A Method to Derive an Expression for Summations of Natural Numbers (i) Raised to a Positive Integer Exponent (P) as (i) is Indexed from 1 to N.

Before stepping into the method described in this paper, we need to first examine the origin of the identities used in the development of this method.

Lets first define a function G(x) as being the difference in another function F(x) as the value of x increases by 1.

So $$G(x) = F(x+1) - F(x)$$

This expression can be manipulated by use of simple mathematical properties to change its appearance as below:

$F(x + 1) - F(x) = G(x)$	Given relationship
$F(x + 1) = G(x) + F(x)$	Addition of F(x) to both sides of the expression
$F(x + 1) = F(x) + G(x)$	Commutative property of addition
$F(x + 2) - F(x + 1) = G(x + 1)$	Given relationship
$F(x + 2) = G(x + 1) + F(x + 1)$	Addition of F(x+1) to both sides of the expression
$F(x + 2) = F(x + 1) + G(x + 1)$	Commutative property of addition
$F(x + 2) = F(x) + G(x) + G(x + 1)$	Substitution of F(x+1)

The above manipulation of the expression can be repeated until we have

$$F(x + m) = F(x) + G(x) + G(x + 1) + G(x + 2) + ... + G(x + m - 1)$$

and this form of the expression can be abbreviated as

$$F(x + m) = F(x) + \sum_{i = x}^{(x + m - 1)} G(i)$$

If we let $N = x + m - 1$ then $N + 1 = x + m$ and the above expression simplifies further to

$$F(N + 1) = F(x) + \sum_{i = x}^{N} G(i)$$

and if you take the special case where x = 1, then we have a form of the relationship that is key to development of this method

$$F(N + 1) = F(1) + \sum_{i = 1}^{N} G(i)$$

and the final form is obtained by subtracting F(1) from both sides

$$\sum_{i = 1}^{N} G(i) = F(N + 1) - F(1)$$

The final form of the expression for the relationship between G(x) and F(x) makes it apparent that if we are given the expression for F(x), we can then calculate the sum of G(i) without generating each value resulting from i = 1 to N. This observation focuses on the objective of this paper and starts us off with some direction to solving this problem.

Since the function G(x) is determined from the expression of F(x), we should investigate resultant expressions of G(x) from known F(x) expressions. For simplicity, let's assume that F(x) is a polynomial in standard form. A table below shows that an easy to prove pattern exists.

Known F(x)	Resultant G(x) = F(x + 1) - F(x)
constant (A, B, C,...)	0
x	1
Ax	A
Ax + B	A + 0
x^2	2x + 1
Ax^2	A(2x + 1)
$Ax^2 + Bx + C$	A(2x + 1) + B + 0
x^p	$(x + 1)^p - x^p$ = binomial expansion of $(x + 1)^p$ minus the first term

We can see from the pattern above that if F(x) is expressed as a polynomial, each term within the expression has an additive effect on expression for G(x). Below is a list of observations about the pattern which will prove useful later.

1) G(x) can be determined by evaluating each term of F(x) as a separate function and then adding the accumulative differences from all of the terms together.

$F(x) = T_1(x) + T_2(x) + ... + T_{d+1}(x)$ where d is the highest degree of x in F(x), and
$G(x) = T_1(x + 1) - T_1(x) + T_2(x + 1) - T_2(x) + ... + T_{d+1}(x + 1) - T_{d+1}(x)$

2) If $T(x) = x^p$ then $T(x + 1) - T(x)$ is equal to the binomial expansion of $(x + 1)^p$ minus x^p
or

$$T(x + 1) - T(x) = \sum_{i=1}^{p} \left(p! / [i!\,(p - i)!]\right) x^{(p - i)}$$

3) The coefficient of T(x) will factor out of T(x + 1) - T(x) and remain a factor for the difference.

If $T(x) = Ax^p$ then $T(x + 1) - T(x) = A[(x + 1)^p - x^p]$ which is equal to the product of A and the binomial expansion of $(x + 1)^p$ minus the first term.

Now we have the problem of working in the other direction. Given an expression for G(x), how do we determine the expression for F(x)?

If we still think of G(x) as a function determined from another function F(x), we can back track how the expression for G(x) took on its form. Before G(x) obtained a standard polynomial form, several terms had to be expanded, combined with like terms, and then written in descending powers of x. The crude form of G(x) may be perceived as

$$G(x) = A_1[(x + 1)^p - x^p] + A_2[(x + 1)^{(p - 1)} - x^{(p - 1)}] + ... + A_{(d + 1)}$$

The process of expanding and combining like terms can make each part of this expression undergo many changes. However, the first term in the standardized form of G(x) was never combined with a like term in the process, because none of the other parts of the expression produced a power of x high enough to combine with it. If you take the time to investigate, you will learn that as you track the generation of terms in G(x), terms with lower degrees of x are the result of combining many like terms and terms with higher degrees of x are the result of combining fewer like terms.

The origin of the first term in G(x) is from the $A_1[(x+1)^p - x^p]$ part of the crude expression. Since x^p cancels, the degree of the first term in standardized G(x) must be equal to (p - 1). We now know the degree of the expression for F(x), which is equal to p. Since the first term in G(x) was never combined with another term, the coefficient of this term is only the result of multiplying A_1 by the coefficient from the binomial expansion of $(x+1)^p$. Since we know the value of p we can determine the coefficient from the binomial expansion and then the value of A_1, which is the coefficient of the first term in F(x). Once we have the values for p and A_1, we can subtract the effect of the first term of F(x) from the generation of G(x). We then have,

$$G(x) - [T_1(x+1) - T_1(x)] = A_2[(x+1)^{(p-1)} - x^{(p-1)}] + A_3[(x+1)^{(p-2)} - x^{(p-2)}] + \dots + A_{(d+1)}$$

The new terms obtained in the expression for $G(x) - [T_1(x+1) - T_1(x)]$ are the result of expanding and combining like terms from the remaining portion of the crude expression of G(x). As before, the new first term, was never combined with a like term in this process. Since the characteristics of generating the terms of $G(x) - [T_1(x+1) - T_1(x)]$ and $G(x) - \sum [T(x+1) - T(x)]$ are the same as for G(x), we can repeat the procedure described above until all of the terms in the expression of F(x) are determined.

Example: Find an expression for $\sum_{x=1}^{N} x^3$

If we let $G(x) = x^3$ we only need to determine an expression for F(x) to solve this problem, since we have the identity

$$\sum_{x=1}^{N} G(x) = F(N+1) - F(1)$$

x^3	Given expression for G(x) tells us that p = 4
$- \quad \underline{{}^1/_4\,(4x^3 + 6x^2 + 4x + 1)}$	Expansion of $(x+1)^4$ minus x^4, letting $A_1 = {}^1/_4$ makes the first term drop out
$- {}^3/_2\,x^2 - x - {}^1/_4$	New expression after subtraction of $[T_1(x+1) - T_1(x)]$
$- \quad \underline{-{}^1/_2\,(3x^2 + 3x + 1)}$	Expansion of $(x+1)^3$ minus x^3, letting $A_2 = -{}^1/_2$ makes the next term drop out
${}^1/_2\,x + {}^1/_4$	New expression after subtraction of $[T_2(x+1) - T_2(x)]$
$- \quad \underline{{}^1/_4\,(2x + 1)}$	Expansion of $(x+1)^2$ minus x^2, letting $A_3 = {}^1/_4$ makes the next term drop out
0	Stop, since the difference remaining is zero, all of the terms of F(x) have been found

As determined above

$$F(x) = {}^1/_4\,x^4 - {}^1/_2\,x^3 + {}^1/_4\,x^2$$

Using this expression in the identity above, we will obtain the solution to the problem.

$$\sum_{x=1}^{N} x^3 = [\,{}^1/_4\,(N+1)^4 - {}^1/_2\,(N+1)^3 + {}^1/_4\,(N+1)^2\,] - [\,{}^1/_4\,(1)^4 - {}^1/_2\,(1)^3 + {}^1/_4\,(1)^2\,]$$

In my experience, F(1) has always been equal to zero, but I have not attempted a deductive proof.

Simplification of the solution by expanding binomials, combining like terms, and factoring what doesn't cancel, provides the final polished solution below

$$\sum_{x=1}^{N} x^3 = \frac{N^2 (N+1)^2}{4}$$

Below is a listing of a few polynomial representations of the summation of x raised to the integer exponent p, if x is indexed from 1 to N.

$$\sum_{x=1}^{N} x^p$$

p	Polynomial expression in method generated form					
1	$\frac{1}{2}\ (N+1)^2$	$-\ \frac{1}{2}\ (N+1)$				
2	$\frac{1}{3}\ (N+1)^3$	$-\ \frac{1}{2}\ (N+1)^2$	$+\ \frac{1}{6}\ (N+1)$			
3	$\frac{1}{4}\ (N+1)^4$	$-\ \frac{1}{2}\ (N+1)^3$	$+\ \frac{1}{4}\ (N+1)^2$			
4	$\frac{1}{5}\ (N+1)^5$	$-\ \frac{1}{2}\ (N+1)^4$	$+\ \frac{1}{3}\ (N+1)^3$	$-\ \frac{1}{30}\ (N+1)$		
5	$\frac{1}{6}\ (N+1)^6$	$-\ \frac{1}{2}\ (N+1)^5$	$+\ \frac{5}{12}\ (N+1)^4$	$-\ \frac{1}{12}\ (N+1)^2$		
6	$\frac{1}{7}\ (N+1)^7$	$-\ \frac{1}{2}\ (N+1)^6$	$+\ \frac{1}{2}\ (N+1)^5$	$-\ \frac{1}{6}\ (N+1)^3$	$+\ \frac{1}{42}\ (N+1)$	
7	$\frac{1}{8}\ (N+1)^8$	$-\ \frac{1}{2}\ (N+1)^7$	$+\ \frac{7}{12}\ (N+1)^6$	$-\ \frac{7}{24}\ (N+1)^4$	$+\ \frac{1}{12}\ (N+1)^2$	
8	$\frac{1}{9}\ (N+1)^9$	$-\ \frac{1}{2}\ (N+1)^8$	$+\ \frac{2}{3}\ (N+1)^7$	$-\ \frac{7}{15}\ (N+1)^5$	$+\ \frac{2}{9}\ (N+1)^3$	$-\ \frac{1}{30}\ (N+1)$
9	$\frac{1}{10}\ (N+1)^{10}$	$-\ \frac{1}{2}\ (N+1)^9$	$+\ \frac{3}{4}\ (N+1)^8$	$-\ \frac{7}{10}\ (N+1)^6$	$+\ \frac{1}{2}\ (N+1)^4$	$-\ \frac{3}{20}\ (N+1)^2$