle of size F_n by $F(n + 1)$ can be decomposed into squares of sizea F_n, F_{n-1}, and so on to F_1=1, from which the identity follows by comparing areas.

Other identities

There are numerous other identities which can be derived using various methods. Some of the most noteworthy are:[26]

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{n-r}F_r^2 \text{(Catalan's identity)}$$
$$F_n^2 - F_{n+1}F_{n-1} = (-1)^{n-1} \text{(Cassini's identity)}$$
$$F_m F_{n+1} - F_{m+1}F_n = (-1)^n F_{m-n} \text{ (d'Ocagne's identity)}$$
$$F_{2n} = F_{n+1}^2 - F_{n-1}^2 = F_n(F_{n+1} + F_{n-1}) = F_n L_n$$

where L_n is the n'th Lucas Number. The last is an identity for doubling n; other identities of this type are

$$F_{3n} = 2F_n^3 + 3F_n F_{n+1} F_{n-1} = 5F_n^3 + 3(-1)^n F_n \text{ by Cassini's identity.}$$
$$F_{3n+1} = F_{n+1}^3 + 3F_{n+1}F_n^2 - F_n^3$$
$$F_{3n+2} = F_{n+1}^3 + 3F_{n+1}^2 F_n + F_n^3$$
$$F_{4n} = 4F_n F_{n+1}(F_{n+1}^2 + 2F_n^2) - 3F_n^2(F_n^2 + 2F_{n+1}^2)$$

These can be found experimentally using lattice reduction, and are useful in setting up the special number field sieve to factorize a Fibonacci number.

More generally,[26]

$$F_{kn+c} = \sum_{i=0}^{k} \binom{k}{i} F_{c-i} F_n^i F_{n+1}^{k-i},$$

of which a special case is

$$F_{2n+k} = F_k F_{n+1}^2 + 2F_{k-1}F_{n+1}F_n + F_{k-2}F_n^2.$$

Doubling identities of this type can be used to calculate F_n using O(log n) long multiplication operations of size n bits. The number of bits of precision needed to perform each multiplication doubles at each step, so the performance is limited by the final multiplication; if the fast Schönhage–Strassen multiplication algorithm is used, this is O(n log n log log n) bit operations.

Power series

The generating function of the Fibonacci sequence is the power series

$$s(x) = \sum_{k=0}^{\infty} F_k x^k.$$

This series has a simple and interesting closed-form solution for $|x| < \frac{1}{\varphi}$:[27]

$$s(x) = \frac{x}{1 - x - x^2}.$$

This solution can be proven by using the Fibonacci recurrence to expand each coefficient in the infinite sum defining $s(x)$:

$$\begin{aligned} s(x) &= \sum_{k=0}^{\infty} F_k x^k \\ &= F_0 + F_1 x + \sum_{k=2}^{\infty} \left(F_{k-1} + F_{k-2}\right) x^k \\ &= x + \sum_{k=2}^{\infty} F_{k-1} x^k + \sum_{k=2}^{\infty} F_{k-2} x^k \\ &= x + x \sum_{k=0}^{\infty} F_k x^k + x^2 \sum_{k=0}^{\infty} F_k x^k \\ &= x + x s(x) + x^2 s(x). \end{aligned}$$

Solving the equation $s(x) = x + xs(x) + x^2 s(x)$ for $s(x)$ results in the closed form solution.

In particular, math puzzle-books note the curious value $\frac{s\left(\frac{1}{10}\right)}{10} = \frac{1}{89}$,[28] or more generally

$$\sum_{n=1}^{\infty} \frac{F_n}{10^{(k+1)(n+1)}} = \frac{1}{10^{2k+2} - 10^{k+1} - 1}$$

for all integers $k \geq 0$.

More generally,

$$\sum_{n=0}^{\infty} \frac{F_n}{k^n} = \frac{k}{k^2 - k - 1}.$$

Reciprocal sums

Infinite sums over reciprocal Fibonacci numbers can sometimes be evaluated in terms of theta functions. For example, we can write the sum of every odd-indexed reciprocal Fibonacci number as

$$\sum_{k=0}^{\infty} \frac{1}{F_{2k+1}} = \frac{\sqrt{5}}{4} \vartheta_2^2 \left(0, \frac{3 - \sqrt{5}}{2}\right),$$

and the sum of squared reciprocal Fibonacci numbers as

$$\sum_{k=1}^{\infty} \frac{1}{F_k^2} = \frac{5}{24} \left(\vartheta_2^4 \left(0, \frac{3 - \sqrt{5}}{2}\right) - \vartheta_4^4 \left(0, \frac{3 - \sqrt{5}}{2}\right) + 1\right).$$

If we add 1 to each Fibonacci number in the first sum, there is also the closed form

$$\sum_{k=0}^{\infty} \frac{1}{1 + F_{2k+1}} = \frac{\sqrt{5}}{2},$$

and there is a nice *nested* sum of squared Fibonacci numbers giving the reciprocal of the golden ratio,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sum_{j=1}^{k} {F_j}^2} = \frac{\sqrt{5}-1}{2}.$$

Results such as these make it plausible that a closed formula for the plain sum of reciprocal Fibonacci numbers could be found, but none is yet known. Despite that, the reciprocal Fibonacci constant

$$\psi = \sum_{k=1}^{\infty} \frac{1}{F_k} = 3.359885666243\ldots$$

has been proved irrational by Richard André-Jeannin.

Millin series gives a remarkable identity:[29]

$$\sum_{n=0}^{\infty} \frac{1}{F_{2^n}} = \frac{7-\sqrt{5}}{2}$$

which follows from the closed form for its partial sums as N tends to infinity:

$$\sum_{n=0}^{N} \frac{1}{F_{2^n}} = 3 - \frac{F_{2^N-1}}{F_{2^N}}.$$

Primes and divisibility

Divisibility properties

Every 3rd number of the sequence is even and more generally, every kth number of the sequence is a multiple of F_k. Thus the Fibonacci sequence is an example of a divisibility sequence. In fact, the Fibonacci sequence satisfies the stronger divisibility property

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}\,.$$

Fibonacci primes

A *Fibonacci prime* is a Fibonacci number that is prime. The first few are:

2, 3, 5, 13, 89, 233, 1597, 28657, 514229, … (sequence A005478 in OEIS).

Fibonacci primes with thousands of digits have been found, but it is not known whether there are infinitely many.[30]

F_{kn} is divisible by F_n, so, apart from $F_4 = 3$, any Fibonacci prime must have a prime index. As there are arbitrarily long runs of composite numbers, there are therefore also arbitrarily long runs of composite Fibonacci numbers.

With the exceptions of 1, 8 and 144 ($F_1 = F_2$, F_6 and F_{12}) every Fibonacci number has a prime factor that is not a factor of any smaller Fibonacci number (Carmichael's theorem).[31]

144 is the only nontrivial square Fibonacci number.[32] Attila Pethő proved[33] in 2001 that there are only finitely many perfect power Fibonacci numbers. In 2006, Y. Bugeaud, M. Mignotte, and S. Siksek proved that only 8 and 144 are non-trivial perfect powers.[34]

No Fibonacci number greater than $F_6 = 8$ is one greater or one less than a prime number.[35]

Any three consecutive Fibonacci numbers, taken two at a time, are relatively prime: that is,

$\gcd(F_n, F_{n+1}) = \gcd(F_n, F_{n+2}) = 1$.

More generally,

$\gcd(F_n, F_m) = F_{\gcd(n, m)}$.[36][37]

Prime divisors of Fibonacci numbers

The divisibility of Fibonacci numbers by a prime p is related to the Legendre symbol $\left(\frac{p}{5}\right)$ which is evaluated as follows:

$$\left(\frac{p}{5}\right) = \begin{cases} 0 & \text{if } p = 5 \\ 1 & \text{if } p \equiv \pm 1 \pmod 5 \\ -1 & \text{if } p \equiv \pm 2 \pmod 5. \end{cases}$$

If p is a prime number then $F_p \equiv \left(\frac{p}{5}\right) \pmod p$ and $F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod p$.[38][39]

For example,

$$\begin{aligned} \left(\tfrac{2}{5}\right) &= -1, & F_3 &= 2, & F_2 &= 1, \\ \left(\tfrac{3}{5}\right) &= -1, & F_4 &= 3, & F_3 &= 2, \\ \left(\tfrac{5}{5}\right) &= 0, & F_5 &= 5, & & \\ \left(\tfrac{7}{5}\right) &= -1, & F_8 &= 21, & F_7 &= 13, \\ \left(\tfrac{11}{5}\right) &= +1, & F_{10} &= 55, & F_{11} &= 89. \end{aligned}$$

It is not known whether there exists a prime p such that $F_{p-\left(\frac{p}{5}\right)} \equiv 0 \pmod{p^2}$. Such primes (if there are any) would be called Wall–Sun–Sun primes.

Also, if $p \neq 5$ is an odd prime number then:[40]

$$5F_{(p\pm1)/2}^2 \equiv \begin{cases} \frac{5\left(\frac{p}{5}\right)\pm5}{2} \pmod p & \text{if } p \equiv 1 \pmod 4 \\ \\ \frac{5\left(\frac{p}{5}\right)\mp3}{2} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Examples of all the cases:

$$p = 7 \equiv 3 \pmod 4,\ \left(\tfrac{7}{5}\right) = -1, \frac{5\left(\frac{7}{5}\right)+3}{2} = -1 \text{ and } \frac{5\left(\frac{7}{5}\right)-3}{2} = -4.$$
$$F_3 = 2 \text{ and } F_4 = 3.$$
$$5F_3^2 = 20 \equiv -1 \pmod 7 \text{ and } 5F_4^2 = 45 \equiv -4 \pmod 7$$

$$p = 11 \equiv 3 \pmod 4,\ \left(\tfrac{11}{5}\right) = +1, \frac{5\left(\frac{11}{5}\right)+3}{2} = 4 \text{ and } \frac{5\left(\frac{11}{5}\right)-3}{2} = 1.$$
$$F_5 = 5 \text{ and } F_6 = 8.$$
$$5F_5^2 = 125 \equiv 4 \pmod{11} \text{ and } 5F_6^2 = 320 \equiv 1 \pmod{11}$$

$$p = 13 \equiv 1 \pmod 4,\ \left(\tfrac{13}{5}\right) = -1, \frac{5\left(\frac{13}{5}\right)-5}{2} = -5 \text{ and } \frac{5\left(\frac{13}{5}\right)+5}{2} = 0.$$
$$F_6 = 8 \text{ and } F_7 = 13.$$
$$5F_6^2 = 320 \equiv -5 \pmod{13} \text{ and } 5F_7^2 = 845 \equiv 0 \pmod{13}$$

$$p = 29 \equiv 1 \pmod 4,\ \left(\tfrac{29}{5}\right) = +1, \frac{5\left(\frac{29}{5}\right)-5}{2} = 0 \text{ and } \frac{5\left(\frac{29}{5}\right)+5}{2} = 5.$$
$$F_{14} = 377 \text{ and } F_{15} = 610.$$
$$5F_{14}^2 = 710645 \equiv 0 \pmod{29} \text{ and } 5F_{15}^2 = 1860500 \equiv 5 \pmod{29}$$

For odd n, all odd prime divisors of F_n are ≡ 1 (mod 4), implying that all odd divisors of F_n (as the products of odd prime divisors) are ≡ 1 (mod 4).[41]

For example, $F_1 = 1$, $F_3 = 2$, $F_5 = 5$, $F_7 = 13$, $F_9 = 34 = 2\times17$, $F_{11} = 89$, $F_{13} = 233$, $F_{15} = 610 = 2\times5\times61$

All known factors of Fibonacci numbers F(i) for all i < 50000 are collected at the relevant repositories.[42][43]

Periodicity modulo *n*

It may be seen that if the members of the Fibonacci sequence are taken mod *n*, the resulting sequence must be periodic with period at most n^2-1. The lengths of the periods for various *n* form the so-called Pisano periods (sequence A001175 in OEIS). Determining the Pisano periods in general is an open problem, although for any particular *n* it can be solved as an instance of cycle detection.

Right triangles

Starting with 5, every second Fibonacci number is the length of the hypotenuse of a right triangle with integer sides, or in other words, the largest number in a Pythagorean triple. The length of the longer leg of this triangle is equal to the sum of the three sides of the preceding triangle in this series of triangles, and the shorter leg is equal to the difference between the preceding bypassed Fibonacci number and the shorter leg of the preceding triangle.

The first triangle in this series has sides of length 5, 4, and 3. Skipping 8, the next triangle has sides of length 13, 12 (5 + 4 + 3), and 5 (8 – 3). Skipping 21, the next triangle has sides of length 34, 30 (13 + 12 + 5), and 16 (21 – 5). This series continues indefinitely. The triangle sides *a*, *b*, *c* can be calculated directly:

$$a_n = F_{2n-1}$$
$$b_n = 2F_nF_{n-1}$$
$$c_n = F_n^2 - F_{n-1}^2.$$

These formulas satisfy $a_n^2 = b_n^2 + c_n^2$ for all *n*, but they only represent triangle sides when $n > 2$.

Any four consecutive Fibonacci numbers F_n, F_{n+1}, F_{n+2} and F_{n+3} can also be used to generate a Pythagorean triple in a different way[44]:

$$a = F_nF_{n+3}\,;\; b = 2F_{n+1}F_{n+2}\,;\; c = F_{n+1}^2 + F_{n+2}^2\,;\; a^2 + b^2 = c^2\,.$$

Example 1: let the Fibonacci numbers be 1, 2, 3 and 5. Then:

$$a = 1 \times 5 = 5$$
$$b = 2 \times 2 \times 3 = 12$$
$$c = 2^2 + 3^2 = 13$$
$$5^2 + 12^2 = 13^2\,.$$

Magnitude

Since F_n is asymptotic to $\varphi^n/\sqrt{5}$, the number of digits in F_n is asymptotic to $n \log_{10} \varphi \approx 0.2090\, n$. As a consequence, for every integer $d > 1$ there are either 4 or 5 Fibonacci numbers with *d* decimal digits.

More generally, in the base *b* representation, the number of digits in F_n is asymptotic to $n \log_b \varphi$.

Applications

The Fibonacci numbers are important in the computational run-time analysis of Euclid's algorithm to determine the greatest common divisor of two integers: the worst case input for this algorithm is a pair of consecutive Fibonacci numbers.[45]

Yuri Matiyasevich was able to show that the Fibonacci numbers can be defined by a Diophantine equation, which led to his original solution of Hilbert's tenth problem.

The Fibonacci numbers are also an example of a complete sequence. This means that every positive integer can be written as a sum of Fibonacci numbers, where any one number is used once at most. Specifically, every positive

integer can be written in a unique way as the sum of *one or more* distinct Fibonacci numbers in such a way that the sum does not include any two consecutive Fibonacci numbers. This is known as Zeckendorf's theorem, and a sum of Fibonacci numbers that satisfies these conditions is called a Zeckendorf representation. The Zeckendorf representation of a number can be used to derive its Fibonacci coding.

Fibonacci numbers are used by some pseudorandom number generators.

Fibonacci numbers are used in a polyphase version of the merge sort algorithm in which an unsorted list is divided into two lists whose lengths correspond to sequential Fibonacci numbers – by dividing the list so that the two parts have lengths in the approximate proportion φ. A tape-drive implementation of the polyphase merge sort was described in *The Art of Computer Programming*.

Fibonacci numbers arise in the analysis of the Fibonacci heap data structure.

The Fibonacci cube is an undirected graph with a Fibonacci number of nodes that has been proposed as a network topology for parallel computing.

A one-dimensional optimization method, called the Fibonacci search technique, uses Fibonacci numbers.[46]

The Fibonacci number series is used for optional lossy compression in the IFF 8SVX audio file format used on Amiga computers. The number series compands the original audio wave similar to logarithmic methods such as μ-law.[47][48]

In music, Fibonacci numbers are sometimes used to determine tunings, and, as in visual art, to determine the length or size of content or formal elements. It is commonly thought that the third movement of Béla Bartók's *Music for Strings, Percussion, and Celesta* was structured using Fibonacci numbers.

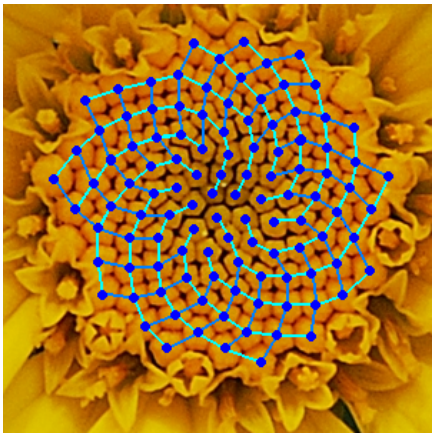
Since the conversion factor 1.609344 for miles to kilometers is close to the golden ratio (denoted φ), the decomposition of distance in miles into a sum of Fibonacci numbers becomes nearly the kilometer sum when the Fibonacci numbers are replaced by their successors. This method amounts to a radix 2 number register in golden ratio base φ being shifted. To convert from kilometers to miles, shift the register down the Fibonacci sequence instead.[49]

In nature

Further information: Patterns in nature

Fibonacci sequences appear in biological settings,[7] in two consecutive Fibonacci numbers, such as branching in trees, arrangement of leaves on a stem, the fruitlets of a pineapple,[8] the flowering of artichoke, an uncurling fern and the arrangement of a pine cone.[9] In addition, numerous poorly substantiated claims of Fibonacci numbers or golden sections in nature are found in popular sources, e.g., relating to the breeding of rabbits, the seeds on a sunflower, the spirals of shells, and the curve of waves.[50] The Fibonacci numbers are also found in the family tree of honeybees.[51]

Przemysław Prusinkiewicz advanced the idea that real instances can in part be understood as the expression of certain algebraic constraints on free groups, specifically as certain Lindenmayer grammars.[52]



Yellow Chamomile head showing the arrangement in 21 (blue) and 13 (aqua) spirals. Such arrangements involving consecutive Fibonacci numbers appear in a wide variety of plants.

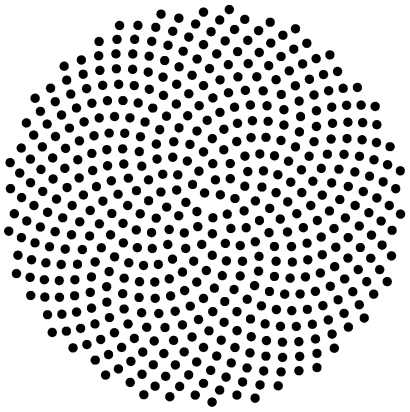


Illustration of Vogel's model for n=1 ... 500

A model for the pattern of florets in the head of a sunflower was proposed by H. Vogel in 1979.[53] This has the form

$$\theta = \frac{2\pi}{\phi^2} n,\ r = c\sqrt{n}$$

where n is the index number of the floret and c is a constant scaling factor; the florets thus lie on Fermat's spiral. The divergence angle, approximately 137.51°, is the golden angle, dividing the circle in the golden ratio. Because this ratio is irrational, no floret has a neighbor at exactly the same angle from the center, so the florets pack efficiently. Because the rational approximations to the golden ratio are of the form $F(j):F(j+1)$, the nearest neighbors of floret number n are those at $n \pm F(j)$ for some index j which depends on r, the distance from the center. It is often said that sunflowers and similar arrangements have 55 spirals in one direction and 89 in the other (or some other pair of adjacent Fibonacci numbers), but this is true only of one range of radii, typically the outermost and thus most conspicuous.[54]

The bee ancestry code

Fibonacci numbers also appear in the description of the reproduction of a population of idealized honeybees, according to the following rules:

- If an egg is laid by an unmated female, it hatches a male or drone bee.
- If, however, an egg was fertilized by a male, it hatches a female.

Thus, a male bee will always have one parent, and a female bee will have two.

If one traces the ancestry of any male bee (1 bee), he has 1 parent (1 bee), 2 grandparents, 3 great-grandparents, 5 great-great-grandparents, and so on. This sequence of numbers of parents is the Fibonacci sequence. The number of ancestors at each level, F_n, is the number of female ancestors, which is F_{n-1}, plus the number of male ancestors, which is F_{n-2}.[55] (This is under the unrealistic assumption that the ancestors at each level are otherwise unrelated.)

Generalizations

The Fibonacci sequence has been generalized in many ways. These include:

- Generalizing the index to negative integers to produce the Negafibonacci numbers.
- Generalizing the index to real numbers using a modification of Binet's formula.[26]
- Starting with other integers. Lucas numbers have $L_1 = 1$, $L_2 = 3$, and $L_n = L_{n-1} + L_{n-2}$. Primefree sequences use the Fibonacci recursion with other starting points in order to generate sequences in which all numbers are composite.
- Letting a number be a linear function (other than the sum) of the 2 preceding numbers. The Pell numbers have $P_n = 2P_{n-1} + P_{n-2}$.
- Not adding the immediately preceding numbers. The Padovan sequence and Perrin numbers have $P(n) = P(n-2) + P(n-3)$.
- Generating the next number by adding 3 numbers (tribonacci numbers), 4 numbers (tetranacci numbers), or more. The resulting sequences are known as *n-Step Fibonacci numbers*.[56]
- Adding other objects than integers, for example functions or strings—one essential example is Fibonacci polynomials.

Notes

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[15] The website (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fibtable.html) has the first 300 F_n factored into primes and links to more extensive tables.

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- " Sloane's A000045 : Fibonacci Numbers (http://oeis.org/A000045)", *The On-Line Encyclopedia of Integer Sequences*. OEIS Foundation.
- Periods of Fibonacci Sequences Mod m (http://www.mathpages.com/home/kmath078/kmath078.htm) at MathPages
- Scientists find clues to the formation of Fibonacci spirals in nature (http://www.physorg.com/news97227410.html)
- Implementation to calculate Fibonacci sequence in Lisp (http://wikinternet.com/wordpress/code/lisp/fibonacci-number/)

Fibonacci retracement

In finance, **Fibonacci retracements** is a method of technical analysis for determining support and resistance levels. They are named after their use of the Fibonacci sequence. Fibonacci retracement is based on the idea that markets will retrace a predictable portion of a move, after which they will continue to move in the original direction.

Fibonacci retracement levels shown on the USD/CAD currency pair. In this case, price retraced approximately 38.2% of a move down before continuing.

The appearance of retracement can be ascribed to ordinary price volatility as described by Burton Malkiel, a Princeton economist in his book *A Random Walk Down Wall Street*, who found no reliable predictions in technical analysis methods taken as a whole. Malkiel argues that asset prices typically exhibit signs of random walk and that one cannot consistently outperform market averages. Fibonacci retracement is created by taking two extreme points on a chart and dividing the vertical distance by the key Fibonacci ratios. 0.0% is considered to be the start of the retracement, while 100.0% is a complete reversal to the original part of the move. Once these levels are identified, horizontal yjyjyjyjyyjyjyjyjjused to identify possible support and resistance levels.

Fibonacci ratios

Fibonacci ratios are mathematical relationships, expressed as ratios, derived from the Fibonacci sequence. The key Fibonacci ratios are 0%, 23.6%, 38.2%, and 100%.

$$F_{100\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{0} = 1$$

The key Fibonacci ratio of 0.618 is derived by dividing any number in the sequence by the number that immediately follows it. *For example: 8/13 is approximately 0.6154, and 55/89 is approximately 0.6180.*

$$F_{61.8\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{-1} \approx 0.618034$$

The 0.382 ratio is found by dividing any number in the sequence by the number that is found two places to the right. *For example: 34/89 is approximately 0.3820.*

$$F_{38.2\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{-2} \approx 0.381966$$

The 0.236 ratio is found by dividing any number in the sequence by the number that is three places to the right. *For example: 55/233 is approximately 0.2361.*

$$F_{23.6\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{-3} \approx 0.236068$$

The 0 ratio is :

$$F_{0\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{-\infty} = 0$$

Other ratios

The 0.764 ratio is the result of subtracting 0.236 from the number 1.

$$F_{76.4\%} = 1 - \left(\frac{1+\sqrt{5}}{2}\right)^{-3} \approx 0.763932$$

The 0.786 ratio is :

$$F_{78.6\%} = \left(\frac{1+\sqrt{5}}{2}\right)^{-\frac{1}{2}} \approx 0.786151$$

The 0.500 ratio is derived from dividing the number 1 (second number in the sequence) by the number 2 (third number in the sequence).

$$F_{50\%} = \frac{1}{2} = 0.500000$$

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Malkiel, Burton (2011). *A random walk down Wall Street: the time-tested strategy for successful investing*. OCLC 50919959.

External links

- *What is Fibonacci retracement, and where do the ratios that are used come from?* [1] at investopedia.com
- Fibonacci Retracements [2] at stockcharts.com
- *How to draw Fibonacci retracement, and how to analyze it?* [3] at lollymotion.com

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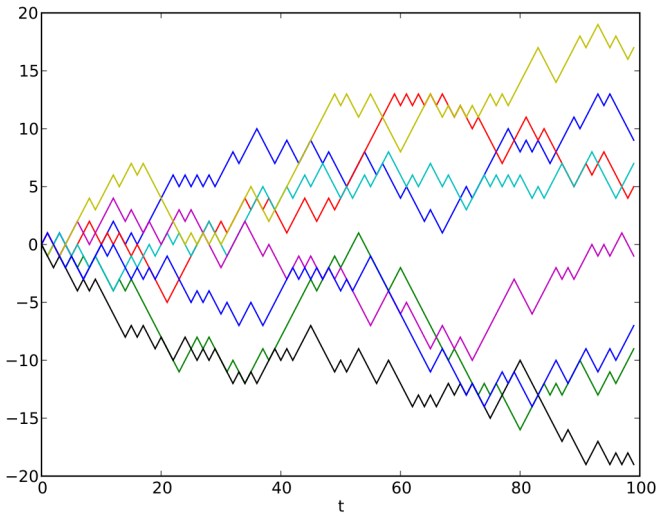
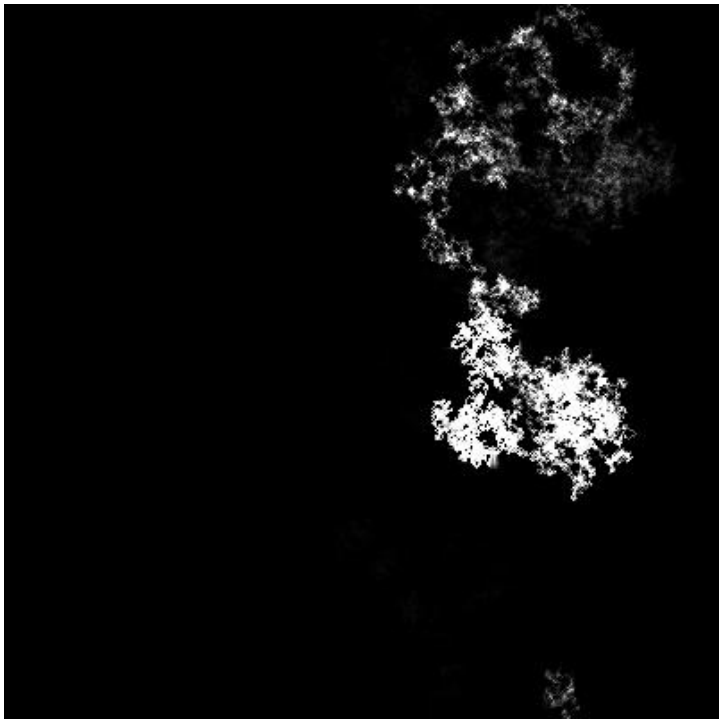
Random walk

A **random walk** is a mathematical formalization of a path that consists of a succession of random steps. For example, the path traced by a molecule as it travels in a liquid or a gas, the search path of a foraging animal, the price of a fluctuating stock and the financial status of a gambler can all be *modeled* as random walks, although they may not be truly random in reality. The term *random walk* was first introduced by Karl Pearson in 1905.[1] Random walks have been used in many fields: ecology, economics, psychology, computer science, physics, chemistry, and biology.[2][3][4][5][6][7][8][9] Random walks explain the observed behaviors of processes in these fields, and thus serve as a fundamental model for the recorded stochastic activity.



Example of eight random walks in one dimension starting at 0. The plot shows the current position on the line (vertical axis) versus the time steps (horizontal axis).



An animated example of a Brownian motion-like random walk with toroidal boundary conditions.

Various different types of random walks are of interest. Often, random walks are assumed to be Markov chains or Markov processes, but other, more complicated walks are also of interest. Some random walks are on graphs, others on the line, in the plane, or in higher dimensions, while some random walks are on groups. Random walks also vary with regard to the time parameter. Often, the walk is in discrete time, and indexed by the natural numbers, as in $X_0, X_1, X_2, \ldots$. However, some walks take their steps at random times, and in that case the position X_t is defined for the continuum of times $t \geq 0$. Specific cases or limits of random walks include the Lévy flight. Random walks are related to the diffusion models and are a fundamental topic in discussions of Markov processes. Several properties of random walks, including dispersal distributions, first-passage times and encounter rates, have been extensively studied.

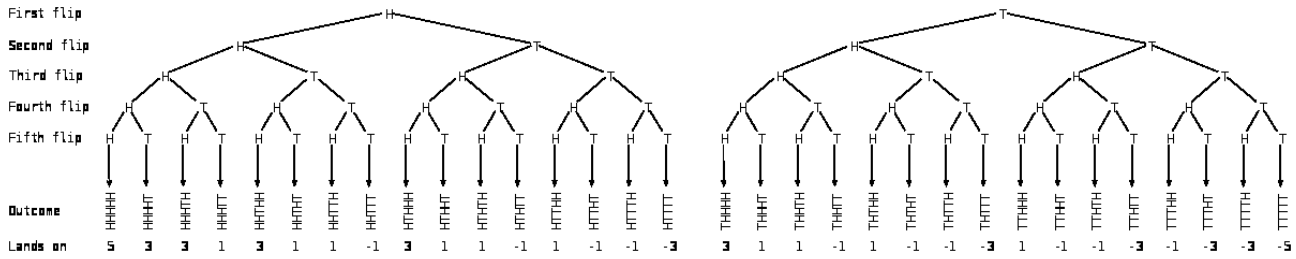
Lattice random walk

A popular random walk model is that of a random walk on a regular lattice, where at each step the location jumps to another site according to some probability distribution. In a **simple random walk**, the location can only jump to neighboring sites of the lattice. In **simple symmetric random walk** on a locally finite lattice, the probabilities of the location jumping to each one of its immediate neighbours are the same. The best studied example is of random walk on the *d*-dimensional integer lattice (sometimes called the hypercubic lattice) $\mathbb{Z}^d$.

One-dimensional random walk

An elementary example of a random walk is the random walk on the integer number line, $\mathbb{Z}$, which starts at 0 and at each step moves +1 or −1 with equal probability.

This walk can be illustrated as follows. A marker is placed at zero on the number line and a fair coin is flipped. If it lands on heads, the marker is moved one unit to the right. If it lands on tails, the marker is moved one unit to the left. After five flips, it marker could now be on 1, −1, 3, −3, 5, or −5. With five flips, three heads and two tails, in any order, will land on 1. There are 10 ways of landing on 1 (by flipping three heads and two tails), 10 ways of landing on −1 (by flipping three tails and two heads), 5 ways of landing on 3 (by flipping four heads and one tail), 5 ways of landing on −3 (by flipping four tails and one head), 1 way of landing on 5 (by flipping five heads), and 1 way of landing on −5 (by flipping five tails). See the figure below for an illustration of the possible outcomes of 5 flips.



All possible random walk outcomes after 5 flips of a fair coin

To define this walk formally, take independent random variables $Z_1, Z_2, \ldots$, where each variable is either 1 or −1, with a 50% probability for either value, and set $S_0 = 0$ and $S_n = \sum_{j=1}^{n} Z_j$. The series $\{S_n\}$ is called the **simple random walk on** $\mathbb{Z}$. This series (the sum of the sequence of −1s and 1s) gives the distance walked, if each part of the walk is of length one. The expectation $E(S_n)$ of S_n is zero. That is, the mean of all coin flips approaches zero as the number of flips increase. This follows by the finite additivity property of expectation:

$$E(S_n) = \sum_{j=1}^{n} E(Z_j) = 0.$$

A similar calculation, using the independence of the random variables and the fact that $E(Z_n^2) = 1$, shows that:

$$E(S_n^2) = \sum_{j=1}^{n} E(Z_j^2) = n.$$

This hints that $E(|S_n|)$, the expected translation distance after *n* steps, should be of the order of $\sqrt{n}$. In fact,

$$\lim_{n\to\infty} \frac{E(|S_n|)}{\sqrt{n}} = \sqrt{\frac{2}{\pi}}.$$

Derivation of $\sqrt{n}$ dispersion proportionality

If we have the situation where the probabilities of moving either left or right are equal, $p = 1/2$ and $q = 1/2$, the probability of taking k steps to the right out of a total of N steps is given by

$$P(k;N) = \binom{N}{k}\left(\frac{1}{2}\right)^N$$

since there are $\binom{N}{k}$ possible ways of taking k and $N-k$ steps to the right and left, respectively. The probability of taking any of these independent steps is 1/2, and so we have the product $1/2^N$. Now, the expectation value of taking k steps is

$$\begin{aligned}\langle k\rangle &= \sum_{k=0}^{N} kP(k;N)\\ &= \sum_{k=0}^{N} k\binom{N}{k}\left(\frac{1}{2}\right)^N\\ &= \left(\frac{1}{2}\right)^N \sum_{k=0}^{N} N\binom{N-1}{k-1}\\ &= \left(\frac{1}{2}\right)^N N2^{N-1}\\ &= \frac{N}{2}.\end{aligned}$$

It is generally the case that $\langle k\rangle = Np$. Note the identity we have used with the binomial coefficient $k\binom{N}{k}$. We use it again below. We must then calculate the expectation value of k^2:

$$\begin{aligned}\langle k^2\rangle &= \sum_{k=0}^{N} k^2P(k;N)\\ &= \left(\frac{1}{2}\right)^N \sum_{k=0}^{N}(k-1+1)N\binom{N-1}{k-1}\\ &= \left(\frac{1}{2}\right)^N N\sum_{k=0}^{N}\left[(N-1)\binom{N-2}{k-2}+\binom{N-1}{k-1}\right]\\ &= \left(\frac{1}{2}\right)^N N\left[(N-1)2^{N-2}+2^{N-1}\right]\\ &= \frac{N^2}{4}+\frac{N}{4}.\end{aligned}$$

It is generally the case that $\langle k^2\rangle = (Np)^2 + Npq$. The dispersion is

$$\begin{aligned}\langle \Delta k\rangle &= \sqrt{\langle k^2\rangle - \langle k\rangle^2}\\ &\propto \sqrt{N}.\end{aligned}$$

This result shows that diffusion is ineffective for mixing because of the way the square root behaves for large N.

How many times will a random walk cross a boundary line if permitted to continue walking forever? A simple random walk on $\mathbb{Z}$ will cross every point an infinite number of times. This result has many names: the *level-crossing phenomenon*, *recurrence* or the *gambler's ruin*. The reason for the last name is as follows: a gambler with a finite amount of money will always lose when playing *a fair game* against a bank with an infinite amount of money. The gambler's money will perform a random walk, and it will reach zero at some point, and the game will be over.

If *a* and *b* are positive integers, then the expected number of steps until a one-dimensional simple random walk starting at 0 first hits *b* or –*a* is *ab*. The probability that this walk will hit *b* before –*a* is $a/(a+b)$, which can be derived from the fact that simple random walk is a martingale.

Some of the results mentioned above can be derived from properties of Pascal's triangle. The number of different walks of *n* steps where each step is +1 or −1 is clearly 2^n. For the simple random walk, each of these walks are

equally likely. In order for S_n to be equal to a number k it is necessary and sufficient that the number of +1 in the walk exceeds those of −1 by k. Thus, the number of walks which satisfy $S_n = k$ is precisely the number of ways of choosing $(n + k)/2$ elements from an n element set (for this to be non-zero, it is necessary that $n + k$ be an even number), which is an entry in Pascal's triangle denoted by $\binom{n}{(n+k)/2}$. Therefore, the probability that $S_n = k$ is equal to $2^{-n}\binom{n}{(n+k)/2}$. Pascal's triangle in terms of factorials and using Stirling's formula, one can obtain good estimates for these probabilities for large values of n.

This relation with Pascal's triangle is demonstrated for small values of n. At zero turns, the only possibility will be to remain at zero. However, at one turn, there is one chance of landing on −1 or one chance of landing on 1. At two turns, a marker at 1 could move to 2 or back to zero. A marker at −1, could move to −2 or back to zero. Therefore, there is one chance of landing on −2, two chances of landing on zero, and one chance of landing on 2.

n	−5	−4	−3	−2	−1	0	1	2	3	4	5
$P[S_0 = k]$						1					
$2P[S_1 = k]$					1		1				
$2^2P[S_2 = k]$				1		2		1			
$2^3P[S_3 = k]$			1		3		3		1		
$2^4P[S_4 = k]$		1		4		6		4		1	
$2^5P[S_5 = k]$	1		5		10		10		5		1

The central limit theorem and the law of the iterated logarithm describe important aspects of the behavior of simple random walk on $\mathbb{Z}$. In particular, the former entails that as n increases, the probabilities (proportional to the numbers in each row) approach a normal distribution.

As a Markov chain

A one-dimensional **random walk** can also be looked at as a Markov chain whose state space is given by the integers $i = 0, \pm 1, \pm 2, \ldots$. For some number p satisfying $0 < p < 1$, the transition probabilities (the probability $P_{i,j}$ of moving from state i to state j) are given by

$$P_{i,i+1} = p = 1 - P_{i,i-1}.$$

Gaussian random walk

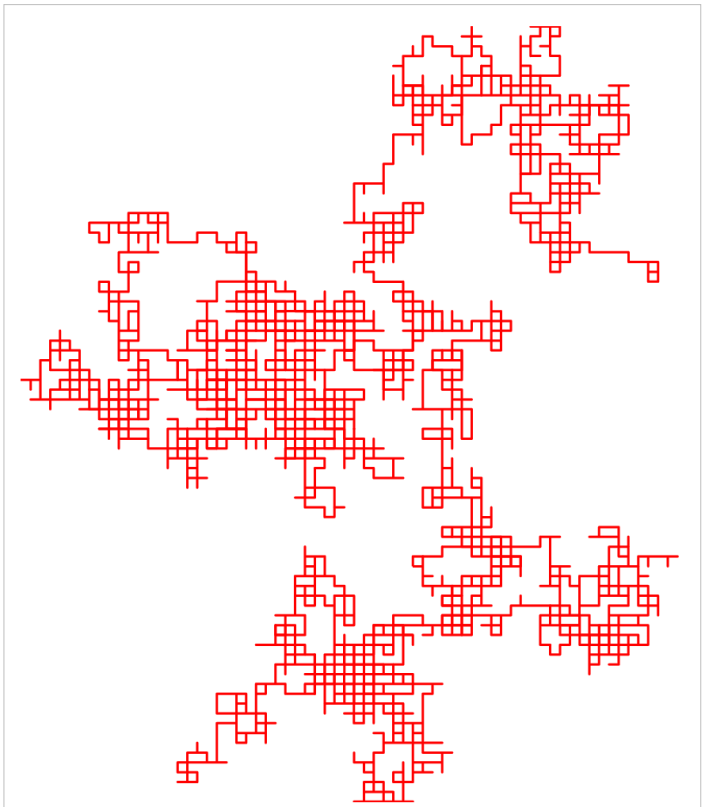
A random walk having a step size that varies according to a normal distribution is used as a model for real-world time series data such as financial markets. The Black–Scholes formula for modeling option prices, for example, uses a gaussian random walk as an underlying assumption.

Here, the step size is the inverse cumulative normal distribution $\Phi^{-1}(z, \mu, \sigma)$ where $0 \le z \le 1$ is a uniformly distributed random number, and μ and σ are the mean and standard deviations of the normal distribution, respectively.

For steps distributed according to any distribution with zero mean and a finite variance (not necessarily just a normal distribution), the root mean square translation distance after n steps is

$$\sqrt{E|S_n^2|} = \sigma\sqrt{n}.$$

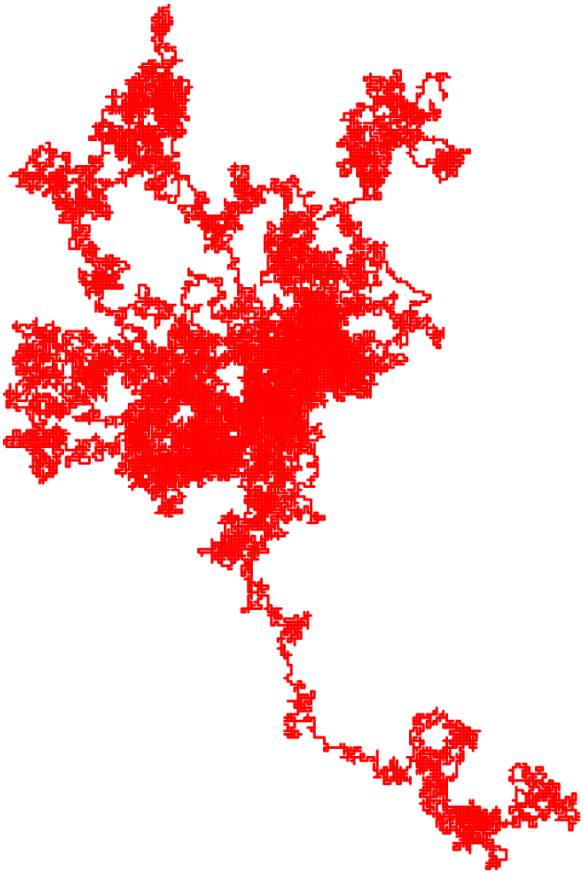
Higher dimensions



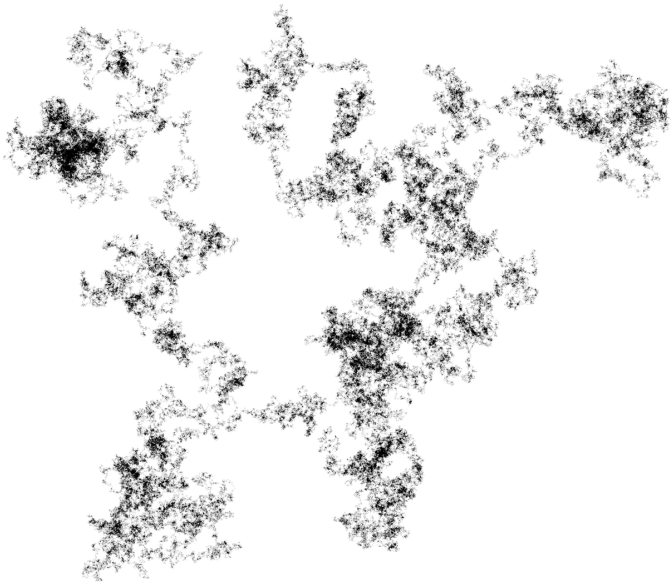
Random walk in two dimensions

Imagine now a drunkard walking randomly in an idealized city. The city is effectively infinite and arranged in a square grid, and at every intersection, the drunkard chooses one of the four possible routes (including the one he came from) with equal probability. Formally, this is a random walk on the set of all points in the plane with integer coordinates. Will the drunkard ever get back to his home from the bar? It turns out that he will. This is the high dimensional equivalent of the level crossing problem discussed above. The probability of returning to the origin decreases as the number of dimensions increases. In three dimensions, the probability decreases to roughly 34%. A derivation, along with values of $p(d)$ are discussed here: Pólya's Random Walk Constants [10].

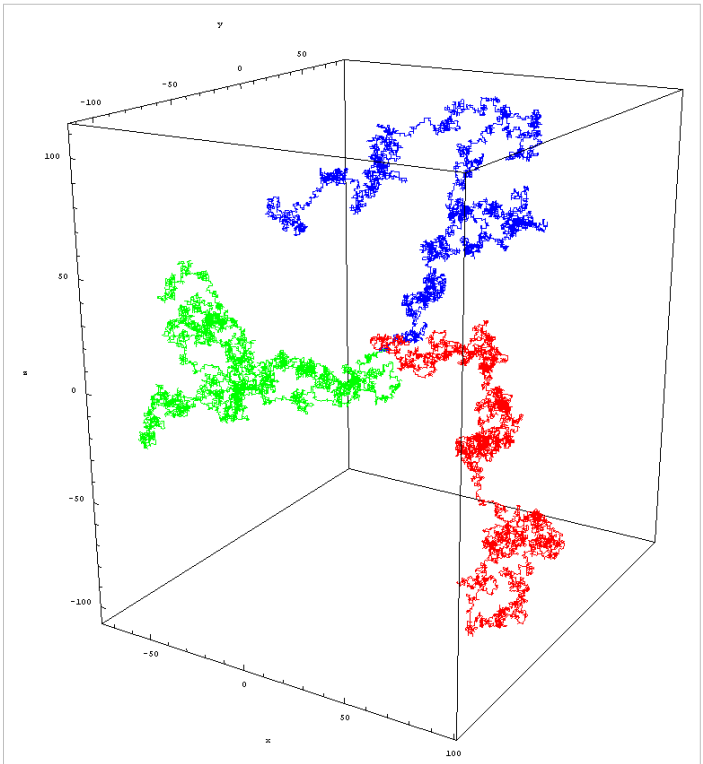
The trajectory of a random walk is the collection of sites it visited, considered as a set with disregard to *when* the walk arrived at the point. In one dimension, the trajectory is simply all points between the minimum height the walk achieved and the maximum (both are, on average, on the order of $\sqrt{n}$). In higher dimensions the set has interesting geometric properties. In fact, one gets a discrete fractal, that is a set which exhibits stochastic self-similarity on large scales, but on small scales one can observe "jaggedness" resulting from the grid on which the walk is performed. The two books of Lawler referenced below are a good source on this topic.



Random walk in two dimensions with more, and smaller, steps

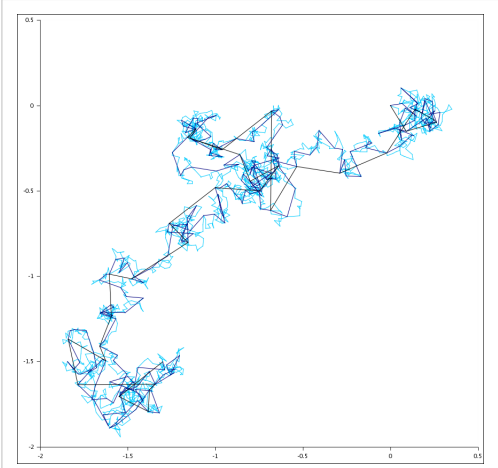


Random walk in two dimensions with two million even smaller steps. This image was generated in such a way that points that are more frequently traversed are darker. In the limit, for very small steps, one obtains Brownian motion.



Three random walks in three dimensions

Relation to Wiener process


Simulated steps approximating a Wiener process in two dimensions

A Wiener process is a stochastic process with similar behaviour to Brownian motion, the physical phenomenon of a minute particle diffusing in a fluid. (Sometimes the Wiener process is called "Brownian motion", although this is strictly speaking a confusion of a model with the phenomenon being modeled.)

A Wiener process is the scaling limit of random walk in dimension 1. This means that if you take a random walk with very small steps you get an approximation to a Wiener process (and, less accurately, to Brownian motion). To be more precise, if the step size is ε, one needs to take a walk of length L/ε^2 to approximate a Wiener process walk of length L. As the step size tends to 0 (and the number of steps increases proportionally) random walk converges to a Wiener process in an appropriate sense. Formally, if B is the space of all paths of length L with the maximum topology, and if M is the space of measure over B with the norm topology, then the convergence is in the space M. Similarly, a Wiener process in several dimensions is the scaling limit of random walk in the same number of dimensions.

A random walk is a discrete fractal (a function with integer dimensions; 1, 2, ...), but a Wiener process trajectory is a true fractal, and there is a connection between the two. For example, take a random walk until it hits a circle of radius r times the step length. The average number of steps it performs is r^2. This fact is the *discrete version* of the fact that a Wiener process walk is a fractal of Hausdorff dimension 2.

In two dimensions, the average number of points the same random walk has on the *boundary* of its trajectory is $r^{4/3}$. This corresponds to the fact that the boundary of the trajectory of a Wiener process is a fractal of dimension 4/3, a

fact predicted by Mandelbrot using simulations but proved only in 2000 by Lawler, Schramm and Werner.[11]

A Wiener process enjoys many symmetries random walk does not. For example, a Wiener process walk is invariant to rotations, but random walk is not, since the underlying grid is not (random walk is invariant to rotations by 90 degrees, but Wiener processes are invariant to rotations by, for example, 17 degrees too). This means that in many cases, problems on random walk are easier to solve by translating them to a Wiener process, solving the problem there, and then translating back. On the other hand, some problems are easier to solve with random walks due to its discrete nature.

Random walk and Wiener process can be *coupled*, namely manifested on the same probability space in a dependent way that forces them to be quite close. The simplest such coupling is the Skorokhod embedding, but other, more precise couplings exist as well.

The convergence of a random walk toward the Wiener process is controlled by the central limit theorem. For a particle in a known fixed position at $t = 0$, the theorem tells us that after a large number of independent steps in the random walk, the walker's position is distributed according to a normal distribution of total variance:

$$\sigma^2 = \frac{t}{\delta t}\varepsilon^2,$$

where t is the time elapsed since the start of the random walk, ε is the size of a step of the random walk, and δt is the time elapsed between two successive steps.

This corresponds to the Green function of the diffusion equation that controls the Wiener process, which demonstrates that, after a large number of steps, the random walk converges toward a Wiener process.

In 3D, the variance corresponding to the Green's function of the diffusion equation is:

$$\sigma^2 = 6\,D\,t$$

By equalizing this quantity with the variance associated to the position of the random walker, one obtains the equivalent diffusion coefficient to be considered for the asymptotic Wiener process toward which the random walk converges after a large number of steps:

$$D = \frac{\varepsilon^2}{6\delta t}$$ (valid only in 3D)

Remark: the two expressions of the variance above correspond to the distribution associated to the vector $\vec{R}$ that links the two ends of the random walk, in 3D. The variance associated to each component R_x, R_y or R_z is only one third of this value (still in 3D).

Anomalous diffusion

In disordered systems such as porous media and fractals σ^2 may not be proportional to t but to t^{2/d_w}. The exponent d_w is called the anomalous diffusion exponent and can be larger or smaller than 2.[12]

Applications

The following are some applications of random walk:

Antony Gormley's *Quantum Cloud* sculpture in London was designed by a computer using a random walk algorithm.

- In economics, the "random walk hypothesis" is used to model shares prices and other factors. Empirical studies found some deviations from this theoretical model, especially in short term and long term correlations. See share prices.
- In population genetics, random walk describes the statistical properties of genetic drift
- In physics, random walks are used as simplified models of physical Brownian motion and diffusion such as the random movement of molecules in liquids and gases. See for example diffusion-limited aggregation. Also in physics, random walks and some of the self interacting walks play a role in quantum field theory.
- In mathematical ecology, random walks are used to describe individual animal movements, to empirically support processes of biodiffusion, and occasionally to model population dynamics.
- In polymer physics, random walk describes an ideal chain. It is the simplest model to study polymers.
- In other fields of mathematics, random walk is used to calculate solutions to Laplace's equation, to estimate the harmonic measure, and for various constructions in analysis and combinatorics.
- In computer science, random walks are used to estimate the size of the Web. In the World Wide Web conference-2006 [13], bar-yossef et al. published their findings and algorithms for the same.
- In image segmentation, random walks are used to determine the labels (i.e., "object" or "background") to associate with each pixel.[14] This algorithm is typically referred to as the random walker segmentation algorithm.

In all these cases, random walk is often substituted for Brownian motion.

- In brain research, random walks and reinforced random walks are used to model cascades of neuron firing in the brain.
- In vision science, fixational eye movements are well described by a random walk.[15]
- In psychology, random walks explain accurately the relation between the time needed to make a decision and the probability that a certain decision will be made.[16]
- Random walks can be used to sample from a state space which is unknown or very large, for example to pick a random page off the internet or, for research of working conditions, a random worker in a given country.
  - When this last approach is used in computer science it is known as Markov Chain Monte Carlo or MCMC for short. Often, sampling from some complicated state space also allows one to get a probabilistic estimate of the space's size. The estimate of the permanent of a large matrix of zeros and ones was the first major problem tackled using this approach.
- Random walks have also been used to sample massive online graphs such as online social networks.
- In wireless networking, a random walk is used to model node movement.
- Motile bacteria engage in a biased random walk.
- Random walks are used to model gambling.

- In physics, random walks underlie the method of Fermi estimation.

Variants of random walks

A number of types of stochastic processes have been considered that are similar to the pure random walks but where the simple structure is allowed to be more generalized. The *pure* structure can be characterized by the steps being being defined by independent and identically distributed random variables.

Random walk on graphs

A random walk of length *k* on a possibly infinite graph *G* with a root *0* is a stochastic process with random variables $X_1, X_2, \ldots, X_k$ such that $X_1 = 0$ and X_{i+1} is a vertex chosen uniformly at random from the neighbors of X_i. Then the number $p_{v,w,k}(G)$ is the probability that a random walk of length *k* starting at *v* ends at *w*. In particular, if *G* is a graph with root *0*, $p_{0,0,2k}$ is the probability that a $2k$ -step random walk returns to *0*.

Assume now that our city is no longer a perfect square grid. When our drunkard reaches a certain junction he picks between the various available roads with equal probability. Thus, if the junction has seven exits the drunkard will go to each one with probability one seventh. This is a random walk on a graph. Will our drunkard reach his home? It turns out that under rather mild conditions, the answer is still yes. For example, if the lengths of all the blocks are between *a* and *b* (where *a* and *b* are any two finite positive numbers), then the drunkard will, almost surely, reach his home. Notice that we do not assume that the graph is planar, i.e. the city may contain tunnels and bridges. One way to prove this result is using the connection to electrical networks. Take a map of the city and place a one ohm resistor on every block. Now measure the "resistance between a point and infinity". In other words, choose some number *R* and take all the points in the electrical network with distance bigger than *R* from our point and wire them together. This is now a finite electrical network and we may measure the resistance from our point to the wired points. Take *R* to infinity. The limit is called the *resistance between a point and infinity*. It turns out that the following is true (an elementary proof can be found in the book by Doyle and Snell):

Theorem: *a graph is transient if and only if the resistance between a point and infinity is finite. It is not important which point is chosen if the graph is connected.*

In other words, in a transient system, one only needs to overcome a finite resistance to get to infinity from any point. In a recurrent system, the resistance from any point to infinity is infinite.

This characterization of recurrence and transience is very useful, and specifically it allows us to analyze the case of a city drawn in the plane with the distances bounded.

A random walk on a graph is a very special case of a Markov chain. Unlike a general Markov chain, random walk on a graph enjoys a property called *time symmetry* or *reversibility*. Roughly speaking, this property, also called the principle of detailed balance, means that the probabilities to traverse a given path in one direction or in the other have a very simple connection between them (if the graph is regular, they are just equal). This property has important consequences.

Starting in the 1980s, much research has gone into connecting properties of the graph to random walks. In addition to the electrical network connection described above, there are important connections to isoperimetric inequalities, see more here, functional inequalities such as Sobolev and Poincaré inequalities and properties of solutions of Laplace's equation. A significant portion of this research was focused on Cayley graphs of finitely generated groups. For example, the proof of Dave Bayer and Persi Diaconis that 7 riffle shuffles are enough to mix a pack of cards (see more details under shuffle) is in effect a result about random walk on the group S_n, and the proof uses the group structure in an essential way. In many cases these discrete results carry over to, or are derived from manifolds and Lie groups.

A good reference for random walk on graphs is the online book by Aldous and Fill [17]. For groups see the book of Woess. If the transition kernel $p(x, y)$ is itself random (based on an environment ω) then the random walk is

called a "random walk in random environment". When the law of the random walk includes the randomness of ω, the law is called the annealed law; on the other hand, if ω is seen as fixed, the law is called a quenched law. See the book of Hughes or the lecture notes of Zeitouni.

We can think about choosing every possible edge with the same probability as maximizing uncertainty (entropy) locally. We could also do it globally – in maximal entropy random walk (MERW) [18] we want all paths to be equally probable, or in other words: for each two vertexes, each path of given length is equally probable. This random walk has much stronger localization properties.

Self-interacting random walks

There are a number of interesting models of random paths in which each step depends on the past in a complicated manner. All are more complex for solving analytically than the usual random walk; still, the behavior of any model of a random walker is obtainable using computers. Examples include:

- The self-avoiding walk (Madras and Slade 1996).[19]

The self-avoiding walk of length n on Z^d is the random n-step path which starts at the origin, makes transitions only between adjacent sites in Z^d, never revisits a site, and is chosen uniformly among all such paths. In two dimensions, due to self-trapping, a typical self-avoiding walk is very short,[20] while in higher dimension it grows beyond all bounds. This model has often been used in polymer physics (since the 1960s).

- The loop-erased random walk (Gregory Lawler).[21][22]
- The reinforced random walk (Robin Pemantle 2007).[23]
- The exploration process.
- The multiagent random walk.[24]

Long-range correlated walks

Long-range correlated time series are found in many biological, climatological and economic systems.

- Heartbeat records[25]
- Non-coding DNA sequences[26]
- Volatility time series of stocks[27]
- Temperature records around the globe[28]

Heterogeneous random walks in one dimension

Heterogeneous random walks in one dimension can have either discrete time or continuous time. The interval is also either discrete or continuous, and it is either finite or without bounds. In a discrete system, the connections are among adjacent states. The dynamics are either Markovian, Semi-Markovian, or even not-Markovian depending on the model. Heterogeneous random walks in 1D have jump probabilities that depend on the location in the system, and/or different jumping time (JT) probability density functions (PDFs) that depend on the location in the system.

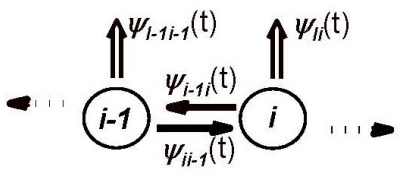


Figure 1 A part of a semi-Markovian chain in 1D with directional JT-PDFs. A way for simulating such a random walk is when first drawing a random number out of a uniform distribution that determines the propagation direction according to the transition probabilities, and then drawing a random time out of the relevant JT-PDF.

Known important results in simple systems include:

- In a symmetric Markovian random walk, the Green's function (also termed the PDF of the walker) for occupying state *i* is a Gaussian in the position and has a variance that scales like the time. This result holds in a system with discrete time and space, yet also in a system with continuous time and space. This result is for systems without bounds.

- When there is a simple bias in the system (i.e. a constant force is applied on the system in a particular direction), the average distance of the random walker from its starting position is linear with time.
- When trying reaching a distance *L* from the starting position in a finite interval of length *L* with a constant force, the time τ for reaching this distance is exponential with the length *L*: $\tau \approx e^L$, when moving against the force, and is linear with the length *L*: $\tau \approx L$, when moving with the force. Without force: $\tau \approx L^2$.

In a completely heterogeneous semi Markovian random walk in a discrete system of *L* (>1) states, the Green's function was found in Laplace space[29][30][31] (the Laplace transform of a function is defined with, $\bar{f}(s) = \int_0^\infty e^{-st} f(t) dt$). Here, the system is defined through the jumping time (JT) PDFs: $\psi_{ij}(t)$ connecting state *i* with state *j* (the jump is from state *i*). The solution is based on the path representation of the Green's function, calculated when including all the path probability density functions of all lengths:

$$\bar{G}_{ij}(s) = \bar{\Gamma}_{ij}(s) \frac{\bar{\Phi}(s, \tilde{L})}{\bar{\Phi}(s, L)} \bar{\Psi}_i(s). \quad \textbf{(1)}$$

Here, $\bar{\Psi}_i(s) = \Sigma_j \bar{\Psi}_{ij}(s)$ and $\bar{\Psi}_{ij}(s) = \frac{1 - \bar{\psi}_{ij}(s)}{s}$. Also, in Eq. (**1**),

$$\bar{\Gamma}_{ij}(s) = \Pi_{c=0}^{i \mp 1} \bar{\psi}_{c \pm 1 c}(s), \quad \textbf{(2)}$$

and,

$$\bar{\Phi}(s, L) = 1 + \Sigma_{c=1}^{[L/2]} (-1)^c \bar{h}(s, c; L) \quad \textbf{(3)}$$

with,

$$\bar{h}(s, i; L) = \Pi_{c=1}^{i} \Sigma_{k_c = 2 + k_{c-1}}^{L - 1 - 2(i-c)} \bar{f}_{k_c}(s) \quad \textbf{(4)}$$

and,

$$\bar{f}_{k_j}(s) = \bar{\psi}_{k_j k_j + 1}(s) \bar{\psi}_{k_j + 1 k_j}(s). \quad \textbf{(5)}$$

For *L=1*, $\bar{\Phi}(s; L) = 1$. The symbol [*L*/2] that appears in the upper bound in the Σ in eq. (**3**) is the floor operation (round towards zero). Finally, the factor $\Phi(s, \tilde{L})$ in eq. (**1**) has the same form as in $\bar{\Phi}(s; L)$ in eqs. (**3**)-(**5**), yet it is calculated on a lattice $\tilde{L}$. Lattice $\tilde{L}$ is constructed from the original lattice by taking out from it the states *i* and *j* and the states between them, and then connecting the obtained two fragments. For cases in which a fragment is a single state, this fragment is excluded; namely, lattice $\tilde{L}$ is the longer fragment. When each fragment is a single state, $\bar{\Phi}(s; \tilde{L}) = 1$.

Equations (**1**)-(**5**) hold for any 1D semi-Markovian random walk in a L-state chain, and form the most general solution in an explicit form for random walks in 1d.

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External links

- Pólya's Random Walk Constants (http://mathworld.wolfram.com/PolyasRandomWalkConstants.html)
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