ANALOG OF THE RIEMANN HYPOTHESIS FOR SOME DIRICLET SERIES

[Dedicated to the memory of Professor S.M.Voronin]

I.SH. JABBAROV

Department of Algebra and Geometry

Ganja State University

AZ2000? Shah Ismail Hatai avenue 187, Ganja

Azerbaijan

Abstract

In this paper the famous Riemann Hypothesis is proven. It was proven that the zeta-function allows uniform approximation in the critical strip by some partial products of Euler form. The basic moment of the work is an investigation of special curves in an infinite dimensional unite cube.

2000 Mathematics Subject Classification 11M26.

Keywords and phrases: Riemann zeta - function, zeroes of the zeta-function, measure theory.

1. Introduction.

In his work [20] B. Riemann had studied the analytical properties of the zeta -function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}; \ \mathrm{Re}\, s > 1$$

has been previously introduced by Euler as a function of real variable only (see [4,20,24]). He had connected the question of distribution of prime numbers with the location of the complex zeroes of the zeta-function. By him was formulated the famous Hypothesis which states that the all of complex (non-real) zeroes of the zeta-function are on the critical line $\sigma = 1/2$. It is equivalent to non-vanishing of the zeta-function on the half plane $\sigma > 1/2$.

About other results, including mean-value theorems, density theorems for zeroes, and etc., see [4, 6, 8, 12, 17, 19, 22, 24].

The Euler product

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}; \mathrm{Re}\, s > 1$$

makes possible learning of zeroes of the zeta-function in the critical strip. Most wide areas of free from the zeroes of the zeta-function obtained by the method of Vinogradov (see [12]).

As a most significant achievement in the zeta-function theory arose the results of S. M. Voronin [see 25-32] which were a further continuation and development of the results of H. Bohr, R. Courant and B. Jessen (see [2, 16, 24]). The Universality Theory given by him for the zeta-function discovered the important property of the zeta-function. In compliance with the Universality Theory every non vanishing in and on the circle $|s| \le r < 1/4$ analytical function can be approximated by $\zeta(3/4 + s + iT)$ with any small error by taking of appropriate value of T. The behavior of the zeta- function and other functions in those works of S. M. Voronin studied by him by using of finite products of the form

$$\prod_p \left(1 - e^{-2\pi i \vartheta_p} p^{-s}\right)^{-1}$$

where p takes on values from some finite set of prime numbers. The results of S.M. Voronin ([25-32]) and of the works [1,14-16] show that the Dirichlet series defined by Euler products also must have like properties.

In the present work we show that the Dirichlet series with the Euler product having analytical continuation to the critical strip without singularities, in some natural conditions, has not zeroes in this strip where the series has mean values, and the primes over which are taken the products, distributed by a suitable way (see formulation of the theorem below). The family of such series includes many of widely used Dirichlet series as the zeta-function, Dirichlet L-functions, or L-functions of some algebraic extensions with commutative Galois groups and etc. The expression "analog of the Riemann Hypothesis" is used in the meaning of non vanishing of the Dirichlet series in the right half of the critical strip $0 < \sigma < 1$.

Let us to introduce into consideration the infinite dimensional unite cube Ω = [0, 1] × [0, 1] × …. Sequences $\vartheta = (\vartheta_p)$, with the components indexed by prime numbers, are elements of this cube.

We let be given a following infinite product taken over all prime numbers

$$F(s) = \prod_p f_p(p^{-s}), \operatorname{Re} s > 1; \qquad (1)$$

here $f_p(z)$ is a rational function of a variable z having not poles in the circle $|z| < 1$,

$$f_p(z) = 1 + \sum_{m=1}^{\infty} a_p^m z^m,$$

and for any positive small δ the inequality

$$\left|a_p^m\right| \le c(\delta) p^{m\delta}; c(\delta) \ge 1 \qquad (2)$$

is satisfied uniformly by p.

Theorem. *Let the function $F(s)$ has not singularities in the half plane $\sigma > 1/2$, with exception of finite number of poles on the line $\sigma = 1$. Further, let every factor of the product (1) have not zeroes in the half plane $\sigma > 1/2$, and for any small positive number λ there exist a constant $c_0(\lambda)>0$ and a number $h_0 > 0$ such that for any $h > h_0$ the following inequality is satisfied:*

$$\sum_{h < p \le h(1+\log^{-10} h)} \left|a_p^1\right| p^{-(1-\lambda)} \ge c_0(\lambda) h^{\lambda/4}. \qquad (3)$$

If in the circle $|s - \sigma_0| \le r < r_0 = \min(1-\sigma_0, \sigma_0 - 1/2), 1/2 < \sigma_0 < 1$ $F(s+it_0)$ has not zeros for some real t_0, then there exists a sequence (ϑ_n) $(\vartheta_n \in \Omega)$ and a sequence (m_n) of integers that for every real t

$$\lim_{n\to\infty} F_n(s+it, \vartheta_n) = F(s+it+it_0),$$

uniformly by s in this circle; here $\vartheta_n = (\vartheta_p^n)$, and

$$F_n(s+it, \theta_n) = \prod_{p \le m_n} f_p(e^{-2\pi i(\vartheta_p^n + \gamma_p)} p^{-s-it}); \gamma_p = \frac{t_0 \log p}{2\pi}.$$

Corollary. *The analog of the Riemann Hypothesis is true, i. e.*

$$F(s) \neq 0$$

when $1/2 < \sigma < 1$.

2. Supplementary statements.

The following variant of S.M. Voronin`s lemma, being proven by him in [28] for the zeta-function, is one of the basic arguments of the present work.

Lemma 1. *Let g(s) be an analytical function in the circle* $|s| < r$ *(*$0 < r < 1/4$*), remaining continuous and non vanishing when* $|s| \le r$*. Then for any* $\varepsilon > 0$ *and for any* $y > 2$ *there is an element* $\bar{\theta} = (\theta_p)_{p \in M}$ *and a finite set M of prime numbers containing all of primes with* $p \le y$ *such that:*

1) $0 \le \theta_p \le 1$ *for* $p \in M$ *;*

2) $\theta_p = \theta_p^0$ *are already given numbers for* $p \le y$*;*

3) $\max\limits_{|s| \le r} \left| g(s) - F_M\left(s + \sigma_0; \bar{\theta}\right) \right| \le \varepsilon$ *;*

here $F_M\left(s + \sigma_0; \bar{\theta}\right)$ *is defined by the equality*

$$F_M\left(s + \sigma_0; \bar{\theta}\right) = \prod_{p \in M} f_p\left(e^{-2\pi i \vartheta_p} p^{-s-\sigma_0}\right).$$

Proof. The proof of the lemma 1 will be conducted by the method of the work [12] of S.M. Voronin. The series (2) of the work [24, p.241] we define as

$$u_k(s) = \log f_{p_k}\left(e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0}\right)$$

where φ_k is an argument of the coefficient $a_{p_k}^1$. We have (for the sufficiently large values of *k*):

$$u_k(s) = \log(1 + a_{p_k}^1 e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0}) + \log(1 + a_{p_k}^1 e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0})^{-1} f_{p_k}(e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0}).$$

By using of an expansion of the logarithmic function into power series, we may write

$$u_k(s) = a_{p_k}^1 e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0} + v_k(s), \qquad (4)$$

where

$$v_k(s) = O(p_k^{2\delta + 2r - 2\sigma_0}) + \log\left(1 + \sum_{n=2}^{\infty} b_n (e^{-2\pi i(\varphi_k + \vartheta_k)} p_k^{-s-\sigma_0})^n\right),$$

and the coefficients b_n defined by the following equality

$$b_n = a_p^n - a_p^{n-1}a_p^1 + a_p^{n-2}\left(a_p^1\right)^2 - \cdots + (-1)^{n-2}a_p^2\left(a_p^1\right)^{n-2}.$$

Therefore,

$$\left|b_n\right| \le (n-1)c^{n-1}(\delta)p^{\delta n}.$$

Since $r < r_0$, we may take δ satisfying the inequality $2\delta + 2r - 2\sigma_0 < -1$. Then, the definition of $u_n(s)$ and (4) show, with the last inequality, that the series

$$\sum_{n=1}^{\infty}\eta_n(s);\ \eta_n(s) = a_{p_n}^1 e^{-2\pi i(\varphi_n + \vartheta_n)} p_n^{-s-\sigma_0} \qquad (5)$$

differs from the series $\sum u_n(s)$ by an absolutely converging series. Therefore, it is sufficiently to show that for any $\varphi(s) \in H_2^{(\gamma r)}$ ($0 < \gamma < 1$ is any) there is a permutation of the series (5) converging to the $\varphi(s)$ (the definition of the Hardy space $H_2^{(\gamma r)}$ was given in [24, p.323]). Further, we are considering (5), following by [28], and noting that

$$\sum_{k=1}^{\infty}\left\|\eta_k(s)\right\|^2 < \infty.$$

We have to define the numbers θ_k and we take any given values for them since $p_k \le y$, and $\theta_k = \rho(k)/4$ otherwise, with the values $\rho(k)$ which will be precisely defined below. Let

$$\Delta(x) = \iint_{|s|\le R} e^{-(s+\sigma_0)x}\overline{\varphi(s)}d\sigma dt.,\ R < r_0.$$

Then

$$(\eta_k(s), \varphi(s)) = \mathrm{Re}\int_{|s|\le R} e^{-2\pi i\rho(k)/4} p_k^{-(s+\sigma_0)}\overline{\varphi(s)}d\sigma dt = \mathrm{Re}\left[\left|a_{p_k}^1\right| e^{-2\pi i\rho(k)/4}\Delta(\log p_k)\right].$$

From the results of the work [24, p.243] we deduce that

$$\left|\Delta(x)\right| \le \pi R^2 e^{-x/2}.$$

So $\left|(\eta_k(s), \varphi(s))\right| \to 0$, as $k \to \infty$. Therefore, we must set such θ_k that the series (5) could have two sub series divergent to $+\infty$ and $-\infty$, correspondingly. From the conditions (2) and (3) it follows that the set of primes satisfying the condition $h < p \le h(1 + \log^{-10} h)$ (for given h) can be dissected into the union of sets P_1, P_2, P_3, P_4 for every of which the following inequality holds

$$\sum_{p\in P_i,\, h<p\le h(1+\log^{-10}h)} \left|a_p^1\right| p^{-(1-\lambda)} \ge 0.1c_0(\delta,\lambda)h^{\lambda/4}, i=1,2,3,4.$$

Really, the addends of the last sum have the bound $c(\delta)h^{-(1-\delta-\lambda)}$. As h is taken sufficiently large, we can suppose this bound to be less than $h^{-0.8}$. It is possible to take such addends sum of which is no less than $0.1c_0(\delta,\lambda)h^{\lambda/4}$ (because the full sum satisfies the condition (3)); since the taken sum is greater than $0.2c_0(\delta,\lambda)h^{\lambda/4}$, we can omit, beginning from the greatest addends, several of them while their sum is not less than $0.2c_0(\delta,\lambda)h^{\lambda/4}$. So we separate such a subset of primes P_1 that

$$0.1c_0(\delta,\lambda)h^{\lambda/4} \le \sum_{p\in P_1,\, h<p\le h(1+\log^{-10}h)} \left|a_p^1\right| p^{-(1-\lambda)} \le 0.2c_0(\delta,\lambda)h^{\lambda/4}.$$

Now we have

$$\sum_{p\notin P_1,\, h<p\le h(1+\log^{-10}h)} \left|a_p^1\right| p^{-(1-\lambda)} \ge 0.8c_0(\delta,\lambda)h^{\lambda/4}.$$

Repeating like reasoning we get suitable subsets P_1, P_2, P_3, P_4. We set $\rho(k)=i-1$ for every $p_k \in P_i$.

By following [28] we find a segment τ_j in $[x_j-1, x_j+1]$ of the length of greater than $0.01(x_j+1)^{-8}$ for which at least one of the inequalities

$$\left|\operatorname{Re}\Delta(x_j)\right| > 0.1e^{-(1-\delta_0)x_j}, \left|\operatorname{Im}\Delta(x_j)\right| > 0.1e^{-(1-\delta_0)x_j}$$

is satisfied in every point of the segment τ_j (note that [see 24, p.244] in the first case $\operatorname{Re}\Delta(x)$ does not change its sign in this segment).Therefore, if we suppose $\lambda < \delta_0/2$, we shall have

$$\sum_{\substack{p_k\in P_1\\ \log p_k\in\gamma_j}} \operatorname{Re}\left[\left|a_{p_k}^1\right| e^{-2\pi i\rho(k)/4}\Delta(\log p_k)\right] >> e^{-(1-\delta_0)x_j} e^{(1-\lambda)x_j} >> e^{\delta_0 x_j/2}$$

in the notations of [28] when the inequality $\operatorname{Re}\Delta(x_j) > 0.1e^{-(1-\delta_0)x_j}$ is fulfilled. Analogously, we can prove the inequality

$$-\sum_{\substack{p_k\in P_3\\ \log p_k\in\gamma_j}} \operatorname{Re}\left[\left|a_{p_k}^1\right| e^{-2\pi i\rho(k)/4}\Delta(\log p_k)\right] >> e^{-(1-\delta_0)x_j} e^{(1-\lambda)x_j} >> e^{\delta_0 x_j/2},$$

provided the condition $\operatorname{Re}\Delta(x_j) < -0.1e^{-(1-\delta_0)x_j}$ is satisfied. By like way we can consider the case when the inequality $\left|\operatorname{Im}\Delta(x_j)\right| > e^{-(1-\delta_0)x_j}$ is fulfilled (in this case we use the subsets P_2 and P_4).

Let now $\tau_j = [\alpha, \alpha + \beta]$. Then we apply the inequalities above by taking $h = e^{\alpha}$. Consequently, some permutation of the series

$$\sum_{n=1}^{\infty} (\eta_k(s), \varphi(s))$$

converges conditionally. So (see [24,28]), there exist a permutation of the series $\sum_{p_n > y} u_n(s)$ converging to the $\varphi(s) - \sum_{p_n \le y} u_n(s)$ uniformly in any compact sub domain of the circle $|s| < r$. Taking sufficiently large partial sums of this series, we get a suitable result. Lemma 1 is proved.

Lemma2. *Let the series of analytical functions*

$$\sum_{n=1}^{\infty} f(s)$$

be given in the one-connected domain G *of the complex* s *− plane, absolutely convergent almost everywhere in the* G *in Lebesgue meaning, and the function*

$$\Phi(\sigma, t) = \sum_{n=1}^{\infty} |f_n(s)|$$

is a summable function in the G. *Then given series uniformly converges in any compact sub domain of the G; particularly, the sum of this series will be an analytical function in the G.*

Proof. It is enough to show that the theorem is true for any rectangle C in the domain *G*. Let C be a rectangle in the G and C' is another rectangle lying directly in the interior of the C, moreover, the sides of them are parallel to the axis. We can suppose that on contour of these rectangles the series is convergent almost everywhere in correspondence with the theorem of G.Fubini (see. [7, p.208]). We deduce from the theorem of Lebesgue on a bounded convergence (see. [21, p.293]):

$$\frac{1}{2\pi i} \int_C \frac{\Phi_0(s)}{s - \xi} ds = \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_C \frac{f_n(s)}{s - \xi} ds,$$

where the integrals are taken in Lebesgue meaning and $\Phi_0(s) = \Phi_0(\sigma, t)$ is a sum of given series on the points of convergence. Because on the right hand side of the equality the integrals exist in the Riemann meaning, we get by applying Cauchy's formula

$$\Phi_1(\xi) = \frac{1}{2\pi i}\int_C \frac{\Phi_0(s)}{s-\xi}ds = \sum_{n=1}^{\infty} f_n(\xi);$$

here $\Phi_1(\xi) = \Phi_0(\xi)$ almost everywhere and ξ is any point on or in the contour. Further, the series in the C is bounded by following way

$$|f_n(\xi)| \le \frac{1}{2\pi}\int_C \frac{|f_n(s)|}{|s-\xi|}|ds| \le \frac{1}{2\pi\delta}\int_C |f_n(s)||ds|$$

if δ denotes the minimal distance between the sides of the C and C'. The series

$$\sum_{n=1}^{\infty}\int_C |f_n(s)||ds|$$

converges in agree with the theorem of Lebesgue on the monotone convergence (see [21, p.290]). Therefore, the series $\sum_{n=1}^{\infty} f_n(\xi)$ converges uniformly in the C'. The lemma 2 is proved.

3. Main auxiliary results.

Definition 1. *Let $\sigma : N \to N$ is any one to one mapping of the set of natural numbers. If there exist a natural number m such that $\sigma(n) = n$ for every n >m then we say that σ is a finite permutation. We call the subset $A \subset \Omega$ to be finite – symmetrical if for any element $\theta = (\theta_n) \in A$ we have $\sigma\theta = (\theta_{\sigma(n)}) \in A$, where σ a finite permutation is.*

Let Σ to denote the set of all finite permutations. It is a group which contains any group S_n of *n*- degree permutations as a subgroup (we shall consider every *n*-degree permutation σ ($n = 1,2,...$) as a finite permutation for which $\sigma(m) = m$ as $m > n$). Let $\omega \in \Omega$, $\Sigma(\omega) = \{\sigma\omega \mid \sigma \in \Sigma\}$, and $\Sigma'(\omega)$ means the closed set of all limit points of the sequence $\Sigma(\omega)$. For a real *t* we denote $\{t\Lambda\} = (\{t\lambda_n\})$ when $\Lambda = (\lambda_n)$. Below we denote by μ a product of linear Lebesgue measures defined in the segment [0,1]: $\mu = m \times m \times \cdots$.

Lemma 3. *Let $A \subset \Omega$ be a finite – symmetrical subset of zero measure and $\Lambda =$ (λ_n) be an unbounded monotonically increasing sequence of positive numbers* *any subset of components of which is linearly independent over the field of rational numbers. Let* $B \supset A$ *be any open set with* $\mu(B) < \varepsilon$ *and*

$$E_0 = \{0 \le t \le 1 \mid (\{t\Lambda\}) \in A \wedge \Sigma'(\{t\Lambda\}) \subset B\}$$

Then we have $m(E_0) \le 6c\varepsilon$ *with an absolute constant c, and m means the linear Lebesgue measure.*

Proof. Let ε be any small positive number. Since the numbers λ_n are linearly independent, we for any finite permutation σ have $(\{t_1\lambda_n\}) \neq \{t_2\lambda_m\}$ when $t_1 \neq t_2$. Really, in the other case we could have the equality $\{t_1\lambda_m\} = \{t_2\lambda_m\}$ for some sufficiently large natural *m,* i.e. $(t_1 - t_2)\lambda_m = k, \quad k \in Z$. Further, by writing the same equality for other integer $r > m$ we have the relation

$$\frac{k_1}{\lambda_r} - \frac{k}{\lambda_m} = \frac{k_1\lambda_m - k\lambda_r}{\lambda_r\lambda_m} = 0$$

which contradicts the linear independence of the numbers λ_n. So, for any pair of different numbers t_1 and t_2 $(\{t_1\lambda_n\}) \notin \{(\{t_2\lambda_{\sigma(n)}\}) \mid \sigma \in \Sigma\}$. We can find a family of open spheres (in the Tichonov topology) such that each of them does not contain any other of $B_1, B_2, \ldots$ (the sphere being consisted in the other one may be omitted) and

$$A \subset B \subset \bigcup_{j=1}^{\infty} B_j, \sum \mu(B_j) < 1.5\varepsilon.$$

Now we take the permutation $\sigma \in \Sigma$ defined by the equalities $\sigma(1) = n_1, \ldots, \sigma(k) = n_k$ with the natural numbers n_j set by following way. At first we take N such that

$$\mu(B'_N) < 2\varepsilon_1,$$

where the B'_N is a projection of the sphere B_1 into the first N axes and $\mu(B_1) = \varepsilon_1$. We cover the B'_N by cubes with the rib δ and summarized measure not exceeding $3\varepsilon_1$. Let us to write $k = N$ and define the numbers $n_1, \ldots, n_k$ by using of following inequalities

$$\lambda_{n_1} > 1,\ \lambda_{n_2}^{-1} < \frac{1}{4}\delta\lambda_{n_1}^{-1},\ \lambda_{n_3}^{-1} < \frac{1}{9}\delta\lambda_{n_2}^{-1}, \ldots, \lambda_{n_k}^{-1} < \frac{1}{k^2}\delta\lambda_{n_{k-1}}^{-1}, \delta < 1. \qquad (6)$$

Now we take any cube with the rib δ and center $(\alpha_m)_{1\le m\le k}$. Then the point $(\{t\lambda_{n_m}\})$ would lie in this cube if it were the case

$$\left|\{t\lambda_{n_m}\}-\alpha_m\right|\le\delta/2 .$$

From the definition of the fractional part we may write, for some integral *r* taking *m=1:*

$$\frac{r+\alpha_1-\delta/2}{\lambda_{n_1}}\le t\le\frac{r+\alpha_1+\delta/2}{\lambda_{n_1}}, \qquad (7)$$

The measure of a set of such *t* does not exceed the value $\delta\lambda_{n_1}^{-1}$. The number of such intervals corresponding to the different values of $r=[t\lambda_{n_1}]\le\lambda_{n_1}$ does not exceed $[\lambda_{n_1}]+2\le\lambda_{n_1}+2$. The total measure of these intervals is as

$$\le(\lambda_{n_1}+2)\delta\lambda_{n_1}^{-1}\le(1+2\lambda_{n_1}^{-1})\delta .$$

Now we examine one of the intervals (6), and taking *m=2* we can write

$$\frac{s+\alpha_2-\delta/2}{\lambda_{n_2}}\le t\le\frac{s+\alpha_2+\delta/2}{\lambda_{n_2}} \qquad (8)$$

with $s=[t\lambda_{n_2}]\le\lambda_{n_2}$. Since we take the conditions (6) and (8) simultaneously, we must estimate the total measures of intervals (8) having nonempty intersections with the intervals (7) by using of the conditions (6). The number of intervals with the length $\lambda_{n_2}^{-1}$ having nonempty intersection with one of the intervals of the view (7) does not exceed the value

$$[\delta\lambda_{n_1}^{-1}\lambda_{n_2}]+2\le\delta\lambda_{n_1}^{-1}\lambda_{n_2}+2 .$$

Therefore, the measure of a set of such *t,* for all of which the conditions (7) and (8) are satisfied simultaneously, does not exceed

$$(\lambda_{n_1}+2)(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\delta\lambda_{n_2}^{-1} .$$

One may continue this reasoning by taking all of conditions of the form

$$\frac{l+\alpha-\delta/2}{\lambda_{n_m}}\le t\le\frac{l+\alpha+\delta/2}{\lambda_{n_m}}, \quad m=1,\dots,k .$$

Then we find the following estimation for the measure $m(\delta)$ of a set of such *t* for which the points $(\{t\lambda_{n_m}\})$ lie into the cubes with the rib of δ :

$$m(\delta) \le (2+\lambda_{n_1})(2+\delta\lambda_{n_1}^{-1}\lambda_{n_2})\cdots(2+\delta\lambda_{n_{k-1}}^{-1}\lambda_{n_k})\delta\lambda_{n_k}^{-1} \le \delta^k \prod_{m=1}^{\infty}(1+2m^{-2}).$$

Therefore, by summing over all of such cubes we get, as an upper bound for the measure of a set of such t for which $(\{t\lambda_{n_m}\}) \in B_1$, the value $\le 3c\varepsilon_1$, $c>0$.

Note that the sequence $\Lambda = (\lambda_n)$, defined above, depends on δ. We shall fix, for every of defined above spheres B_k, some sequence Λ_k by using of conditions (6). Considering all of such spheres we denote $\Sigma_0 = \{\Lambda_k \mid k = 1,2,...\}$. Since the set A is finite-symmetrical, the measure of interested us values of t can be estimated by using of any sequence Λ_k because, as it was noted above, the sets $\Sigma(\{t\Lambda\})$ for various values of t have an empty intersection.

Further, for any point t of the E_0, the set $\Sigma(\{t\Lambda\})$ has a non-empty intersection with finite number of spheres B_k only. Really, if else, then some limit point (which is contained by the open set B) of $\Sigma(\Lambda)$ belong say to B_s. Let d is a distance from θ to the bound of B_s. Then, for infinitely many indexes n_k beginning from some k the all of spheres B_{n_k} would belong into the spheres with radius < $d/2$ and the center θ. So, for sufficiently large k the all of such spheres would belong into B_s, which is contradiction. Consequently, the set E_0 can be represented as a union of subsets E_k, $k=1,2,\dots$ where

$$E_k = \{t \in E_0 \mid \Sigma(\{t\Lambda\}) \cap \bigcup_{m>k} B_m = \varnothing\}.$$

Then,

$$t \in E_k \Rightarrow \sum(\{t\Lambda\}) \subset \bigcup_{k\le m} B_k,\ E_0 = \bigcup_{k=1}^{\infty} E_k;\ E_k \subset E_{k+1} (k \ge 1).$$

So we have

$$m(E_0) \le \limsup_{\Lambda\in\Sigma_0} m(E(\Lambda)) \le \sum_k \limsup_{\Lambda\in\Sigma_0} m(E^{(k)}(\Lambda)) \le$$

$$\le 3c(\varepsilon_1 + \varepsilon_2 + \cdots) = 3c\varepsilon,$$

where $E(\Lambda) = \{t \in E_0 \mid (\{t\Lambda\}) \in B\}$ and $E^{(k)}(\Lambda) = \{t \in E_0 \mid \{t\Lambda\} \in B_k\}$. The proof of the lemma 3 is completed.

Lemma 4. *Let the conditions of the theorem be satisfied. Then there exist a* *sequence of points* (θ_k) ($\theta_\kappa \in \Omega$) *and natural numbers* (m_n) *such that*

$$\lim_{k\to\infty} F_k\left(s,\, \theta_k\right) = F\left(s_0 + s\right)$$

in the circle $|s| \le r < r_0$ *uniformly by s.*

Proof. Let $y > 2$ be a whole positive number which will be precisely defined below. We suppose

$$y_0 = y,\, y_1 = 2y_0, \ldots, y_m = 2y_{m-1} = 2^m y_0, \ldots .$$

From the lemma 1 it follows that for the given ε and a whole number $y > 2$ there exist a set M_1 of primes such that M_1 contains all of the primes $p \le y$ and

$$\max_{|s|\le r}\left|F(s_0+s)-\eta_1(s_1)\right| \le \varepsilon;\, \eta_1(s_1) = \prod_{p\in M_1} f_p(e^{-2\pi i(\vartheta_p^0+\gamma_p)} p^{-s_1}),\, s_1 = \sigma_0 + s\,.$$

Moreover, $\vartheta_p^0 = 0$ when $p \le y$. Now we denote

$$F_1(s_1;\vartheta) = \prod_{p\le m_1} f_p\left(e^{-2\pi i(\vartheta_p+\gamma_p)} p^{-s_1}\right)\,,$$

$$h_1(s_1;\vartheta) = F_1(s_1;\vartheta)\cdot \eta_1^{-1}(s_1) - 1;$$

here $\vartheta_p = \vartheta_p^0$ when $p\in M_1$ and $m_1 = \max_{m\in M_1} m$. If $r+\delta+\mu < r_0$, then by taking $s_1' = s_1 + it_0$ we can find a constant $c(\delta,\mu)$ ($\mu > 0$) such that

$$\int_{\Omega_1}\left(\iint_{|s|\le r+\mu}\left|h_1(s_1;\vartheta)\right|^2 d\sigma dt\right)d\vartheta \le \iint_{|s|\le r+\mu}\left(\int_{\Omega_1}\left|h_1(s_1;\vartheta)\right|^2 d\vartheta\right)d\sigma dt \le$$

$$\le \pi(r+\mu)^2 \max_{|s|\le r+\mu}\int_{\Omega_1}\left|\sum_{n>y} a_n(\vartheta) n^{-s_1'}\right|^2 d\vartheta \le \frac{4c(\delta,\mu)(r+\mu)^2}{1-2\sigma_0+2r+2\delta+2\mu} y^{1-2\sigma_0+2r+2\delta+2\mu};$$

the summation under the sign of integral is taken over such natural numbers *n* in the canonical factorizations of which take part only the primes *p*, $p\notin M_1, p \le m_1$,

$$a_n(\vartheta) = \prod_{p|n} a_p^{\alpha_p} e^{2\pi i \alpha_p(\vartheta_p+\gamma_p)};\, n = \prod p^{\alpha_p}$$

and Ω_1 is a projection of Ω into the subspace of coordinate axes θ_p with $p\notin M_1$. Then it will be found a point $\vartheta_1' = \left(\vartheta_p\right)_{p\notin M_1}$ such that

$$\iint_{|s|\le r+\mu}\left|h_1(s_1;\vartheta_1')\right|^2 d\sigma dt \le \frac{4\pi\, c(\delta,\mu)(r+\mu)^2}{1+2r-2\sigma_0+2\delta+2\mu} y^{1+2r-2\sigma_0+2\delta+2\mu}\,,$$

or

$$\max_{|s|\le r}|h_1(s_1;\vartheta_1')| \le \sqrt{2}\mu^{-1}\left(\frac{1}{2\pi}\iint_{|s|\le r+\mu}|h_1(s_1;\vartheta_1')|^2 d\sigma dt\right)^{1/2} \le c_1(\delta,\mu)y^{1/2+r-\sigma_0+\delta+\mu}$$

(see[19, p. 345]) and $c_1(\delta,\mu)>0$ is a constant. So, setting $\theta_1 = (\theta_0, \theta_1')$, $\theta_0 = \left(\vartheta_p^0\right)_{p\in M_1}$, $y = y_0$ we shall have

$$\max_{|s|\le r}\{|F(s_1+it_0)-F_1(s_1;\vartheta_1)|\} \le \max_{|s|\le r}\{|F(s_1+it_0)-\eta_1(s_1)|+|\eta_1(s_1)|\cdot|h_1(s_1;\vartheta_1')|\} \le$$

$$\le \varepsilon + (A+1)c_1(\delta,\lambda)y_0^{1/2+r-\sigma_0+\delta+\mu},$$

on condition that y_0 would satisfy the inequality

$$c_1(\delta,\mu)y_0^{1/2+r-\sigma_0+\delta+\mu}(A+1)\le\varepsilon;\ A=\max_{|s|\le r}|F(s_1)|.$$

We replace now ε by $\varepsilon/2$. There will be find a set of primes M_2 containing all of the prime numbers $\le 2y_0 = y_1$ and satisfying, by the lemma 1, the inequality

$$\max_{|s|\le r}|F(s_1+it_0)-\eta_2(s_1)|\le\varepsilon/2,$$

where

$$\eta_2(s_1)=\prod_{p\in M_2} f_p(e^{-2\pi i(\vartheta_p^0+\gamma_p)}p^{-s_1}),$$

and $\vartheta_p^1 = 0$ if $p \le y_1$. By like way we find a point $\vartheta_2' \in \Omega_2$ (Ω_2 is a projection of Ω into the subspace of the coordinate axes ϑ_p, $p \notin M_2$) such that

$$\max_{|s|\le r}|F(s_1+it_0)-F_2(s_1;\vartheta_2)|\le\varepsilon;\ \vartheta_2=(\vartheta_1,\vartheta_2');$$

$$F_1(s_1;\vartheta)=\prod_{p\le m_1} f_p\left(e^{-2\pi i(\vartheta_p+\gamma_p)}p^{-s_1}\right),\ m_1=\max_{m\in M_1} m.$$

Really,

$$|F_2(s_1)-\eta_2(s_1)|=|\eta_2(s_1)|\cdot|h_2(s_1;\vartheta)|;\ h_2(s_1;\vartheta)=F_2(s_1;\vartheta)\cdot\eta_2^{-1}(s_1)-1.$$

Now we get by taking the mean value

$$\max_{|s|\le r}|h_2(s_1;\vartheta_2')|\le\sqrt{2}\mu^{-1}\left(\frac{1}{2\pi}\iint_{|s|\le r+\mu}|h_2(s_1;\vartheta_2')|^2 d\sigma dt\right)^{1/2}\le c_1(\delta,\mu)(2y_0)^{1/2+r-\sigma_0+\delta+\mu}.$$

Therefore,

$$\max_{|s|\le r}\{|F(s_1+it_0)-F_2(s_1;\vartheta_1)|\}\le\max_{|s|\le r}\{|F(s_1+it_0)-\eta_2(s_1)|+|\eta_2(s_1)|\cdot|h_2(s_1;\vartheta_2')|\}\le$$

$$\le\varepsilon/2+(A+1)c_1(\delta,\mu)(2y_o)^{1/2+r-\sigma_0+\delta+\mu};\ \vartheta_2=(\vartheta_1,\vartheta_2').$$

By repeating this calculus we find $\vartheta_{k+1}=(\vartheta_k, \vartheta'_{k+1}) \in \Omega,\ \vartheta_k = (\vartheta_p^k)_{p\in M_{k+1}}$, for every $\kappa > 1$, such that $\vartheta_p^k = 0$ when $p \le y_\kappa$ and

$$\max_{|s|\le r} \left|F(s_1 + it_0) - F_{k+1}(s_1; \vartheta_{k+1})\right| \le 2^{1+k(1/2+r-\sigma_0+\delta+\mu)} \varepsilon \,;$$

here

$$F_{k+1}(s_1;\vartheta) = \prod_{p\le m_{k+1}} f_p\left(e^{-2\pi i(\vartheta_p+\gamma_p)} p^{-s_1}\right),\ m_{k+1} = \max_{m\in M_{k+1}} m$$

Consequently, we have the equality

$$\lim_{k\to\infty} F_k(s_1;\vartheta_k) = F(s_1 + it_0)$$

uniformly by s ($|s| \le r$). Lemma 4 is proved.

4. Proof of the theorem.

Since in any bounded domain the $F(s)$ could have only finite number of zeroes, we can find a circle $K: |s-\sigma_0| \le r < r_0 = \min(1-\sigma_0, \sigma_0 - 1/2)$ not containing zeroes of the $F(s)$. Now we consider the integral

$$B_k = \int_\Omega \left(\iint_{|s|\le r} \left|F_{k+1}(\sigma_0 + s; \theta_{k+1} + \theta) - F_k(\sigma_0 + s;\ \theta_k + \theta)\right| d\sigma d\tau \right) d\theta\,,$$

for $\kappa = 0, 1,\ldots$ and put $F_0(\sigma_0 + s; \theta_0 + \theta) = 0$ if $\kappa=0$. Applying the Schwartz inequality and changing the order of the integration we find as above:

$$B_k^2 \le 4\pi r^2 \iint_{|s|\le r} d\sigma d\tau \int_\Omega \left| \prod_{p\le 2^{k-1} y_0} f_p(e^{-2\pi i(\vartheta_p^n+\gamma_p)} p^{-s-\sigma_0}) \right|^2 \prod_{p\le 2^{k-1}} d\theta_p \cdot \sum_{n > 2^{k-1} y_0} n^{2r+2\mu+2\delta-2\sigma_0} \le$$

$$\le c(\delta,\mu)\left(2^{k-1} y_0\right)^{1+2r+2\delta+2\mu-2\sigma_0}$$

with some constant $c(\delta,\mu) > 0$. Since $2r + 2\delta + 2\mu + 1 - 2\sigma_0 < 0$, from this estimation it follows the convergence of the series below almost everywhere in the Ω

$$\sum_{k=1}^{\infty} \iint_{|s|\le r} \left|F_k(\sigma_0 + s; \theta_k + \theta) - F_{k-1}(\sigma_0 + s; \theta_{k-1} + \theta)\right| d\sigma d\tau; s = \sigma + i\tau\,. \quad (9)$$

Let A be the set of divergence of the series (9) It is a finite symmetrical subset. By the theorem of Yegorov (see [7, p. 166]) the series above is converging almost uniformly in the outside of some subset $\Omega'_1,\ \mu(\Omega'_1) = 0$. We can suppose the set $A \cup \Omega'_1$ to be finite symmetrical (if else one can take all permutations of all its elements). We can find some countable family of spheres B_r with the total measure of does not exceeding ε the union of which contains the set $A \cup \Omega_1'$. For any natural number n we define the set $\Sigma_n'(t\Lambda)$ as a set of all limit points of the sequence $\Sigma_n(\omega) = \{\sigma\omega \mid \sigma \in \Sigma \wedge \sigma(1) = 1 \wedge \cdots \wedge \sigma(n) = n\}$. Let $\Lambda = (\lambda_n), \lambda_n = (1/2\pi)\log p_n$, where p_n denote the n-th prime number and $B^{(n)} = \{t \mid \{t\Lambda\} \in A \wedge \Sigma_n'(\{t\Lambda\}) \subset \bigcup_r B_r\}, n = 1,2,\ldots$. We have $B^{(n)} \subset B^{(n+1)}$. The set $\Sigma_n'(\{t\Lambda\})$ is a closed set. It is clear that if we shall restrict the sequences $\{t\Lambda\}$ taking the components $\{t\lambda_n\}$ with indexes greater than n only and denote by $\{t\Lambda\}'$ the restricted sequence then the set $\Sigma'(\{t\Lambda\}')$ also will be a closed set. Now we consider the products $[0,1]^n \times \{\{t\Lambda\}'\}$ for every t. We have

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A.$$

The example below shows that from this fact it does not follow the equality $A = \Omega$. Let $I = [0,1], U = [0,1/2], V = [1/2;1]$ and

$$X_0 = U \times U \times \cdots,\ X_1 = V \times U \times \cdots,\ X_2 = I \times V \times U \times \cdots, \ldots,\ X_{s+1} = I^s \times V \times U \times \cdots, \ldots.$$

It is clear that $\mu(X_s) = 0$ for all s. If

$$X = \bigcup_{s=0}^{\infty} X_s$$

then $\mu(X) = 0$. As it is seen from the construction of X, the equality $X = [0,1]^s \times X$ is satisfied for every s.

Since the set $[0,1]^n \times \{\{t\Lambda\}'\}$ is a closed set, there exists only a finite set R of natural numbers such that $[0,1]^n \times \{\{t\Lambda\}'\} \subset \bigcup_{r\in R} B_r$. Consider the set of restricted points θ' of the spheres B_r. Let $B_r' = \{\theta' \mid \theta \in B_r\}$. Then the intersection of them being an open set contains the point $\{t\Lambda\}'$. So we have

$$[0,1]^n \times \{\{t\Lambda\}'\} \subset [0,1]^n \times \bigcap_{r\in R} B_r' \subset \bigcup_{r\in R} B_r,$$

for every considered t. The analogical relation is true if we would exchange the point $\{t\Lambda\}$ by any limit point ω of the sequence $\Sigma(\{t\Lambda\})$, because $\omega \in B_r$. If by B' we denote the union of all open sets of the view $\bigcap_{r\in R} B_r'$, then we get the relation

$$\{t\Lambda\} \in [0,1]^n \times \{\{t\Lambda\}'\} \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r$$

for each considered values of t, or

$$\{\omega\} \in [0,1]^n \times \{\omega'\} \subset A \subset [0,1]^n \times B' \subset \bigcup_r B_r$$

for any limit point of ω. From this it follows that $\mu(B') \le \varepsilon$. The set B' is an open set and $\Sigma'(\{t\Lambda\}') \subset B'$. Now we can apply the lemma 4 and get the bound $m(B^{(n)}) \le 6c\varepsilon$. So we have $m(B) \le 6c\varepsilon$.

Consequently, taking *n=y_k*, *k=1, 2, 3,...* for every *k* we find such a limit point $\omega_k \in \Omega \setminus \bigcup_r B_r$ of the sequence $\Sigma_{y_k}(\{t\Lambda\})$ for which the series

$$\sum_{l=1}^{\infty} \iint_{|s| \le r} \left| F_l(\sigma_0 + s; \theta_l + \omega_k) - F_{l-1}(\sigma_0 + s; \theta_{l-1} + \omega_k) \right| d\sigma d\tau$$

is convergent. Since the set $\Omega \setminus \bigcup_r B_r$ is closed, the limit point $\overline{\omega} = (\{t\Lambda\})$ of the sequence (ω_k) will belong into $\Omega \setminus \bigcup_r B_r$. So the series below

$$\sum_{l=1}^{\infty} \iint_{|s| \le r} \left| F_l(\sigma_0 + s; \theta_l + i\{t\Lambda\}) - F_{l-1}(\sigma_0 + s; \theta_{l-1} + i\{t\Lambda\}) \right| d\sigma d\tau$$

is convergent for all values of $t \notin B$ i.e. for the values of *t* with exception of their set of a measure of $12c\varepsilon$. Since ε is any, the latest result shows the convergence of the series (9) for the almost all *t* (it is clear that the condition $0 \le t \le 1$ now can be omitted) . Then, by the lemma 2, for any given $\delta_0 < 1$ the sequence

$$F_k(\sigma_0 + s; \theta_k + i\{t\Lambda\}),$$

for all such *t* converges in the circle $|s| \le r\delta_0$ $(\delta_0 < 1)$ uniformly to some analytical function $f(s_1; t)$:

$$\lim_{k\to\infty} F_k(\sigma_0 + s + it; \theta_k) = f(s_1; t). \quad (10)$$

In spite of the equality (10), when we use *t* as a variable we must note that the logarithms of the left and right hand sides of it may differ one from other by theirs arguments. Therefore, we cannot use the principle of analytical continuation. For the completing the proof of the theorem, we take any large positive number *T*. Now we note that there exist a finite number of open circles $\Delta_1,...,\Delta_m$ every of which does not contain any other and the union of which contains the segment $1/2 + \lambda < \sigma \le 1 - \lambda, \lambda > 0, t = t_0$ lying in the critical strip. Since the set of taken values of *t* is an everywhere dense in the interval [*T*,-*T*], the union of the circles

C(t)=$\{\sigma_0+it+s:|s|\le r\delta_0\}$ contains the rectangle $\sigma_0 - r\delta_0^2 \le \operatorname{Re} s_1 \le \sigma_0 + r\delta_0^2,\ -T \le \operatorname{Im} s_1 \le T$ in which the conditions of the lemma 2 are satisfied by the series

$$F_1(s_1;\vartheta_1) + \left(F_2(s_1;\vartheta_2) - F_1(s_1;\vartheta_1)\right) + \cdots. \quad (11)$$

Therefore, by the lemma 2, this series defines the analytical function in the considering rectangle which coincides with the $F(s_1+it_0)$ in the circle C(0).

For applying of the principle of analytical continuation, we must take an one – connected open domain where the both of the functions $\log F_*(s)$ and $\log F(s)$ are regular (here the function $F_*(s)$ is a sum of the series (11)). Let $\rho_1, \dots, \rho_L$ are all the possible zeroes of the function $F(s)$ in the considering rectangle on the contour of which the function $F(s)$ has not any zeroes. We take cross cuts over the segments $1/2 \le \operatorname{Re} s \le \operatorname{Re} \rho_l,\quad \operatorname{Im} s = \operatorname{Im} \rho_l, l = 1,...,L.$ In the open domain of the considering rectangle not containing the segments the function $\log F_*(s)$ and $\log F(s)$ are regular. Therefore, the equality $F_*(s) = F(s)$ is satisfied in the all open domain defined above. Now we get the equality $F_*(s) = F(s)$ in the all rectangle because both of those functions are regular. The proof of the theorem is completed.

6. Proof of the corollary.

The deduction of the corollary comes out from the theorem of Rouch`e (see [19,p.137]. Let t be any real number. At first we shall show that for any $0<r<r_0$ in the circle $C=\{s|\ |s-\sigma_0-it| \le r\}$ $F(s)$ has not zeroes. Let

$$m = \min_{s\in C} |F(s)|.$$

By the theorem we can find such $n=n(t)$ for which in and on the contour C the following inequality holds

$$|F(s+it_0) - F_n(s;\vartheta_n)| \le 0.25m.$$

Then on the contour of the C the inequality

$$|F_n(s;\vartheta_n) - F(s+it_0)| \le |F(s+it_0)|$$

is satisfied. By the theorem of Rouch`e the functions $F(s+it_0)$ and $F_n(s;\vartheta_n)$ have the same number of zeroes in the C. But $F_n(s;\vartheta_n)$ has not any zeroes in the circle C. Therefore, $F(s+it_0)$ also has not zeroes in the C. Since t is any, we deduce from this that the strip $-r < \operatorname{Re} s - \sigma_0 < r$ is free from the zeroes of the function $F(s)$.

The conclusions have been made above show that the corresponding strip is free from the zeroes of *F(s)*. For any $\mu > 0$ there can be found finite number of open strips the union of which covers the strip $1/2 + \mu \le \operatorname{Re} s \le 1 - \mu$. Therefore, the function $F(s)$ has not any zeroes in this strip. Since μ is any, then the corollary is proven.

References

[1] B. Bagchi. A joint universality theorem for Dirichlet L –functions. Math. Zeit., 1982,v. 181, p.319-335.

[2] H. Bohr, R.Courant. Neue Anwendungen der Theorie der Diophantischen Approximationen auf die Riemannsche Zetafunction. J.reine angew. Math. (1944) p.249-274.

[3] K.Chandrasekharan. Introduction to Analytic Number Theory. M.: Mir, 1974 (rus).

[4] J.B.Conrey. The Riemann Hypothesis. Notes of AMS.march, 2003.

[5] R.Courant. Differential and integral calculus. M. Nauka, 1967,510 pp. (rus).

[6] H.Davenport. Multiplicative number theory. M.:Nauka 1971(rus).

[7] N.Dunford and J.T.Schwartz. Linear operators. Part I: General theory. M. PFL, 1962, 896 p.

[8] K. Chandrasekharan. Arithmetical functions. M. Nauka, 1975., 270 pp. (rus).

[9] L.Euler. Introduction to the analyses of infinitesimals. – M. :ONTI, 1936. (rus).

[10] E. Hewitt and K.A.Ross. Abstract Harmonic Analysis. v.I, Nauka, 1975, 656 pp. (rus).

[11] I. Sh. Jabbarov. On some Dirichlet series. MAK 98, Materials of first regional math. conf., Publ. of Altay State Univ., Barnaul, 1998, (rus).

[12] А.А. Карацуба. Основы аналитической теории чисел. М. Наука, 1983, (rus).

[13] E. Landau Euler and Functionalgleichung der Riemannschen Zeta – Funktion. Bibliotheca mathematica, (3), Bd.7., H.1,1906, Leipzig, ss. 69 -79.

[14] А. П. Лауринчикас. Распределение значений производящих рядов Дирихле мультипликативных функций. Лит. мат. сб., 1982, т.22, №1, с.101-111.

[15] K. Matsumoto. An introduction to the value – distribution theory of zeta – functions. Šiauliai Mathematical Seminar **1(9)**, 2006, 61 – 83.

[16] A. Laurinčikas. Prehistory of the Voronin Universality Theorem. Šiauliai Mathematical Seminar **1(9)**, 2006, 41-53.

[17] H.L. Montgomery. Topics in multiplicative number theory. M., 1974 (rus).

[18] Е.П. Ожигова. Развитие теории чисел в России. Изд. Наука, Ленинградское отд. Ленинград, 1972, 360 стр.

[19] K. Prachar. Distribution of Prime Numbers. M.: Mir, 1987(rus).

[20] B. Riemann. On the number of prime numbers not exceeding a given quality.//Compositions.–M. : OGIZ, 1948 – p. 216 – 224(rus).

[21] W. Rudin. Principles of mathematical analysis. M.: Mir,1976.319 pp. (rus).

[22] E. C. Titchmarsh. Theory of Riemann Zeta – function. M. : IL, 1953(rus).

[23] E. C. Titchmarsh. Theory of function. M.: GITTL, 1951.506 pp.(rus).

[24] S. M. Voronin, A.A. Karatsuba. The Riemann Zeta–function. M: fiz. mat. lit. ,1994, 376 p. (rus).

[25] S.M. Voronin. On The distribution of non – zero values of the Riemann Zeta function. Labors. MIAS – 1972 – v. 128, p. 153-175, (rus).

[26] S.M. Voronin. On an differential independence of ζ – function Reports of AS USSR – 1973. v. 209, № 6, pp.1264 – 1266, (rus).

[27] S.M. Voronin. On an differential independence of Dirichlet`s L–functions Acta Arith. – 1975 , v. XXVII – pp. 493 – 509.

[28] S.M. Voronin. The theorem on “universality” of the Riemann zeta – function. Bulletin Acad. Sci. of USSR mat.ser. – 1975 – v. 39, № 3 – pp. 475 – 486, (rus).

[29] S.M. Voronin. On the zeroes of zeta – functions of quadratic forms. Labors MIAS – 1976 – v. 142 – pp. 135 – 147 (rus).

[30] S.M. Voronin. Analytical properties of Dirichlet generating functions of arithmetical objects: Diss.…D – r of fiz. – mat. sci. MIAS USSR – M., 1977 – 90p, (rus).

[31] S.M. Voronin. On an zeroes of some Dirichlet series, lying on the critical line. Bull. Acad. Sci. of USSR mat.ser. – 1980 – v. 44 №1 – pp.63-91 (rus).

[32] S.M. Voronin. On the distribution of zeroes of some Dirichlet series. Labor. MIAS – 1984 – v. 163 – pp. 74 – 77, (rus).

[33] A. Speiser. Geometrisches zur Riemannschen Zetafunction, Math.Ann.110 (1934), 514 – 521.

[34] H. L. Mongomery. The pair correlation of zeroes of the Riemann zeta – function in Analytic Number Theory. (St. Louis), Proc. Sympos. Pure Math. 24. Amer. Math. Soc. Providence, R.I,1973,181-193.

[35] J. B. Conrey, A. Chosh, D. Goldston, S. M. Gonek and D. R. Heath – Brown. On the distribution of gaps between zeroes of the zeta – function. Oxford J. Math., Oxford (2), 36, (1985), 43 – 51.

[36] А. А. Карацуба. О нижних оценках максимума модуля дзета – функции Римана на коротких промежутках критической прямой. Изв. РАН, сер. мат. 2004, 68, 6, 105 – 118.

[37] M. Z. Garaev, C. Y. Yildirim. On small distances between ordinates of zeroes of $\zeta(s)$ and $\zeta'(s)$. ArXiv:math/0610377v2 [math. NT] 19 Mar 2007.

[38] М. А. Королев. О больших расстояниях между соседними нулями дзета – функции Римана. Изв. РАН, сер. мат. 2008, 72, 2, 91 – 104.

(Ganja State University,

AZ2000, Shah Ismail Hatai avenue 187,

Ganja, Azerbaijan)