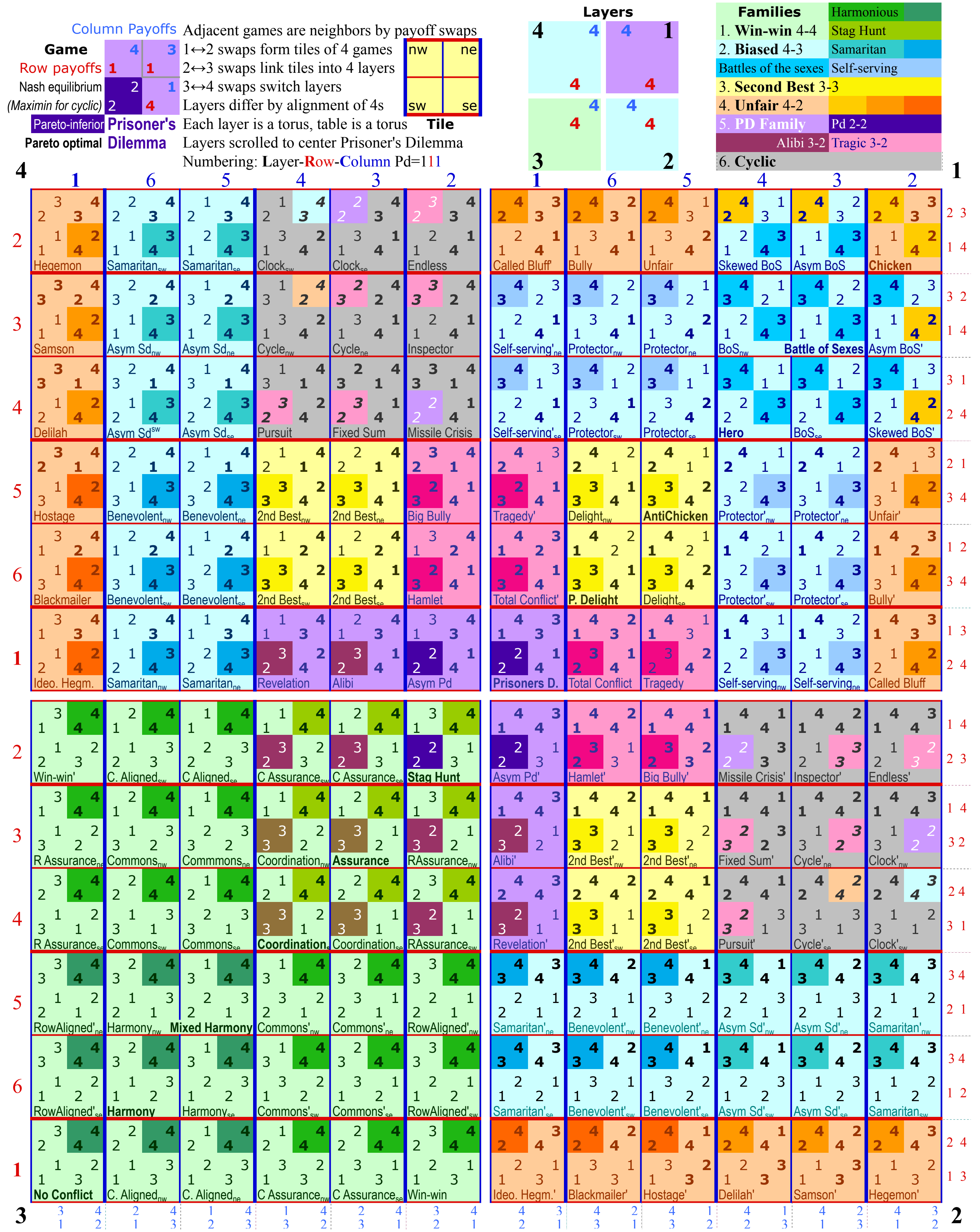
Topology of 2x2 Ordinal Games with Payoff Families



Adjacent games are neighbors by payoff swaps
1↔2 swaps form tiles of 4 games
2↔3 swaps link tiles into 4 layers
3↔4 swaps switch layers
Layers differ by alignment of 4s
Each layer is a torus, table is a torus
Layers scrolled to center Prisoner's Dilemma
Numbering: Layer-Row-Column Pd=111

Families
1. **Win-win** 4-4
2. **Biased** 4-3
Battles of the sexes
3. **Second Best** 3-3
4. **Unfair** 4-2
5. **PD Family**
Alibi 3-2
6. **Cyclic**

Harmonious
Stag Hunt
Samaritan
Self-serving
Pd 2-2
Tragic 3-2

Layer 4

	1	6	5	4	3	2
2	Hegemon	Samaritan$_{sw}$	Samaritan$_{se}$	Clock$_{sw}$	Clock$_{se}$	Endless
3	Samson	Asym Sd$_{nw}$	Asym Sd$_{ne}$	Cycle$_{nw}$	Cycle$_{ne}$	Inspector
4	Delilah	Asym Sd^{sw}	Asym Sd$_{se}$	Pursuit	Fixed Sum	Missile Crisis
5	Hostage	Benevolent$_{nw}$	Benevolent$_{ne}$	2nd Best$_{nw}$	2nd Best$_{ne}$	Big Bully
6	Blackmailer	Benevolent$_{sw}$	Benevolent$_{se}$	2nd Best$_{sw}$	2nd Best$_{se}$	Hamlet
1	Ideo. Hegm.	Samaritan$_{nw}$	Samaritan$_{ne}$	Revelation	Alibi	Asym Pd

Layer 1

	1	6	5	4	3	2	
2	Called Bluff'	Bully	Unfair	Skewed BoS	Asym BoS	**Chicken**	2 3 1 4
3	Self-serving'$_{ne}$	Protector$_{nw}$	Protector$_{ne}$	BoS$_{nw}$	**Battle of Sexes**	Asym BoS'	3 2 1 4
4	Self-serving'$_{se}$	Protector$_{sw}$	Protector$_{se}$	**Hero**	BoS$_{se}$	Skewed BoS'	3 1 2 4
5	Tragedy'	Delight$_{nw}$	**AntiChicken**	Protector'$_{nw}$	Protector'$_{ne}$	Unfair'	2 1 3 4
6	Total Conflict'	**P. Delight**	Delight$_{se}$	Protector'$_{sw}$	Protector'$_{se}$	Bully'	1 2 3 4
1	**Prisoners D.**	Total Conflict	Tragedy	Self-serving$_{nw}$	Self-serving$_{ne}$	Called Bluff	1 3 2 4

Layer 3

	1	6	5	4	3	2
2	Win-win'	C. Aligned$_{sw}$	C Aligned$_{se}$	C Assurance$_{sw}$	C Assurance$_{se}$	**Stag Hunt**
3	R Assurance$_{ne}$	Commons$_{nw}$	Commmons$_{ne}$	Coordination$_{nw}$	**Assurance**	RAssurance$_{nw}$
4	R Assurance$_{se}$	Commons$_{sw}$	Commons$_{se}$	**Coordination**	Coordination$_{se}$	RAssurance$_{sw}$
5	RowAligned'$_{ne}$	Harmony$_{nw}$	**Mixed Harmony**	Commons'$_{nw}$	Commons'$_{ne}$	RowAligned'$_{nw}$
6	RowAligned'$_{se}$	**Harmony**	Harmony$_{se}$	Commons'$_{sw}$	Commons'$_{se}$	RowAligned'$_{sw}$
1	**No Conflict**	C. Aligned$_{nw}$	C. Aligned$_{ne}$	C Assurance$_{nw}$	C Assurance$_{se}$	Win-win
3	3 4 1 2	2 4 1 3	1 4 2 3	1 4 3 2	2 4 3 1	3 4 2 1

Layer 2

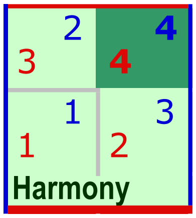
2	Asym Pd'	Hamlet'	Big Bully'	Missile Crisis'	Inspector'	Endless'	1 4 2 3
3	Alibi'	2nd Best'$_{nw}$	2nd Best'$_{ne}$	Fixed Sum'	Cycle'$_{ne}$	Clock'$_{nw}$	1 4 3 2
4	Revelation'	2nd Best'$_{sw}$	2nd Best'$_{se}$	Pursuit'	Cycle'$_{se}$	Clock'$_{sw}$	2 4 3 1
5	Samaritan'$_{ne}$	Benevolent'$_{nw}$	Benevolent'$_{ne}$	Asym Sd'$_{nw}$	Asym Sd'$_{ne}$	Samaritan'$_{nw}$	3 4 2 1
6	Samaritan'$_{se}$	Benevolent'$_{sw}$	Benevolent'$_{se}$	Asym Sd'$_{sw}$	Asym Sd'$_{se}$	Samaritan$_{sw}$	3 4 1 2
1	Ideo. Hegm.'	Blackmailer'	Hostage'	Delilah'	Samson'	Hegemon'	2 4 1 3
2	4 3 2 1	4 2 3 1	4 1 3 2	4 1 2 3	4 2 1 3	4 3 1 2	

To find a game: Make ordinal 1<2<3<4. Put column with Row's 4 right, row with Column's 4 up. Find layer by alignment of 4s; then intersection of Row & Column payoffs.

Based on Robinson & Goforth 2005 *The Topology of 2x2 Games: A New Periodic Table.* See www.cs.laurentian.ca/dgoforth/home.html www.bryanbruns.com v2.6

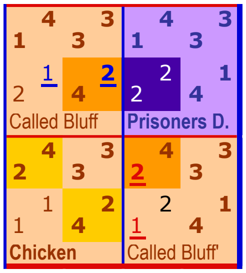
A Brief Overview of the Topology of 2x2 Ordinal Games

The Robinson-Goforth topology conveniently and elegantly displays relationships between ordinal 2x2 games, where two players each have two strategies and four differently ranked preferences for the outcomes.



Harmony. *366 Row or Column's choice benefits the other, and leads to win-win*

Swaps in the two lowest payoffs (1↔2) form *tiles* of four games closest to each other in the payoff space. Swaps in middle payoffs (2↔3) and additional low swaps form *layers*. The four layers differ by the alignment of 4s. Each layer is a torus. Scrolling Prisoner's Dilemma (Game 111) to the center helps visualize the structure of the topology (see figures on facing page).



Prisoner's Dilemma Tile*: Swaps in lowest two payoffs transform Prisoner's Dilemma into Chicken.*

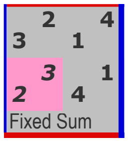
Prisoner's Dilemma: 111 If both prisoners both keep silent, they only get a light sentence, their second-best outcome (3,3). If only one confesses, he goes free and the other gets a long sentence. But the dominant strategy for each is to confess (defect), ending up with both getting a medium-length sentence, the second-worst outcome (2,2).

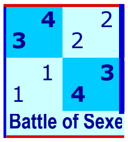
Chicken: 122 If both play a Hawk strategy, both die (1,1). If both play Dove, both get second-best (3,3). Two Hawk-Dove Nash Equilibria yield very unequal outcomes (4,2 and 2,4).

The twelve symmetric games, where players face identical payoffs, form a diagonal axis from southwest to northeast. Row payoffs are the same across rows, and column payoffs the same down columns. The sixty-six asymmetric games on either side of this axis of symmetry switch positions for Row and Column, leaving 78 unique games.

The lower three rows in each layer have dominant strategies for Row (with higher payoffs whatever Column does), as to the left three columns for Column. Based on dominant strategies, three-fourths of games have a single Nash Equilibrium, (a pair strategies that are best replies to each other). *Battles of the Sexes* and *Stag Hunts*, including asymmetric variants, have no dominant strategies and two equilibria. *Cyclic* games have no dominant strategies and (for ordinal games) no equilibria.

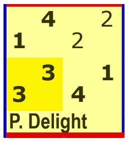
Fixed Rank-sum*. 443 Ordinal equivalent of Zero-sum; gains for one are matched by losses for the other. Cyclic, since one player always has an incentive to move. Maximin strategies avoid the worst payoff.*



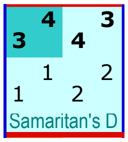


Battle of the Sexes*. 133 A couple prefers doing something together, but each has a different first choice*

Prisoner's Dilemma Family games have a Pareto-superior outcome that both players would prefer to the Nash Equilibrium. The family can extend to include *Tragic* games, also with a poor equilibrium but lacking a better alternative. In *Second-Best* games such as Prisoner's Delight (Game 144) both players can achieve their second-ranked preference (3,3). *Biased* games with high but unequal (4-3) equilibria form the largest payoff family. In *Samaritan* games, the largest subfamily, a player with a dominant strategy gets their second-ranked payoff. In *Self-serving* games, the dominant strategy gets the best of a biased equilibrium. *Unfair* games have highly unequal (4-2) equilibria.



Prisoner's Delight*. 166 Incentives lead to second-best payoffs*



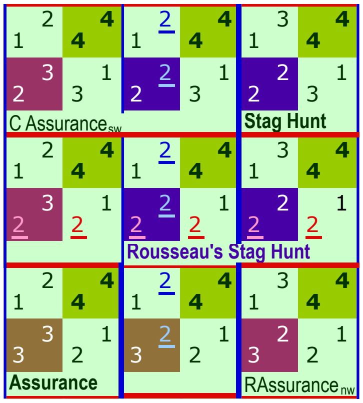
Samaritan's Dilemma*. 262 Column could help himself, without Row's aid. Row prefers that both invest but prefers to help whatever Row does, while Column most prefers to be helped and do less.*

Swaps in high payoffs (3↔4) link layers. In six *hotspots*, the alignment of high payoffs makes swaps for Row or Column connect the same two tiles, double-linking two layers, as in the Layer 1-3 Hotspot where Battle of the Sexes games turn into Stag Hunts. In six *pipes*, high swaps for Row or Column link to different tiles, weaving together four tiles on four layers. At the center of the display, high swaps, one for each player, convert Prisoner's Dilemma into an Asymmetric Prisoner's Dilemma (412 or 221) and then into Stag Hunt (322). Most games can be converted to win-win through one or two swaps.

Most games have mixed interests: a strategy may help or hurt other player. Games of pure cooperation or conflict are less frequent, and fixed-sum (zero-sum) games even rarer. In *Type* games, such as Unfair (152), incentives make one player kind and the other cruel.

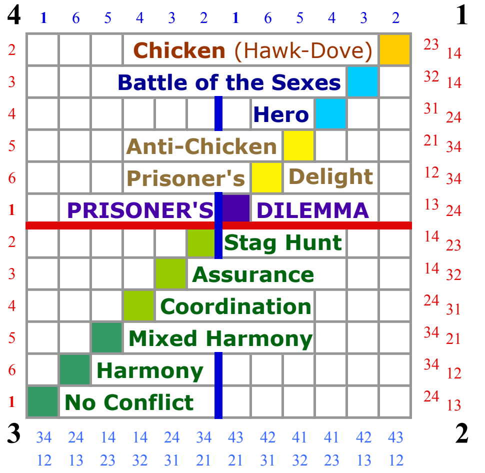
Games with ties lie between the strict ordinal games, linked by *half-swaps* that make (or break) ties in preferences. For games with interval-scale or real payoffs, normalized versions can be mapped into the topology. The topology, and its index coordinates for uniquely identifying games, can also extend to multiple players, moves, plays, and preference forms.

Rousseau's Stag Hunt*. A hunter could choose to cooperate in the risky hunt for a stag, or could catch a hare regardless of what the other hunter does. The game with ties lies between Assurance (333) and the strict Stag Hunt (322).*



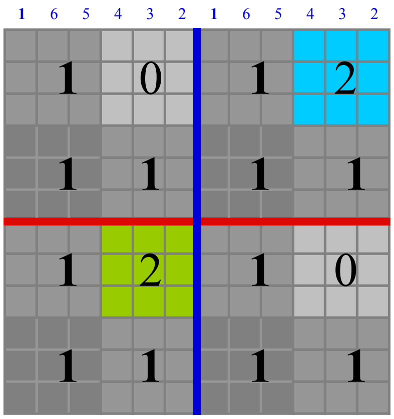
Structures in the Topology of 2x2 Ordinal Games

a. Twelve Symmetric Games on Diagonal



Row payoffs same across row, column same down column

b. Dominant Strategies and Nash Equilibria

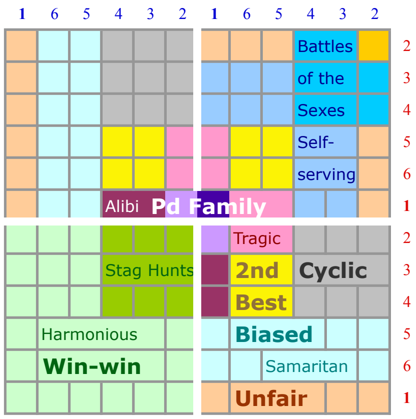


Row has dominant strategies in lower 3 rows,

column in left 3 columns, of each layer

Number of Nash Equilbria: 0, 1, or 2

c. Payoff Familes and Subfamilies

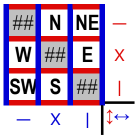


Based on payoffs at Nash Equilibria

Most games are asymmetric

Most equilibria have unequal payoffs

d. High swaps (3↔4) link layers



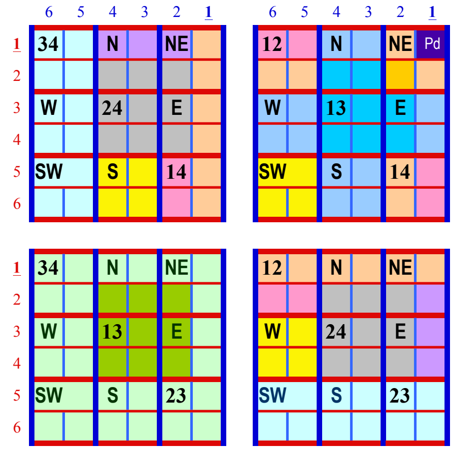
moving to equivalently located tiles

Hotspots (##) double-link 2 tiles

(eg. 13: Coordination-Battle of Sexes)

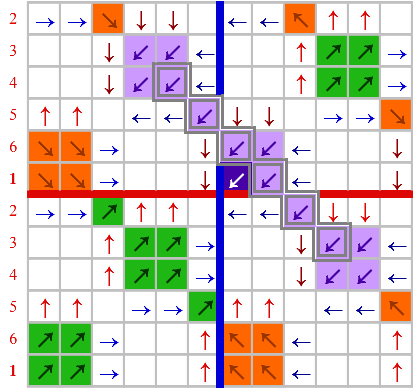
Pipes (N,NE,E,W,SW,S) link four tiles on four layers

(Prisoner's Dilemma scrolled to northeast to unify tiles)

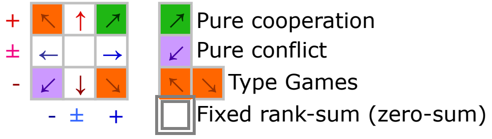


High swaps realign 4s & switch row or column in tile

e. Interests Aligned, Mixed, or Opposed



Inducement correspondences

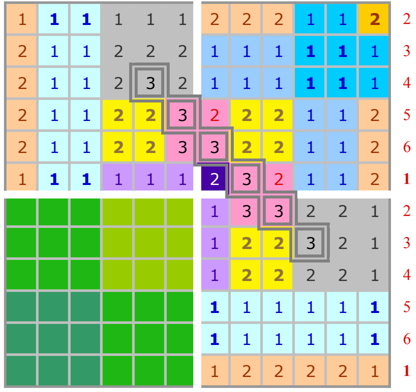


see Schelling 1963 *The Strategy of Conflict*

Greenberg 1990 *The Theory of Social Situations*

Robinson & Goforth 2005 *The Topology of 2x2 Games* Ch. 8

f. Remediability: Swaps to Win-win



2	1	1	1	1	1	1	2	2	2	1	1	2
3	2	1	1	2	2	2	1	1	1	1	1	1
4	2	1	1	2	3	2	1	1	1	1	1	1
5	2	1	1	2	2	3	2	2	2	1	1	2
6	2	1	1	2	2	3	3	2	2	1	1	2
1	1	1	1	1	1	1	2	3	2	1	1	2
2							1	3	3	2	2	1
3							1	2	2	3	2	1
4							1	2	2	2	2	1
5							1	1	1	1	1	1
6							1	1	1	1	1	1
1							1	2	2	2	2	1

\# Number of swaps to reach win-win

(may include 2↔3 and 1↔2 swaps)

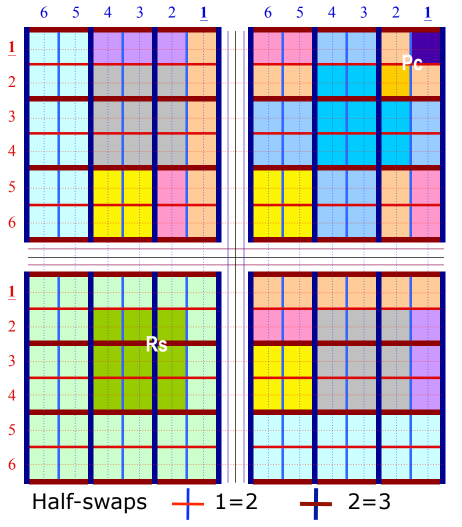
for Pareto-efficient pathways

(no swap leaves a player at a lower rank)

\# **Bold** = each player has a pathway

Fixed rank-sum games are hardest to remedy

g. Games with Ties are within the Topology



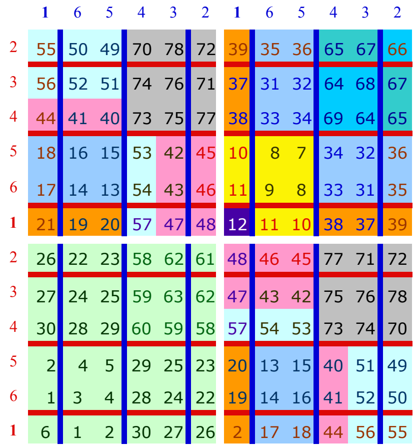
Ordinal games are at each intersection (nodes/vertices)

Games with ties (non-strict) lie between strict

ordinal games (as do other normalized games)

See Robinson, Goforth & Cargill 2007

h. Rapoport, Guyer & Gordon Taxonomy

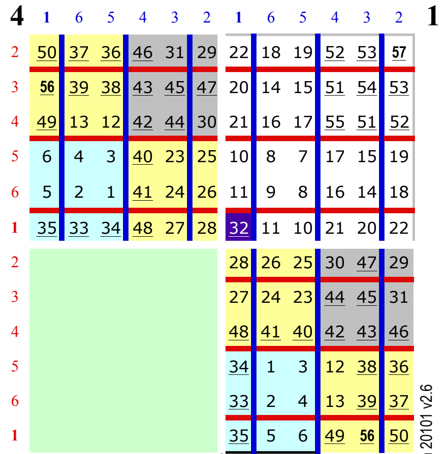


	1	6	5	4	3	2	1	6	5	4	3	2
2	55	50	49	70	78	72	39	35	36	65	67	66
3	56	52	51	74	76	71	37	31	32	64	68	67
4	44	41	40	73	75	77	38	33	34	69	64	65
5	18	16	15	53	42	45	10	8	7	34	32	36
6	17	14	13	54	43	46	11	9	8	33	31	35
1	21	19	20	57	47	48	12	11	10	38	37	39
2	26	22	23	58	62	61	48	46	45	77	71	72
3	27	24	25	59	63	62	47	43	42	75	76	78
4	30	28	29	60	59	58	57	54	53	73	74	70
5	2	4	5	29	25	23	20	13	15	40	51	49
6	1	3	4	28	24	22	19	14	16	41	52	50
1	6	1	2	30	27	26	2	17	18	44	56	55

No equilibria	Unstable
No conflict	Stable
Strongly stable deficient	Strongly stable
2 equilibria, eq. solution	Force-vulnerable
2 equilibria, non-eq. solution	Threat-vulnerable

Adapted from Robinson & Goforth 2003; *see* Rapoport et al. 1976 *The 2x2 Games*

i. Brams Typology and Game Numbers



	1	6	5	4	3	2	1	6	5	4	3	2
2	50	37	36	46	31	29	22	18	19	52	53	57
3	56	39	38	43	45	47	20	14	15	51	54	53
4	49	13	12	42	44	30	21	16	17	55	51	52
5	6	4	3	40	23	25	10	8	7	17	15	19
6	5	2	1	41	24	26	11	9	8	16	14	18
1	35	33	34	48	27	28	32	11	10	21	20	22
2							28	26	25	30	47	29
3							27	24	23	44	45	31
4							48	41	40	42	43	46
5							34	1	3	12	38	36
6							33	2	4	13	39	37
1							35	5	6	49	56	50

3 Number of non-myopic equilibria (NMEs)

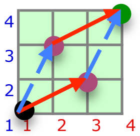
3 2 1

See Brams 1994 *Theory of Moves*

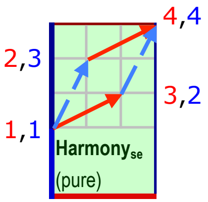
2

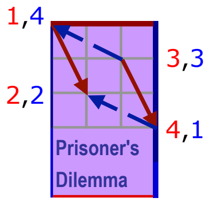
Not cyclic
Weakly cyclic
Moderately cyclic
Strongly cyclic
No conflict

Topology of 2x2 Ordinal Games: Order Graphs



Order Graph



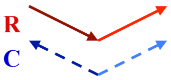


Payoffs are at intersection of lines, arrows show inducement vectors

Row moves right, given Column's strategy

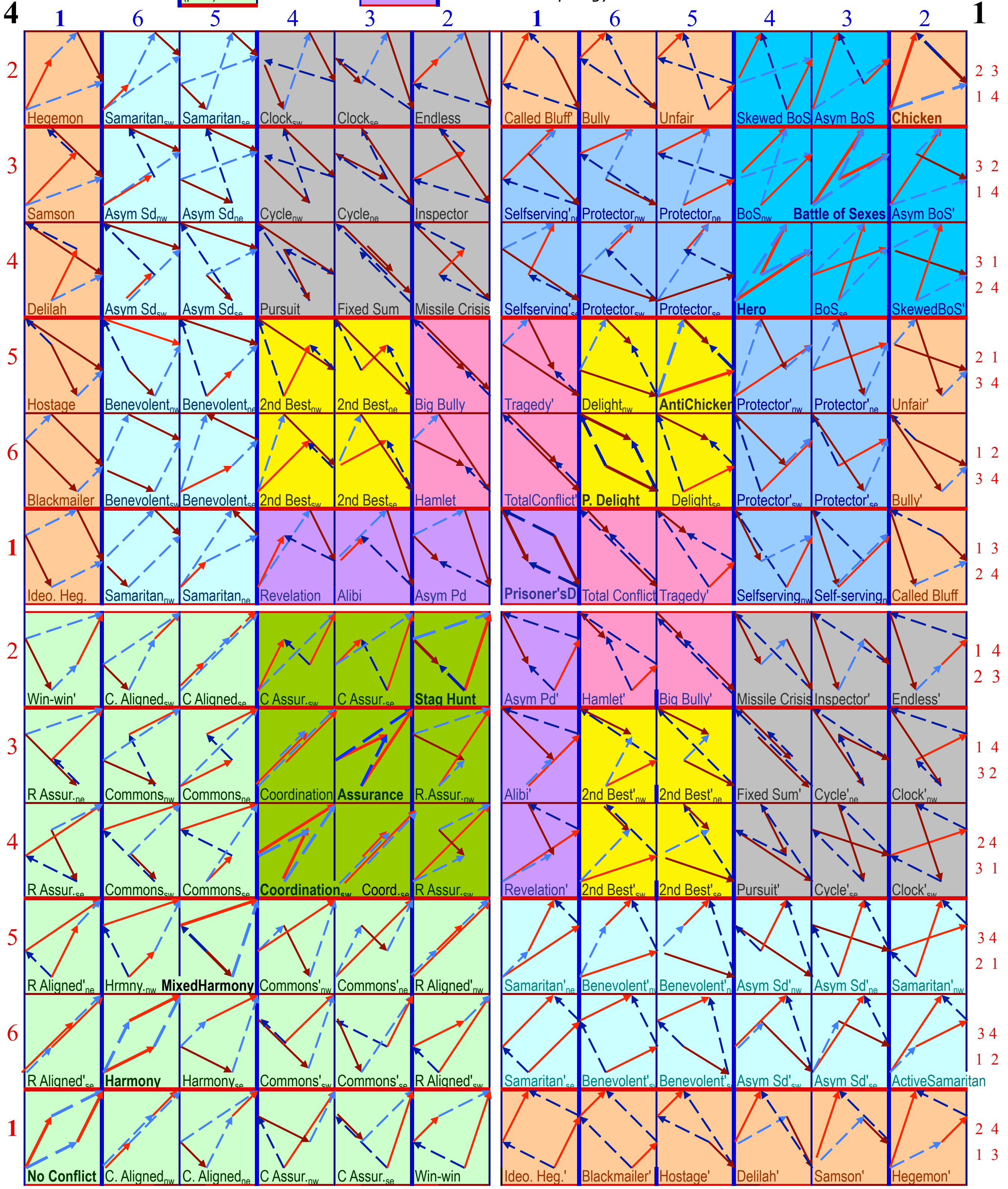
Column moves up, given Row's strategy

Row up/down, Column left/right show impact on other player's payoff



Based on Robinson and Goforth 2005

The Topology of 2x2 Games: A New Periodic Table



4	1	6	5	4	3	2	1	6	5	4	3	2	1
2	Hegemon	Samaritan$_{sw}$	Samaritan$_{se}$	Clock$_{sw}$	Clock$_{se}$	Endless	Called Bluff'	Bully	Unfair	Skewed BoS	Asym BoS	**Chicken**	2 3 1 4
3	Samson	Asym Sd$_{nw}$	Asym Sd$_{ne}$	Cycle$_{nw}$	Cycle$_{ne}$	Inspector	Selfserving'$_{ne}$	Protector$_{nw}$	Protector$_{ne}$	BoS$_{nw}$	**Battle of Sexes**	Asym BoS'	3 2 1 4
4	Delilah	Asym Sd$_{sw}$	Asym Sd$_{se}$	Pursuit	Fixed Sum	Missile Crisis	Selfserving'$_{se}$	Protector$_{sw}$	Protector$_{se}$	**Hero**	BoS$_{se}$	SkewedBoS'	3 1 2 4
5	Hostage	Benevolent$_{nw}$	Benevolent$_{ne}$	2nd Best$_{nw}$	2nd Best$_{ne}$	Big Bully	Tragedy'	Delight$_{nw}$	**AntiChicken**	Protector'$_{nw}$	Protector'$_{ne}$	Unfair'	2 1 3 4
6	Blackmailer	Benevolent$_{sw}$	Benevolent$_{se}$	2nd Best$_{sw}$	2nd Best$_{se}$	Hamlet	TotalConflict'	**P. Delight**	Delight$_{se}$	Protector'$_{sw}$	Protector'$_{se}$	Bully'	1 2 3 4
1	Ideo. Heg.	Samaritan$_{nw}$	Samaritan$_{ne}$	Revelation	Alibi	Asym Pd	**Prisoner'sD**	Total Conflict	Tragedy'	Selfserving$_{nw}$	Self-serving$_{n}$	Called Bluff	1 3 2 4
2	Win-win'	C. Aligned$_{sw}$	C Aligned$_{se}$	C Assur.$_{sw}$	C Assur.$_{se}$	**Stag Hunt**	Asym Pd'	Hamlet'	Big Bully'	Missile Crisis	Inspector'	Endless'	1 4 2 3
3	R Assur.$_{ne}$	Commons$_{nw}$	Commons$_{ne}$	Coordination	**Assurance**	R.Assur.$_{nw}$	Alibi'	2nd Best'$_{nw}$	2nd Best'$_{ne}$	Fixed Sum'	Cycle'$_{ne}$	Clock'$_{nw}$	1 4 3 2
4	R Assur.$_{se}$	Commons$_{sw}$	Commons$_{se}$	**Coordination$_{sw}$**	Coord.$_{se}$	R Assur.$_{sw}$	Revelation'	2nd Best'$_{sw}$	2nd Best'$_{se}$	Pursuit'	Cycle'$_{se}$	Clock'$_{sw}$	2 4 3 1
5	R Aligned'$_{ne}$	Hrmny.$_{nw}$	**MixedHarmony**	Commons'$_{nw}$	Commons'$_{ne}$	R Aligned'$_{nw}$	Samaritan'$_{ne}$	Benevolent'$_{nw}$	Benevolent'$_{n}$	Asym Sd'$_{nw}$	Asym Sd'$_{ne}$	Samaritan'$_{nw}$	3 4 2 1
6	R Aligned'$_{se}$	**Harmony**	Harmony$_{se}$	Commons'$_{sw}$	Commons'$_{se}$	R Aligned'$_{sw}$	Samaritan'$_{se}$	Benevolent'$_{sw}$	Benevolent'$_{s}$	Asym Sd'$_{sw}$	Asym Sd'$_{se}$	ActiveSamaritan	3 4 1 2
1	**No Conflict**	C. Aligned$_{nw}$	C. Aligned$_{ne}$	C Assur.$_{nw}$	C Assur.$_{se}$	Win-win	Ideo. Heg.'	Blackmailer'	Hostage'	Delilah'	Samson'	Hegemon'	2 4 1 3
3	3 4 1 2	2 4 1 3	1 4 2 3	1 4 3 2	2 4 3 1	3 4 2 1	4 3 2 1	4 2 3 1	4 1 3 2	4 1 2 3	4 2 1 3	4 3 1 2	2