General Solution to find the Area of the Curves of type $\mathbf{x}^n + y^n = a^n$

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Abstract

In High School we are taught to find the area of a curve using an Definite Integral. For example area of a circle is πr^2 where r is the radius of the circle and the equation of a circle is $x^2 + y^2 = r^2$. Similarly there is geometrical figure known as an asteroid, its equation is $x^{2/3} + y^{2/3} = r^{2/3}$ and by working out we find its area to be $\frac{3\pi r^2}{8}$. So I tried to find a general solution of this type of curves[1]. Although it has a lot of mathematics but a student who is clear with basic calculus will be able to understand it without much difficulty.

1 The Calculations

Well lets start with the basic equation $x^n + y^n = a^n$ our aim is to find the area in terms of a definate integral.

$$x^n + y^n = r^n \tag{1}$$

r is the radius of the curve

$$y = (r^n - x^n)^{1/n} \tag{2}$$

As the area of a curve is given by

$$\int_0^r y dx \tag{3}$$

$$\int_0^r (r^n - x^n)^{1/n} dx \tag{4}$$

Let $x^n = r^n \sin^2 \theta$

[1]When I say general solution it means that I am trying to solve the general equation of this type of curves in this case the general equation is $x^n + y^n = r^n$

$x = r \sin^{2/n} \theta$

$dx = \frac{2}{n} r \cos\theta \sin^{\frac{2-n}{n}} \theta$

The integral then becomes

$$\frac{2r^2}{n} \int_0^{\frac{\pi}{2}} (\sin^{\frac{2-n}{n}} \theta \cos^{\frac{2+n}{n}} \theta) d\theta \tag{5}$$

Now let $\sin\theta =$ u

du= $\cos\theta$ dθ

Then the integral becomes

$$\frac{2r^2}{n} \int_0^1 (u^{\frac{2-n}{n}} \sqrt[n]{1-u^2}) du \tag{6}$$

Well thats the equation in which if we put the value of n we get the area.

Lets put $n = 2$ this the main equation will be $x^2 + y^2 = r^2$ then the integral will become

$$r^2 \int_0^1 \sqrt{1-u^2} du \tag{7}$$

and after some calculations we get the answer as πr^2.

So the main equation is

$$\frac{2r^2}{n} \int_0^1 u^{\frac{2-n}{n}} \sqrt[n]{1-u^2} du \tag{8}$$

2 The Main Idea

Well the main idea behind this calculation is to find an equation in which if we plug the value of n we get the answer in a few steps. In this way we can find the area of a curve easily although if the student is not familiar with β functions and the Γ functions solutions to some n's is a difficult task.