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Weak gravitational field of the electromagnetic radiation in a ring laser

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Abstract

The gravitational field due to the circulating flow of electromagnetic radiation of a unidirectional ring laser is found by solving the linearized Einstein field equations at any interior point of the laser ring. The general relativistic spin equations are then used to study the behavior of a massive spinning neutral particle at the center of the ring laser. It is found that the particle exhibits the phenomenon known as inertial frame-dragging. © 2000 Elsevier Science B.V. All rights reserved.

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In classical Newtonian mechanics matter is the sole generator of the gravitational field. One of the interesting consequences of general relativity is the prediction that light is also a source of gravity. The gravitational field of a noncirculating beam of light was studied many years ago by Tolman [1]. This was done by using the weak field approximation to Einstein's gravitational field equations. Tolman then determined the acceleration of a stationary particle in the neighborhood of the light beam. He found that the acceleration experienced by the particle was twice as great as that expected on the basis of Newtonian theory for the gravitational field of a massive rod of similar length and density. This would seem to imply that, in some ways, light may be even more effective than matter in generating a gravitational field. More recently, Scully [2] has given a general relativistic treatment of the gravitational interaction between two parallel laser beams in the weak field approximation.

The present work is a generalization of these earlier investigations to the gravitational field produced by a circulating flow of electromagnetic radiation. Recent advances in laser technology have provided a means of producing such a flow using a ring laser [3]. This device is capable of generating an intense, coherent, and continuously circulating beam of light. In this letter,the gravitational field at any interior point of a ring laser is found by solving the linearized Einstein field equations. The general relativistic spin equations are then used to study the behavior of a massive spinning neutral particle at the center of the laser ring. It is shown that such a

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particle manifests the behavior known as inertial frame-dragging[1].

The usual linearized Einstein gravitational field equations[2], with the Hilbert gauge condition $\partial_\mu\left(h^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}h\right)=0$, have the form

$$\partial_\lambda\partial^\lambda\left(h^{\mu\nu}-\tfrac{1}{2}\eta^{\mu\nu}h\right)=-k\tau^{\mu\nu} \qquad (1)$$

where $h^{\mu\nu}$ is the gravitational field tensor, $\tau^{\mu\nu}$ is the energy-momentum tensor, $h=h^\sigma_\sigma$ with the Lorentz metric $\eta_{\mu\nu}=(1,-1,-1,-1)$. The energy-momentum tensor for the electromagnetic field is given by

$$\tau^{\mu\nu}=-\tfrac{1}{4\pi}\left(f^{\mu\alpha}f^\nu_\alpha-\tfrac{1}{4}\eta^{\mu\nu}f^{\alpha\beta}f_{\alpha\beta}\right) \qquad (2)$$

where $f^{\mu\nu}$ is the Maxwell field tensor. Since the trace of Eq. (2) is $\tau=\tau^\mu_\mu=0$ then Eq. (1), which can be rewritten as $\partial_\lambda\partial^\lambda h^{\mu\nu}=-k\left(\tau^{\mu\nu}-\frac{1}{2}\eta^{\mu\nu}\tau\right)$, reduces to

$$\partial_\lambda\partial^\lambda h^{\mu\nu}=-k\tau^{\mu\nu} \qquad (3)$$

A sketch of the stationary (i.e. nonrotating) ring laser configuration to be considered is shown in Fig. 1. A laser at L generates a beam that enters the ring through a half-silvered mirror at A. The beam is then reflected from the corner mirrors at B, C, and D. The length of each side of the ring is a. The following results are independent of the state of polarization of the light beam and depend only on the energy density and direction of the radiation flow. Consequently, there is no loss in generality in choosing a specific polarization state. The state chosen here is one that is polarized perpendicular to the plane of the ring laser. Taking the plane of the ring to lie in the

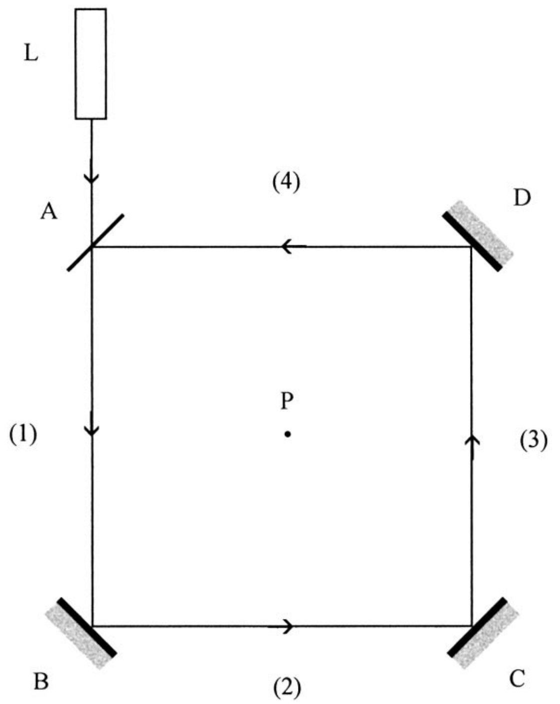

Fig. 1. A sketch of the ring laser geometry.

xy plane, the Maxwell field tensor components for the beam are given by

$$f^{30}_{(1)}=f^{30}_{(2)}=f^{30}_{(3)}=f^{30}_{(4)}=E_z \qquad (4)$$

$$f^{13}_{(1)}=-f^{13}_{(3)}=-B_y\,,\quad f^{32}_{(2)}=-f^{32}_{(4)}=B_x \qquad (5)$$

with all other $f_{(n)}{}^{\mu\nu}=0$ and where indices in parentheses (n); $n=1,\ldots,4$ indicate the particular beam path in Fig. 1. For paths (1), (3) and (2), (4) the time averaged energy density of the radiation is given, respectively, by

$$\rho=\tfrac{1}{16\pi}\left(E_z^2+B_y^2\right),\quad \rho=\tfrac{1}{16\pi}\left(E_z^2+B_x^2\right).$$

Using Eqs. (4) and (5) in Eq. (2) yields the non-vanishing time averaged energy-momentum tensor components

$$\tau^{00}_{(1)}=\tau^{11}_{(1)}=\tau^{01}_{(1)}=\rho \qquad (6)$$

$$\tau^{00}_{(2)}=\tau^{22}_{(2)}=\tau^{02}_{(2)}=\rho \qquad (7)$$

$$\tau^{00}_{(3)}=\tau^{11}_{(3)}=-\tau^{01}_{(3)}=\rho \qquad (8)$$

$$\tau^{00}_{(4)}=\tau^{22}_{(4)}=-\tau^{02}_{(4)}=\rho \qquad (9)$$

[1] The post Newtonian phenomenon known as inertial frame-dragging is usually associated with the gravitational field generated by rotating matter. For example, it is predicted that a satellite in a polar orbit around the earth should be dragged around by the gravitational field created by the earth's rotation. The frame-dragging effect should also be displayed in the precession of a gyroscope falling freely in the gravitational field of the rotating earth. See, for example, [4]. This reference combines a detailed discussion of inertial frame-dragging produced by the gravitational field of rotating matter with a comprehensive survey of many of the proposed experiments to test this effect.

[2] See, for example, [5]. In the present Letter, $h_{\mu\nu}$ is chosen to be dimensionless so that $k=8\pi G/c^4$ with gravitational constant $G=6.67\times10^{-11}\ \mathrm{m^3\,kg^{-1}\,s^{-2}}$ and speed of light $c=3\times10^8\ \mathrm{m/s}$.

The solution of the linearized Einstein field equations, Eq. (3), for the ring configuration in Fig. 1 is given by

$$h^{\mu\nu}(\boldsymbol{x}) = -k\sum_{n=1}^{4}\int d^3x' G_{(n)}(\boldsymbol{x},\boldsymbol{x}')\tau^{\mu\nu}_{(n)}(\boldsymbol{x}') \tag{10}$$

with the Green functions

$$G_{(1)}(\boldsymbol{x},\boldsymbol{x}') = \frac{\delta(x'_2)\,\delta(x'_3)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{11}$$

$$G_{(2)}(\boldsymbol{x},\boldsymbol{x}') = \frac{\delta(x'_1-a)\,\delta(x'_3)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{12}$$

$$G_{(3)}(\boldsymbol{x},\boldsymbol{x}') = \frac{\delta(x'_2-a)\,\delta(x'_3)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{13}$$

$$G_{(4)}(\boldsymbol{x},\boldsymbol{x}') = \frac{\delta(x'_1)\,\delta(x'_3)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|} \tag{14}$$

Using Eqs. (6)–(9) and Eq. (11)–(14) in Eq. (10) yields the non-vanishing gravitational field components

$$h^{00} = -\frac{k\rho}{4\pi}\left[\Phi_{(1)}+\Phi_{(2)}+\Phi_{(3)}+\Phi_{(4)}\right] \tag{15}$$

$$h^{01} = -\frac{k\rho}{4\pi}\left[\Phi_{(1)}-\Phi_{(3)}\right] \tag{16}$$

$$h^{02} = -\frac{k\rho}{4\pi}\left[\Phi_{(2)}-\Phi_{(4)}\right] \tag{17}$$

$$h^{11} = -\frac{k\rho}{4\pi}\left[\Phi_{(1)}+\Phi_{(3)}\right] \tag{18}$$

$$h^{22} = -\frac{k\rho}{4\pi}\left[\Phi_{(2)}+\Phi_{(4)}\right] \tag{19}$$

where

$$\Phi_{(1)} = \int_0^a dx'\left[(x-x')^2+y^2+z^2\right]^{-\frac{1}{2}} = \ln\left\{\frac{-x+a+\left[(x-a)^2+y^2+z^2\right]^{\frac{1}{2}}}{-x+\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}\right\} \tag{20}$$

$$\Phi_{(2)} = \int_0^a dy'\left[(x-a)^2+(y-y')^2+z^2\right]^{-\frac{1}{2}} = \ln\left\{\frac{-y+a+\left[(y-a)^2+(x-a)^2+z^2\right]^{\frac{1}{2}}}{-y+\left[(x-a)^2+y^2+z^2\right]^{\frac{1}{2}}}\right\} \tag{21}$$

$$\Phi_{(3)} = \int_0^a dx'\left[(x-x')^2+(y-a)^2+z^2\right]^{-\frac{1}{2}} = \ln\left\{\frac{-x+a+\left[(x-a)^2+(y-a)^2+z^2\right]^{\frac{1}{2}}}{-x+\left[x^2+(y-a)^2+z^2\right]^{\frac{1}{2}}}\right\} \tag{22}$$

$$\Phi_{(4)} = \int_0^a dy'\left[x^2+(y-y')^2+z^2\right]^{-\frac{1}{2}} = \ln\left\{\frac{-y+a+\left[(y-a)^2+x^2+z^2\right]^{\frac{1}{2}}}{-y+\left(x^2+y^2+z^2\right)^{\frac{1}{2}}}\right\} \tag{23}$$

Eqs. (15)–(19) together with Eqs. (20)–(23) provide the complete metric for the gravitational field at any interior point of the ring laser.

The general relativistic spin equations for a neutral massive spinning particle, in the weak field and slow motion approximation, are given by [6]

$$\frac{dS_i}{dt} - c\Gamma^k_{i0}S_k + \Gamma^0_{i0}S_j\mathrm{v}^j - \Gamma^k_{il}S_k\mathrm{v}^l + c^{-1}\Gamma^0_{ik}\mathrm{v}^k\mathrm{v}^jS_j = 0 \tag{24}$$

Consider the case of a stationary neutral massive spinning particle at the center of the ring laser in Fig. 1. Substituting Eqs. (15)–(23) in Eq. (24), with $h_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}h^{\alpha\beta}$, and evaluating the general result at point P in the center, $(\frac{a}{2},\frac{a}{2},0)$, of the laser ring, yields the simple result

$$\frac{d\boldsymbol{S}}{dt} = \dot{\boldsymbol{\Omega}}\times\boldsymbol{S} \tag{25}$$

where $\dot{\boldsymbol{\Omega}} = \left(0,0,\sqrt{2}\,ck\rho/\pi a\right)$. Eq. (25) has exactly the form (see footnote 1) required for general relativistic gravitational frame-dragging. Thus, Eq. (25) shows that a stationary neutral massive spinning particle at the center of the ring laser in Fig. 1 will tend to precess in a counterclockwise direction with a rate of precession given by

$$\dot{\Omega} = \frac{8\sqrt{2}\,G\rho}{ac^3} \tag{26}$$

with radiation linear density ρ and beam length a. It is straightforward to show that reversing the direction of the radiation flow in the laser ring of Fig. 1 results in a precession of the particle in the opposite sense.

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