Qual é a energia média, <E>, de um oscilador harmónico clássico à temperatura absoluta T?

A energia total de um oscilador harmónico clássico, E, é dada por

$$E = T + U \Leftrightarrow E = \frac{1}{2}mv^2 + \frac{1}{2}Kr^2 \Leftrightarrow E = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) + \frac{1}{2}K(x^2 + y^2 + z^2)$$

$$\langle E \rangle = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dv_x dv_y dv_z dx dy dz}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dv_x dv_y dv_z dx dy dz}$$

Ou rearranjando este integral de grau 6 de outra forma, temos que

$$< E >= \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\overbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dx dy dz}^{I_1}\, dv_x dv_y dv_z}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dx dy dz}_{I_2}\, dv_x dv_y dv_z}$$

Em primeiro lugar, vamos calcular os seguintes integrais triplos:

$$I_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dx dy dz$$

$$I_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dx dy dz$$

Recorrendo às coordenadas esféricas, em que o elemento de volume dV é dado por

$$dV = dx \cdot dy \cdot dz = dr \cdot r\, d\theta \cdot r \sin\theta\, d\varphi = r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

$$I_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dxdydz =$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

$$I_2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} dxdydz = = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_B T}}\right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

Como $e^{-\left[\frac{1}{2}mv^2\right]}$ é independente com r, θ e φ (isto é, $e^{-\left[\frac{1}{2}mv^2\right]}$ não varia em função r, θ e φ), então podemos passar $o\ termo\ e^{-\left[\frac{1}{2}mv^2\right]}$ para fora do integral triplo.

$$I_1 = e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_B T}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{\left[\frac{1}{2}mv^2 + \frac{1}{2}Kr^2\right] e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_B T}}\right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

$$I_2 = e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_B T}} \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_B T}}\right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}}\right\}r^2dr\cdot\sin\theta\,d\theta\cdot d\varphi=$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\{\left[\frac{1}{2}mv^2r^2+\frac{1}{2}Kr^4\right]e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}}\right\}dr\cdot\sin\theta\,d\theta\cdot d\varphi$$

$$=\int_0^{2\pi}d\varphi\int_0^{\pi}\sin\theta\,d\theta\int_0^{\infty}\left\{\left[\frac{1}{2}mv^2r^2+\frac{1}{2}Kr^4\right]e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}}\right\}dr$$

$$=[\varphi]_0^{2\pi}\cdot[-\cos\theta]_0^{\pi}\cdot\left\{\frac{1}{2}mv^2\int_0^{\infty}r^2\,e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}}dr+\frac{1}{2}K\int_0^{\infty}r^4\,e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}}dr\right\}=$$

$$=2\pi\cdot 2\cdot\left\{\frac{1}{2}mv^2\cdot\left(\frac{2K_BT}{K}\right)^{\frac{3}{2}}\cdot\frac{\sqrt{\pi}}{4}+\frac{1}{2}K\cdot\left(\frac{2K_BT}{K}\right)^{\frac{5}{2}}\cdot\frac{3\sqrt{\pi}}{8}\right\}$$

$$=\frac{mv^2}{2}\cdot\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}+\left(\frac{2}{K}\right)^{\frac{3}{2}}\cdot(K_BT)^{\frac{5}{2}}\cdot\frac{3(\pi)^{\frac{3}{2}}}{2}\Longrightarrow$$

$$\Longrightarrow I_1=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_BT}}\right\}dxdydz$$

$$=e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}\cdot\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}+\left(\frac{2}{K}\right)^{\frac{3}{2}}\cdot(K_BT)^{\frac{5}{2}}\cdot\frac{3(\pi)^{\frac{3}{2}}}{2}\right\}\Longrightarrow I_1$$

$$=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_BT}}\right\}dxdydz$$

$$=\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}\cdot e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}$$

Agora precisamos de justificar alguns destes cálculos.

Substituição de variável: $t^2 = \frac{Kr^2}{2K_BT} \Leftrightarrow t = r \cdot \sqrt{\frac{K}{2K_BT}} \Longrightarrow dt = dr \cdot \sqrt{\frac{K}{2K_BT}} \Leftrightarrow dr = dt \cdot \sqrt{\frac{2K_BT}{K}}$

$$t^2 = \frac{Kr^2}{2K_BT} \Longrightarrow r^4 = \frac{4 \cdot (K_BT)^2 \cdot t^4}{K^2}$$

$$\int_0^\infty r^2\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr = \int_0^\infty \frac{2 \cdot (K_BT) \cdot t^2}{K} e^{-t^2} \sqrt{\frac{2K_BT}{K}}\, dt = \left(\frac{2K_BT}{K}\right)^{\frac{3}{2}} \cdot \int_0^\infty t^2 e^{-t^2} dt$$

$$\int_0^\infty r^4\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr = \int_0^\infty \left(\frac{4 \cdot (K_BT)^2 \cdot t^4}{K^2}\right) e^{-t^2} \cdot \sqrt{\frac{2K_BT}{K}}\, dt = \left(\frac{2K_BT}{K}\right)^{\frac{5}{2}} \int_0^\infty t^4 e^{-t^2} dt$$

Agora precisamos calcular estes dois integrais recorrendo à técnica da integração por partes.

$$\int_0^\infty t^2 e^{-t^2} dt = \left[\frac{-te^{-t^2}}{2}\right]_0^\infty - \left(\int_0^\infty \frac{-e^{-t^2}}{2} dt\right) = \underbrace{\left[\frac{-te^{-t^2}}{2}\right]_0^\infty}_{0} + \overbrace{\frac{1}{2}\left(\int_0^\infty e^{-t^2} dt\right)}^{\frac{\sqrt{\pi}}{4}} = \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty t^4 e^{-t^2} dt = \left[\frac{-t^3 e^{-t^2}}{2}\right]_0^\infty - \left(\int_0^\infty \frac{-3t^2 e^{-t^2}}{2} dt\right) = \underbrace{\left[\frac{-t^3 e^{-t^2}}{2}\right]_0^\infty}_{0} - \left(\underbrace{\left[\frac{3te^{-t^2}}{4}\right]_0^\infty}_{0} - \left(\int_0^\infty \frac{3e^{-t^2}}{4} dt\right)\right) =$$

$$= \underbrace{\left[\frac{-t^3 e^{-t^2}}{2}\right]_0^\infty}_{0} - \left(\underbrace{\left[\frac{3te^{-t^2}}{4}\right]_0^\infty}_{0} - \overbrace{\frac{3}{4}\left(\int_0^\infty e^{-t^2} dt\right)}^{\frac{3\sqrt{\pi}}{8}}\right) = \frac{3\sqrt{\pi}}{8}$$

$$\int_{-\infty}^{+\infty} e^{-t^2} dt = 2 \cdot \int_0^\infty e^{-t^2} dt = \sqrt{\pi} \Leftrightarrow \int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}$$

$$\int_0^\infty r^2\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr = \sqrt{\frac{2K_BT}{K}} \int_0^\infty t^2 e^{-t^2} dt = \left(\frac{2K_BT}{K}\right)^{\frac{3}{2}} \cdot \frac{\sqrt{\pi}}{4}$$

$$\int_0^\infty r^4\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr = \left(\frac{2K_BT}{K}\right)^{\frac{5}{2}} \int_0^\infty t^4 e^{-t^2} dt = \left(\frac{2K_BT}{K}\right)^{\frac{5}{2}} \cdot \frac{3\sqrt{\pi}}{8}$$

$$\frac{1}{2}mv^2 \int_0^\infty r^2\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr + \frac{1}{2}K \int_0^\infty r^4\, e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} dr = \frac{mv^2}{2} \cdot \left(\frac{2K_BT}{K}\right)^{\frac{3}{2}} \cdot \frac{\sqrt{\pi}}{4} + \sqrt{\frac{2}{K}}\,(K_BT)^{\frac{3}{2}} \cdot \frac{3\sqrt{\pi}}{8}$$

Agora, vamos calcularmo integral I_2.

$$I_2 = e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}} \int_0^{2\pi}\int_0^{\pi}\int_0^\infty \left\{ e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^\infty \left\{ e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta \int_0^\infty \left\{ r^2 e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} dr$$

$$= [\varphi]_0^{2\pi} \cdot [-\cos\theta]_0^{\pi} \cdot \int_0^\infty \left\{ r^2 e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} dr = 2\pi \cdot 2 \cdot \int_0^\infty \left\{ r^2 e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} dr$$

$$= 4\pi \cdot \sqrt{\frac{2K_BT}{K}} \underbrace{\int_0^\infty t^2 e^{-t^2} dt}_{\frac{\sqrt{\pi}}{4}} = (\pi)^{\frac{3}{2}} \cdot \sqrt{\frac{2K_BT}{K}}$$

$$I_2 = e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}} \int_0^{2\pi}\int_0^{\pi}\int_0^\infty \left\{ e^{\frac{-\left[\frac{1}{2}Kr^2\right]}{K_BT}} \right\} r^2 dr \cdot \sin\theta\, d\theta \cdot d\varphi = e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}} \cdot \left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}$$

$$I_1 = \left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}} \cdot e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}} \cdot \left\{ \frac{mv^2}{2} + \frac{3K_BT}{2} \right\}$$

$$\langle E\rangle = \frac{\overbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_BT}}\right\}}^{I_1} dxdydz\, dv_xdv_ydv_z}{\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{e^{\frac{-\left[\frac{1}{2}mv^2+\frac{1}{2}Kr^2\right]}{K_BT}}\right\}}_{I_2} dxdydz\, dv_xdv_ydv_z} =$$

$$= \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}\cdot e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}\right)dv_xdv_ydv_z}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}\right)dv_xdv_ydv_z}$$

$$= \frac{\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}\cdot\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}\right)dv_xdv_ydv_z}{\left(\frac{2\pi K_BT}{K}\right)^{\frac{3}{2}}\cdot\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_xdv_ydv_z} =$$

$$= \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}\right)dv_xdv_ydv_z}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_xdv_ydv_z}$$

Então temos que

$$\langle E\rangle = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}\right)dv_xdv_ydv_z}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_xdv_ydv_z}$$

$$= \frac{\left\{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_z{}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_y{}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left(v_x{}^2+v_y{}^2+v_z{}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_xdv_ydv_z\right\}}{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_z{}^2\right]}{K_BT}}\right)dv_z\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_y{}^2\right]}{K_BT}}\right)dv_y\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_x}$$

Agora, precisamos genertalizar e em vez de calcular os integrais um a um, vamos fazer o seguinte:

Para i,j=x,y,z, temos $v_i, v_j = v_x, v_y, v_z$ e o primeiro integral do numerador de $\langle E\rangle$ em função de v_x fica

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\left\{\frac{mv_x{}^2+\left(\sum_{j=y,z}mv_j{}^2\right)}{2}+2K_BT\cdot\frac{3}{4}\right\}\right)dv_x = \int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\frac{mv_x{}^2}{2}dv_x + e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\frac{\left(\sum_{j=y,z}mv_j{}^2\right)}{2} + e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot 2K_BT\cdot\frac{3}{4}\right)dv_x = \underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\frac{mv_x{}^2}{2}\right)dv_x}_{I_3} + \underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\frac{\left(\sum_{j=y,z}mv_j{}^2\right)}{2}\right)dv_x}_{I_4} + \frac{3K_BT}{2}\cdot\underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\right)dv_x}_{I_5}$$

$$I_3 = \int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\cdot\frac{mv_x{}^2}{2}\right)dv_x \to Substituição\ de\ variável\begin{cases} h^2 = \frac{mv_x{}^2}{2K_BT} \Leftrightarrow h = \sqrt{\frac{m}{2K_BT}}v_x \\ dh = \sqrt{\frac{m}{2K_BT}}dv_x \Leftrightarrow dv_x = \sqrt{\frac{2K_BT}{m}}dh \\ v_x{}^2 = \frac{2K_BT}{m}h^2 \Rightarrow \frac{mv_x{}^2}{2} = K_BTh^2 \end{cases}$$

Então temos que

$$I_3 = \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m {v_x}^2\right]}{K_B T}} \cdot \frac{m {v_x}^2}{2} \right) dv_x \,_3 = \int_{-\infty}^{+\infty} \left(e^{-h^2} \cdot K_B T h^2 \right) \cdot \sqrt{\frac{2K_B T}{m}} dh \Leftrightarrow I_3 = \sqrt{\frac{2K_B T}{m}} \cdot K_B T h^2 \cdot \underbrace{\int_{-\infty}^{+\infty} h^2 e^{-h^2} dh}_{\frac{\sqrt{\pi}}{2}}$$

$$\Leftrightarrow I_3 = \sqrt{\frac{2K_B T}{m}} \cdot K_B T \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{2\pi K_B T}{m}} \cdot \frac{K_B T}{2}$$

Então temos que

$$I_3 = \sqrt{\frac{2\pi K_B T}{m}} \cdot \frac{K_B T}{2}$$

$$I_4 = \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m {v_x}^2\right]}{K_B T}} \cdot \sum_{j \neq i} m {v_j}^2 \right) dv_x \rightarrow Substituição\ de\ variável \begin{cases} h^2 = \frac{m {v_x}^2}{2K_B T} \Leftrightarrow h = \sqrt{\frac{m}{2K_B T}} v_x \\ dh = \sqrt{\frac{m}{2K_B T}} dv_x \Leftrightarrow dv_x = \sqrt{\frac{2K_B T}{m}} dh \\ {v_x}^2 = \frac{2K_B T}{m} h^2 \end{cases}$$

$$I_4 = \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m {v_x}^2\right]}{K_B T}} \cdot \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right) \right) dv_x = \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right) \cdot \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m {v_x}^2\right]}{K_B T}} \right) dv_x$$

$$= \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right) \cdot \int_{-\infty}^{+\infty} e^{-h^2} \sqrt{\frac{2K_B T}{m}} dh = \sqrt{\frac{2K_B T}{m}} \cdot \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right) \cdot \underbrace{\left(\int_{-\infty}^{+\infty} e^{-h^2} dh \right)}_{\sqrt{\pi}}$$

$$= \sqrt{\frac{2K_B T}{m}} \cdot \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right) \cdot \sqrt{\pi} = \sqrt{\frac{2\pi K_B T}{m}} \cdot \left(\frac{\sum_{j=y,z} m {v_j}^2}{2} \right)$$

Então temos que

$$I_4 = \sqrt{\frac{2K_BT}{m}} \cdot \left(\frac{\sum_{j=y,z} m{v_j}^2}{2}\right) \cdot \sqrt{\pi} = \sqrt{\frac{2\pi K_BT}{m}} \cdot \left(\frac{\sum_{j=y,z} m{v_j}^2}{2}\right)$$

e como é óbvio

$$I_5 = \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right) dv_x \longrightarrow Substituição\ de\ variável \begin{cases} h^2 = \dfrac{m{v_x}^2}{2K_BT} \Leftrightarrow h = \sqrt{\dfrac{m}{2K_BT}}\, v_x \\ dh = \sqrt{\dfrac{m}{2K_BT}}\, dv_x \Leftrightarrow dv_x = \sqrt{\dfrac{2K_BT}{m}}\, dh \\ {v_x}^2 = \dfrac{2K_BT}{m} h^2 \end{cases}$$

$$I_5 = \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right) dv_x = \sqrt{\frac{2K_BT}{m}} \underbrace{\int_{-\infty}^{+\infty} e^{-h^2} dh}_{\sqrt{\pi}} = \sqrt{\frac{2K_BT}{m}} \cdot \sqrt{\pi} = \sqrt{\frac{2\pi K_BT}{m}}$$

De seguida, temos que:

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\left\{\frac{m{v_x}^2}{2}+\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)+2K_BT\cdot\frac{3}{4}\right\}\right)dv_x$$

$$=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\frac{m{v_x}^2}{2}+e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)+e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\frac{3K_BT}{2}\right)dv_x$$

$$=\underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\frac{m{v_x}^2}{2}\right)dv_x}_{I_3}+\underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\cdot\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)\right)dv_x}_{I_4}+\frac{3K_BT}{2}$$

$$\cdot\underbrace{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)dv_x}_{I_5}=\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{K_BT}{2}+\sqrt{\frac{2\pi K_BT}{m}}\cdot\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)+\frac{3K_BT}{2}\cdot\sqrt{\frac{2\pi K_BT}{m}}=$$

$$=\sqrt{\frac{2\pi K_BT}{m}}\left\{2K_BT+\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)\right\}$$

Fazendo Para i=x,y,z, temos $v_i=v_x,v_y,v_z$ e

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_i}^2\right]}{K_BT}}\right)dv_i=\int_{-\infty}^{+\infty}e^{-h^2}\cdot\sqrt{\frac{2K_BT}{m}}dh=\sqrt{\frac{2K_BT}{m}}\cdot\underbrace{\int_{-\infty}^{+\infty}e^{-h^2}dh}_{\sqrt{\pi}}=\sqrt{\frac{2\pi K_BT}{m}}$$

Conclusão:

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_i}^2\right]}{K_BT}}\right)dv_i=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_x=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z=\sqrt{\frac{2\pi K_BT}{m}}$$

E o denominador de $\langle E\rangle\ fica\ igual\ a$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_x=\left[\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_i}^2\right]}{K_BT}}\right)dv_i\right]^3=\left[\sqrt{\frac{2\pi K_BT}{m}}\right]^3=\left(\frac{2\pi K_BT}{m}\right)^{\frac{3}{2}}$$

$$\langle E\rangle = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\cdot\left\{\frac{mv^2}{2}+\frac{3K_BT}{2}\right\}\right)dv_xdv_ydv_z}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_xdv_ydv_z}$$

$$= \frac{\left\{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_z{}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_y{}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}mv_x{}^2\right]}{K_BT}}\right)\cdot\left(m\left(v_x{}^2+v_y{}^2+v_z{}^2\right)+2K_BT\cdot\frac{3}{4}\right)\right\}dv_xdv_ydv_z\right\}}{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_z{}^2\right]}{K_BT}}\right)dv_z\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_y{}^2\right]}{K_BT}}\right)dv_y\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_x}$$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_z{}^2\right]}{K_BT}}\right)dv_z\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv_y{}^2\right]}{K_BT}}\right)dv_y\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}mv^2\right]}{K_BT}}\right)dv_x = \left(\frac{2\pi K_BT}{m}\right)^{\frac{3}{2}}$$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_xdv_y=$$

$$=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\sqrt{\frac{2\pi K_BT}{m}}\left\{2K_BT+\left(\frac{\sum_{j=y,z}m{v_j}^2}{2}\right)\right\}\right\}dv_y$$

$$=\int_{-\infty}^{+\infty}\left\{2K_BT\cdot\sqrt{\frac{2\pi K_BT}{m}}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)+\sqrt{\frac{2\pi K_BT}{m}}\cdot m{v_y}^2\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)+\sqrt{\frac{2\pi K_BT}{m}}\cdot m{v_z}^2\right.$$

$$\left.\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\right\}dv_y=\int_{-\infty}^{+\infty}2K_BT\cdot\sqrt{\frac{2\pi K_BT}{m}}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y$$

$$+\int_{-\infty}^{+\infty}\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\int_{-\infty}^{+\infty}\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_z}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y$$

$$=2K_BT\cdot\sqrt{\frac{2\pi K_BT}{m}}\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\sqrt{\frac{2\pi K_BT}{m}}\cdot\int_{-\infty}^{+\infty}\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\sqrt{\frac{2\pi K_BT}{m}}$$

$$\cdot\frac{m{v_z}^2}{2}\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y$$

$$\int_{-\infty}^{+\infty}\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y\longrightarrow Substituição:\begin{cases}h^2=\frac{m{v_y}^2}{2K_BT}\Leftrightarrow h=\sqrt{\frac{m}{2K_BT}}v_y\\ dh=\sqrt{\frac{m}{2K_BT}}dv_y\Leftrightarrow dv_y=\sqrt{\frac{2K_BT}{m}}dh\\ {v_y}^2=\frac{2K_BT}{m}h^2\Longrightarrow K_BTh^2=\frac{m{v_y}^2}{2}\end{cases}$$

$$\int_{-\infty}^{+\infty} \frac{m{v_y}^2}{2} \cdot \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = K_BT \cdot \int_{-\infty}^{+\infty} h^2 \cdot \left(e^{-h^2}\right) \cdot \sqrt{\frac{2K_BT}{m}}\, dh = K_BT \cdot \sqrt{\frac{2K_BT}{m}} \cdot \underbrace{\int_{-\infty}^{+\infty} h^2 e^{-h^2} dh}_{\frac{\sqrt{\pi}}{2}} = K_BT \cdot \sqrt{\frac{2K_BT}{m}} \cdot \frac{\sqrt{\pi}}{2} \Longrightarrow \int_{-\infty}^{+\infty} \frac{m{v_y}^2}{2} \cdot \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = \frac{K_BT}{2} \cdot \sqrt{\frac{2\pi K_BT}{m}}$$

$$\sqrt{\frac{2\pi K_BT}{m}} \cdot \int_{-\infty}^{+\infty} \frac{m{v_y}^2}{2} \cdot \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = \sqrt{\frac{2\pi K_BT}{m}} \cdot \frac{K_BT}{2} \cdot \sqrt{\frac{2\pi K_BT}{m}} = \frac{K_BT}{2} \cdot \left(\frac{2\pi K_BT}{m}\right)$$

$$\int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = \int_{-\infty}^{+\infty} e^{-h^2} \cdot \sqrt{\frac{2K_BT}{m}}\, dh = \sqrt{\frac{2K_BT}{m}} \cdot \underbrace{\int_{-\infty}^{+\infty} e^{-h^2} dh}_{\sqrt{\pi}} = \sqrt{\frac{2\pi K_BT}{m}}$$

$$\sqrt{\frac{\pi}{m}} \cdot (2K_BT)^{\frac{3}{2}} \cdot \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = \sqrt{\frac{\pi}{m}} \cdot (2K_BT)^{\frac{3}{2}} \cdot \sqrt{\frac{2\pi K_BT}{m}} = 2K_BT \cdot \left(\frac{2\pi K_BT}{m}\right)$$

$$\sqrt{\frac{2\pi K_BT}{m}} \cdot \frac{m{v_z}^2}{2} \cdot \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right) dv_y = \sqrt{\frac{2\pi K_BT}{m}} \cdot \frac{m{v_z}^2}{2} \cdot \sqrt{\frac{2\pi K_BT}{m}} = \frac{2\pi K_BT}{m} \cdot \left(\frac{m{v_z}^2}{2}\right)$$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+2K_BT\cdot\frac{3}{4}\right)\right\}dv_x\right\}dv_y=$$

$$=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\sqrt{\frac{2\pi K_BT}{m}}\left\{2K_BT+\frac{\left(\sum_{j=y,z}m{v_j}^2\right)}{2}\right\}\right\}dv_y$$

$$=\int_{-\infty}^{+\infty}\left\{\sqrt{\frac{\pi}{m}}\cdot(2K_BT)^{\frac{3}{2}}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)+\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)+\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_z}^2}{2}\right.$$

$$\left.\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\right\}dv_y=\int_{-\infty}^{+\infty}\sqrt{\frac{\pi}{m}}\cdot(2K_BT)^{\frac{3}{2}}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y$$

$$+\int_{-\infty}^{+\infty}\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\int_{-\infty}^{+\infty}\sqrt{\frac{2\pi K_BT}{m}}\cdot\frac{m{v_z}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y$$

$$=\sqrt{\frac{\pi}{m}}\cdot(2K_BT)^{\frac{3}{2}}\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\sqrt{\frac{2\pi K_BT}{m}}\cdot\int_{-\infty}^{+\infty}\frac{m{v_y}^2}{2}\cdot\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y+\sqrt{\frac{2\pi K_BT}{m}}$$

$$\cdot\frac{m{v_z}^2}{2}\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)dv_y=2K_BT\cdot\left(\frac{2\pi K_BT}{m}\right)+\frac{K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)+\frac{2\pi K_BT}{m}\cdot\left(\frac{m{v_z}^2}{2}\right)\Longrightarrow$$

$$\Longrightarrow\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+2K_BT\cdot\frac{3}{4}\right)\right\}dv_x\right\}dv_y=$$

$$=\frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)+\frac{2\pi K_BT}{m}\cdot\left(\frac{m{v_z}^2}{2}\right)$$

$$\Longrightarrow\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_x\right\}dv_y$$

$$=\frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)+\frac{2\pi K_BT}{m}\cdot\left(\frac{m{v_z}^2}{2}\right)$$

$$\Longrightarrow \int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_x\right\}dv_y$$

$$=\left(\frac{2\pi K_BT}{m}\right)\cdot\left(\frac{5K_BT}{2}+\left(\frac{m{v_z}^2}{2}\right)\right)$$

$$\left\{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_x\right\}dv_y\right\}dv_z$$

$$=\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\left(\frac{5K_BT}{2}+\left(\frac{m{v_z}^2}{2}\right)\right)dv_z$$

$$=\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)\cdot\left(\frac{5K_BT}{2}+\left(\frac{m{v_z}^2}{2}\right)\right)dv_z$$

$$=\frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z+\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(\frac{m{v_z}^2}{2}\right)\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z$$

$$\int_{-\infty}^{+\infty}\left(\frac{m{v_z}^2}{2}\right)\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z \longrightarrow Substituição: \begin{cases} h^2=\frac{m{v_z}^2}{2K_BT}\Leftrightarrow h=\sqrt{\frac{m}{2K_BT}}v_z \\ dh=\sqrt{\frac{m}{2K_BT}}dv_z \Leftrightarrow dv_z=\sqrt{\frac{2K_BT}{m}}dh \\ {v_z}^2=\frac{2K_BT}{m}h^2 \Longrightarrow K_BTh^2=\frac{m{v_z}^2}{2} \end{cases}$$

$$\int_{-\infty}^{+\infty}\left(\frac{m{v_z}^2}{2}\right)\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \int_{-\infty}^{+\infty}(K_BTh^2)\left(e^{-h^2}\right)\sqrt{\frac{2K_BT}{m}}dh = K_BT\cdot\sqrt{\frac{2K_BT}{m}}\cdot\underbrace{\int_{-\infty}^{+\infty}h^2e^{-h^2}dh}_{\frac{\sqrt{\pi}}{2}} = K_BT\cdot\sqrt{\frac{2K_BT}{m}}\cdot\frac{\sqrt{\pi}}{2}\Longrightarrow$$

$$\int_{-\infty}^{+\infty}\left(\frac{m{v_z}^2}{2}\right)\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \frac{K_BT}{2}\cdot\sqrt{\frac{2\pi K_BT}{m}}\Longrightarrow\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(\frac{m{v_z}^2}{2}\right)\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \frac{K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)^{\frac{3}{2}}$$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z \longrightarrow Substituição:\begin{cases} h^2 = \frac{m{v_z}^2}{2K_BT}\Leftrightarrow h = \sqrt{\frac{m}{2K_BT}}v_z \\ dh = \sqrt{\frac{m}{2K_BT}}dv_z \Leftrightarrow dv_z = \sqrt{\frac{2K_BT}{m}}dh \\ {v_z}^2 = \frac{2K_BT}{m}h^2 \Longrightarrow K_BTh^2 = \frac{m{v_z}^2}{2}\end{cases}$$

$$\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \int_{-\infty}^{+\infty}e^{-h^2}\sqrt{\frac{2K_BT}{m}}dh = \sqrt{\frac{2K_BT}{m}}\cdot\underbrace{\int_{-\infty}^{+\infty}e^{-h^2}dh}_{\sqrt{\pi}} = \sqrt{\frac{2\pi K_BT}{m}}\Leftrightarrow$$

$$\Leftrightarrow\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \sqrt{\frac{2\pi K_BT}{m}}\Leftrightarrow$$

$$\Leftrightarrow\frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\sqrt{\frac{2\pi K_BT}{m}}\Leftrightarrow$$

$$\Leftrightarrow\frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)dv_z = \frac{5K_BT}{2}\cdot\left(\frac{2\pi K_BT}{m}\right)^{\frac{3}{2}}$$

Então temos que

$$\left\{\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)\int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_y}^2\right]}{K_BT}}\right)\left\{\int_{-\infty}^{+\infty}\left\{\left(e^{\frac{-\left[\frac{1}{2}m{v_x}^2\right]}{K_BT}}\right)\cdot\left(\frac{m\left({v_x}^2+{v_y}^2+{v_z}^2\right)}{2}+\frac{3K_BT}{2}\right)\right\}dv_x\right\}dv_y\right\}dv_z =$$

$$= \int_{-\infty}^{+\infty}\left(e^{\frac{-\left[\frac{1}{2}m{v_z}^2\right]}{K_BT}}\right)\cdot\left(\frac{2\pi K_BT}{m}\right)\cdot\left(\frac{5K_BT}{2}+\left(\frac{m{v_z}^2}{2}\right)\right)dv_z =$$

$$= \left(\frac{2\pi K_B T}{m}\right) \cdot \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m v_z{}^2\right]}{K_B T}}\right) \cdot \left(\frac{5K_B T}{2} + \left(\frac{m v_z{}^2}{2}\right)\right) dv_z =$$

$$= \frac{5K_B T}{2} \cdot \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}} + \frac{K_B T}{2} \cdot \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}} =$$

$$= \frac{6K_B T}{2} \cdot \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}}$$

Então que o numerador de $\langle E \rangle$ é dado por

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{\left[\frac{1}{2} m v^2 + \frac{1}{2} K r^2\right] e^{\frac{-\left[\frac{1}{2} m v^2 + \frac{1}{2} K r^2\right]}{K_B T}}\right\} dv_x dv_y dv_z dx dy dz = \frac{6K_B T}{2} \cdot \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}}$$

e como é ovio de calcular

$$\int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m v_z{}^2\right]}{K_B T}}\right) dv_z \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m v_y{}^2\right]}{K_B T}}\right) dv_y \int_{-\infty}^{+\infty} \left(e^{\frac{-\left[\frac{1}{2} m v^2\right]}{K_B T}}\right) dv_x = \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}}$$

Então temos que

$$\langle E \rangle = \frac{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{\left[\frac{1}{2} m v^2 + \frac{1}{2} K r^2\right] e^{\frac{-\left[\frac{1}{2} m v^2 + \frac{1}{2} K r^2\right]}{K_B T}}\right\} dv_x dv_y dv_z dx dy dz}{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \left\{e^{\frac{-\left[\frac{1}{2} m v^2 + \frac{1}{2} K r^2\right]}{K_B T}}\right\} dv_x dv_y dv_z dx dy dz} \Leftrightarrow$$

$$\Leftrightarrow \langle E \rangle = \frac{\frac{6K_B T}{2} \cdot \left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}}}{\left(\frac{2\pi K_B T}{m}\right)^{\frac{3}{2}}} \Leftrightarrow \langle E \rangle = \frac{6K_B T}{2} \Leftrightarrow \langle E \rangle = 3K_B T$$

$$U = N_A \cdot \langle E \rangle \Leftrightarrow U = N_A \cdot 3K_B T \Leftrightarrow U = {}^{3 \overbrace{N_A K_B}^{R = N_A K_B} T}\!\big/_{mole} \Leftrightarrow U = {}^{3RT}\!\big/_{mole}$$

Calor específico a volume constante: $C_v = \frac{dU}{dT} \Leftrightarrow C_v = \frac{d}{dT}\left({}^{3RT}/_{mole}\right) \Leftrightarrow C_v = {}^{3R}/_{mole}$

$$C_v = {}^{3R}/_{mole} \Longrightarrow \text{Lei de Dulong e Petit}$$