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New Trends in Nanotechnology and Nonlinear Dynamical Systems

The Rational Matrices

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Abstract: In this paper, we give the research about rational properties of two matrices with the same dimension. We will see that the properties here are analogous to the case in real numbers. This study is about their properties and some examples are given.

Keywords: matrix, division, matrices division, rational matrices.

1. INTRODUCTION

Let us start with the definition as follows.

Let A and $B \in M(\mathbb{R})$ with $B \neq 0$. Then the matrix $\left[\frac{{}^{A}_{B}i_j}{|B|}\right]_{n\times n}$ is called the division of A by B and denoted by $A \div B$ or $\frac{A}{B}$. The set of rational matrices are shown by

$$\mathbb{Q}(\mathcal{M}) = \left\{\frac{A}{B}\middle| A,B \in \mathcal{M}(\mathbb{R}), |B| \neq 0\right\}.$$

The term rational in the set $\mathbb{Q}(\mathcal{M}) = \left\{\frac{A}{B}\middle| A,B \in \mathcal{M}(\mathbb{R}), |B| \neq 0\right\}$ [1] refers to the fact that a rational matrix represents a ratio of two matrices. The set of rational matrices is include by $\mathcal{M}(\mathbb{R})$. A rational matrix is a matrix like $\frac{A}{B}$, where A, B are matrices. If $B = 0$ then this division is not defined. $\frac{0}{A} = 0 \in \mathbb{Q}(\mathcal{M})$ and $\frac{A}{A} = I \in \mathbb{Q}(\mathcal{M})$.

We will give the Lemma without proof[1].

2. THE RATIONAL MATRICES

Lemma 1. Let $A, B, C \in \mathcal{M}(\mathbb{R}), |C| \neq 0$. Then

i. $\left[{}^{A}_{B}i_j\right] = \frac{1}{|C|}\left[{}^{C}_{B}i_j\right]\left[{}^{A}_{C}i_j\right]$.

ii. $\left[{}^{A}_{B}i_j\right] = A\left[{}^{I}_{B}i_j\right]$.

We can easily get

Lemma 2. Let $\frac{A}{B}, \frac{C}{B} \in \mathbb{Q}(\mathcal{M})$. Then $\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}$.

Proof. Let $\frac{A}{B}, \frac{C}{B} \in \mathbb{Q}(\mathcal{M})$

$$\begin{aligned}\frac{A}{B} + \frac{C}{B} &= \frac{1}{|B|}\left[{}^{A}_{B}i_j\right] + \frac{1}{|B|}\left[{}^{C}_{B}i_j\right] \\ &= \frac{1}{|B|}\left(\left[{}^{A}_{B}i_j\right] + \left[{}^{C}_{B}i_j\right]\right) \\ &= \frac{1}{|B|}\left(\left[{}^{A}_{B}i_j + {}^{C}_{B}i_j\right]\right) \\ &= \left[{}^{(A+C)}_{B}i_j\right] = \frac{A+C}{B}.\end{aligned}$$

Theorem 1. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$. Then

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|\left|{}^{C}_{D}i_j\right|}\left[\frac{\left({}^{A}_{B}i_j\right)}{\left({}^{C}_{D}i_j\right)}i_j\right].$$

Proof. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$.

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|}\frac{\left[{}^{A}_{B}i_j\right]}{\left[{}^{C}_{D}i_j\right]} = \frac{|D|}{|B|\left|{}^{C}_{D}i_j\right|}\left[\frac{\left({}^{A}_{B}i_j\right)}{\left({}^{C}_{D}i_j\right)}i_j\right].$$

Corollary 1. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M})$. Then

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \frac{|D|}{|B|}\left[{}^{D}_{C}i_j\right]\left[{}^{A}_{B}i_j\right].$$

Theorem 2. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M}), |C| \neq 0$. Then

$$\left(\frac{A}{B}\right) \div \left(\frac{C}{D}\right) = \left(\frac{D}{C}\right)\left(\frac{A}{B}\right).$$

Proof. Let $\frac{A}{B}, \frac{C}{D} \in \mathbb{Q}(\mathcal{M}), |C| \neq 0$.

$$\frac{\frac{A}{B}}{\frac{C}{D}} = \left[\frac{C}{D}\right]^{-1}\left[\frac{A}{B}\right] = \left[\frac{D}{C}\right]\left[\frac{A}{B}\right].$$

[1] Square matrix of reel numbers is denoted with $M(\mathbb{R})$.

Example 1. Let $A=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, $C=\begin{bmatrix}2 & 1\\ 1 & 1\end{bmatrix}$ and $D=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}$ $\frac{A}{B}=\begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}$

$$\frac{C}{D}=\begin{bmatrix}-1 & -4\\ 1 & 3\end{bmatrix} \Rightarrow \frac{\frac{A}{B}}{\frac{C}{D}}=\begin{bmatrix}5 & 4\\ -1 & -1\end{bmatrix},$$

$$\frac{C}{D}=\begin{bmatrix}-1 & -4\\ 1 & 3\end{bmatrix} \Rightarrow \frac{D}{C}=\begin{bmatrix}3 & 4\\ -1 & -1\end{bmatrix},$$

$$\left(\frac{D}{C}\right)\left(\frac{A}{B}\right)=\begin{bmatrix}3 & 4\\ -1 & -1\end{bmatrix}\begin{bmatrix}-1 & 0\\ 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}5 & 4\\ -1 & -1\end{bmatrix}=\frac{\frac{A}{B}}{\frac{C}{D}}.$$

Lemma 3. Let $\frac{A}{B}\in\mathbb{Q}(M), C\in M(\mathbb{R})$. Then

i. $\left(\frac{A}{B}\right)=\frac{AC}{B}$.

ii. $\left(\frac{A}{B}\right)A=\frac{A^2}{B}$.

iii. If $|C|\neq 0$, also $\frac{A}{B}=\left[\frac{C}{B}\right]\left[\frac{A}{C}\right]$.

Proof. If $A=\left[a_{ij}\right], B=\left[b_{ij}\right]$ and $C=\left[c_{ij}\right]$ then

i. $\frac{AC}{B}=$

$$\begin{bmatrix}\sum_{s=1}^{n}\left(\sum_{k=1}^{n}a_{sk}c_{k1}\right)C_{s1}(B) & \cdots & \left(\sum_{k=1}^{n}a_{sk}c_{kn}\right)C_{s1}(B)\\ \vdots & \vdots & \vdots\\ \sum_{s=1}^{n}\left(\sum_{k=1}^{n}a_{sk}c_{k1}\right)C_{sn}(B) & \cdots & \sum_{s=1}^{n}\left(\sum_{k=1}^{n}a_{sk}c_{k1}\right)C_{sn}(B)\end{bmatrix}$$

$$\begin{bmatrix}\sum_{k=1}^{n}a_{sk}C_{k1}(B) & \cdots & \sum_{k=1}^{n}a_{sk}C_{k1}(B)\\ \vdots & \vdots & \vdots\\ \sum_{k=1}^{n}a_{sk}C_{k1}(B) & \cdots & \sum_{k=1}^{n}a_{sk}C_{k1}(B)\end{bmatrix}\begin{bmatrix}c_{11} & \cdots & c_{1n}\\ \vdots & \vdots & \vdots\\ c_{n1} & \cdots & c_{nn}\end{bmatrix}$$

$=\left(\frac{A}{B}\right)C$ ■

ii. If $C=A$ from i)

$$\left(\frac{A}{B}\right)A=\left[\frac{{}^{A}_{B}i_{j}}{|B|}\right]A=\frac{1}{|B|}\left[{}^{A}_{B}i_{j}\right]A=\frac{1}{|B|}\left[{}^{A^2}_{B}i_{j}\right]=\frac{A^2}{B}.$$

iii. If $|C|\neq 0$ from Lemma 1.

$$\frac{A}{B}=\left[\frac{\left({}^{A}_{B}i_{j}\right)_{ji}}{|B|}\right]=\frac{1}{|B|}\frac{1}{|C|}\left[\left({}^{C}_{B}i_{j}\right)_{ji}\right]\left[{}^{A}_{C}i_{j}\right]=\left[\frac{C}{B}\right]\left[\frac{A}{C}\right].$$

Theorem 3. Let $\frac{A}{B},\frac{C}{D},\frac{E}{F}\in\mathbb{Q}(M)$. Then

i. $\frac{A}{B}+\frac{C}{D}=\frac{C}{D}+\frac{A}{B}$.

ii. $\frac{A}{B}+\left(\frac{C}{D}+\frac{E}{F}\right)=\left(\frac{A}{B}+\frac{C}{D}\right)+\frac{E}{F}$.

iii. $\frac{A}{B}+0=\frac{A}{B}, 0+\frac{A}{B}=\frac{A}{B}$.

iv. $\left(\frac{A}{B}\right)\left(\frac{C}{D}\right)$ is rational matrix.

Let us the following lemma without proof.

Lemma 4. Let $\frac{A}{B},\frac{C}{D}\in\mathbb{Q}(M)$. Then

$$\left(\frac{A}{B}\right)\left(\frac{C}{D}\right)=\frac{A\left(\frac{C}{D}\right)}{B}.$$

For $\frac{A}{B}\in\mathbb{Q}(M)$ and $n\in\mathbb{Z}^{+}$. Is then

$$\left(\frac{A}{B}\right)^{n}=\frac{1}{|B|^{n}}\left[{}^{A}_{B}i_{j}\right]^{n}$$

Example 2. Let and .

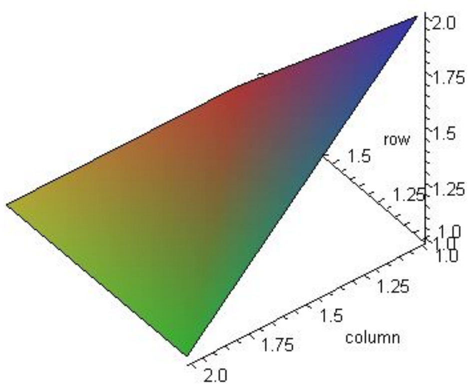

Fig. 1. Matrix A

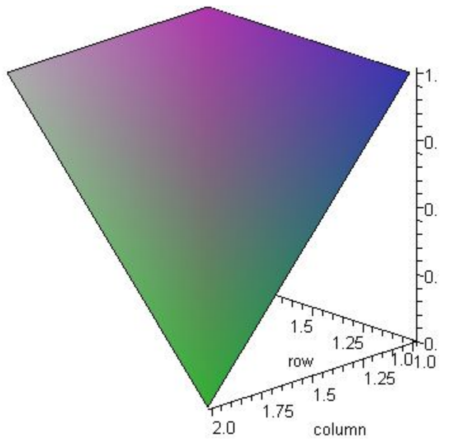

Fig. 2. Matrix B.

Then

$$\frac{A}{B}=\begin{bmatrix}\begin{vmatrix}2&0\\1&1\end{vmatrix}&\begin{vmatrix}1&0\\1&1\end{vmatrix}\\\begin{vmatrix}1&2\\1&1\end{vmatrix}&\begin{vmatrix}1&1\\1&1\end{vmatrix}\end{bmatrix}=\begin{bmatrix}2&1\\-1&0\end{bmatrix}$$

$$\left(\frac{A}{B}\right)^2=\begin{bmatrix}3&2\\-2&1\end{bmatrix}\text{ and }\frac{A^2}{B^2}=\begin{bmatrix}5&3\\-7&-4\end{bmatrix},$$

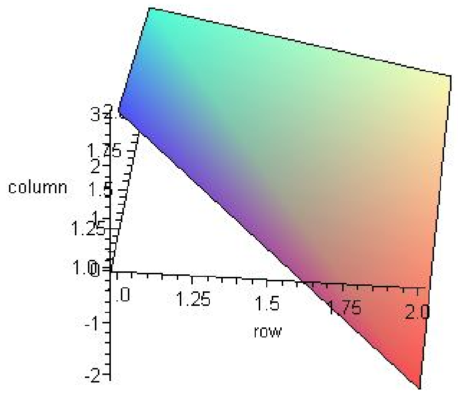

Fig. 3. Matrix $\left(\frac{A}{B}\right)^2$.

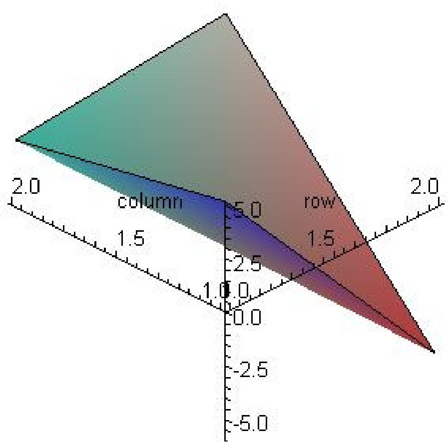

Fig. 4. Matrix $\frac{A^2}{B^2}$.

then

$$\left(\frac{A}{B}\right)^2\neq\frac{A^2}{B^2}.$$

Note. In general, for rational matrices we will get,

$$\left(\frac{A}{B}\right)^n\neq\frac{A^n}{B^n}.$$

Theorem 4. If $\frac{A}{B}\in\mathbb{Q}(\mathcal{M})$ and $n\in\mathbb{Z}^+$, then

i. $\left(\frac{A}{B}\right)^{-1}=\frac{B}{A}=\frac{I}{\frac{A}{B}}$.

ii. $\left(\frac{A}{B}\right)^{-n}=\left(\left(\frac{A}{B}\right)^{-1}\right)^n=\left(\frac{B}{A}\right)^n$.

Proof. i. For the proof see paper [Keleş, H. (2010)].
ii. It is see clearly.

Theorem 5. Let $\frac{A}{B},\frac{C}{D}\in\mathbb{Q}(\mathbb{R})$. Then

$$\left(\frac{A}{B}\right)\div\left(\frac{C}{D}\right)=\frac{|D|}{|B||\frac{C}{D}i_j|}\left[\frac{(\frac{A}{B}i_j)}{(\frac{C}{D}i_j)}i_j\right].$$

Proof. If $\frac{A}{B},\frac{C}{D}\in\mathbb{Q}(\mathbb{R})$.

$$\left(\frac{A}{B}\right)\div\left(\frac{C}{D}\right)=\frac{|D|}{|B|}\frac{[\frac{A}{B}i_j]}{[\frac{C}{D}i_j]}=\frac{|D|}{|B||\frac{C}{D}i_j|}\left[\frac{(\frac{A}{B}i_j)}{(\frac{C}{D}i_j)}i_j\right].$$

REFERENCES

Keleş, H. (2010). *Matrix Division*. International Association of Engineers (IAENG. London. U.K.).
Keleş, H. (2008). *Lineer Cebire Giriş*. Akademi Yayınevi. Trabzon.Turkey. 2008.
Bolian, L. and Hong-Jian L. ., (2000). *Matrices in Combinatorics and Graph Theory*. Kluwer Academic Publishers. The Netherlands.