

IV. *A second Letter from Mr. Colin M^c Laurin, Professor of Mathematicks in the University of Edinburgh and F. R. S. to Martin Folkes, Esq; concerning the Roots of Equations, with the Demonstration of other Rules in Algebra; being the Continuation of the Letter published in the Philosophical Transactions, N^o 394.*

Edinburgh, April 19th, 1729.

S I R,

IN the Year 1725, I wrote to you that I had a Method of demonstrating Sir *Isaac Newton's* Rule concerning the impossible Roots of Equations, deduced from this obvious Principle, that the Squares of the Differences of real Quantities must always be positive; and some time after, I sent you the first Principles of that Method, which were published in the *Philosophical Transactions* for the Month of May, 1726. The Design I have for some Time had of publishing a Treatise of Algebra, where I propos'd to treat this and several other Subjects in a new Manner, made me think it unnecessary to send you the remaining Part of that Paper. But some Reasons have now determin'd me to send you with the Continuation of my former Method, a short Account of two other Methods in which I have treated the same Subject, and some Observations on Equations that I take to be new, and which will, perhaps, be more acceptable to you than what relates to the imaginary Roots themselves. Besides Sir *Isaac Newton's* Rule, there arises from the following general

ral Propositions, a great Variety of new Rules, different from his, and from any other hitherto published, for discovering when an Equation has imaginary Roots. I shall particularly explain one that is more useful for that Purpose, than any that have been hitherto published.

Suppose there is an Equation of (*n*) Dimensions of this Form,

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + Fx^{n-6} - Gx^{n-7} + Hx^{n-8} - Ix^{n-9} + Kx^{n-10} \&c. = 0.$$

And that the Roots of this Equation are, *a, b, c, d, e, f, g, h, i, k, l, &c.* then shall $A = a + b + c + d + e + f \&c.$ and therefore I call *a, b, c, d, e, f, &c.* *Parts* or *Terms* of the Coefficient A. For the same Reason I call *ab, ac, ad, ae, bc, bd, cd, &c.* *Parts* or *Terms* of the Coefficient B; *abc, abd, abe, acd, bcd, &c.* *Parts* or *Terms* of C; *abcd, abce, abcf, &c.* *Parts* or *Terms* of the Coefficient D, and so on. By the *Dimensions* of any Coefficient; I mean the Number of Roots or Factors that are multiplied into each other in its Parts, which is always equal to the Number of Terms in the Equation that precede that Coefficient. Thus A is a Coefficient of one Dimension, B of two, C of three, and so of the rest. I call a Part or Term of a Coefficient C *similar* to a Part or Term of any Coefficient G, when the Part of G involves all the Factors of the Part of C: Thus *abc, abcdefg* are similar Parts of C and G; after the same manner *abcd, abcdef* are similar Parts of D and F, the Part of F involving all the Factors of the Part of D. Those I call *dissimilar* Parts that involve no common Root or Factor: Thus *abc*, and *defgh* are dissimilar Parts of the Coefficients C and F. The Sum of all the

the Products that can be made by multiplying the Parts of any Coefficient C by all the similar Parts of G, I express by $C'G'$ placing a small Line over each Coefficient: After the same manner $D'F'$ expresses the Sum of all the Products that can be made by multiplying the similar Parts of D and F by each other; and $C' \times C'$ expresses the Sum of the Squares of the Parts of the Coefficient C, but $C' \times C$; expresses the Sum of the Products that can be made by multiplying any two Parts of C by one another. These Expressions being understood, and the five Propositions in *Phil. Transf.* N^o 394, being premised, next follows

P R O P. VI.

If the Difference of the Dimensions of any two Coefficients C and G be called (m) then shall the Product of these Coefficients multiplied by one ano-

$$\text{ther be equal to } C'G' + \overline{m+2} \times B'H' + \frac{m+3}{1} \times \frac{m+4}{2} A'I' + \frac{m+4}{1} \times \frac{m+5}{2} \times \frac{m+6}{3} \times I \times K.$$

Where B and H are the Coefficients adjacent to the Coefficients C and G, A and I the Coefficients adjacent to B and H, I and K the Coefficients adjacent to B and H.

It is known that $C = abc + abd + abe + abf + abg, \&c.$ and $G = abcdefg + abcdefh + abcdefi + bcdefgh, \&c.$ and it is manifest,

1. That in the Product CG each Term of $C'G'$ will arise once as $a^2b^2c^2defg$. But

2. Any Term of $B'H'$ as $a^2b^2cdefgh$ may be the Product of abc , and $abdefgh$, or of abd and $abcdefgh$, or of abe and $abcdfgh$, or of abf and $abcdegh$,

$abcdegh$, or of abg and $abcdefh$, or lastly of abb and $abcdefg$; so that it may be the Product of any Term of C that involves with ab one of the Roots, c, d, e, f, g, h , multiplied by that Term of G, which involves ab and the other five; that is, it may arise in the Product CG as often as there are Roots in $a^2b^2cdefgh$ besides a and b , or in general, as often as there are Units in the Difference of the Dimensions of B and H, that is, $m + 2$ times; because m expresses the Difference of the Dimensions of C and G, and consequently in expressing the Value of CG the Coefficient of the second Term B'H' must be $m + 2$.

3. Any Term of AI, as $a^2bcdefghi$, may be the Product of any Part of C that involves the Root a with any two of the rest b, c, d, e, f, g, h, i (the Number of which is the Difference of the Dimensions of A and I, which is in general equal to $m + 4$) multiplied by the Part of G that involves a and the other six; and therefore $a^2bcdefghi$ or any other Term of A'I' must arise as often as different Products of two Quantities can be taken from Quantities whose Number is $m + 4$, that is $\frac{m + 4}{2} \times \frac{m + 4 - 1}{2}$ times or $\frac{m + 3}{1} \times \frac{m + 4}{2}$ times; and consequently in expressing the Value of CG the Coefficient of the third Term A'I' must be $\frac{m + 3}{1} \times \frac{m + 4}{3}$.

4. Any Term of $1 \times K$ as $abcdefghik$, may be the Product of any Part of C that involves three of its Factors, and of the Part of G that involves the rest, and therefore may arise in the Product CG as often as different

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Products of three Quantities can be taken out of Quantities whose Number is $m + 6$ that is, $\overline{m + 6} \times \frac{m + 5}{2} \times \frac{m + 4}{3}$ times, and therefore the Coefficient of the fourth Term in the Value of CG must be $\frac{m + 4}{1} \times \frac{m + 5}{2} \times \frac{m + 6}{3}$.

In general, in expressing the Value of the Product of any two Coefficients C and G, if x express the Order of any Term of this Value as $A' I'$, that is, the Number of Terms that precede it, the Coefficient of that Term must be $\frac{2x + m}{1} \times \frac{2x + m - 1}{2} \times \frac{2x + m - 2}{3}$ &c. taking as many Factors as there are Units in x .

COR. I. If it is required to find by this Proposition the Square of any Coefficient E, then suppose $m = 0$, the Difference of the Dimensions of the Coefficients in this Case vanishing, and we shall have $E^2 = E' \times E' + 2 D' F' + 3 \times \frac{4}{2} \times C' G' + 4 \times \frac{5}{2} \times \frac{6}{3} \times B' H'$ &c. $= E' \times E' + 2 D' F' + 6 C' G' + 20 B' H' + 70 A' I' + 252 K$. Therefore if $E' \times E'$, express the Sum of the Products of any two parts of E multiplied by each other, we shall have $E^2 = E' \times E' + 2 E' \times E'$, and therefore $E' \times E' = D' F' + 3 C' G' + 10 B' H' + 35 A' I' + 126 K$.

K

COR.

of the Parts of E by each other, that is, than $E' \times E_i$; but $2 E' \times E_i = E^2 - E' E' =$ (by the first Theorem in the last Corollary) $2 DF - 2 CG + 2 BH - 2 A' I + 2 K$, and therefore since $\frac{l-1}{2l} \times E^2$ must always exceed $E' E_i$, it follows that $\frac{l-1}{2l} E^2$ must always be greater than $DF - CG + BH - AI + K$ when the Roots of the Equation are real Quantities.

SCHOL. In following my Method this was the first general Propofition presented it self. For having first observed that if l expreffes the Number of any Quantities, the Square of their Sum multiplied by $\frac{l-1}{2l}$ must always exceed the Sum of the Products made by multiplying any two of them by each other; and that the Excess was the Sum of the Squares of the Differences of the Quantities divided by $2l$, it was easy to see in the Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} \&c. = 0$. Since B is the Sum of the Products of any two of the Parts of A, that if l expreffes the Number of the Roots of the Equation, $\frac{l-1}{2l} \times A^2$ must always exceed B; and this is one Part of the 5th Propofition. In the next Place, I compared the Sum of the Products of any two Parts of B with AC, and found that it was not equal to AC but to $AC - D$ from which I inferred, that if l expreffes

the Number of the Parts of B then $\frac{l-r}{2l} \times B^2$ must always exceed $AC - D$; and these easily suggested this general Proposition.

P R O P. VIII.

Let r express the Dimensions of the Coefficient C , and s the Difference of the Dimensions of the Coefficients C and G , then B and H being Coefficients adjacent to C and G , $\frac{n-r-s}{s+1} \times r C' G'$ shall always be greater than $s+1 \times s+2 \times B'H'$ when the Roots of the Equation are all real Quantities affected with the same Sign.

For taking the Differences of all those Parts of the Coefficient C that are similar in all their Factors but one, as $abc, abb, abi, \&c.$ and multiplying the Square of each Difference by such Parts of the Coefficient D (which is of s Dimensions): as are dissimilar to both the Parts of C in that Difference, the Sum of all those Squares thus multiplied, will consist of Terms of $C'G'$ taken positively, and of Terms of $B'H'$ taken negatively. By multiplying in this manner $\overline{abc-abb}^2 + \overline{abc-abi}^2 + \overline{abc-abk}^2 \&c. + \overline{abc-acb}^2 + \overline{abc-aci}^2 + \overline{abc-ack}^2 \&c. + \overline{abc-bcb}^2 + \overline{abc-bce}^2 + \overline{abc-bck}^2 \&c.$ by $defg$ the Term of D , that is dissimilar to all those Parts of C , you will find that $a^2b^2c^2defg$ will arise in the Sum of the Products $r \times n - r - s$ times: For those Products may be also expressed thus $defga^2b^2 \times \overline{c-b}^2 + \overline{c-i}^2 + \overline{c-k}^2$ &c. $+ defga^2c^2 \times \overline{b-b}^2 + \overline{b-i}^2 + \overline{b-k}^2 \&c. + defgb^2c^2$

$defgb^2c^2 \times \overline{a-b}^2 + \overline{a-i}^2 + \overline{a-k}^2$ &c. where the Number of the Differences $c-b, c-i, c-k,$ &c. whose Squares are multiplied by $defga^2b^2$ is manifestly equal to the Number of the Roots of the Equation that do not enter $a^2b^2c^2defg$ or $abcddefg$, that is, to the Excess of the Number of the Roots of the Equation above the Dimensions of $abcddefg$, a Term of C , that is to $\overline{n-r-s}$. But in collecting all the said Products, $\overline{n-r-s} \times a^2b^2c^2defg$ must arise as often as there are Units in r : Because the Terms which are subtracted from abc may differ from it in the Root c , as $abb, abi, abk,$ &c. or in the Root b , as $acb, aci, ack,$ &c. or in the Root a as bcb, bci, bck ; that is, $\overline{n-r-s} \times a^2b^2c^2defg$ must arise as often as there are Dimensions in abc , a Term of C , or as often in general as there are units in r , which expresses the Dimensions of C : Therefore the Term $a^2b^2c^2defg$ will arise in the Sum of the above-mentioned Products $r \times \overline{n-r-s}$ times.

The Negative Part must consist of the Terms of $B'H'$ doubled; each of which, as $2a^2b^2c^2defg$ may arise as often as there can be Differences $c-d, c-e, c-f, c-g, d-e,$ &c. assumed amongst the Terms c, d, e, f, g whose Number is equal to $s+2$ that is, $\overline{s+2} \times \frac{s+1}{2}$ times; and therefore $a^2b^2c^2defg$ or any other Part of $B'H'$ must arise in the negative Part $\overline{s+1} \times \overline{s+2}$ times; and since the whole aggregate must be positive it follows $\overline{n-r-s} \times r C'G'$ must always exceed $\overline{s+1} \times \overline{s+2} \times B'H'$.

COR.

COR. I. Suppose we are to compare $E'E'$ the Sum of the Squares of the Parts of E with $D'F'$ the Sum of the Products of the similar Parts of D and F ; in this Case s vanishes, and therefore $\overline{n-r} \times r E'E'$ must exceed $2 D'F'$. Let $\overline{n-r} \times r = m$ and consequently $\overline{n-r-1} \times r-1 = m-n+1$; $\overline{n-r-2} \times r-2 = m-2n+4$; $\overline{n-r-3} \times r-3 = m-3n+9$; $\overline{n-r-4} \times r-4 = m-4n+16$. Since it is plain that $\overline{n-r-q} \times r-q = \overline{n-r} \times r - qn + q^2$. Then by this Proposition, supposing

$$\begin{aligned} \overline{m} \times E'E' - 2 D'F' &= a' \\ \overline{m-n+1} \times D'F' - 12 C'G' &= b' \\ \overline{m-2n+4} \times C'G' - 30 B'H' &= c' \\ \overline{m-3n+9} \times B'H' - 56 A'I' &= d' \\ \overline{m-4n+16} \times A'I' - 90 K' &= e' \end{aligned}$$

The Quantities a', b', c', d', e' , must be always positive when the Roots of the Equation are real Quantities affected with the same Sign. The Coefficients prefixed to the negative Parts are the Numbers 2, 12, 30, 56, 90, whose Differences equally increase by the same Number 8.

COR. II. Supposing as before, that $\overline{n-r} \times r = m$; and also that $\overline{m} \times \overline{m-n+1} = m'$; $m' \times \overline{m-2n+4} = m''$; $m'' \times \overline{m-3n+9} = m'''$ &c. it may be demonstrated after the manner of this Proposition, that if

$$\begin{aligned} \overline{m} E'E' - 2 D'F' &= a' \\ \overline{m'} E'E' - 2 \times 12 C'G' &= a'' \\ \overline{m''} E'E' - 2 \times 12 \times 30 B'H' &= a''' \\ \overline{m'''} E'E' - 2 \times 12 \times 30 \times 56 A'I' &= a'''' \text{ \&c.} \end{aligned}$$

Then

Then shall a', a'', a''' , &c. be always positive when the Roots are real Quantities, whether they be affected with the same, or with different Signs. The Negative Coefficients arise by multiplying those in the preceding Corollary, 2, 12, 30, 56, 90, by one another.

P R O P. IX.

Let a', b', c', d', e' , and m express the same Quantities as in the Corollaries of the last Proposition, and $m E^2$ —

$$\overline{m + n + 1} \times D F = a' + b' + 2c' + 5d' + 14e'.$$

For by Cor. ii Prop. vi.

$E^2 = E' E' + 2 D' F' + 6 C' G' + 20 B' H' + 70 A' I' + 252 K$,
and by the same

$D F = - - - D' F' + 4 C' G' + 15 B' H' + 56 A' I' + 210 K$;

therefore $m E^2 - \overline{m + n + 1} \times D F = m E' E' +$

$\overline{m - n - 1} \times D' F' + \overline{m - 2n - 2} \times 2 C' G'$

$+ \overline{m - 3n - 3} \times 5 B' H' + \overline{m - 4n - 4} \times$

$14 A' I' + \overline{m - 5n - 5} \times 42 K =$ (by substituting

successively for $m E' E'$, $\overline{m - n + 1} \times D' F'$,

$\overline{m - 2n + 4} \times C' G'$, $\overline{m - 3n + 9} \times B' H'$,

$\overline{m - 4n + 16} \times A' I'$ their Values deduced from the

first Corollary of the last Proposition) $= a' + b' +$

$2c' + 5d' + 14e'$, where the Coefficients prefixed

to a', b', c', d', e' , are the Differences of the Coefficients

of $E' E'$, $D' F'$, $C' G'$, $B' H'$, $A' I'$ and K in the Values

of E^2 and $D F$ taken from Cor. ii. Prop. vi. being $1 - 0$,

$2 - 1$, $6 - 4$, $20 - 15$, $70 - 56$ and $252 - 210$.

COR.

COR. Since $m = \frac{n-r}{r} \times r$ therefore $m + n + 1$
 $= \frac{n-r+1}{n-r} \times r + 1$; and consequently $\frac{r}{r+1} \times$
 $\frac{n-r}{n-r+1} \times E^r$ must always be greater than DF the
 Product of the Coefficients adjacent to E; and hence
 the Fractions are deduced, that in Sir *Isaac Newton's*
 Rule are placed over the Terms of the Equation, which
 multiplied by the Square of the Terms under them,
 must always exceed the Products of the adjacent Terms
 of the Equation, when the Roots are real Quantities:
 For it is manifest that the Fraction to be placed over
 the Term $E x^{n-r}$ according to that Rule is the Quo-
 tient of $\frac{n-r}{r+1}$ divided by $\frac{n-r+1}{r}$.

P R O P. X.

*The same Expressions being allowed as in the pre-
 ceeding Propositions, it will be found in the same man-
 ner that as* $m E^r - \frac{m+n+1}{m+n+1} \times DF = a' + b' + 2c' + 5d' + 14e'$ fo
 $\frac{m-n+1}{m-n+1} \times DF - \frac{m+2n+4}{m+2n+4} \times CG = -b' + 3c' + 9d' + 28e'$
 $\frac{m-2n+4}{m-2n+4} \times CG - \frac{m+3n+9}{m+3n+9} \times BH = -c' + 5d' + 20e'$
 $\frac{m-3n+9}{m-3n+9} \times BH - \frac{m+4n+16}{m+4n+16} \times AI = -d' + 7e'$
 $\frac{m-4n+16}{m-4n+16} \times AI - \frac{m+5n+25}{m+5n+25} \times K = -e$.

These Theorems are easily deduced from the Theo-
 rems given in the second Corollary of Prop. vi. and the
 first Corollary of the viiith Proposition; and the Co-
 efficients prefixed to a', b', c', d', e' , are the Differences
 of the Coefficients of the corresponding Terms in the
 Values of E^2, DF, CG, BH, AI and K in Cor. ii.
 Prop. vi.

COR.

COR. Hence the Products of any two Coefficients, as DF and AI may be compared together when the Sum of the Dimensions of D and F is equal to the Sum of the Dimensions of A and I. Let the Dimensions of A and F be equal to s and m respectively, and

$$\text{let } p = \frac{n-s}{s+1} \times \frac{n-s-1}{s+2} \times \frac{n-s-2}{s+3} \text{ \&c. taking}$$

as many Factors as there are Units in the Difference

$$\text{of the Dimensions of D and A. Let } q = \frac{n-m}{m+1} \times \frac{n-m-1}{m+2} \times \frac{n-m-2}{m+3} \text{ \&c. taking as many Factors}$$

as you took in the Value of p . Then shall $\frac{q}{p} \times$

DF always exceed AI when the Roots of the Equation are real Quantities affected with the same Sign; and this Rule obtains, though the Roots are affected with different Signs when the Coefficients D and F are equal.

P R O P. XI.

The same Things being supposed as in the preceding Propositions.

$$1. \left. \begin{aligned} mE^2 - \overline{m+1} \times 2 DF + \overline{m+4} \times 2 CG - \overline{m+9} \times 2 BH + \overline{m+16} \times 2 AI - \overline{m+25} \times 2 K \end{aligned} \right\} = a'$$

$$2. \left. \begin{aligned} \overline{m-2} \times DF - \overline{m-4} \times 4 CG + \overline{m-9} \times 9 BH - \overline{m-16} \times 16 AI + \overline{m-25} \times 25 K \end{aligned} \right\} = b'$$

$$3. \left. \begin{aligned} \overline{m-2} \times 4 CG - \overline{m-2} \times 9 \times 6 BH + \overline{m-16} \times 20 AI + \overline{m-25} \times 50 K \end{aligned} \right\} = c'$$

$$4. \overline{m-3} \times 9 BH - \overline{m-3} \times 16 \times 8 AI + \overline{m-25} \times 35 K = d'$$

$$5. \overline{m-4} \times 16 \times AI - \overline{m-4} \times 25 \times 10 K = e'$$

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These Theorems follow easily from the third Corollary of the vith Proposition. The first easily appears thus, $a' = m E' E' - 2 D' F' =$ (by that Corollary)
 $m E^2 - 2 m DF + 2 m CG - 2 m BH + 2 m AI - 2 m K.$
 $- 2 DF + 8 CG - 18 BH + 32 AI - 50 K.$

$$= \overline{m E^2} - \overline{m + 1} \times 2 DF + \overline{m + 4} \times 2 CG - \overline{m + 9} \times 2 BH + \overline{m + 16} \times 2 AI - \overline{m + 25} \times 2 K.$$

The other Theorems are deduced from the same Corollary compared with Cor. i. Prop. viii.

P R O P. XII.

The same Things being supposed as in the second Corollary of the viiith Proposition.

$$1. \left. \begin{aligned} m E^2 - \overline{m + 1} \times 2 DF + \overline{m + 4} \times 2 CG - \overline{m + 9} \times \\ 2 BH + \overline{m + 16} \times 2 AI - \overline{m + 25} \times 2 K \end{aligned} \right\} = a'.$$

$$2. \left. \begin{aligned} m' E^2 - 2 m' DF + \overline{m' - 12} \times 2 CG - \overline{m' - 72} \times \\ 2 BH + \overline{m' - 240} \times 2 AI - \overline{m' - 600} \times 2 K \end{aligned} \right\} = a''.$$

$$3. \left. \begin{aligned} m'' E^2 - 2 m'' DF + 2 m'' CG - \overline{m'' + 360} \times \\ 2 BH + \overline{m'' + 360} \times 8 \times 2 AI - \overline{m'' + 360} \times 35 \times 2 K \end{aligned} \right\} = a'''.$$

$$4. \left. \begin{aligned} m''' \times E^2 - 2 DF + 2 CG - 2 BH + \overline{m''' - 750} \times 28 \times \\ AI - \overline{m'''} - 7200 \times 28 \times 2 K \end{aligned} \right\} = a''''.$$

Etc.

These Theorems follow from the third Corollary of the vith Proposition compared with the second Corollary of the eighth Proposition. The first is the same with the first of the last Proposition. The second is demonstrated by substituting in $m' E' E' - 24 C' G' =$

m''' . The Values of $E' E'$ and $C' G'$ given in the third
 Cor.

Cor. of the vith Proposition. The third is found by substituting in $m' E' E' - 720 B' H' = a'''$ the Values of $E' E'$ and $B' H'$; and by a like Substitution these Theorems may be continued.

A General COROLLARY.

From these Propositions a great Variety of Rules may be deduced for discovering when an Equation has imaginary Roots. The Foundation of Sir *Isaac Newton's* Rule is demonstrated in the ninth Proposition, and its Corollary. The seventh Proposition shews that if

$\frac{l-1}{2l} \times E^2$ does not exceed $DF - CG + BH - AI$

+ K , some of the Roots of the Equation must be imaginary; and sometimes this Rule will discover impossible Roots in an Equation, that do not appear by Sir *Isaac Newton's* Rule. These are the only two Rules that have been hitherto published. But the Rules that arise from the Theorems in the eleventh and twelfth Propositions, are preferable to both; because any imaginary Roots that can be discovered by the viith or ixth always appear from the xith and xiith Propositions; and impossible Roots will often be discovered by the xith and xiith Propositions in an Equation, that do not appear in that Equation when examined by the viith and ixth Propositions. The Advantage which the Rules deduced from the xith Proposition, have above those deduced from the preceding Propositions, will be manifest by considering that in the xith Proposition we have the Values of the Quantities a', b', c', d', e' , separately; whereas in the preceding Propositions, we have only the Values of certain Aggregates of these Quantities

joined with the same Signs. Now it is obvious that if these Quantities be separately found positive, any such Aggregates of them must be positive; but these Aggregates may be positive, and yet some of the Quantities a', b', c', d', e' , themselves may be found negative: From which it follows, that if the Roots of the Equation are all affected with the same Sign, and no impossible Roots appear by Proposition xith, none will appear by the preceding Propositions; but that some imaginary Roots may be discovered by Proposition xith, when none appear in the Equation examined by the Propositions that precede the xith. If some of the Roots of the Equation are positive, and some negative (which always easily appears by considering the Signs of the Terms of the Equation) then the xiith Proposition will be in many Cases more apt to discover imaginary Roots in an Equation than those that precede it.

The Rule that flows from the first Theorem of the xith Proposition, obtains when the Roots of the Equation are affected with different Signs, as well as when they all have the same Sign, and it is this; Multiply the Number of the Terms in an Equation that precedes any Term, as $E x^{m-1}$ by the Number of Terms that follow it in the same Equation, and call the Product m . Suppose that $+D, -C, +B, -A, +I$ are the Coefficients preceding the Term $E x^{m-1}$, and that $+F, -G, +H, -I, +K$ are the Coefficients that follow

it; then if $\frac{1}{2} m E^2$ does not exceed $\overline{m+1} \times D F$
 $-\overline{m+4} \times C G + \overline{m+9} \times B H - \overline{m+16} \times A I$
 $+ \overline{m+25} \times K$ the Equation must have some imaginary Roots; where the Coefficients $m+1, m+4, m+9,$
 &c.

&c. are found by adding to m the Squares of the Numbers 1, 2, 3, 4, &c. which shew the Distances of the Coefficients to which they are prefixed, from the Coefficient E. The second Theorem of the xiith Proposition shews, that if $\frac{1}{2}m' E^2$ does not exceed $m' D F$ — $\frac{m' - 12 \times CG}{m' - 72 \times BH} - \frac{m' - 240}{m' - 600 \times K}$, the Equation must have some Roots imaginary.

For an Example, If the four Roots of the Biquadratic Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ are real Quantities, it will follow equally from the vth, viith, ixth, and xith Propositions, that $\frac{3}{8}A^2$ must be greater than B, and that $\frac{3}{8}C^2$ must exceed BD. The viith further shews that $\frac{5}{12}B^2$ must exceed AC — D; the ixth demonstrates that $\frac{4}{9}B^2$ must exceed AC; but our Rule deduced from Prop xi. shews that $2B^2$ must exceed $5AC - 8D$, the excess being $\frac{1}{2}a'$, and the Rule deduced from the second Theorem of the xiith Proposition shews that B^2 must always exceed $2AC + 4D$, the Excess being $\frac{1}{4}a''$. It appears from several preceding Propositions, that if the Roots of the Equation have all the same Sign, then AC must exceed 16D: Let the Excesses $5B^2 - 12AC + 12D = p$, $4B^2 - 9AC = q$, $AC - 16D = s$; and it is plain that $a' (= 4B^2 - 10AC + 16D) = q - s$

$-s = \frac{2}{5} \times \overline{2p - s}$; and that $a'' = q + s = \frac{2}{5} \times \overline{2p + 4s}$. Let us suppose,

1. That s is positive, then it is manifest that if either p or q be negative, a' must also be found negative, and consequently that when the viith or ixth Propositions shew any Roots to be imaginary, the xith Proposition must discover them at the same time. But as a'

($= q - s = \frac{2}{5} \times \overline{2p - s}$) may be found negative

when p and q are both positive, it follows that the Rule we have deduced from the xith Proposition may discover imaginary Roots in an Equation, that do not appear by the preceding Propositions: Thus if you examine the Equation $x^4 - 6x^3 + 10x^2 - 7x + 1$ by Sir *Isaac Newton's* Rule, or by our viith Proposition, no imaginary Roots appear in it from

either. But since $2B^2 - 5AC + 8D (= \frac{1}{2}a') = 200 - 210 + 8 = -2$ is in this Equation negative, it is manifest that two Roots of the Equation must be imaginary. Let us suppose

2. That s is negative, and that from the Signs of the Terms of the Equation, it appears that some Roots are positive and some negative; then in Order to see if the Equation has any imaginary Roots, the most useful Rule is that we deduced from the second Theorem of Prop. xii. *viz.* that if B^2 does not exceed $2AC + 4D$ some of the Roots of the Equation must be imaginary: For the Excess of B^2 above $2AC + 4D$ being

$\frac{1}{4}a'' = \frac{1}{4} \times \overline{q + s} = \frac{1}{10} \times \overline{2p + 4s}$, and s

being

being negative, it is manifest, that if q or p be negative $\frac{1}{4} a''$ must be negative; and that $\frac{1}{4} a''$ may be negative when q and p are both positive; that is, This Rule must always discover some Roots to be imaginary when the viith or ixth Propositions discover any impossible Roots in an Equation; and will very often discover such Roots in an Equation when these Propositions discover none. For Example, if you examine the Equation $x^4 + 5x^3 + 6x^2 - x - 12 = 0$, you will discover no imaginary Roots in it by the viith or ixth Propositions; and though $AC - 16D (= s)$ be negative, it does not follow, that the Equation has any impossible Roots, because it appears from the Signs of the Terms, that the Equation has Roots affected with different Signs. But since $B^2 - 2AC - 4D (= 36 + 10 - 48 = -2)$ is negative, it appears from our Rule, that the Equation must have some imaginary Roots.

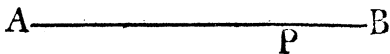
I might shew in the next Place, how the Rules deduced from the xith and xiith Propositions may be extended so as to discover when more than two Roots of an Equation are imaginary, and in general to determine the Number of imaginary Roots in any Equation; but as it would require a long Discussion, and some *Lemmata* to demonstrate this strictly, I shall only observe that these xith and xiith Propositions will be found to be still the most useful of all those we have given for that Purpose. To give one Example of this; If we are to examine the Equation $x^4 - 4ax^3 + 6a^2x^2 - 4ab^2x + b^4 = 0$ by Sir *Isaac Newton's* Rule, it is found to have four impossible Roots when a is greater than b ; for though the Square of the second Term multiplied

ed by $\frac{3}{8}$ be equal to the Product of the first and third

Terms, yet in that Case, in applying Sir *Isaac Newton's* Rule, the Sign — ought to be placed under the second Term, and the same is to be said of the Square of the fourth Term. The Rule deduced from the viiith Proposition shews four Roots imaginary, when a is greater than b , and also when b^2 is greater than $15 a^2$; but a Rule founded on the xith Proposition, shews the four Roots to be imaginary always when a exceeds b , or when b^2 exceeds $9 a^2$; from which the Excellency of this Rule above these two is manifest. I have said so much of Biquadratic Equations, that I must leave it to those that are willing to take the Trouble, to make like Remarks on the higher Sorts of Equations.

In investigating the preceeding Propositions, when I found my self obliged to go through so intricate Calculations, I often attempted to find some more easy Way of treating this Subject. The following was of considerable Use to me, and may perhaps be entertaining to you. By it, I investigate some *maxima* in a very easy Manner, that could not be demonstrated in the common Way with so little Trouble.

LEMMA V. Let the given Line AB be divided any where in P and the Rectangle of the Parts AP and PB will be a *maximum* when these Parts are equal.

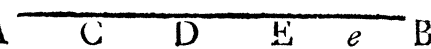


This is manifest from the Elements of *Euclid*.

LEMMA VI. If the Line AB is divided into any Number of Parts AB, CD, DE, EB, the Product of all those Parts multiplied into one another will be a

MAX-

maximum when the Parts are equal amongst themselves. For let the Point D be where you will, it is manifest that if DB be bisected in E, the Product $AC \times CD \times DE \times EB$ will be

greater than $AC \times CD \times De \times eB$  A $\overline{C \quad D \quad E \quad e}$ B

because by the last Lemma $DE \times EB$ is greater than $De \times eB$; and for the same reason AD and CE must be bisected in C and D; and consequently all the Parts AC, CD, DE, EB must be equal amongst themselves, that their Product may be a *maximum*.

LEMMA VII. The Sum of the Products that can be made by multiplying any two Parts of AB by one another is a *maximum* when the Parts are equal. The Sum of these Products is $AC \times CB + CD \times DB + DE \times EB$: Now that $DE \times EB$ may be a *maximum*, DB must be bisected in E by the vth Lemma, and for the same reason AD and CE must be bisected in C and D, that is all the Parts, AC, CD, DE, EB must be equal, that the Sum of all these Products may be a *maximum*.

LEMMA VIII. The Sum of the Products of any three Parts of the Line AB is a *maximum*, when all the Parts are equal. For that Sum is $AC \times CD \times DE + EB \times AC \times CD + AC \times DE + CD \times DE$; and supposing the Point E given, it is manifest that AE must be equally trisected in C and D that $AC \times CD \times DE$ may be a *maximum* by Lemma vi. and that $AC \times CD + AC \times DE + CD \times DE$ may be a *maximum* by Lemma viith. From which it is manifest that all the Parts AC, CD, DE, EB must be equal, that the Sum of the Products of any three of them may be a *maximum*.

LEMMA IX. It is manifest that this way of reasoning is general, and that the Sum of any Quantities being given, the Sum of all the Products that can be

made by multiplying any given Number of them by one another, must be a *maximum* when these Quantities are equal. But the Sum of the Squares, or of any pure Powers of these Quantities, is a *minimum*, when the Quantities are equal.

T H E O R E M.

Suppose $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} + D x^{n-4} - E x^{n-5} \&c. = 0$, *to be an Equation that has not all its Roots equal to one another: Let* r *express the Dimensions of any Coefficient* D , *and let*

$$l = n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \&c. \text{ taking as ma-}$$

ny Factors as there are Units in r ; *then shall* $\frac{l}{n^r} \times A^r$

be always greater than D , *if the Roots of the Equation are real Quantities affected with the same Sign.*

This may be demonstrated from the preceding Propositions: But to demonstrate it from the last Lemma, let us assume an Equation that has all its Roots equal to one another, and the Sum of all its Roots equal to A , the Sum of the Roots of the proposed Equation.

This Equation will be $x - \frac{1}{n} A \Big|^n = 0$, or

$$x^n - A x^{n-1} + n \times \frac{n-1}{2} \times \frac{A^2}{n^2} x^{n-2} - n \times$$

$$\frac{n-1}{2} \times \frac{n-2}{3} \times \frac{A^3}{n^3} x^{n-3} \&c. = 0 \text{ and if } r \text{ ex-}$$

press the Dimensions of the Coefficient of any Term of this Equation (or the Number of Terms which

pre-

precede it) it is manifest that the Term it self will be $I \times \frac{A^r}{n^r} x^{n-r}$: But by the Supposition $D x^{n-r}$ is the

Corresponding Term in the proposed Equation, and D must be the Sum of all the Products that can be made by multiplying as many Roots of that Equation by one another, as there are Units in r ; and $\frac{I A^r}{n^r}$ must be the

Sum of the like Products of the Roots of the other Equation; which must be the greater Quantity by the preceding Lemmata, because its Roots are equal amongst themselves, and their Sum is equal to the Sum of the Roots of the proposed Equation; and the Sum of such Products is a *maximum* when the Roots are equal amongst themselves. By pursuing this Method,

it may be demonstrated that $\frac{2B}{n \times n - 1} \Big| \frac{r}{2} \times I$ must always

exceed the Coefficient prefixed to the Term x^{n-r} in an Equation whose Roots are all real Quantities affected with the same Sign; providing that r be a Number

greater than 2; and also that $\frac{2 \times 3 C}{n \times n - 1 \times n - 2} \Big| \frac{r}{3} \times I$

must exceed the same Coefficient, if r be any Number greater than 3.

It is easy to continue these Theorems.

The third Method which I mentioned in the Beginning of this Letter, is deduced from the Consideration of the Limits of the Roots of Equations; and though it is explained by some Authors already, yet as I demonstrate

monstrate and apply it to this Subject in a different Manner, I shall add a short Account of it.

LEMMA X. If you transform the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ into one that shall have each of its Roots less than the respective Values of x by a given Difference e ; suppose $y = x - e$ or $x = e + y$ and the transformed Equation, the Order of the Terms being inverted, will have this Form.

$$\begin{array}{r} e^4 + 4 e^3 y + 6 e^2 y^2 + 4 e y^3 + y^4 = 0. \\ - A e^4 - 3 A e^3 y - 3 A e y^3 - A y^4 \\ + B e^2 + 2 B e y + B y^2 \\ - C e - C y \\ + D \end{array}$$

Where it is manifest,

1. That the first Term $e^4 - A e^3 + B e^2 - C e + D$ is the Quantity that arises by substituting e in Place of x in the proposed Equation $x^4 - A x^3 + B x^2 - C x + D$.

2. That the Coefficient of the second Term $4 e^3 - 3 A e^2 + 2 B e - C$ is the Quantity that arises by multiplying each Part of the first $e^4 - A e^3 + B e^2 - C e + D$ by the Index of e in that Part, and dividing the Product by e .

3. That the Coefficient of the third Term $6 e^2 - 3 A e + B$ is the Quantity that arises from the preceding Coefficient $4 e^3 - 3 A e^2 + 2 B e - C$ by multiplying each Part by the Index of e in it, and dividing the Product by $2 e$.

4. That the Coefficient of the fourth Term arises in like Manner from the preceding, only you now divide by $3 e$; and in general, the Coefficient of any Term may be deduced from the Coefficient of that Term which precedes it, by multiplying each Part of the

the preceding Coefficient by the Index of e in that Part, and dividing the Product by e and by the Index of y , in the Term whose Coefficient is required.

LEMMA XI. If any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$ be transformed in the same Manner, by supposing $x = y - e$ or $x = e + y$, and consequently $x^n = \overline{e + y}^n$, $Ax^{n-1} = A \times \overline{e + y}^{n-1}$, $Bx^{n-2} = B \times \overline{e + y}^{n-2} \&c.$ The transformed Equation will have this Form, the Order of the Terms being inverted,

$$\begin{aligned}
 & e^n + ne^{n-1}y + n \times \frac{n-1}{2} \times e^{n-2}y^2 \&c. = 0 \\
 & - Ae^{n-1} - \overline{n-1} \times Ae^{n-2}y - \overline{n-1} \times \frac{n-2}{2} \times Ae^{n-3}y^2 \&c. \\
 & + Be^{n-2} + \overline{n-2} \times Be^{n-3}y + \overline{n-2} \times \frac{n-3}{2} \times Be^{n-4}y^2 \&c. \\
 & - Ce^{n-3} - \overline{n-3} \times Ce^{n-4}y - \overline{n-3} \times \frac{n-4}{2} \times Ce^{n-5}y^2 \&c. \\
 & \quad \&c. \qquad \qquad \quad \&c. \qquad \qquad \quad \&c.
 \end{aligned}$$

Where it is manifest,

1. That the first Term $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3} \&c.$ is the Quantity that arises by substituting e in the Place of x in the proposed Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c.$

2. That the Coefficient of the second Term $ne^{n-1} - \overline{n-1} \times Ae^{n-2} + \overline{n-2} \times Be^{n-3} - \overline{n-3} \times Ce^{n-4} \&c.$ is deduced from the preceding $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3} \&c.$ by multiplying each of its Parts by the Index of e in that Part, and dividing by e .

3. That

3. That the Coefficient of the third Term is deduced from the Coefficient of the second Term, by multiplying after the same manner, each of its Parts by the Index of e and dividing by $2 e$. In general, the Coefficient of any Term y^r is deduced from the Coefficient of the preceding Term, that is of y^{r-1} by multiplying every Part of that Coefficient by the Index of e in it, and dividing the Product by $r e$.

LEMMA XII. If you substitute any two Quantities K and L in the Place of x in $x^4 - Ax^3 + Bx^2 - Cx + D$, and the Quantities that result from these Substitutions be affected with contrary Signs, the Quantities K and L must be *Limits* of one or more real Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$. That is, one of these Quantities must be greater, and the other less than one or more Roots of that Equation.

For if you suppose that a, b, c, d , are the Roots of that Equation, then it is plain from the *Genesis* of Equations, that $x^4 - Ax^3 + Bx^2 - Cx + D = \underbrace{x - a} \times \underbrace{x - b} \times \underbrace{x - c} \times \underbrace{x - d}$; and therefore K and L being substituted for x in $\underbrace{x - a} \times \underbrace{x - b} \times \underbrace{x - c} \times \underbrace{x - d}$, the Product becomes in the one Case positive, and in the other negative; so that one of the Factors $x - a, x - b, x - c, x - d$ must have a Sign when K is substituted for x in it, contrary to the Sign which it is affected with when L is substituted in it for x , suppose that Factor to be $x - b$; and since $K - b$ and $L - b$ are Quantities whereof the one is positive, and the other negative, it is manifest that b one of the Roots of the Equation must be less than one, and greater than the other of the two Quantities

tities K and L : So that K and L must be the *Limits* of the Root b .

I say further, that the Root whereof K and L are *Limits*, must be a real Root of the Equation ; for the Product of the Factors that involve impossible Roots in an Equation can never have its Signs changed by substituting any real Quantity whatsoever in place of x ; because the Number of such Roots is always an even Number, and the Product of any two of these Roots such as $x - m - \sqrt{-n}$, and $x - m + \sqrt{-n}$ is $\overline{x - m}^2 + n^2$ which must be always positive, whatever Quantity be substituted for x while n remains positive, that is, while these two Roots are impossible.

LEMMA XIII. If you substitute K and L for x in $x^n - Ax^{n-1} + Bx^{n-2}$ &c. and the Quantities that result be affected with contrary Signs, then shall K and L be the *Limits* of one or more real Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2}$ &c. = 0. This may be demonstrated after the same Manner as the last Lemma.

THEOREM I. If a, b, c, d are the Roots of the Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, they shall be the *Limits* of the Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

Suppose a to be the least Root of the biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, b the second Root, c the third, and d the fourth, and the Values of y in the Equation in the x^{th} Lemma, will be $a - e$, $b - e$, $c - e$, $d - e$; then by substituting successively a, b, c, d for e in that Equation of y , one of the Values of y will vanish in every Substitution, and the first Term of the Equation of y , *viz.* $e^4 - Ae^3 + Be^2 - Ce + D$ vanishing, the Equation will be reduced to a Cubick of this Form.

$$\begin{array}{r}
 4e^3 + 6e^2y + 4ey^2 + y^3 = 0 \\
 - 3Ae^2 - 3Aey - Ay^2 \\
 + 2Be + By \\
 - C
 \end{array}$$

And consequently $4e^3 - 3Ae^2 + 2Be - C$ must be the Product of the three remaining Values of y having its Sign changed; that is, it must be equal to $-\frac{b-a}{a} \times \frac{c-a}{a} \times \frac{d-a}{a}$ when e is supposed equal to a , it must be $-\frac{a-b}{a} \times \frac{c-b}{a} \times \frac{d-b}{a}$ when $e = b$; it must be $-\frac{a-c}{a} \times \frac{b-c}{a} \times \frac{d-c}{a}$ when $e = c$; and it must be $-\frac{a-d}{a} \times \frac{b-d}{a} \times \frac{c-d}{a}$ when $c = d$. Now it is manifest that these Products $\frac{b-a}{a} \times \frac{c-a}{a} \times \frac{d-a}{a}$, $\frac{a-b}{a} \times \frac{c-b}{a} \times \frac{d-b}{a}$, $\frac{a-c}{a} \times \frac{b-c}{a} \times \frac{d-c}{a}$, $\frac{a-d}{a} \times \frac{b-d}{a} \times \frac{c-d}{a}$ must be affected with the Signs $+, -, +, -$ respectively; the first being the Product of three positive Quantities, the second the Product of one negative and two positives, the third the Product of two negatives and one positive, and the fourth the Product of three negatives. Therefore since by substituting a, b, c, d for e in the Quantity $4e^3 - 3Ae^2 + 2Be - C$, it becomes alternately a positive and a negative Quantity, it follows from the last Lemma that a, b, c, d must be the Limits of the Roots of the Equation $4e^3 - 3Ae^2 + 2Be - C = 0$, or of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$.

COR. It follows from this Theorem, that if a, b and c are the three Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$, they must be Limits betwixt a, b, c, d the Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$: For if a, b, c, d are Limits of the
 Roots

Roots $a', b',$ and c' ; these Roots conversely must be *Limits* betwixt a, b, c and d .

THEOREM II. Multiply the Terms of any Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ by any Arithmetical Series of Quantities $l + 4m, l + 3m, l + 2m, l + m, l$, and the Roots of the Biquadratick a, b, c, d will be the *Limits* of the Roots of the Equation that results from that Multiplication that is of the Equation.

$$lx^4 - lAx^3 + lBx^2 - lCx + lD = 0 \\ + 4mx^4 - 3mAx^3 + 2mBx^2 - mCx$$

Suppose that substituting the Roots a, b, c, d of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ successively, for x in $4x^3 - 3Ax^2 + 2Bx - C$, the Quantities that result are $-R, +S, -T, +Z$; while $x^4 - Ax^3 + Bx^2 - Cx + D$ is in every Substitution equal to nothing; and it is manifest that the Quantity

$$+lx^4 - lAx^3 + lBx^2 - lCx + lD \\ + 4mx^4 - 3mAx^3 + 2mBx^2 - mCx$$

will become (when a, b, c, d are substituted successively in it for x) equal to $-mRx, +mSx, -mTx, +mZx$; where the Signs of these Quantities being alternately negative and positive, it follows that a, b, c, d must be *Limits* of that Equation by Lemma xii.

COR. Hence it follows, that a, b, c and d are *Limits* of the Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$, and conversely, that the Roots of this Cubick are *Limits* of the Roots of the biquadratick Equation $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, for multiplying the Terms of this biquadratick Equation by the Arithmetical Progression $0, -1, -2, -3, -4$, the Cubick $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ arises.

THEOREM III. *In general, the Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$, are the Limits of the Roots of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3} \&c. = 0$, or of any Equation that is deduced from it by multiplying its Terms by any Arithmetical Progression $l \mp d, l \mp 2d, l \mp 3d \&c.$ and conversely the Roots of this new Equation will be the Limits of the Roots of the proposed Equation $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$.*

This Theorem is demonstrated from the xith and xiiith Lemmata in the same manner as the preceding Theorems were demonstrated from the xth and xiith. From these Theorems it is easy to infer all that is delivered by the Writers of Algebra on this Subject.

THEOREM IV. *The Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$ will have as many imaginary Roots as the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} - \overline{n-2} \times Bx^{n-3} \&c. = 0$, or the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} \&c. = 0$.*

Suppose that any Root of the Equation $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3} \&c. = 0$, as p becomes imaginary, and the two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$, which by Theorem III. ought to be its *Limits*, cannot both be real Quantities; for it is manifest from the Demonstration of Theorem I. that if they are real Quantities, then being substituted for x in $nx^{n-1} - \overline{n-1} \times Ax^{n-2} + \overline{n-2} \times Bx^{n-3} \&c.$ the Quantities that result must have contrary Signs, and consequently the Root p , whereof they are *Limits*, must be a real Root; which

which is against the Supposition. The same is true of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} \&c. = 0$, for the same Reason.

COR. The biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$, will have two imaginary Roots, if two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ be imaginary; or if two Roots of the Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ be imaginary. But two Roots of the Equation $4x^3 - 3Ax^2 + 2Bx - C = 0$ must be imaginary, when two Roots of the Quadratick $6x^2 - 3Ax + B = 0$, or of the Quadratick $3Ax^2 - 4Bx + 3C = 0$ are imaginary, because the Roots of these quadratick Equations are the *Limits* of the Roots of that Cubick, by the third Theorem; and for the same reason two Roots of the Cubick Equation $Ax^3 - 2Bx^2 + 3Cx - 4D = 0$ must be imaginary, when the Roots of the quadratick $3Ax^2 - 4Bx + 3C = 0$, or of the quadratick $Bx^2 - 3Cx + 6D = 0$ are impossible. Therefore two Roots of the Biquadratick $x^4 - Ax^3 + Bx^2 - Cx + D = 0$ must be imaginary when the Roots of any one of these three quadratick Equations $6x^2 - 3Ax + B = 0$, $3Ax^2 - 4Bx + 3C = 0$, $Bx^2 - 3Cx + 6D = 0$ become imaginary; that is, when $\frac{3}{8}A^2$ is less than B , $\frac{4}{9}B^2$ less than

AC , or $\frac{3}{8}C^2$ less than BD .

COR. II. By proceeding in the same manner, you may deduce from any Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$, as many quadratick Equations as there are Terms excepting the first and last whose Roots must be all real Quantities, if the

proposed Equation has no imaginary Roots. The Quadratick deduced from the three first Terms $x^n - Ax^{n-1} + Bx^{n-2}$ will manifestly have this Form, $\frac{n \times n - 1 \times n - 2 \times n - 3}{n - 2 \times n - 3 \times n - 4} \&c. \times x^2 - \frac{n - 1 \times n - 2 \times n - 3}{n - 4 \times n - 5} \&c. \times Ax + \frac{n - 2 \times n - 3}{n - 4 \times n - 5} \&c. \times B = 0$, continuing the Factors in each till you have as many as there are Units in $n - 2$. Then dividing the Equation by all the Factors $n - 2, n - 3 \&c.$ which are found in each Coefficient, the Equation will become $n \times n - 1 \times x^2 - n - 1 \times 2 Ax + 2 \times 1 \times B = 0$, whose Roots will be imaginary by Prop. i. when $n \times n - 1 \times 2 \times 4 B$ exceeds $\frac{n - 1}{2 n} \times 4 A^2$, or when B exceeds $\frac{n - 1}{2 n} A^2$, so that the

proposed Equation must have some imaginary Roots when B exceeds $\frac{n - 1}{2 n} A^2$; as we demonstrated after

another Manner in the v^{th} Proposition. The Quadratick Equation deduced in the same Manner from the three first Terms of the Equation $Ax^{n-1} - 2Bx^{n-2} + 3Cx^{n-3} \&c. = 0$, will have this Form $\frac{n - 1 \times n - 2 \times n - 3}{n - 2 \times n - 3 \times n - 4} \&c. \times Ax^2 - \frac{n - 2 \times n - 3 \times n - 4}{n - 5 \times n - 6} \&c. \times 2Bx + \frac{n - 3 \times n - 4 \times n - 5}{n - 6 \times n - 7} \&c. \times 3C = 0$; which by dividing by the Factors common to all the Terms, is reduced to $n - 1 \times n - 2 \times Ax^2 - n - 2 \times 4Bx + 6C = 0$, whose Roots must be imaginary when $\frac{2}{3} \times \frac{n - 2}{n - 1} \times B^2$ is less than AC ; and therefore in that case some Roots of the proposed Equation must be imaginary.

CoR. III. In general, let $Dx^{n-r+1} - Ex^{n-r} + Fx^{n-r-1}$ be any three Terms of the Equation, $x^n -$
 Ax

$Ax^{n-1} + Bx^{n-2} \&c. = 0$, that immediately follow one another, multiply the Terms of this Equation first by the Progression $n, n-1, n-2, \&c.$ then by the Progression $n-1, n-2, n-3, \&c.$ then by $n-2, n-3, n-4, \&c.$ till you have multiplied by as many Progressions as there are Units in $n-r-1$: Then multiply the Terms of the Equation that arises, as often by the Progression $0, 1, 2, 3 \&c.$ as there are Units in $r-1$, and you will at length arrive at a Quadratick of this Form,

$$\begin{aligned} & \frac{n-r+1 \times n-r \times n-r-1 \times n-r-2 \&c. \times r-1}{x^{r-2} \times r-3 \times r-4 \&c. D x^2} \\ & - \frac{n-r \times n-r-1 \times n-r-2 \times n-r-3 \&c.}{x^r \times r-1 \times r-2 \times r-3 \&c. \times E x} \\ & + \frac{n-r-1 \times n-r-2 \times n-r-3 \times n-r-4 \&c.}{x^r + 1 \times r \times r-1 \times r-2 \&c. \times F} = 0, \end{aligned}$$

and dividing by the Factors $n-r-1, n-r-2, \&c.$ and $r-1, r-2 \&c.$ which are found in each Coefficient, this Equation will be reduced to $\frac{n-r+1}{x^{n-r} \times 2 \times 1 \times D x^2} - \frac{n-r \times 2 \times r \times 2}{2 \times 1 \times r + 1 \times r} E x +$

imaginary (by Prop i.) when $\frac{n-r}{n-r+1} \times \frac{r}{r+1} \times E^2$ is less than $D F$. From which it is manifest that if you divide each Term of this Series of Fractions $\frac{n}{1}, \frac{n-1}{2},$

$\frac{n-2}{3}, \frac{n-3}{4}, \&c. \frac{n-r+1 \cdot n-r}{r}, \frac{n-r}{r+1}$ by that which precedes it, and place the Quotients above the Terms of the Equation $x^n - A x^{n-1} + B x^{n-2} - C x^{n-3} \&c. =$

&c. = 0, beginning with the second: Then if the Square of any Term multiplied by the Fraction over it be found less than the Product of the adjacent Terms, some of the Roots of that Equation must be imaginary Quantities. There remain many things that might be added on this Subject, but I am afraid you will think I have said as much of it as it deserves; and therefore I shall only add the Demonstration of some Algebraick Rules and Theorems that are very easily deduced from the xith Lemma.

I. The Rule for discovering when two or more Roots of an Equation are equal, immediately follows from that Lemma, Suppose that two Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} \&c. = 0$ are equal, and two Values of y (which is equal always to $x - e$) will be equal. Suppose that e is equal to one of those two equal Values of x ; and two Values of y will vanish, and consequently y^2 must enter each of the Terms of the Equation of y ; and therefore in this Case the first and second Term of the Equation of y in Lemma xith must vanish, that is, $e^n - Ae^{n-1} + Be^{n-2} - Ce^{n-3} \&c. = 0$ and $\frac{ne^{n-1} - n - 1 \times Ae^{n-2} + n - 2 \times Be^{n-3} - n - 3 \times Ce^{n-4} \&c. = 0$ at the same time; and consequently these two Equations must have one Root common, which must be one of those Values of x that were supposed equal to each other. It is manifest therefore that when two Values of x are equal in the Equation $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$, one of them must be a Root of the Equation $nx^{n-1} - \frac{n-1}{n-1} \times Ax^{n-2} + \frac{n-2}{n-2} \times Bx^{n-3} \&c. = 0$.

If three Values of x be supposed equal amongst themselves and to e , then three Values of y ($= x - e$) will vanish, and the first three Terms of the Equation of y

in

in Lemma xi. will vanish, and therefore $\frac{n \times n - 1}{1} \times e^{n-2} - \frac{n-1}{1} \times \frac{n-2}{1} \times A e^{n-3} + \frac{n-2}{1} \times \frac{n-3}{1} \times B e^{n-4} \&c. = 0$; and one of the equal Values of x will be a Root of this last Equation, and two of them will be Roots of the Equation $\frac{n \times n - 1}{1} \times A x^{n-2} + \frac{n-2}{1} \times B x^{n-3} \&c. = 0$. In general, it appears that if the Equation $x^n - A x^{n-1} + B x^{n-2} \&c. = 0$ have as many Roots equal amongst themselves as there are Units in S , then shall as many of those be Roots of the Equation $\frac{n \times n - 1}{1} \times A x^{n-2} + \frac{n-2}{1} \times B x^{n-3} \&c. = 0$ as there are Units in $S - 1$; as many of them shall be Roots of the Equation $\frac{n \times n - 1}{1} \times \frac{n-2}{1} \times A x^{n-3} + \frac{n-2}{1} \times \frac{n-3}{1} \times B x^{n-4} \&c. = 0$, as there are Units in $S - 2$; and so on.

II. The general Rule which Sir *Isaac Newton* has given in the *Article de limitibus Equationum* for finding a *Limit* greater than any of the Values of x immediately follows from the xith Lemma; for it is manifest that if e be such a Quantity as substituted in all the Coefficients of the Equation of y , viz. in $e^n - A e^{n-1} + B e^{n-2} \&c. n e^{n-1} - \frac{n-1}{1} \times A e^{n-2} + \frac{n-2}{1} \times B e^{n-3} \&c. n \times \frac{n-1}{2} \times e^{n-2} - \frac{n-1}{1} \times \frac{n-2}{2} \times A e^{n-3} + \frac{n-2}{1} \times \frac{n-3}{2} \times B e^{n-4} \&c.$ gives the

Quantities that result all positive; then there being no Changes of the Signs of the Equation of y in this case, all its Values must be negative; and since y is always equal to $x - e$ it follows that e must be a greater Quantity than any of the Values of x ; that is, it must be a

Limit greater than any of the Roots of the Equation $x^n - Ax^{n-1} + Bx^{n-2} \&c. = 0$.

III. From this xith Lemma some important Theorems in the Method of *Series*, and of *Fluxions*, and the Resolution of Equations are demonſtrated with great Facility; it is obvious that the Coefficient of the ſecond Term of the Equation of y in that Lemma is the *Fluxion* of the firſt Term divided by the *Fluxion* of e ; the Coefficient of the third Term is the ſecond *Fluxion* of that firſt Term divided by $2\dot{e}^2$; ſuppoſing e to flow uniformly. The third Term is the third *Fluxion* of the firſt Term divided by $2 \times 3 \dot{e}^3$; and ſo on. Therefore ſuppoſing $e^n - Ae^{n-1} + Be^{n-2} \&c. = c$, the

Equation for determining y will be $c + \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{1 \times 2 \dot{e}^2}y^2$

$+ \frac{\ddot{\ddot{c}}}{1 \times 2 \times 3 \dot{e}^3}y^3 \&c. = 0$; and hence, when e is near

the true Value of x , Theorems may be deduced for approximating to y , and conſequently to x , which is ſuppoſed equal to $y + e$.

IV. Let AP (= x) be the Abſciſs and PM (= z) the Ordinate of any Curve BLM; and ſuppoſe any other Abſciſs AK = e and Ordinate KL = c , then

ſhall z (= PM) = $c \mp \frac{\dot{c}}{\dot{e}}y + \frac{\ddot{c}}{2\dot{e}^2}y^2 \mp \frac{\ddot{\ddot{c}}}{2 \times 3 \dot{e}^3}y^3$

$+ \frac{\ddot{\ddot{\ddot{c}}}}{2 \times 3 \times 4 \dot{e}^4}y^4 \&c.$

For let z be ſuppoſed equal to any Series conſiſting of given Quantities, and the Powers of x , as to $Ax^n + Bx^r + Cx^s \&c.$ and ſubſtituting $e \mp y$ for x , we ſhall find after the manner of the xith Lemma,

+ $\frac{\ddot{z}}{2x^2}y^2$ &c. The Area KLMP is equal to the Flu-
ent of xy or of cy , but

$$cy = xy \pm \frac{\dot{z}}{x}y\dot{y} + \frac{\ddot{z}}{2x^2}y^2\dot{y} \pm \frac{\ddot{\dot{z}}}{2 \times 3 x^2}y^3\dot{y} \&c.$$

and $xy = cy \mp \frac{\dot{c}}{e}y\dot{y} + \frac{\ddot{c}}{2e^2}y^2\dot{y} \mp \frac{\ddot{\dot{c}}}{2 \times 3 e^2}y^3\dot{y} \&c.$

And consequently by finding the Fluents

$$KLMP = cy \mp \frac{\dot{c}}{2e}y^2 + \frac{\ddot{c}}{2 \times 3 e^2}y^3 \mp \frac{\ddot{\dot{c}}}{2 \times 3 \times 4 e^3}y^4 \&c.$$

$$\text{or } KLMP = xy \pm \frac{\dot{z}}{2x}y^2 + \frac{\ddot{z}}{2 \times 3 x^2}y^3 \pm \frac{\ddot{\dot{z}}}{2 \times 3 \times 4 x^3}y^4 \&c.$$

This last is the Theorem published by the learned Mr. *Bernouilli* in the *Acta Lipsæ* 1694. It is now high Time to conclude this long Letter; I beg you may accept of it as a Proof of that Respect and Esteem with which

I am,

S I R,

Your most Obedient,

Most Humble Servant,

Colin Mac Laurin.