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Real and complex analysis
Notes for a course given at the
University of Sheffield 2007

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## Contents

## 1. Reminder on metric spaces

### 1.1. **Metric spaces.**

**Definition 1.1.1.** A *metric space* $(X, d)$ is a set $X$ together with a *distance* function $d : X \times X \longrightarrow \mathbb{R}$ satisfying the following properties ($P, Q, R \in X$):

(i) $d(P, Q) \geq 0$ and $d(P, Q) = 0$ if and only if $P = Q$ (positivity).
(ii) $d(P, Q) = d(Q, P)$ (symmetry).
(iii) $d(P, R) \leq d(P, Q) + d(Q, R)$ (triangle inequality).

**Remark 1.1.2.** This concept of a distance is made according to the natural distance between two points in "space" (the world).

**Example 1.1.3.** The real numbers are a metric space with the distance function given by $d(a, b) := |a - b|$. In a similar way the complex numbers form a metric space with the distance function given by

$$d(z, w) := |z - w| = \sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2}\,.$$

(Exercise: check this).

In the next section we will see that every finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ can be made into a metric space using a norm.

**Example 1.1.4.** Let $(X, d)$ be a metric space and let $Y \subseteq X$ be a subset. Then $Y$ is immediately again a metric space by setting $d_Y(P, Q) := d(P, Q)$.

With the concept of the distance one can define further important notions.

**Definition 1.1.5.** Let $X$ and $Y$ be two metric spaces. A function $f : X \longrightarrow Y$ is called *continuous* in $P \in X$ if for every real number $\varepsilon > 0$ there exists a $\delta > 0$ such that for every point $Q \in X$ with $d(P, Q) < \delta$ we have $d(f(P), f(Q)) < \varepsilon$. It is continuous if it is continuous in every point.

The meaning of continuous is that one can control the distance between the two image points $f(P)$ and $f(Q)$ by controlling the distance between the points $P$ and $Q$.

**Definition 1.1.6.** For a real number $\delta > 0$ and a point $P \in X$ in a metric space one calls the set
$$B(P, \delta) := \{Q \in X : d(P, Q) < \delta\}$$
the *open ball neighborhood* of $P$ of *radius* $\delta$.

A subset $U \subseteq X$ is called *open* if for every point $P \in U$ there exists a real number $\delta > 0$ such that $B(P, \delta) \subseteq U$.

A subset $A \subseteq X$ is called *closed* if its complement $X - A$ is open.

The continuity of $f : X \longrightarrow Y$ in $P$ is equivalent to the fact that for every $\varepsilon > 0$ there exist $\delta > 0$ such that $f(B(P, \delta)) \subseteq B(f(P), \varepsilon)$.

**Example 1.1.7.** The balls $B(P, \delta)$ in a metric space are open. To show this let $Q \in B(P, \delta)$ be a point. This means that $d(P, Q) = \varepsilon < \delta$. We **claim** that
$$B(Q, \delta - \varepsilon) \subseteq B(P, \delta)\,.$$
For this let $R \in B(Q, \delta - \varepsilon)$. Then $d(R, Q) < \delta - \varepsilon$ and so by the triangle inequality we have
$$d(P, R) \leq d(P, Q) + d(Q, R) < \varepsilon + \delta - \varepsilon = \delta\,,$$
hence $R \in B(P, \delta)$.

**Proposition 1.1.8.** *Let $X$ be a metric space. Then the system of all open subsets of $X$ fulfills the following properties :*

(i) *The whole space $X$ and the empty subset $\emptyset$ are open.*
(ii) *The intersection of two open subsets is again open.*
(iii) *Let $U_i$, $i \in I$, be an (arbitrary) collection of open subsets. Then also their union $\bigcup_{i \in I} U_i$ is open.*

*Proof.* As exercise 1.4. □

A set with a system of subsets fulfilling these properties is called a *topological space*; hence metric spaces are topological spaces. Properties of a space which can be formulated with the notion of open subsets are called *topological properties*.

**Definition 1.1.9.** Let $X$ be a metric space. A *sequence* $x_n$ in $X$ is a mapping $\mathbb{N} \longrightarrow X$, $n \mapsto x_n$, so that for every natural number $n \in \mathbb{N}$ a uniquely determined point $x_n \in X$ is given. We say that the sequence $x_n$, $n \in \mathbb{N}$, *converges* to $x \in X$if for every $\delta > 0$ ($\in \mathbb{R}$) there exists a natural number $n_0$ (depending on $\delta$) such that for all $m \geq n_0$ we have $d(x_m, x) < \delta$.

This means that $x_n$ gets arbitrarily close to $x$. The easiest example is $x_n = 1/n$ in $X = \mathbb{R}$, which converges to 0.

**Proposition 1.1.10.** *The following are equivalent for a mapping $f : X \longrightarrow Y$ between two metric spaces.*

(i) *$f$ is continuous.*

(ii) *For every sequence* $P_n \in X$ *converging to* $P$ *also the image sequence* $f(P_n) \in Y$ *converges to* $f(P)$.
(iii) *For every open subset* $U \subseteq Y$ *also the preimage* $f^{-1}(U)$ *is open in* $X$.

*Proof.* (i) $\Rightarrow$ (ii). Let $P_n \in X$ be a sequence converging to $P$. We have to show that $f(P_n) \in Y$ converges to $f(P)$. Let $\varepsilon > 0$ be given. By the continuous property there exists a $\delta > 0$ such that $f(B(P,\delta)) \subseteq B(f(P),\varepsilon)$. By the convergence property there exists an $n_0$ such that for all $m \geq n_0$ we have $d(P, P_m) < \delta$, or equivalently $P_m \in B(P,\delta)$. Then also $f(P_m) \in B(f(P),\varepsilon)$ for $m \geq n_0$, which means that $f(P_n)$ converges to $f(P)$.

(ii) $\Rightarrow$ (i). (This is an exercise in the negation of statements) Suppose that (ii) holds, but that $f$ is not continuous in $P \in X$. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ it is not true that $f(B(P,\delta)) \subseteq B(f(P),\varepsilon)$. Then in particular for every natural number $n \in \mathbb{N}$ we have $f(B(P,1/n)) \not\subseteq B(f(P),\varepsilon)$. This means that there exists (at least one) a point $P_n \in B(P,1/n)$, but with $f(P_n) \notin B(f(P),\varepsilon)$. This gives a sequence $P_n$ converging to $P$, but such that the image sequence $f(P_n)$ does not converge to $f(P)$. This contradicts (ii).

(i) $\Rightarrow$ (iii). Let $U \subseteq Y$ be an open subset and let $W := f^{-1}(U)$ be its preimage, let $P \in W$ be a point so that $f(P) \in U$. By the openness property there exists an $\varepsilon > 0$ such that $B(f(P),\varepsilon) \subseteq U$. By (i) there exists $\delta > 0$ such that $f(B(P,\delta)) \subseteq B(f(P),\varepsilon)$. Hence $P \in B(P,\delta) \subseteq f^{-1}(U) = W$ is open.

(iii) $\Rightarrow$ (i). Let $P \in X$ and $\varepsilon > 0$ be given. The subset $B(f(P),\varepsilon)$ is an open subset by Example 1.1.7, so by (iii) also its preimage $f^{-1}(B(f(P),\varepsilon))$ is open. As $P$ belongs to this set, this means in particular by the definition of openness that there exists some $\delta$ with $B(P,\delta) \subseteq f^{-1}(B(f(P),\varepsilon))$. But this is the definition of continuous in the sense of (i). $\square$

**Definition 1.1.11.** Let $X$ and $Y$ two metric spaces and $f : X \longrightarrow Y$ a mapping. $f$ is called a *homeomorphism*, if it is continuous, bijective and if the inverse mapping $f^{-1} : Y \longrightarrow X$ is also continuous.

Two spaces are called *homeomorphic* if there exists a homeomorphism between them.

## 1.2. **The norm in a vector space.**

We use the symbol $\mathbb{K}$ to denote either the real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$. In both cases the *modulus* is defined and makes these fields to metric spaces. In the following we will consider $\mathbb{K}$-vector spaces, i.e. vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

**Definition 1.2.1.** A *norm* on a $\mathbb{K}$-vector space $V$ is a mapping

$$|| \ || : V \longrightarrow \mathbb{R}\,, v \longmapsto ||v||$$

fulfilling the following properties:

(i) $||v|| \geq 0$ for all $v \in V$ and $||v|| = 0$ if and only if $v = 0$ (positivity).
(ii) For all $a \in \mathbb{K}$ and all $v \in V$ we have $||a \cdot v|| = |a| \cdot ||v||$ (homogeneity).
(iii) $||v + w|| \leq ||v|| + ||w||$ for all $v, w \in V$ (triangle inequality).

A vector space endowed with a norm is called a *normed vector space.*

**Fact 1.2.2.** *Every normed vector space* $(V, || \, ||)$ *is also a metric space by declaring the distance of two points* $v$ *and* $w$ *to be*

$$d(v, w) := ||v - w|| \,.$$

*Proof.* As exercise 2.1. $\square$

We will be only interested in metric spaces which are subspaces of a normed vector space. In particular we will be interested in open subsets $G \subseteq V$.

**Example 1.2.3.** The most important norm is the *Euclidean norm* on $\mathbb{R}^n$ and $\mathbb{C}^n$. On $\mathbb{R}^n$, this is

$$||z|| := \sqrt{z_1^2 + \cdots + z_n^2} \,.$$

This gives the ordinary distance from the point $z$ to the origin. This is a consequence of the Pythagorean theorem. Sometimes we write $|| \, ||_{Euc}$.

On $\mathbb{C}^n$, the Euclidean norm is

$$||z|| := \sqrt{|z_1|^2 + \cdots + |z_n|^2} \,.$$

If we identify $\mathbb{C}^n \cong \mathbb{R}^{2n}$, then the real-Euclidean and the complex-Euclidean norm coincide. (This norm comes from an inner product.)

**Example 1.2.4.** The *sum-norm* on $\mathbb{K}^n$ is defined as

$$||x||_1 := \sum_{i=1}^{n} |x_i| \,.$$

This is indeed a norm (sometimes called the *taxicab norm*). (Prove this. Show that it does not come from an inner product.)

**Example 1.2.5.** The *maximum norm* on $\mathbb{K}^n$ is given by

$$||x||_{\max} := \max\{|x_i| : \, i = 1, \ldots, n\}.$$

is also a norm. (Exercise)

**Remark 1.2.6.** For a finite-dimensional $\mathbb{K}$-vector space $V$, a norm defines a metric and hence a topology. The norm and the metric may differ, but the topology is always the same as we will see. So the topology, i.e. the set of open subsets and the concept of continuity, is independent of the choice of a norm.

**Lemma 1.2.7.** *Let* $(V, \| \, \|)$ *be a normed vector space. Then the norm* $\| \, \| : V \longrightarrow \mathbb{R}$ *is continuous.*

*Proof.* As exercise 2.2. $\square$

**Lemma 1.2.8.** *Let* $W$ *be a normed* $\mathbb{K}$*-vector space and let* $(\mathbb{K}^n, || \, ||_{Euc})$ *be the Euclidean space. Let* $L : \mathbb{K}^n \longrightarrow W$ *be a linear map. Then* $L$ *is* bounded*, which means that there exist a real number* $M$ *such that* $\|L(x)\| \leq M \cdot \|x\|$ *for all* $x \in \mathbb{K}^n$*. It follows that* $L$ *is also continuous.*

*Proof.* For this set $T := \max\{\|L(e_i)\| : i = 1, \ldots, n\}$, where $e_i$ are the unit vectors. Then for $x = c_1e_1 + \ldots + c_ne_n$ we have

$$\begin{aligned} \|L(x)\| &= \|c_1L(e_1) + \ldots + c_nL(e_n)\| \\ &\leq \|c_1L(e_1)\| + \ldots + \|c_nL(e_n)\| \\ &= |c_1|\,\|L(e_1)\| + \ldots + |c_n|\,\|L(e_n)\| \\ &\leq n \cdot \max_{i=1,\ldots,n}\{|c_i|\} \cdot \max_{i=1,\ldots,n}\{\|L(e_i)\|\} \\ &\leq n \cdot \sqrt{c_1^2 + \ldots + c_n^2} \cdot T = ||x||_{Euc} \cdot n \cdot T \end{aligned}$$

Hence $M = nT$ does the job. Now in general, a bounded linear mapping between arbitrary normed vector spaces are continuous. Let $M$ be such a bound, and $v$ and $w \in V$ be given. Then

$$||L(v) - L(w)|| = ||L(v - w)|| \leq M||v - w||\,.$$

So for a given $\varepsilon > 0$ just take $\delta = \varepsilon/M$. □

**Theorem 1.2.9.** *Let $(V, \|\,\|)$ be a finite-dimensional $\mathbb{K}$-vector space of dimension $n$. Then $V$ is linearly homeomorphic to the Euclidean space $(\mathbb{K}^n, ||\,||_{Euc})$.*

This means that there exists a linear bijective mapping $\mathbb{K}^n \longrightarrow V$ which is continuous and such that the inverse mapping is also continuous.

*Proof.* Let $v_1, \ldots, v_n$ be a basis of $V$ and let $L : \mathbb{K}^n \longrightarrow V$ be the linear bijective mapping given by $e_i \mapsto v_i$ (so that $\sum_{i=1}^n c_ie_i$ is sent to $\sum_{i=1}^n c_iv_i$). Then this mapping is continuous by Lemma 1.2.8. So we only have to show that the inverse mapping $L^{-1} : V \longrightarrow \mathbb{K}^n$ is also continuous.

We show first that there exists another bound $m > 0$ with the property that $\|L(x)\| \geq m\,\|x\|$ for all $x \in \mathbb{K}^n$. For this we use that the composed mapping

$$\mathbb{K}^n \xrightarrow{L} V \xrightarrow{\|\,\|} \mathbb{R}$$

is continuous (Lemma 1.2.7). On the *unit sphere* $S = \{x \in \mathbb{K}^n : \|x\| = 1\}$, which is *compact* (as it is closed and bounded), this function has a minimum non-negative value, which we call $m$. By compactness, this minimum is achieved, which means there exists a point $P \in S$ such that $\|L(P)\| = m$. Now $m \neq 0$, for otherwise $L(P) = 0$, which contradicts the fact that $L$ is a bijection. Hence (as $\frac{v}{\|v\|}$ lies on the unit sphere)

$$\|L(v)\| = \left\|\|v\|\,L(\frac{v}{\|v\|})\right\| = \|v\|\left\|L(\frac{v}{\|v\|})\right\| \geq m\,\|v\|\;.$$

For the inverse mapping $L^{-1}$ we look at $Q \in V$. Let $L^{-1}(Q) = P$. Then we know that $\|L(P)\| \geq m \cdot \|P\|$, and replacing $P$ by $L^{-1}(Q)$ gives

$$\|Q\| = \left\|L(L^{-1}(Q))\right\| \geq m\left\|L^{-1}(Q)\right\|$$

or $\|L^{-1}(Q)\| \leq 1/m \cdot \|Q\|$. So we can take $1/m$ as a bound from above for $L^{-1}$ which shows continuity. □

**Theorem 1.2.10.** *Let $V$ be a finite-dimensional normed $\mathbb{K}$-vector space and let $W$ be another normed $\mathbb{K}-$vector space. Let $L : V \longrightarrow W$ be $\mathbb{K}$-linear. Then $L$ is continuous.*

*Proof.* Let $\Psi : \mathbb{K}^n \longrightarrow V$ be a linear homeomorphism as in the proof of Theorem 1.2.9. Hence $L$ is continuous if and only if $L \circ \Psi$ is continuous, but $L \circ \Psi$ is continuous by Lemma 1.2.8. □

## 1.3. **Connected spaces.**

A subset of a metric (or topological) space can be open, or closed, or neither open and closed, or both. For example, the whole space and the empty subset are open and closed.

**Definition 1.3.1.** A metric (or topological) space $X$ is called *connected* if $X \neq \emptyset$ and if $\emptyset$ and $X$ are the only subsets which are open and closed.

**Remark 1.3.2.** $X$ is not connected if and only if there exists a *partition* $X = U_1 \uplus U_2$ with $U_1$ and $U_2$ both open and non empty ($U_1 \uplus U_2$ means the union of the two sets and that they are disjoint, that is, $U_1 \cap U_2 = \emptyset$).

**Proposition 1.3.3.** *The image of a connected space under a continuous mapping is connected.*

*Proof.* Let $f : X \longrightarrow Y$ be continuous, let $X$ be connected and consider the image $f(X) \subseteq Y$, which is a metric space. Suppose that $f(X) = U_1 \uplus U_2$ is a non-trivial disjoint partition into two open (hence also closed) subsets. Then also $X = f^{-1}(U_1) \uplus f^{-1}(U_2)$ is a non-trivial partition into two open subsets, contradicting the assumption on $X$. □

**Definition 1.3.4.** A metric (or topological) space $X$ is called *path-connected* if for every two points $P$ and $Q$ in $X$ there exists a continuous mapping (a *path*) $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = P$ and $\gamma(1) = Q$.

**Example 1.3.5.** Let $V$ be a normed $\mathbb{K}$-vector space. Then every ball $B(P, \delta)$ is path-connected. It is enough to show for every point $Q \in B(P, \delta)$ that there exists a path to $P$. Look at $t \mapsto P + t(Q - P)$, $t \in [0, 1]$, which is continuous (as it is affine-linear, hence continuous by Theorem 1.2.10). This path lies inside the ball, because of

$$\|P + t(Q - P) - P\| = \|t(Q - P)\| = |t| \, \|Q - P\| \leq \|Q - P\| < \delta \, .$$

**Proposition 1.3.6.** *Let $X$ be a metric space which is path-connected. Then $X$ is also connected.*

*Proof.* Assume $X = U_1 \uplus U_2$ with two non-empty open (and hence closed) subsets. Let $P \in U_1$ and $Q \in U_2$. By assumption, there exists a continuous mapping $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = P$ and $\gamma(1) = Q$ and this gives a partition

$$[0, 1] = \gamma^{-1}(U_1) \uplus \gamma^{-1}(U_2)$$

(both open and closed and non-empty). This would however mean that the interval $[0, 1]$ is not connected, which contradicts theorem 1.3.7. $\square$

**Theorem 1.3.7.** *The interval* $[0, 1] \subseteq \mathbb{R}$ *(and every other interval) is connected.*

*Proof.* This is related to the axiomatic treatment of what the real numbers are. Assume that $[0, 1] = U_1 \uplus U_2$, both open and hence closed and non-empty. Let $a \in U_1$ and $b \in U_2$, and assume without loss of generality that $a < b$. We construct inductively two sequences by

$$a_{n+1} := a_n, \text{ if } \frac{a_n + b_n}{2} \in U_2, \text{ and } = \frac{a_n + b_n}{2} \text{ if } \frac{a_n + b_n}{2} \in U_1$$

and

$$b_{n+1} := b_n, \text{ if } \frac{a_n + b_n}{2} \in U_1, \text{ and } = \frac{a_n + b_n}{2} \text{ if } \frac{a_n + b_n}{2} \in U_2 \,.$$

Then, by construction, $a_n$ is a sequence in $U_1$ and $b_n$ is a sequence in $U_2$ (proof by induction), and $a_n < b_n$. Also, $|b_n - a_n| = |b - a|/2^n$, so their distance converges to 0. Hence both sequences are **Cauchy sequences** and they converge in $\mathbb{R}$ (this is the property of the real numbers called *completeness*) to the same limit point $x$. As $U_1$ and $U_2$ are both closed, we have that $x \in U_1 \cap U_2$. This contradicts the disjointness. $\square$

It is also true that every connected subset of $\mathbb{R}$ is an interval, see exercise 3.6.

**Theorem 1.3.8.** *Let $X$ be an open subset of a normed vector space $V$ of finite dimension. Then $X$ is connected if and only if it is path-connected.*

*Proof.* The relation between two points of being path-connected is an equivalence relation (exercise 2.10). Its equivalence classes are open: With $P \in X$ also the whole ball $B(P, \delta) \subseteq X$ (which exists by the open property) is path-connected with $P$ (Example 1.3.5). So the equivalence classes $U_i$, $i \in I$, form a partition of $X$ into open subsets. But then also the complement of such a component is open, as it is a union of open subsets. This means that the component itself is also closed. Hence in a connected space there exists only one equivalence class for path-connected, which means that the space is path-connected. $\square$

## 2. Differentiation in higher dimensions

### 2.1. **The total differential.**

We want to 'differentiate' continuous mappings $\varphi : V \longrightarrow W$ between vector spaces and more general continuous mappings which are only defined on a certain open subset $G \subseteq V$,

$$\varphi : G \longrightarrow W \,.$$

Already in dimension one, many functions are defined (in a natural way) only on a certain, often open subset, like the function $z \mapsto 1/z$ is defined on $\mathbb{R} - \{0\}$ or on $\mathbb{C} - \{0\}$.

If we recall the situation of one variable, say we have a map $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, then the basic idea of a differentiable map and its differentiation is to find a 'tangent to the graph'. This is a line which goes through a point $(x, \varphi(x))$ and which has the same 'slope' as the function. One can say that the tangent is the best *linear approximation* of $\varphi$ for a given point $x \in \mathbb{R}$. As the slope is just again a number, differentiation gives for every point $x$ again a number, hence we get a new function, denoted by $\varphi'$. In the higher dimensional situation this is very different, but the idea of best *linear approximation* is still working.

All vector spaces are finite dimensional and a norm is defined on them.

**Definition 2.1.1.** Let $\varphi : G \longrightarrow W$ be a mapping defined on an open subset $G \subseteq V$. Then $\varphi$ is said to be *differentiable* at a point $P \in G$ if there exists a linear mapping $L : V \longrightarrow W$ fulfilling the property that

$$\varphi(P + v) = \varphi(P) + L(v) + ||v||r(v)$$

with a function $r : B(0, \delta) \longrightarrow W$ which is defined on an open ball neighborhood of 0 and is continuous in 0 with $r(0) = 0$ and the equation holds for all $v \in V$ such that $P + v \in B(P, \delta) \subseteq G$)

This linear mapping, if it exists, is called the *(total) differential* of $\varphi$ at $P$, and is denote by $(D\varphi)_P$.

Equivalent: The expression

$$r(v) = \frac{\varphi(P + v) - \varphi(P) - L(v)}{||v||}$$

has limit 0 for $v \longrightarrow 0$. It is also equivalent to say that the limit (of functions)

$$\lim_{v \longrightarrow 0, v \neq 0} \frac{\|\varphi(P + v) - \varphi(P) - L(v)\|}{\|v\|} = 0$$

exists and is 0.

**Remark 2.1.2.** Note that this notion is defined using continuity, but since all norms yield the same topology, it is independent on the actual norm on the vector space.

This is a theoretical concept rather than a computational concept. We will relate this concept with the concept of partial derivatives, which is better suited for computations, which is however dependent on coordinates (see Example 2.2.9).

**Example 2.1.3.** The function $|\,| : \mathbb{R} \longrightarrow \mathbb{R}$, $t \mapsto |t|$ is not differentiable in $P = 0$. A linear map is given by $v \mapsto cv$, $c \in \mathbb{R}$. We have to consider $\frac{|v| - cv}{|v|}$, which is $1 - c$ for $v > 0$ and which is $1 + c$ for $v < 0$. For $c \neq 0$ this does not have a limit (as it has two different limits from left and right), and for $c = 0$ the limit is $1 \neq 0$.

**Lemma 2.1.4.** *Let $\varphi : G \longrightarrow W$ be a mapping defined on an open subset $G \subseteq V$. Let $P \in G$. Then there exists at most one linear mapping with the property described in Definition 2.1.1. If $\varphi$ is differentiable in $P$, then the differential is uniquely determined.*

*Proof.* Suppose that $\varphi(P+v) = \varphi(P) + L_1(v) + ||v||r_1(v)$ and $\varphi(P+v) = \varphi(P) + L_2(v) + ||v||r_2(v)$ with linear mappings $L_1$ and $L_2$ and functions $r_1, r_2 : B \longrightarrow W$ which are continuous in 0 and with $r_1(0) = r_2(0) = 0$. We have to show that $L_1 = L_2$. For this we subtract the two equations (equation of mappings with values in the vector space $W$, so subtracting means subtracting the values), and we get the equation

$$0 = (L_1 - L_2)(v) + ||v||(r_1(v) - r_2(v)) .$$

Therefore we have to show that the constant 0-mapping has the property that the linear mapping 0 is its only linear approximation. So suppose that

$$0 = L(v) + ||v||r(v)$$

with $L$ linear and $r$ continuous in 0 with $r(0) = 0$. If $L$ would not be the 0 mapping, then there exists a vector $v$ with $L(v) = w \neq 0$. Then for $s \in \mathbb{K}$ we have

$$0 = L(sv) + ||sv||r(sv) = sw + |s| \cdot ||v|| \cdot r(sv) .$$

This implies for $s \neq 0$ that $r(sv) = -sw/|s| \cdot ||v||$. Its norm is the constant $||w||/||v|| \neq 0$. Hence $\lim_{s \longrightarrow 0}(r(sv)) \neq 0$. □

**Proposition 2.1.5.** *Let $L : V \longrightarrow W$ be a linear mapping. Then $L$ is differentiable in every point $P \in V$ and the total differential is $L$ itself for every point.*

*Proof.* We can write immediately

$$L(P+v) = L(P) + L(v) ,$$

so we can take $r = 0$. □

**Example 2.1.6.** If $L : V \longrightarrow W$ is constant, say $L(v) \equiv w \in W$ for all $v \in V$, then it is differentiable with total differential 0 (exercise).

**Proposition 2.1.7.** *Let $\varphi_1, \varphi_2 : G \longrightarrow W$ be mappings which are differentiable in $P \in V$ with differentials $(D\varphi_1)_P$ and $(D\varphi_1)_P$. Then also $\varphi = \varphi_1 + \varphi_2$ is differentiable in $P$ with differential $(D\varphi)_P = (D\varphi_1)_P + (D\varphi_1)_P$. Also, $D(a\varphi)_P = a(D(\varphi))_P$.*

*Proof.* Let $\varphi_1(P+v) = \varphi_1(P) + L_1(v) + ||v||r_1(v)$ and $\varphi_2(P+v) = \varphi_2(P) + L_2(v) + ||v||r_2(v)$ . Then

$$\begin{aligned} (\varphi_1 + \varphi_2)(P+v) &= \varphi_1(P+v) + \varphi_2(P+v) \\ &= \varphi_1(P) + L_1(v) + ||v||r_1(v) + \varphi_2(P) + L_2(v) + ||v||r_2(v) \\ &= (\varphi_1 + \varphi_2)(P) + (L_1 + L_2)(v) + ||v||(r_1(v) + r_2(v)) \end{aligned}$$

This has the required form, and $r_1 + r_2$ is also continuous in 0 with $(r_1 + r_2)(0) = 0$. The second statement is similiar. □

**Proposition 2.1.8.** *Let $\varphi : G \longrightarrow W$ be a mapping which is differentiable in $P \in G$. Then $\varphi$ is also continuous in $P$.*

*Proof.* By definition, we have $\varphi(P+v) = \varphi(P) + L(v) + ||v||r(v)$. The right hand side is continuous (by Definition 2.1.1 and Theorem 1.2.10) in $v = 0$ with value $\varphi(P)$, hence $\varphi$ is continuous in $P$. □

The beauty of the concept of total differential is seen by the following version of the chain rule.

**Theorem 2.1.9.** *Let $V$, $W$ and $U$ be $\mathbb{K}$-vector spaces, let $G \subseteq V$ and $D \subseteq W$ be open subsets and let $\varphi : G \longrightarrow W$ and $\psi : D \longrightarrow U$ be mappings with $\varphi(G) \subseteq D$. Suppose that $\varphi$ is differentiable in $P \in G$ and that $\psi$ is differentiable in $\varphi(P) \in D$. Then the composition $\psi \circ \varphi : G \longrightarrow U$ is differentiable in $P$ with differential*

$$D(\psi \circ \varphi)_P = (D\psi)_{\varphi(P)} \circ (D\varphi)_P \,.$$

*Proof.* By assumption we have (setting $Q := \varphi(P)$)

$$\varphi(P+v) = \varphi(P) + L(v) + ||v||r(v)$$

and

$$\psi(Q+w) = \psi(Q) + M(w) + ||w||s(w)$$

with linear mappings $L : V \longrightarrow W$ and $M : W \longrightarrow U$ and mappings $r : B \longrightarrow W$ and $s : B' \longrightarrow U$ (defined on certain open neighborhoods of 0 in $V$ and $W$), both continuous in 0 with value 0 at 0.

Then we get

$$\begin{aligned}
(\psi \circ \varphi)(P+v) &= \psi(\varphi(P+v)) \\
&= \psi(\varphi(P) + L(v) + ||v|| \cdot r(v)) \\
&= \psi(\varphi(P)) + M(L(v) + ||v|| \cdot r(v)) \\
&\quad + ||L(v) + ||v|| \cdot r(v)|| \cdot s(L(v) + ||v|| \cdot r(v)) \\
&= \psi(\varphi(P)) + M(L(v)) + M(||v|| \cdot r(v)) \\
&\quad + ||L(v) + ||v|| \cdot r(v)|| \cdot s(L(v) + ||v|| \cdot r(v)) \\
&= \psi(\varphi(P)) + (M \circ L)(v) + ||v||M(r(v)) \\
&\quad + ||(||v||L(\frac{v}{||v||}) + ||v|| \cdot r(v))|| \cdot s(L(v) + ||v|| \cdot r(v)) \\
&= \psi(\varphi(P)) + (M \circ L)(v) \\
&\quad + ||v||\Big(M(r(v)) + ||L(\frac{v}{||v||}) + r(v)|| \cdot s(L(v) + ||v|| \cdot r(v))\Big)
\end{aligned}$$

In the first equation we treat $L(v) + ||v|| \cdot r(v)$ as $w$. The last two equations hold only for $v \neq 0$. The expression

$$t(v) := M(r(v)) + ||L(\frac{v}{||v||}) + r(v)|| \cdot s(L(v) + ||v|| \cdot r(v))$$

is our candidate for the error function. The first summand $M(r(v))$ is continuous in $v = 0$ with value 0, so we only have to deal with the second summand. The $||\ ||$-expression is bounded in a neighborhood of 0, so the continuity depends only on the factor on the right. But for $v \longrightarrow 0$ the expression $L(v) + ||v|| \cdot r(v)$ has limit 0. Hence also $s(L(v) + ||v|| \cdot r(v))$ is continuous in 0 with limit 0 there. $\square$

## 2.2. **Directional and partial derivatives.**

Let $f : \mathbb{K}^n \longrightarrow \mathbb{K}$ be a mapping written as

$$(x_1, \ldots, x_n) \longmapsto f(x_1, \ldots, x_n)\,.$$

Then by considering for an index $i$ the other variables $x_j$, $j \neq i$, as constant, one gets a mapping $\mathbb{K} \longrightarrow \mathbb{K}$ depending only on $x_i$ (respectively, the other variables are considered as parameters). If this function is differentiable as a function in one variable, then we say that $f$ is *partially differentiable with respect to* $x_i$, and the derivative is denoted by $\partial f / \partial x_i$. The advantage of partial derivatives are on the computational side. It depends however on the choice of a basis. The partial derivatives are themselves mappings $\mathbb{K}^n \longrightarrow \mathbb{K}$. We want to understand the relationship between partial derivatives and the total differential.

**Definition 2.2.1.** Let $\varphi : G \longrightarrow W$ be a mapping, $G \subseteq V$. Let $P \in G$ be a point and let $v \in V$ be a fixed vector. Then we say that $\varphi$ is *differentiable in direction* $v$ if the limit

$$\lim_{s \longrightarrow 0, s \neq 0} \frac{\varphi(P + sv) - \varphi(P)}{s}$$

exists. This limit (if it exists) is a vector in $W$. In this case this limit is called *the derivative of* $\varphi$ *in direction* $v$. It is denoted by $(D_v(\varphi))_P$.

The existence of $(D_v(\varphi))_P$ depends only on the mapping $\mathbb{K} \supseteq B \longrightarrow W$ given by $s \longrightarrow \varphi(P+sv)$ (where the interval (in the real case) or the open ball (in the complex case) is such that $s \in B$ implies that $P + sv \in G$).

Total differentiability implies directional differentiability.

**Proposition 2.2.2.** *Let* $\varphi : G \longrightarrow W$ *be a mapping which is differentiable in the point* $P \in G$. *Then* $\varphi$ *is in* $P$ *differentiable in every direction* $v$, *and we have* $(D_v(\varphi))_P = (D_P\varphi)(v)$.

*Proof.* Note that $D_P\varphi$ is a linear mapping $V \longrightarrow W$, so the result of applying it to a vector $v \in V$ gives a vector in $W$. By assumption we have

$$\varphi(P + v) = \varphi(P) + L(v) + ||v||r(v)$$

(with the usual conditions on $r$). So in particular we have

$$\varphi(P + sv) = \varphi(P) + sL(v) + |s|\,||v||r(sv)\,.$$

Hence

$$\lim_{s \longrightarrow 0, s \neq 0} \frac{\varphi(P + sv) - \varphi(P)}{s} = \lim_{s \longrightarrow 0, s \neq 0} \frac{sL(v) + |s|\,||v||r(sv)}{s}$$

$$= \lim_{s \longrightarrow 0, s \neq 0} (L(v) + \frac{|s|}{s}||v||r(sv)) = L(v)\,,$$

since $\lim_{s \longrightarrow 0} r(sv) = 0$ and $\frac{|s|}{s}||v||$ is bounded. □

The partial derivatives are essentially the directional derivatives in the direction of the basis vectors. So, in particular, partial derivatives make only sense if a basis is chosen on the vector space where a mapping is defined.

**Definition 2.2.3.** Let a mapping $\varphi : \mathbb{K}^n \supseteq G \longrightarrow \mathbb{K}^m$ be given as

$$\varphi(x_1, \ldots, x_n) = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

Let $P \in G = (x_1, \ldots, x_n)$ be a point. For fixed $i$ and $j$ we can consider the mapping

$$x_i \longmapsto f_j(x_1, \ldots, x_n)\,,$$

which is a function in one variable, and where the other variables $x_j$, $j \neq i$, are fixed. If this function is differentiable, then we say that $f_j$ is *partially differentiable* with respect to $x_i$, and denote the derivative (which is an element in $\mathbb{K}$) by

$$\frac{\partial f_j}{\partial x_i}(P)$$

(called the partial derivative).

We say that the map $\varphi$ is *partially differentiable* in $P$ if for all $i$ and all $j$ the partial derivatives exist.

**Lemma 2.2.4.** *Let a mapping $\varphi : \mathbb{K}^n \supseteq G \longrightarrow \mathbb{K}^m$ be given as $\varphi(x_1, \ldots, x_n)$. Then the partial derivative of $f_j$ in a point $P = (x_1, \ldots, x_n)$ is the directional derivation of $f_j$ in direction of the $i$th standard vector $e_i$, and $\varphi$ is partially differentiable if and only if the directional derivatives exist in direction of all standard vectors.*

*Proof.* We look at $f = f_j$. As the partial derivatives are derivatives of functions in one variable, we have that

$$\frac{\partial f_j}{\partial x_i}(P) = \lim_{s \longrightarrow 0, s \neq 0} \frac{f(x_1, \ldots, x_i + s, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{s}\,.$$

But this is exactly the definition of the derivative in direction of the $i$-th standard vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. $\square$

The following easy example show that all directional derivatives may exist (in particular, all partial derivatives), but that the mapping itself is not even continuous (and not differentiable).

**Example 2.2.5.** Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with $f(x,y) := \frac{xy^3}{x^2+y^6}$ for $(x,y) \neq 0$ and $f(0,0) := 0$. For a vector $v \neq 0$, $v = (a,b)$, and a real parameter $s$ we get on the linear subspace $\mathbb{R}v$ the function

$$s \longmapsto \frac{sas^3b^3}{s^2a^2 + s^6b^6} = \frac{s^2ab^3}{a^2 + s^4b^6}\,.$$

For $a \neq 0$ the denominator is always positive and this function is continuous with value 0 for $s = 0$ and also differentiable. For $a = 0$ this function is constant $= 0$, hence also differentiable. Hence in 0 all directional derivatives exist. The function however is not even continuous: for the sequence $(1/m^3, 1/m)$ (which converges to 0) we have $f(1/m^3, 1/m) = \frac{1/m^3 1/m^3}{1/m^6 + 1/m^6} = 1/2$, but $f(0,0) = 0$.

Here the partial derivatives are however not continuous in 0.

**Theorem 2.2.6.** *Let $\varphi : G \longrightarrow \mathbb{K}^m$ be a mapping, $G \subseteq \mathbb{K}^n$. Denote the coordinates of $V = \mathbb{K}^n$ by $x_i$. Let $P \in G$. Suppose that all partial derivatives in $P$ exist and that they are continuous in $P$. Then $\varphi$ is (totally) differentiable in $P$. If the mapping $\varphi$ is given as $f_1, \ldots, f_m$ with respect to a basis of $W$, then the total differential in $P$ is the matrix*

$$((\partial f_j / \partial x_i)(P))_{1 \leq i \leq n, 1 \leq j \leq m} \,.$$

*Proof.* Recall the Mean value theorem:

Let $h : [a, b] \longrightarrow V$ be differentiable. Then there exists $\sigma \in [a, b]$ such that

$$\|h(b) - h(a)\| \leq (b - a) \|h'(\sigma)\| \,.$$

By assumptions the mappings

$$(D_i(\varphi))(x) := (D_{e_i}(\varphi))(x) = (\frac{\partial f_1}{\partial x_i}(x), \ldots, \frac{\partial f_m}{\partial x_i}(x)) \in \mathbb{K}^m$$

exist in $G$ and are continuous in $P$. So the only candidate for the differential is the linear map giving by

$$v = (v_1, \ldots, v_n) \longmapsto \sum_{i=1}^{n} v_i (D_i(\varphi))(P) \,.$$

So we have to show that this map fulfills the defining properties of the total differential. Set $P_i = P + v_1 e_1 + \ldots + v_i e_i$ (depending on $v$). Then (for $v$ small enough) we have

$$\begin{aligned}
\left\| \varphi(P + v) - \varphi(P) - \sum_{i=1}^{n} v_i (D_i(\varphi))(P) \right\| &= \left\| \sum_{i=1}^{n} (\varphi(P_i) - \varphi(P_{i-1}) - v_i (D_i(\varphi))(P)) \right\| \\
&\leq \sum_{i=1}^{n} \|\varphi(P_i) - \varphi(P_{i-1}) - v_i (D_i(\varphi))(P)\| \\
&= \sum_{i=1}^{n} \|\varphi(P_{i-1} + v_i e_i) - \varphi(P_{i-1}) - v_i (D_i(\varphi))(P)\|
\end{aligned}$$

We look at the summands independently. The map (defined on the unit interval)

$$h_i : s \longmapsto \varphi(P_{i-1} + s v_i e_i) - s v_i (D_i(\varphi))(P)$$

is differentiable (because of the existence of the partial derivatives in $G$) and its derivative is the map

$$s \longmapsto v_i (D_i(\varphi))(P_{i-1} + s v_i e_i) - v_i (D_i(\varphi))(P) \,.$$

According to the mean value theorem we get that there exists a real number $0 \leq \sigma_i \leq 1$ such that (this is the norm of $h_i(1) - h_i(0)$)

$$\begin{aligned}
&\|\varphi(P_{i-1} + v_i e_i) - \varphi(P_{i-1}) - v_i (D_i(\varphi))(P)\| \\
&\qquad \leq \|v_i (D_i(\varphi))(P_{i-1} + \sigma_i v_i e_i) - v_i (D_i(\varphi))(P)\| \,.
\end{aligned}$$

Hence summing up we get the estimate that our expression is bounded by

$$\leq \sum_{i=1}^{n} |v_i| \, \|(D_i(\varphi))(P_{i-1} + \sigma_i v_i e_i) - (D_i(\varphi))(P)\|$$

$$\leq \|v\|_{\max} \sum_{i=1}^{n} \|(D_i(\varphi))(P_{i-1} + \sigma_i v_i e_i) - (D_i(\varphi))(P)\| \, .$$

As the partial derivatives are continuous in $P$ we know that this sum gets arbitrarily small for $v$ small enough. Hence the limit of this sum is 0 for $v \longrightarrow 0$. □

**Remark 2.2.7.** This applies immediately for polynomial functions in an arbitrary number of variables, but also for other holomorphic functions like exponential function or the trigonometric functions. As the example above shows one has to be however careful if one inverts such functions.

**Example 2.2.8.** Consider $\mathbb{K}^3 \longrightarrow \mathbb{K}^2$ given by

$$(x, y, z) \longmapsto (xy^2 - z^3, \, \sin(xy) + x^2 \exp(z)) = (f_1, f_2) \, .$$

The partial derivatives of $f_1$ are

$$\partial f_1/\partial x = y^2, \, \partial f_1/\partial y = 2xy, \, \partial f_1/\partial z = -3z^2$$

and the partial derivatives of $f_2$ are

$$\partial f_2/\partial x = y \cos(xy) + 2x \exp(z), \, \partial f_2/\partial y = x \cos(xy), \, \partial f_2/\partial z = x^2 \exp(z) \, .$$

This gives for an arbitrary point $P = (x, y, z)$ the matrix which describes the linear mapping $(D\varphi)_P$,

$$\begin{pmatrix} y^2 & 2xy & -3z^2 \\ y\cos(xy) + 2x\exp(z) & x\cos(xy) & x^2\exp(z) \end{pmatrix} \, .$$

For a special point, say $P = (2, 1, 3)$, we have to insert to get the matrix,

$$\begin{pmatrix} 4 & 4 & -27 \\ \cos(2) + 4\exp(3) & 2\cos(2) & 4\exp(3) \end{pmatrix} \, .$$

The following example stresses that the total differential is independent of a choice of a basis, whereas the partial derivatives are dependent.

**Example 2.2.9.** Consider the mapping

$$f : \mathbb{K}^3 \longrightarrow \mathbb{K}, \text{ given by } (x, y, z) \longmapsto 2xy^2 + x^2z^3 + z^2 \, .$$

It is easy to compute for every point the partial derivatives, yielding

$$\begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}_{(x,y,z)} = \begin{pmatrix} 2y^2 + 2xz^3 \\ 4xy \\ 3x^2z^2 + 2z \end{pmatrix} ,$$

and since they are continuous we have also found the total differential for every point.

Suppose now that we are only interested in the function when restricted to the plane

$$E \subset \mathbb{K}^3, E = \{(x, y, z) \colon 3x + 2y - 5z = 0\} \, .$$

So $E$ is the kernel of $L$, where $L : \mathbb{K}^3 \longrightarrow \mathbb{K}$ is the linear mapping given by $(x, y, z) \mapsto 3x + 2y - 5z$. As a kernel $E$ is itself a (two-dimensional) vector space. So restricting $f$ to this plane gives $f|_E : E \longrightarrow \mathbb{K}$. This mapping can be seen as the composition $E \subset \mathbb{K}^3 \xrightarrow{f} \mathbb{K}$, therefore it is differentiable by the chain rule (2.1.9). If we denote the inclusion of $E$ inside $\mathbb{K}^3$ by $N$ (which is linear), then by the chain rule the total differential of the composed map in a point $P \in E$ is just $(Df)_P \circ N : E \longrightarrow \mathbb{K}$. So this makes all sense and fits together well.

It does however not make sense to talk about partial derivatives for $f|_E : E \longrightarrow \mathbb{K}$, because there is no natural basis on $E$ and so there are no coordinates on it. It is easy to find a basis on $E$ and hence coordinates, but there is no best choice, and the partial derivatives look in every basis different.

One basis is given by $v_1 = (0, 5, 2)$ and $v_2 = (5, 0, 3)$ and another is given by $w_1 = (1, 1, 1)$ and $w_2 = (2, -3, 0)$. With such a basis we can identify $\mathbb{K}^2 \longrightarrow E$ and hence get a numerical description of the mapping $\mathbb{K}^2 \cong E \longrightarrow \mathbb{K}$, and we can compute its partial derivatives.

In the first basis the mapping is

$$(s, t) \longmapsto sv_1 + tv_2 = s(0, 5, 2) + t(5, 0, 3) = (5t, 5s, 2s + 3t)$$

and this is mapped by $f$ to

$$\begin{aligned} & 2(5t)(5s)^2 + (5t)^2(2s + 3t)^3 + (2s + 3t)^2 \\ = \; & 250ts^2 + 25t^2(8s^3 + 36s^2t + 54st^2 + 27t^3) + 4s^2 + 9t^2 + 12st \\ = \; & 250ts^2 + 200s^3t^2 + 900s^2t^3 + 1350st^4 + 675t^5 + 4s^2 + 9t^2 + 12st\,. \end{aligned}$$

The partial derivatives of this composed mapping (call it $g$) with respect to this basis are

$$\partial g/\partial s = 500ts + 600s^2t^2 + 1800st^3 + 1350t^4 + 8s + 12t$$

and

$$\partial g/\partial t = 250s^2 + 400s^3t + 2700s^2t^2 + 5400st^3 + 3375t^4 + 18t + 12s\,.$$

In the second basis $w_1 = (1, 1, 1)$ and $w_2 = (2, -3, 0)$ the mapping is

$$(r, u) \longmapsto rw_1 + uw_2 = r(1, 1, 1) + u(2, -3, 0) = (r + 2u, r - 3u, r)$$

and this is mapped by $f$ to

$$\begin{aligned} & 2(r + 2u)(r - 3u)^2 + (r + 2u)^2r^3 + r^2 \\ = \; & 2r^3 + 4r^2u - 12r^2u - 24ru^2 + 18ru^2 + 36u^3 + r^5 + 4r^4u + 4r^3u^2 + r^2 \\ = \; & 2r^3 - 8r^2u - 6ru^2 + 36u^3 + r^5 + 4r^4u + 4r^3u^2 + r^2\,. \end{aligned}$$

The partial derivatives of this composed mapping (call it $h$) with respect to this basis are

$$\partial h/\partial r = 6r^2 - 16ru - 6u^2 + 5r^4 + 16r^3u + 12r^2u^2 + 2r$$

and

$$\partial h/\partial u = -8r^2 - 12ru + 108u^2 + 4r^4 + 8r^3u\,.$$

Summary: Coordinates are good for computations, but not good for mathematics.

## 2.3. **Real and complex differentiability.**

Let $V$ and $W$ be two complex vector spaces. These vector spaces are then also real vector spaces (with double dimension). For a mapping $G \longrightarrow W$ (where $G \subseteq V$ open) one has to be careful with the concept of *complex differentiability* and *real differentiability*. If we look at the condition

$$\varphi(P+v) = \varphi(P) + L(v) + ||v|| r(v)\,,$$

then complex differentiability in a point $P \in G$ means that $L$ is a complex-linear mapping, whereas real differentiability just means that it is a real-linear mapping. Hence this difference is basically already a problem of linear algebra. Since a complex-linear mapping is also real-linear, it follows that a complex-differentiable mapping is also real-differentiable. If $\varphi$ is real-differentiable, then the real total differentiable $(D\varphi)_P$ is the only possible linear mapping fulfilling the above condition (as the differential is uniquely determined by 2.1.4), hence it is complex-differentiable if and only if $(D\varphi)_P$ is complex-linear.

Let $L : V \longrightarrow W$ be a mapping between complex vector spaces, which is real-linear. Then $L$ is also complex-linear if and only if $L(iv) = iL(v)$ holds for all $v \in V$. It is enough to check this condition for a complex basis of $V$. For a real total differential $(D\varphi)_P$ this is the condition that $(D\varphi)_P(iv) = i(D\varphi)_P(v)$, or, in terms of directional derivatives, $(D_{iv}\varphi)_P = i(D_v\varphi)_P$ (and again it is enough to check this condition for a complex basis of $V$). In the following we discuss several versions on how to characterize complex differentiability.

**Theorem 2.3.1.** *Let $V$ and $W$ be two complex vector spaces and let $\varphi : G \longrightarrow W$ be a mapping which is real-differentiable in $P \in G$. Let $v_j$, $j \in J$, be a complex basis of $V$ with coordinate functions $z_j = x_j + iy_j$. Then $\varphi$ is complex-differentiable in $P$ if and only if*

$$(\frac{\partial \varphi}{\partial y_j})(P) = i(\frac{\partial \varphi}{\partial x_j})(P)$$

*holds for all $j$ (on both sides we have vectors in $W$).*

*Proof.* Let $L = (D\varphi)(P)$ be the real total differential, which is a $2m \times 2n$ matrix with real entries (with respect to a basis of $W$). As $G \longrightarrow W$ is real-differentiable it is in particular real partially differentiable with respect to $x_j$ and $y_j$, and these derivatives give the entries in the matrix. The map $\varphi$ is complex-differentiable if and only if $L$ is (not only real-, but also) complex-linear. As (by Proposition 2.2.2)

$$(\partial\varphi/\partial y_j)(P) = (D_{iv_j}\varphi)(P) = (D\varphi)_P(iv_j) \quad \text{and} \quad (\partial\varphi/\partial x_j)(P) = (D_{v_j}\varphi)_P = (D\varphi)_P(v_j)$$

we get that the $\mathbb{C}$-linearity of the differential (namely $(D\varphi)_P(iv_j) = i(D\varphi)_P(v_j)$) is equivalent to

$$(\frac{\partial \varphi}{\partial y_j})(P) = i(\frac{\partial \varphi}{\partial x_j})(P)\,.$$

□

For $W = \mathbb{C}$ we get the following special case, called the **Cauchy-Riemann differential equations**.

**Corollary 2.3.2.** *Let $G \subseteq \mathbb{C}^n$ open, and let $\varphi : G \longrightarrow \mathbb{C}$ be a mapping which is real-differentiable in $P \in G$. Write $\varphi = g + ih$ with real-valued functions $g, h : G \longrightarrow \mathbb{R}$. Write $z_j = x_j + iy_j$, $j = 1, \ldots, n$, for the coordinates. Then $\varphi$ is complex-differentiable in $P$ if and only if*

$$(\frac{\partial g}{\partial y_j})(P) = -(\frac{\partial h}{\partial x_j})(P) \quad \textit{and} \quad (\frac{\partial h}{\partial y_j})(P) = (\frac{\partial g}{\partial x_j})(P)$$

*holds for all $j$.*

*Proof.* We have

$$(\frac{\partial \varphi}{\partial y_j})(P) = (\frac{\partial (g + ih)}{\partial y_j})P = (\frac{\partial g}{\partial y_j})(P) + i(\frac{\partial h}{\partial y_j})(P)$$

and similar

$$(\frac{\partial \varphi}{\partial x_j})(P) = (\frac{\partial g}{\partial x_j})(P) + i(\frac{\partial h}{\partial x_j})(P)$$

Hence the condition in Theorem 2.3.1 is that

$$(\frac{\partial g}{\partial y_j})(P) + i(\frac{\partial h}{\partial y_j})(P) = i((\frac{\partial g}{\partial x_j})(P) + i(\frac{\partial h}{\partial x_j})(P)).$$

Comparing the real and the imaginary part gives the claimed conditions

$$(\frac{\partial g}{\partial y_j})(P) = -(\frac{\partial h}{\partial x_j})(P) \quad \text{and} \quad (\frac{\partial h}{\partial y_j})(P) = (\frac{\partial g}{\partial x_j})(P).$$

□

**Remark 2.3.3.** Suppose also that $V = \mathbb{C}$ with coordinate $z = x + iy$. Then the condition for $\varphi(z) = g(z) + ih(z)$ to be complex-differentiable is

$$(\frac{\partial g}{\partial y})(P) = -(\frac{\partial h}{\partial x})(P) \quad \text{and} \quad (\frac{\partial h}{\partial y})(P) = (\frac{\partial g}{\partial x})(P).$$

This means for the real total differential (the $2 \times 2$-matrix)

$$\begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}$$

that in the diagonal there is the same term and the values on the antidiagonal are the negative of each other.

**Example 2.3.4.** Consider the mapping

$$\mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{C} = \mathbb{R}^2$$

given in real coordinates as $(x, y) \mapsto (4x^2 - xy, 2xy^2 + y^3) = (g, h)$. This is clearly differentiable in the real sense, with matrix

$$\begin{pmatrix} 8x - y & x \\ 2y^2 & 4xy + 3y^2 \end{pmatrix}$$

and it is not complex-differentiable.

**Example 2.3.5.** Consider the mapping

$$\mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{C} = \mathbb{R}^2$$

given in real coordinates as $(x, y) \mapsto (x^4 - 6x^2y^2 + y^4, 4x^3y - 4xy^3) = (g, h)$. This is clearly differentiable in the real sense, with matrix

$$\begin{pmatrix} 4x^3 - 12xy^2 & -12x^2y + 4y^3 \\ 12x^2y - 4y^3 & 4x^3 - 12xy^2 \end{pmatrix} .$$

Hence it fulfills the Cauchy-Riemann differential equation and is complex-differentiable. In fact it is the mapping

$$z \longmapsto z^4 = (x^4 - 6x^2y^2 + y^4, 4x^3y - 4xy^3) .$$

The complex differential is just $4z^3$. This complex-linear mapping sends

$$1 \longmapsto 4z^3 = 4(x^3 - 3xy^2 + (3x^2y - y^3)i)$$

and

$$i \longmapsto 4iz^3 = 4((x^3 - 3xy^2)i - (3x^2y - y^3)) .$$

These two vectors are the columns of the corresponding real matrix.

## 2.4. **Higher derivatives.**

For a map $\varphi : G \longrightarrow W$ and a fixed vector $u \in V$ the directional derivative in direction $u$ is (if it exists) itself a mapping

$$D_u(\varphi) : P \longmapsto (D_u(\varphi))(P) .$$

As such it makes sense to ask whether $D_u(\varphi)$ is differentiable in direction $v$. We speak about *higher derivatives*. The following theorem is called the *Theorem of Clairaut* or *Theorem of Schwarz*.

**Theorem 2.4.1.** *Let $\varphi : G \longrightarrow W$ be a map such that for $u, v \in V$ the second directional derivatives $D_vD_u(\varphi)$ and $D_uD_v(\varphi)$ exist and are continuous. Then $D_vD_u(\varphi) = D_uD_v(\varphi)$.*

*Proof.* By looking at the components we may assume that $W = \mathbb{K}$ and that $\mathbb{K} = \mathbb{R}$. We will apply the one dimensional real mean value theorem. Fix a point $P \in G$. We look at $(s, t) \mapsto \varphi(P + su + tv)$ and study this map for $s, t$ small enough. We fix these (for the moment) and look at

$$\sigma \longmapsto \varphi(P + \sigma u + tv) - \varphi(P + \sigma u) .$$

By the mean value theorem we get an $s_1$ (between 0 and $s$) such that

$$\varphi(P + su + tv) - \varphi(P + su) - \varphi(P + tv) + \varphi(P)$$
$$= s(D_u\varphi)(P + s_1u + tv) - (D_u\varphi)(P + s_1u) .$$

Now we apply again the mean value theorem to the map

$$\tau \longmapsto (D_u\varphi)(P + s_1u + \tau v)$$

and get the existence of a $t_1 \leq t$ such that

$$(D_u\varphi)(P + s_1u + tv) - (D_u\varphi)(P + s_1u) = t(D_v(D_u\varphi))(P + s_1u + t_1v) \,.$$

So together we get

$$\varphi(P + su + tv) - \varphi(P + su) - \varphi(P + tv) + \varphi(P) = st(D_v(D_u\varphi))(P + s_1u + t_1v) \,.$$

Doing the same trick in the other direction we get also $s_2$ and $t_2$ such that this expression also equals

$$st(D_u(D_v\varphi))(P + s_2u + t_2v) \,.$$

Hence for given (small enough) $s, t \neq 0$ we deduce that there exist $s_1$, $t_1$, $s_2$ and $t_2$ such that

$$(D_v(D_u\varphi))(P + s_1u + t_1v) = (D_u(D_v\varphi))(P + s_2u + t_2v) \,.$$

The assumption of continuity of both second directional derivatives implies for $s \mapsto 0$ and $t \mapsto 0$ that also $s_1$, $t_1$, $s_2$ and $t_2$ converge to 0, hence we get the identity. □

## 3. Differential forms and path integrals

### 3.1. **Differential forms.**

Let $V$ and $W$ denote two finite dimensional vector spaces over $\mathbb{K}$. Let $G \subseteq V$ be an open subset and let $\varphi : G \longrightarrow W$ be a (totally) **differentiable mapping**. Recall that **the process of differentiation gives for every point** $P \in G$ a $\mathbb{K}$-**linear mapping**

$$(D\varphi)_P : V \longrightarrow W,\; v \longmapsto (D\varphi)_P(v) \,.$$

Note that this is an object of a new nature! To a point we assign a linear mapping. This is quite different to the case of one variable, where the derivative of a function is again a function, and so it is an object of the same kind.

We make a definition for this new object.

**Definition 3.1.1.** Let $G \subseteq V$ be an open subset. **A 1-*form* (or a *differential form of first degree*) with values in $W$ is a mapping**

$$\omega : G \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V, W) \,.$$

Here $\mathrm{Hom}_{\mathbb{K}}(V, W)$ is the set of $\mathbb{K}$-linear mappings, which form itself a $\mathbb{K}$-vector space.

**Question 3.1.2.** A guiding question in this course is: Let a 1-form $\omega$ be given. **When** is it **given** as the **total differential of a differentiable mapping** $\varphi : V \longrightarrow W$, that is, does there exist $\varphi$ with $\omega = D\varphi$?

**Remark 3.1.3.** This is a generalization of the question when does a (continuous) function $g : I \longrightarrow \mathbb{R}$ (where $I \subseteq \mathbb{R}$ is an interval) have a primitive function. The answer to this question is always yes (though it might be very difficult to find a primitive function explicitly). But here the question and its answer are much more involved, and totally *new phenomena* occur:

(1) There will be some (easy to formulate) necessary conditions on a differential form to be a differential ("symmetry", closed form) for the existence of a primitive function. But this will in general not be sufficient.

(2) Using paths, one can reduce the problem to a one-dimensional integration problem. However, the choice of the paths is very important. The result will be the same for two paths if there is a certain topological relation (homotopy) between them.

(3) A primitive mapping might exist *locally, but not globally*. This means that for every $P \in G$ there exist a (small) neighborhood $U$ of this point ($P \in U \subseteq G$, often $U$ will be an open ball) such that if we restrict the form to $U$ there exists a primitive mapping, but for different points these mappings do not 'glue together'. The difference between local and global solutions is a typical feature of higher dimensional analysis.

(4) It will turn out that the analysis depends and reflects topological properties of the open subset $G$. A leading example for this is the difference between $G = \mathbb{R}^2$ (the real plane, which we will often consider as $\mathbb{C}$) and $G = \mathbb{R}^2 - \{0\}$.

(5) The problem of finding a primitive function for a (continuous) complex function $\mathbb{C} \supseteq U \longrightarrow \mathbb{C}$ is already very difficult. So though the process of differentiation is basically the same in the real and the complex case (think of polynomials, the exponential function or the trigonometric functions), the process of integrating is very different in the complex case. A typical example is the complex function $1/z : \mathbb{C} - \{0\} \longrightarrow \mathbb{C}$. What is its integral (its primitive function)? In the real case it is the logarithm, but there is no complex logarithm which is everywhere defined on $\mathbb{C} - \{0\}$. Already all the phenomena mentioned above occur already in this special case.

**Remark 3.1.4.** In this course we will **only consider one forms** and will usually just **talk about differential forms**. As $\mathrm{Hom}_{\mathbb{K}}(V, W)$ is itself a vector space over $\mathbb{K}$ (of dimension $\dim(V)\cdot\dim(W)$), one can talk about **continuous** and **differentiable differential forms**. If $V = \mathbb{K}^n$ and $W = \mathbb{K}^m$, then a linear mapping $V \longrightarrow W$ corresponds to an $m \times n$-matrix. In this case a differential form $\omega$ assigns to every point $P \in G$ a matrix. If $\mathbb{K} = \mathbb{R}$ and $W = \mathbb{R}$, then we talk about ***real-valued differential forms***. If $\mathbb{K} = \mathbb{C}$ and $W = \mathbb{C}$, then we talk about ***complex-valued differential forms***.

For every point $P \in G$ we get a linear mapping $\omega(P) : V \longrightarrow W$. In particular, for $P \in G$ and $v \in V$, the expression

$$\omega(P; v) = (\omega(P))(v)$$

makes sense and is a vector in $W$.

**Notation 3.1.5.** The **most important case** for us will be $V = \mathbb{K}^n$ and $W = \mathbb{K}$. In this case we will **denote** by $dx_j$ **the linear mapping**

$$\mathbb{K}^n \longrightarrow \mathbb{K}\,, (v_1, \ldots, v_n) \longmapsto v_j\,,$$

as well as – for an arbitrary open subset $G \subseteq \mathbb{K}^n$ –, the **differential form**, **which assigns to every point** $P \in G$ **the linear mapping** $dx_j$. So $dx_j$ is the differential form, where the linear mapping is always the $j$-th ***projection*** and does not vary with $P \in G$.

In this case we can **express** a **differential form** as

$$\omega = g_1 dx_1 + \ldots + g_n dx_n \, ,$$

where $g_1, \ldots, g_n$ are $\mathbb{K}$-valued functions defined on $G$. A differential form which is given in this way has to be interpreted in the following way:

$$\omega(P) = (g_1 dx_1 + \ldots + g_n dx_n)(P) = g_1(P) dx_1 + \ldots + g_n(P) dx_n \, .$$

Now $g_i(P) \in \mathbb{K}$ are numbers and this is a linear combination of linear mappings, hence it is itself a linear mapping, namely for $v = (v_1, \ldots, v_n)$ this is

$$(g_1(P) dx_1 + \ldots + g_n(P) dx_n)(v) = g_1(P) v_1 + \ldots + g_n(P) v_n \in \mathbb{K} \, .$$

If $\omega = Df$ is the differential of $f : G \longrightarrow \mathbb{K}$, then

$$Df = \sum_{j=1}^{n} \frac{\partial f}{\partial x} dx_j \, .$$

To see that this is true apply both sides to the standard basis and use Lemma 2.2.4. For a point $P$ and a vector $v = (v_1, \ldots, v_n)$ this gives

$$(Df)_P(v) = \sum_{j=1}^{n} \frac{\partial f}{\partial x}(P) v_j \, .$$

**Definition 3.1.6.** For an open subset $G \subseteq V$ we denote **the set of differential forms** by

$$\Omega(G; W)$$

This is in a natural way itself an (infinite-dimensional) vector space over $\mathbb{K}$.

Even more is true: **we can** not only **multiply** a **differential form** $\omega$ with a constant in $\mathbb{K}$, but **also with a function** $f : G \longrightarrow \mathbb{K}$ to get another differential form. Namely, **the form** $f\omega$ is **defined** to be

$$(f\omega)(P) := f(P)\omega(P) \, .$$

Note that here $\omega(P) : V \longrightarrow W$ is a linear mapping and $f(P) \in \mathbb{K}$ is a scalar, so the product is again a linear mapping.

**Definition 3.1.7. A differential form** $\omega \in \Omega(G; W)$ is called ***exact*** if there **exists a differentiable mapping** $\varphi : G \longrightarrow W$ such that $\omega = D\varphi$. In this case $\varphi$ is called a ***primitive*** **function** or a ***primitive mapping*** for $\omega$.

**Theorem 3.1.8.** *Let $G \subseteq V$ be an* **open connected subset** *and let $\omega$ be a* **differential form** *on $G$ with values in $W$. Then the* **difference between two primitive mappings** *(if they exist) $\varphi_1$ and $\varphi_2$ is* **constant**. *There exists at most one primitive mapping with a prescribed value $\varphi(P) = w$.*

*Proof.* We **consider the difference** $\varphi_1 - \varphi_2 : G \longrightarrow W$, which is also differentiable. **Its total differential is**

$$D(\varphi_1 - \varphi_2) = D\varphi_1 - D\varphi_2 = \omega - \omega = 0\,.$$

So we may **assume** that $\varphi$ is a **differentiable function** and that **its total differential is** 0 everywhere on $G$. Then in particular **all directional derivatives are** 0. **For** $P \in G$ **let** $P \in B \subseteq G$ **be an open ball neighborhood**. Every other point $Q$ in $B$ can be reached from $P$ by going along a **line segment** (see Example 1.3.5). The **value** of $\varphi$ **along this line segment does not change** (by the corresponding result in one variable). Hence $\varphi$ **is constant** $= \varphi(P)$ on $B$. For every value $w \in W$ it follows that the subset

$$\{Q \in G : \varphi(Q) = w\}$$

is **the union of open balls**, hence **open**. But then also its **complement**

$$\bigcup_{u \in W, u \neq w} \{Q \in G : \varphi(Q) = u\}$$

is open. So if this set is $\neq \emptyset$, then it must be already $G$ (**connected**!), and so $\varphi$ is **constant**. $\square$

### 3.2. Pull-back of differential forms.

The notion of the pull-back is important in many branches of mathematics. The general setting is that one has on object $O$ (a function, a differential form, a mapping to it) attached to a basis object $B$ (like a space), and one has a mapping from another basis object $\psi : B' \longrightarrow B$. Then one can often construct out of these data a new object $O'$ over $B'$. This is often called the *pull-back* of $O$ along (or under) the *base change* $B' \longrightarrow B$, and often denoted by $\psi^*(O)$ (the star is always above). Of course, in every specific situation one has to specify what the exact meaning of a pull-back is.

The **pull-back of a function**: Suppose that we have a function $\varphi : M \longrightarrow N$ (an object on $M$), and a mapping $\psi : L \longrightarrow M$. Then this gives easily a function on $L$, namely the composition $\varphi \circ \psi$.

From this one might expect that the pull-back of a differential form $\omega : G \longrightarrow \operatorname{Hom}_{\mathbb{K}}(V, W)$ for a mapping $\psi : G' \longrightarrow G$ ($G' \subseteq V'$ an open subset in another vector space) is just $\omega \circ \psi$. This is however not a good idea. A guiding line is that if $\omega$ is **exact with primitive function** $f$ (so that $Df = \omega$), then the **derivative of the composition** $f \circ \psi$ **should give the pull-back** of $\omega$. That is, the **diagram**

$$\begin{tikzcd} C^1(G,W) \arrow[r, "\psi^*"] \arrow[d, "D"'] & C^1(G',W) \arrow[d, "D"] \\ \Omega(G,W) \arrow[r, "?"] & \Omega(G',W) \end{tikzcd}$$

should **commute** (where $C^1(G, W)$ denotes the set of continuously differentiable mappings).

Therefore we arrive at the following definition.

**Definition 3.2.1.** Let $\psi : G' \longrightarrow G$ be a **differential map** and let $\omega \in \Omega(G, W)$ be a **differential form**. **Then the *pull-back of* $\omega$ *along* $\psi$** is the **form** $\psi^*(\omega)$ **defined by**

$$(\psi^*(\omega))(P', v') := \omega(\psi(P'); (D\psi)_{P'}(v')) .$$

In terms of linear maps, $(\psi^*(\omega))(P')$ is the linear map $\omega(\psi(P')) \circ (D\psi)_{P'} : V' \longrightarrow W$.

**Theorem 3.2.2.** *Let $\omega \in \Omega(G, W)$ be an* **exact differential form with primitive mapping** $f : G \longrightarrow W$*, and let $G' \subseteq V'$ be an* **open subset in another vector space** $V'$*. Let $\psi : G' \longrightarrow G$ be a* **differentiable map**. **Then also $\psi^*\omega$ is exact with primitive map** $f \circ \psi$.

*Proof.* We just have **to differentiate the mapping**

$$G' \longrightarrow W, P' \longmapsto f(\psi(P')) .$$

This is by Theorem 2.1.7 for a **point** $P' \in G'$ and a **vector** $v' \in V'$ the **vector** (in $W$)

$$(D(f \circ \psi))_{P'}(v') = (Df)_{\psi(P')} \circ (D\psi)_{P'}(v') = \omega(\psi(P'), (D\psi)_{P'}(v')) = \psi^*\omega(P', v') ,$$

the last equality follows from the definition of pull-back. □

**Remark 3.2.3.** If everything is given in **coordinates**, then **the pull-back of a differential form** can be **computed** as follows. Let $\omega$ be a **differential form** on $\mathbb{K}^n$ (or on an open subset) given as

$$\omega = g_1(x_1, \ldots, x_n)dx_1 + \ldots + g_n(x_1, \ldots, x_n)dx_n .$$

Suppose that a **mapping** $\psi : \mathbb{K}^k \longrightarrow \mathbb{K}^n$ is given as

$$(u_1, \ldots, u_k) \longmapsto (\psi_1(u_1, \ldots, u_k), \ldots, \psi_n(u_1, \ldots, u_k)) .$$

Then one has to **replace 'symbolically' in the expression for $\omega$ each $x_j$ by $\psi_j(u_1, \ldots, u_k)$, including** $dx_j$ according to the following rule:

$$dx_j = d(\psi_j(u_1, \ldots, u_k)) = \frac{\partial \psi_j(u_1, \ldots, u_k)}{\partial u_1} du_1 + \ldots + \frac{\partial \psi_j(u_1, \ldots, u_k)}{\partial u_k} du_k .$$

The resulting outcome is an expression of the form $\sum_{i=1}^k h_i(u_1, \ldots, u_k)du_i$, which is the pull-back.

## 3.3. **Path integrals.**

Regarding the question whether differential forms are exact, path integrals play an important 'test role'. We have met already **paths** in the context of path-connected spaces. A *path* for us **is a continuous mapping** $\gamma : [a, b] \longrightarrow G \subseteq V$. Most of the time we will also impose the condition that it is ***piecewise differentiable***. This means that there exist points $a < a_1 < \ldots < a_k < b$ such that the restrictions $\gamma : [a_\ell, a_{\ell+1}] \longrightarrow V$ are differentiable.

Note that even for $\mathbb{K} = \mathbb{C}$ the paths are always defined on a **real interval**. Note also that for a one-dimensional space we do not have to distinguish between the several concepts of differentiability.

A special and the easiest (but already very important) case for piecewise differentiable paths is a **piecewise linear path**.

**Definition 3.3.1.** For a **differential form** $\omega \in \Omega(G, W)$ and a **differentiable path** $\gamma : [a, b] \longrightarrow G$ we define

$$\int_\gamma \omega := \int_a^b \gamma^* \omega = \int_a^b \omega(\gamma(t); \gamma'(t))dt$$

and **call** it **the *path integral* (or *line integral*) of** $\omega$ **along the path** $\gamma$.

**Remark 3.3.2.** Here $\gamma^*\omega$ is the **pull-back of the differential form** $\omega$ **along** $\gamma$, $\gamma'(t)$ is the derivative of the path at $t$, which is a vector in $V$, and so $\omega(\gamma(t); \gamma'(t))$ is a vector (depending on the real parameter $t$) in $W$. Therefore on the right we have an **integral depending on a real parameter and having values in** $W$. Hence this **integral is well-defined** and has a value in $W$. If $\gamma$ is only piecewise differentiable, we can still define the path integral by looking at $a < a_1 < \ldots < a_k < b$ such that the restrictions $\gamma : [a_\ell, a_{\ell+1}] \longrightarrow V$ are differentiable and declaring the path integral to be the (finite) sum of the path integrals over these differentiable paths.

**Remark 3.3.3.** In the setting of **physics**, a differential form (or its dual version, a vector field) **describes forces** and the path integral is then the **amount of energy or work** needed to walk along the path.

**Computation 3.3.4.** *Suppose we want to* **compute a path integral**. *Then* $\omega$ *must be given as*

$$\omega(P) = g_1 dx_1 + \ldots + g_n dx_n \,,$$

*where* $x_j$ *are* **coordinates** *in* $V = \mathbb{K}^n$ *and* $g_j : G \longrightarrow W$ *are* **mappings**. *The* **path** *is given as* $t \mapsto (\gamma_1(t), \ldots, \gamma_n(t))$, *where each* $\gamma_j$ *is a mapping* $I \longrightarrow \mathbb{K}$. *The* **derivative of the path** *is then the* $n$*-tuple* $(\gamma_1'(t), \ldots, \gamma_n'(t))$. *The* **pull-back of the differential form** *is then* $t \mapsto$

$$\begin{aligned} \omega(\gamma(t); \gamma'(t)) &= (g_1(\gamma(t))dx_1 + \ldots + g_n(\gamma(t))dx_n)(\gamma_1'(t), \ldots, \gamma_n'(t))^\tau \\ &= g_1(\gamma(t))(\gamma_1'(t)) + \ldots + g_n(\gamma(t))(\gamma_n'(t)) \,. \end{aligned}$$

*The first expression is a linear mapping applied to a certain vector (here* $\tau$ *for transposed) and in the second term we have plucked it in. It needs some time to get used to this calculus and to see what refers to what. To* **remember** *it: in* $g_j(x_1, \ldots, x_n)$ *each* $x_j$ *has to be* **replaced** *by* $\gamma_j(t)$ *and* $dx_j$ *has to be replaced by* $\gamma_j'(t)$.

**Example 3.3.5.** Consider the differential form $(x, y) \mapsto 3dx - xdy$ and the path

$$[0, 5] \longrightarrow \mathbb{R}^2, \, t \longmapsto (t, t^2) \,.$$

The derivative of this path is the vector $(1, 2t)$ (depending on $t$). Hence the pull-back is

$$t \longmapsto \omega((t, t^2); (1, 2t)) = 3 - t(2t) = 3 - 2t^2 \,.$$

**Example 3.3.6.** A trivial example is then the path $\gamma$ is **constant**, say $t \mapsto P$ for all $t \in I$. Then its derivative is 0 and since $\omega(P, -)$ is a linear mapping, applied to 0 gives always 0. Hence also the path integral is 0 (as integrating over the 0 function gives 0).

**Remark 3.3.7.** Important is already the case where $\gamma$ is a **linear path**, given by $t \mapsto P + tv$, where $v \in V$ is a certain vector. Then its derivative is always $v$ (independent of $t$), so in $\omega(\gamma(t); \gamma'(t)) = \omega(P + tv, v)$ only the first argument varies, the second is constant.

**Example 3.3.8.** Consider the differential form

$$(x, y, z) \longmapsto (xy + z^2)dx + zdy + x^3dz$$

and the linear path

$$\gamma : [0, 1] \longrightarrow \mathbb{R}^3,\ t \longmapsto (1, 2, 0) + t(3, 0, 2) = (1 + 3t, 2, 2t)\,.$$

The pull-back is the mapping $t \mapsto$

$$\begin{aligned} & (((1 + 3t)2 + (2t)^2)dx + 2tdy + (1 + 3t)^3dz)(3, 0, 2)^\tau \\ = \ & 3((1 + 3t)2 + (2t)^2) + 2(1 + 3t)^3 \\ = \ & 12t^2 + 18t + 6 + 54t^3 + 54t^2 + 18t + 2 \\ = \ & 54t^3 + 66t^2 + 36t + 8 \end{aligned}$$

Integrating this from 0 to 1 gives

$$\int_0^1 (54t^3 + 66t^2 + 36t + 8)d = (\frac{27}{2}t^4 + 22t^3 + 18t^2 + 8t)|_0^1 = 61\frac{1}{2}\,.$$

**Definition 3.3.9.** Let $G$ be a topological space. For a path $\gamma : [a, b] \longrightarrow G$ we call $\overleftarrow{\gamma}(t) := \gamma(b + a - t)$ (defined on the same interval) the **reverse path** of $\gamma$.

For a **second path** $\beta : [b, c] \longrightarrow G$ with $\beta(b) = \gamma(b)$ we call the path $\gamma\beta : [a, c] \longrightarrow G$, which follows on $[a, b]$ the first path and on $[b, c]$ the second path, the **concatenation of the two paths**.

A **path** is called **closed** if its **start point** and **end point** are **identical**. Such closed paths are also called a *contour* or a *loop*.

**Remark 3.3.10.** For a closed path $\gamma$ (a contour) the path integral is often called a **contour integral** and it is sometimes written as $\oint_\gamma \omega$.

We collect **several rules for the computation of path integrals** in the following theorem.

**Theorem 3.3.11.** *Let $G \subseteq V$ be an open subset in a $\mathbb{K}$-vector space $V$, let $\omega$, $\omega_1$, $\omega_2$ be* **differential forms** *on $G$ with values in a vector space $W$. Let $\gamma : I = [a, b] \longrightarrow G$ be a* **piecewise differentiable path***. Then the following hold.*

(i) $\int_\gamma (c_1\omega_1 + c_2\omega_2) = c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2$ *, where $c_1, c_2 \in \mathbb{K}$ (***Linearity***).*
(ii) *Let $\beta : [b, c] \longrightarrow G$ be a second path with $\beta(b) = \gamma(b)$ and consider the the* **concatenation** *$\gamma\beta : [a, c] \longrightarrow G$. Then $\int_{\gamma\beta} \omega = \int_\gamma \omega + \int_\beta \omega$ (***Additivity***).*
(iii) *Let $\overleftarrow{\gamma}$ be the* **reverse path***. Then $\int_{\overleftarrow{\gamma}} \omega = -\int_\gamma \omega$.*

(iv) *Let* $\psi : \overline{G} \longrightarrow G$ *be* **differentiable** *and let* $\overline{\gamma} : I \longrightarrow \overline{G}$ *be a (piecewise differentiable)* **path** *in* $\overline{G}$ *with the* **composition** $\gamma = \psi \circ \overline{\gamma}$*, which is a path* $I \longrightarrow G$*. Then* $\int_\gamma \omega = \int_{\overline{\gamma}} \psi^*(\omega)$ *(***Substitution***).*

(v) *Suppose that* $V$ *and* $W$ *are normed. Then we have the* **estimate**

$$\left\| \int_\gamma \omega \right\| \leq \int_a^b \|\omega(\gamma(t))\| \cdot \|\gamma'(t)\| \, dt \, .$$

*Proof.* (i).

$$\begin{aligned}
\int_\gamma (c_1\omega_1 + c_2\omega_2) &= \int_a^b \gamma^*(c_1\omega_1 + c_2\omega_2) \\
&= \int_a^b (c_1\omega_1 + c_2\omega_2)(\gamma(t), \gamma'(t))dt \\
&= \int_a^b \Big( c_1\omega_1(\gamma(t), \gamma'(t)) + c_2\omega_2(\gamma(t), \gamma'(t)) \Big) dt \\
&= c_1 \int_a^b \omega_1(\gamma(t), \gamma'(t))dt + c_2 \int_a^b \omega_2(\gamma(t), \gamma'(t))dt \\
&= c_1 \int_\gamma \omega_1 + c_2 \int_\gamma \omega_2 \, .
\end{aligned}$$

In the pre-ultimate step we have used the **linearity of the integral**.

(ii). This follows from the **additivity of an integral with respect to two intervals**.

(iii). This follows from $\overleftarrow{\gamma}' = -\gamma'$.

(iv).

$$\begin{aligned}
\int_\gamma \omega &= \int_a^b \omega(\gamma(t), \gamma'(t))dt \\
&= \int_a^b \omega(\psi(\overline{\gamma}(t)), (\psi \circ \overline{\gamma})'(t))dt \\
&= \int_a^b \omega \left( \psi(\overline{\gamma}(t)), (D\psi)_{\overline{\gamma}(t)}(\overline{\gamma}'(t)) \right) dt \\
&= \int_a^b (\psi^*(\omega)(\overline{\gamma}(t), \overline{\gamma}'(t))dt \\
&= \int_{\overline{\gamma}} (\psi^*\omega) \, .
\end{aligned}$$

Here we have used in the third equation the **chain rule** and in the forth the definition of the **pull-back of a differential form**.

(v). We have

$$\begin{aligned}
\left\| \int_\gamma \omega \right\| &= \left\| \int_a^b \omega(\gamma(t), \gamma'(t))dt \right\| \\
&\leq \int_a^b \|\omega(\gamma(t), \gamma'(t))\| \, dt \\
&\leq \int_a^b \|\omega(\gamma(t))\| \cdot \|\gamma'(t)\| \, dt \, .
\end{aligned}$$

The first estimate follows from the **mean value theorem for integrals** and the second from Exercise 5.12 (the norm of $\omega(\gamma(t))$ is the norm of a linear map). $\square$

### 3.4. **Exact differential forms and path conditions.**

We **characterize exactness** - the existence of a primitive function - in **terms of path integrals**.

**Theorem 3.4.1.** *For a continuous differential form $\omega$ on $G \subseteq V$ with values in $W$ the following are equivalent.*

(i) *$\omega$ is* **exact**.
(ii) *For every* **piecewise differentiable path** $\gamma : [a,b] \longrightarrow G$ *the path integral* $\int_\gamma \omega$ **depends only on the start point** $\gamma(a)$ *and the* **end point** $\gamma(b)$.
(iii) *For every* **closed piecewise differentiable path** $\gamma$ *one has* $\int_\gamma \omega = 0$ *(so all* **contour integrals** *are* **zero***).*

*If these conditions are fulfilled, and if $G$ is* **connected**, *then one can* **choose a point** $P_0 \in G$ *and define a* **primitive map by the integral**

$$\varphi(P) := \int_\gamma \omega \,,$$

*where $\gamma$ is an* **arbitrary (piecewise differentiable) path** *from $P_0$ to $P$.*

*Proof.* (i) $\Rightarrow$ (ii). Let $\gamma : [a,b] \longrightarrow G$ be a continuous piecewise differentiable path and let $\varphi : G \longrightarrow W$ be a primitive map for $\omega$. We may assume that $\gamma$ is differentiable. Then

$$\int_\gamma \omega = \int_\gamma D\varphi = \int_a^b \gamma^*(D\varphi) = \int_a^b D(\gamma^*\varphi) = [\gamma^*\varphi]_a^b = \varphi(\gamma(b)) - \varphi(\gamma(a)) \,.$$

The third equation is the rule on pull-back and differentiation (3.2.2), the forth is the **fundamental theorem of calculus**.

The implication (ii) $\Rightarrow$ (iii) is trivial: let $\gamma : [a,b] \longrightarrow G$ be a closed path with $\gamma(a) = \gamma(b) = P$. The integral over this path is by (ii) the same as the integral over every other path with start point $P$ and end point $P$. But the **integral over the constant path** $c : [a,b] \longrightarrow G$, $c = P$, is 0 (Example 3.3.6).

The implication (iii) $\Rightarrow$ (ii) is also not difficult and follows from a **nice trick**: Let $\gamma_1, \gamma_2 : [a,b] \longrightarrow G$ be two paths with **start point** $P$ and **end point** $Q$. Then

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\overleftarrow{\gamma_2}} \omega = \int_{\gamma_1 \overleftarrow{\gamma_2}} \omega \,.$$

Here $\overleftarrow{\gamma}$ denotes the **reverse path**, and we have used 3.3.11 (ii), (iii). Now $\gamma_1 \overleftarrow{\gamma_2}$ is a **closed path**, hence this path integral is 0 by (iii).

For (ii) $\Rightarrow$ (i) we **have to construct a primitive function** on each connected component of $G$, so we may assume that $G$ is **connected**. We construct a primitive function in the following way: Look at $P_0 \in G$ and set

$$\varphi(P) := \int_\gamma \omega\,,$$

where $\gamma$ is an arbitrary (piecewise continuous) **path** from $P_0$ to $P$. This is well-defined by property (ii). We have to show that this **function** is **differentiable** and that $\omega$ is its **total differential**. We work with **directional derivatives**. Fix a point $P \in G$ and let $v \in V$ be a **vector representing a direction**. Let

$$\gamma_{v,s} : [0, s] \longrightarrow G,\; t \longmapsto P + tv$$

be the **corresponding linear path** (make sure that $s$ is small enough such that $P + tv \in G$ for all $0 \le t \le s$). This is a path from $P$ to $P + sv$. Let $\gamma$ be a path from $P_0$ to $P$. Then

$$\varphi(P + sv) - \varphi(P) = \int_\gamma \omega + \int_{\gamma_{v,s}} \omega - \int_\gamma \omega = \int_{\gamma_{v,s}} \omega = \int_0^s \omega(P + tv; v)dt\,.$$

Hence

$$(D_v(\varphi))(P) = \lim_{s \longrightarrow 0} \frac{\varphi(P + sv) - \varphi(P)}{s} = \lim_{s \longrightarrow 0} \frac{1}{s} \int_0^s \omega(P + tv; v)dt = \omega(P; v)\,.$$

For the last equation observe that $t \mapsto \omega(P + tv; v)$ is defined on a real interval, so this equation is the **fundamental theorem of calculus**. As $\omega$ is by assumption **continuous**, it follows from Theorem 2.2.6 that $\varphi$ is not only directional differentiable, but also **totally differentiable**. $\square$

### 3.5. **Symmetric and closed differential forms.**

We are going now to understand an **easy necessary condition** for a differential form $\omega$ to be **exact**. Consider the form $\omega : G \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V, W)$. As $\mathrm{Hom}_{\mathbb{K}}(V, W)$ is itself a $\mathbb{K}$-vector space, it makes sense to say that $\omega$ **is differentiable** or not. Suppose it is. Then for every point $P \in V$ the **differential** is a **linear mapping**

$$(D\omega)_P : V \longrightarrow \mathrm{Hom}(V, W)\,,$$

where the target vector space is itself a vector space of linear mappings.

A vector $v \in V$ can now occur in **two places**! If $v$ and $u$ belong to $V$, then the first vector $v$ defines (for a fixed point $P$) a linear mapping in $\mathrm{Hom}(V, W)$, and this linear mapping applied to $u$ gives then a vector in $W$. Hence one can consider this whole thing as defining a (**bilinear**) **mapping**

$$(D\omega)_P : V \times V \longrightarrow W,\; (v, u) \longmapsto ((D\omega)_P(v))(u)\,.$$

We write in general $(D\omega)_P(v, u)$ for this, where in general the order of the vectors is important.

**Definition 3.5.1.** We say that a **differentiable differential** form $\omega : G \longrightarrow \mathrm{Hom}(V, W)$ is **symmetric** if its **total differential** $D\omega$ is in every point $P \in G$ **symmetric**, meaning that

$$(D\omega)_P(v, u) = (D\omega)_P(u, v)$$

for all $v, u \in V$.

**Remark 3.5.2.** Even in the case $\mathbb{K} = \mathbb{C}$ the notion **symmetric** refers often to the **underlying real structure**. We will adopt this viewpoint in the next section.

**Lemma 3.5.3.** (i) *The expression* $(D\omega)_P(v, u)$ *is also the* **derivative in direction** $v$ *of the mapping*

$$P \longmapsto \omega(P, u)\,.$$

(ii) *If* $\omega$ *is* **exact** *and* $\omega = D\varphi$*, then* $D\omega(v, u) = D_v D_u(\varphi)$ *(the* **second directional derivative***).*

(iii) *The* **derivative of the mapping** $G \times V \longrightarrow W$, $(P, Q) \mapsto \omega(P, Q)$ *in* **direction** $(v, u)$ *is*

$$(D\omega)_P(v, Q) + \omega(P, u)\,.$$

*Proof.* (i). The mapping $P \longmapsto \omega(P, u)$ is the **composed mapping**

$$G \xrightarrow{\omega} Hom(V, W) \xrightarrow{\mathrm{Ev}_u} W\,,$$

where $\mathrm{Ev}_u$ means the **evaluation** at $u$, that is, the mapping $L \mapsto L(u)$. This **evaluation is linear**. So the derivative in direction $v$ can be computed according to the chain rule as

$$= (\mathrm{Ev}_u \circ (D\omega)_P)(v) = \mathrm{Ev}_u((D\omega)_P(v)) = ((D\omega)_P)(v))(u) = (D\omega)_P(v, u)\,.$$

(ii). By part (i), $(D\omega)_P(v, u)$ is the directional derivative of

$$P \longmapsto \omega(P, u) = (D\varphi)_P(u) = (D_u(\varphi))_P$$

in direction $v$, and this is $(D_v D_u \varphi)(P)$ as claimed.

(iii). We are dealing with the composed mapping

$$G \times V \xrightarrow{(\omega, \mathrm{id})} \mathrm{Hom}_{\mathbb{K}}(V, W) \times V \xrightarrow{\mathrm{Ev}} W\,.$$

By the chain rule and by Exercise 4.4 we get for the directional derivative

$$\begin{aligned} ((D\mathrm{Ev})_{(\omega(P), Q)} \circ (D\omega, \mathrm{id})_{(P,Q)})(v, u) &= (D\mathrm{Ev})_{(\omega(P), Q)}((D\omega)_P(v), u) \\ &= (D\omega)_P(v, Q) + \omega(P, u)\,. \end{aligned}$$

□

**Theorem 3.5.4.** *Let* $\omega : G \longrightarrow \mathrm{Hom}(V, W)$ *be a* **continuously differentiable differential form***. Suppose that* $\omega$ *is* **exact***. Then* $\omega$ *is also* **symmetric***.*

*Proof.* Write $\omega = D\varphi$. By Lemma 3.5.3 (ii) we have $D\omega(v, u) = D_v D_u(\varphi)$. By assumption this is **continuous**, hence by Theorem 2.4.1 we can **interchange** $v$ and $u$, which gives the result. □

In **terms of a basis**, the **symmetry can be checked** as follows.

**Proposition 3.5.5.** *Let $v_1, \ldots, v_n$ be a basis of $V$ with* **coordinate functions** $x_1, \ldots, x_n$. *Let a* **continuously differentiable form** $\omega : G \longrightarrow \mathrm{Hom}(V, W)$ *be given as $\omega = \sum_{j=1}^n g_j dx_j$, where $g_j : G \longrightarrow W$. Then $\omega$ is* **symmetric** *if and only if for all $1 \leq j, i \leq n$*

$$\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i} \,.$$

*Proof.* Note that the form is (continuously) differentiable if and only if all $g_j$ are (continuously) differentiable. By Lemma 3.5.3 (i) we know that $(D\omega)_P(v_j, v_i)$ is the $j$-th partial derivative of the map

$$P \longmapsto \omega(P, v_i) = (\sum_{j=1}^n g_j(P)dx_j)(v_i) = g_i(P)\,.$$

So this is $\frac{\partial g_i(P)}{\partial x_j}$, and the symmetry implies that $\frac{\partial g_i}{\partial x_j} = \frac{\partial g_j}{\partial x_i}$. On the other hand, if this condition is fulfilled, then $(D\omega)_P(v_j, v_i) = (D\omega)_P(v_i, v_j)$ it is enough to check the symmetry on a basis. □

So we know that **exact differential forms are symmetric**. Not all symmetric forms are exact, however they are **locally exact**. This means that locally, that is, in a neighborhood of each point, there exists a primitive function. A primitive function does however not exist globally in general. In fact, **the global existence of a primitive function depends on and reflects topological properties** of $G$.

We study first a special kind of open subsets where a symmetric form is always exact.

**Definition 3.5.6.** A subset $G \subseteq V$ in a finite dimensional $\mathbb{K}$-vector space $V$ is called **star-shaped** with respect to $P \in G$ if for all $Q \in G$ also the connecting **real line segment** between $P$ and $Q$ belong to $G$.

The line segment exists in $V$ and is given as $P + s(Q - P)$ for $s \in [0, 1]$ (a real number, even if $\mathbb{K} = \mathbb{C}$).

Star-shaped subsets are for us the easiest open subsets. A star-shaped subset is in particular (pathwise) connected. A **vector space** itself is star-shaped with respect to every point. Also an **open ball** is star-shaped with respect to every point.The **punctured plane** $\mathbb{C} - \{0\} = \mathbb{R}^2 - \{0\}$ is **not star-shaped** (with respect to no point).

**Theorem 3.5.7.** *Let $G \subseteq V$ be an open subset which is* **star-shaped** *with respect to $Q \in G$. Let $\omega$ be a* **continuously differentiable form** *on $G$. Suppose that $\omega$ is* **symmetric**. *Then $\omega$ is* **exact**, *and a* **primitive function** *is given by*

$$\varphi(P) = \int_\gamma \omega = \int_0^1 \omega(Q + tP; P - Q)dt$$

*(where $\gamma$ is the* **linear path connecting** *$Q$ and $P \in G$).*

*Proof.* We may assume that $Q = 0$, so that $\varphi(P) = \int_0^1 \omega(tP; P)dt$. We want to compute $D_u\varphi$, and for this we may **differentiate under the integral**, so

$$(D_u\varphi) = \int_0^1 D_u(P \longmapsto \omega(tP, P))dt\,.$$

The **derivative in direction** $u$ of the mapping $P \longmapsto \omega(tP, P)$ is by Lemma 3.5.3 (iii) and by **symmetry**

$$(D\omega)_{tP}(tu, P) + \omega(tP, u) = t(D\omega)_{tP}(P, u) + \omega(tP, u)\,.$$

For fixed $P \in G$ and $u \in V$ this is the derivative of the function $t \mapsto t\omega(tP, u)$ (since $Q \mapsto \omega(Q, u)$ has differential $v \mapsto (D\omega)_Q(v, u)$). Thus we get

$$(D_u\varphi)(P) = \int_0^1 (t(D\omega)_{tP}(P, u) + \omega(tP, u))dt = [t\omega(tP, u)]_0^1 = \omega(P, u)\,.$$

□

**Definition 3.5.8.** A differential form $\omega : G \longrightarrow \mathrm{Hom}_{\mathbb{K}}(V, W)$ is called **closed** (or **locally exact**) if for every point $P \in G$ there exists an **open neighborhood** $U$, $P \in U \subseteq G$, such that on $U$ there exists a **primitive function** for $\omega$.

**Theorem 3.5.9.** *A* **continuously differentiable form** *on $G$ is* **closed** *if and only if it is* **symmetric**.

*Proof.* Suppose $\omega$ is locally exact. Then $\omega$ is symmetric by Theorem 3.5.4, since **symmetric** is a **local property**.

Now suppose $\omega$ is symmetric. For every point $P \in G$ there exists an **open ball** $P \in B(P, \varepsilon)$, and balls are **star-shaped**. Hence by 3.5.7 there **exists** a **primitive function** on these balls. □

## 3.6. **Complex-differentiable forms and real closed forms.**

For a complex-valued function $f : G \longrightarrow \mathbb{C}$, $G \subseteq \mathbb{C}$ open, we consider the differential form $f(z)dz$ as a real form.

**Theorem 3.6.1.** *Let $G \subseteq \mathbb{C} = \mathbb{R}^2$ be open. Let $f : G \longrightarrow \mathbb{C}$ be stresscontinuously differentiable in the real sense. Then the* **differential form** $fdz$ *is* **symmetric** *(***in the real sense***) if and only if $f$ is* **complex-differentiable**.

*Proof.* We let $z = x + yi$ and we write $f(z) = f(x, y) = g(x, y) + ih(x, y)$. We then write the **differential form in real terms** as

$$fdz = (g + ih)d(x + iy) = (g + ih)dx + (ig - h)dy\,.$$

Then **symmetry** means that

$$\partial(g + ih)/\partial y = \partial(ig - h)/\partial x\,.$$

But this means that the two conditions (looking at the real part and the imaginary part) $\partial g/\partial y = -\partial h/\partial x$ and $\partial g/\partial x = \partial h/\partial y$ hold. These are precisely the **Cauchy-Riemann differential equations** (see 2.3.2), which **characterizes complex-differentiability**. □

**Remark 3.6.2.** Let $f$ be a **continuously complex-differentiable function** defined on $G \subseteq \mathbb{C}$ and let $\gamma = (\gamma_1, \gamma_2)$ be a (piecewise) **differentiable path** in $G$. Then the **path integral** for such a path for the closed differential form $f(z)dz$ is

$$\int_\gamma f(z)dz = \int_a^b ((Ref)(\gamma_1(t) + i\gamma_2(t)) \cdot \gamma_1'(t))dt + i\int_a^b ((Imf)(\gamma_1(t) + i\gamma_2(t)) \cdot \gamma_2'(t))dt\,.$$

**Example 3.6.3.** Let $f = z^3 : \mathbb{C} \longrightarrow \mathbb{C}$. As a real function in the variables $x$ and $y$ this is the mapping

$$(x, y) \longmapsto (x^3 - 3xy^2, 3x^2y - y^3)$$

and the corresponding differential form is

$$(x, y) \longmapsto (x^3 - 3xy^2)dx + (3x^2y - y^3)dy\,.$$

The symmetry condition is fulfilled, since we get

$$\frac{\partial(x^3 - 3xy^2)}{\partial y} = 6xy$$

and also

$$\frac{\partial(3x^2y - y^3)}{\partial y} = 6xy\,.$$

### 3.7. **Differential forms and vector fields.**

Vector fields, which are mappings $F : G \longrightarrow V$, and differential forms (with values in $\mathbb{K}$) are closely related, in fact they are dual to each other in a precise sense. Vector fields have more intuition on their side, and the path integral along a vector field has a meaning which is easier to understand and has a lot of applications in physics. From a computational point of view there is hardly a difference. The advantage for differential forms is their functorial behaviour (as expressed in the pull-back, for example).

If the vector space $V$ has an inner product $<,>$, then the theory of vector fields and of $\mathbb{K}$-valued differential forms are equivalent. The terminology is however often quite different. A vector field which is 'exact' is called **conservative** or a **gradient field**, and the negative of a 'primitive function' is called a **potential**.

## 4. The fundamental group

The **path integrals** of a **differential form reflect properties of the form**, but also **of the space**. Suppose that $\omega$ is a **closed**, but not **exact form**. Then by theorem 3.4.1 there **exists** a **closed path** (same start and end point) such that the **path integral** along this path is **not** 0. The typical example for this is $G = \mathbb{C} - \{0\}$ and the complex differential form $dz/z$, or, in real terms,

$$(x, y) \longmapsto (\frac{xdx + ydy}{x^2 + y^2}, \frac{-ydx + xdy}{x^2 + y^2})\,.$$

So this form assigns to $(x,y)$ the linear mapping $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by the matrix
$$\begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}.$$
A closed path with non zero path integral is given by $\gamma : [0,1] \longrightarrow G,\ t \mapsto e^{2\pi i t}$, that is, the unit circle, run through one time counter-clockwise. We have computed (Exercise 6.2) that $\int_\gamma = 2\pi i$. If we run $n$ times through the circle, then the path integral is $2n\pi i$. It will turn out that **the value of the path integral of a closed path depends only on how often we effectively run around the hole**.

In this section we deal only with paths (and forget the differential forms for a while) and try to understand which properties of a space (which will always be an open subset $G$ in a vector space $V$) can be seen by looking at the paths in it.

## 4.1. The fundamental group.

**Definition 4.1.1. Two paths** $\gamma_1, \gamma_2 : I = [0,1] \longrightarrow X$ in a metric (or topological) space with **same start point** $\gamma_1(0) = \gamma_2(0) = P$ and **same end point** $\gamma_1(1) = \gamma_2(1) = Q$ $(P, Q \in X)$ are called *homotopic* if there **exists** a *homotopy* between them. This means a **continuous mapping** $H : I \times I \longrightarrow X$ with the following properties.
$$H(0,t) = \gamma_1(t) \quad \text{and} \quad H(1,t) = \gamma_2(t)\,.$$
$$H(s,0) = P \quad \text{and} \quad H(s,1) = Q \quad \text{for all} \quad s\,.$$

**Remark 4.1.2.** For each fixed $s \in I$ the restricted mapping $H(s,-)$ is itself a path. So the idea is that we have a *continuous deformation* which **deforms** the path $\gamma_1$ into the $\gamma_2$. This is sometimes possible and sometimes not. It is clear that if two paths are homotopic using a homotopy with the interval $[0,1]$, then it is also possible (by shrinking or extending) to use a homotopy defined on another interval. We will use this without further ado.

**Lemma 4.1.3.** *The* **relation** *between paths with fixed start points and end points of being* **homotopic** *is an* **equivalence relation**.

*Proof.* This is rather an exercise in having understood the definition of homotopic. For the **reflexivity** consider the homotopy $H(s,t) = \gamma(t)$. For the **symmetry** we go through the homotopy in the other direction, that is, if $H : I \times I \longrightarrow X$ is a homotopy from $\gamma_1$ to $\gamma_2$, then $\tilde{H}(s,t) := H(1-s,t)$ is a homotopy from $\gamma_2$ to $\gamma_1$. **Transitivity**: Let $H_1$ be a homotopy between $\gamma_1$ and $\gamma_2$ and let $H_2$ be a homotopy between $\gamma_2$ and $\gamma_3$. We may assume that the homotopies are defined on $[0,0.5] \times I$. Then be putting the two homotopies together we get a homotopy relating $\gamma_1$ with $\gamma_3$ (note that this is continuous, as on the common border the mapping is $\gamma_2$, see Exercise 6.9). $\square$

**Lemma 4.1.4.** *Let $\gamma : [0,1] \longrightarrow X$ be a* **path** *in a* **metric** *(topological)* **space** *and let $\theta : [0,1] \longrightarrow [0,1]$ be a* **continuous map** *with $\theta(0) = 0$ and $\theta(1) = 1$ (this is called an* **oriented change of parametrization** *or* **reparametrization***). Then $\gamma$ and $\gamma \circ \theta$ are* **homotopic**.

*Proof.* Consider the homotopy given as

$$H(s,t) = \gamma((1-s)t + s\theta(t))$$

For $s = 0$ this is $\gamma$ and for $s = 1$ this is $\gamma \circ \theta$. The continuity is clear, and $(1-s)t+s\theta(t)$ is always **inside** the **unit interval** (the "trajectory" of the paths are always the same, only the velocity changes). □

By 4.1.3 we have a partition of the set of all paths with fixed start point and end point into **equivalence classes**, the *homotopy classes of paths*. The most important case is the case where the start point is also the end point, which is the **case of** the **closed paths**. In this case the **homotopy classes form even a group**!

**Definition 4.1.5.** Let $X$ be a **metric** (topological) **space** and let $P$ be a **point**. Then $\pi(X, P)$ is the **set of homotopy classes of closed paths with start and end point** $P$. It is called the *fundamental group* of $X$ in the point $P$.

**Proposition 4.1.6.** *The* **fundamental group** $\pi(X, P)$ *is in fact a* **group***, where the* **composition** *is given by* concatenation of paths *and where the* **neutral element** *is given by the* **constant path** $\gamma \equiv P$. *The* **inverse element** *of (the homotopy class of) a path is given by the* **inverse path** *(or reverse path), which is the path run in the opposite direction.*

*Proof.* For two paths $\gamma_1$ and $\gamma_2$ (with start point and end point being $P$) the **concatenation** is the path $\gamma_1\gamma_2 : [0,1] \longrightarrow X$ which is defined as

$$\gamma_1\gamma_2(t) = \begin{cases} \gamma_1(2t) \text{ for } 0 \leq t \leq 0.5 \\ \gamma_2(2(t-0.5)) \text{ for } 0.5 \leq t \leq 1 \end{cases} .$$

In the context of homotopy and the fundamental group all paths should be defined on the unit interval So this is the path which first runs with double velocity through $\gamma_1$ and then again with double velocity through $\gamma_2$ [1]

Then we define the **composition** of **two homotopy classes** $[\gamma_1]$ and $[\gamma_2]$ by $[\gamma_1\gamma_2]$. One has to check that this definition is **independent of the choice of representative**. So let $\gamma_1 \sim \beta_1$ and $\gamma_2 \sim \beta_2$. We have to show that then $\gamma_1\gamma_2 \sim \beta_1\beta_2$. If $H_1$ and $H_2$ are homotopies which relate $\gamma_1$ with $\beta_1$ and $\gamma_2$ with $\beta_2$, then one can concatenate the homotopies to a homotopy to show that the concatenations are homotopic.

The **associativity** follows from Lemma 4.1.4, as $(\gamma_1\gamma_2)\gamma_3$ and $\gamma_1(\gamma_2\gamma_3)$ differ only **change of parameter**.

The **constant path** $\beta = P$ is the **neutral element**, since again $\gamma\beta$ and $\gamma$ differ by a change of parameter.

[1]This definition of concatenation is slightly different from the one in 3.3.9.

The **inverse path** is denoted by $\overleftarrow{\gamma}$ and defined by $\overleftarrow{\gamma}(t) := \gamma(1-t)$. The concatenation path $\gamma\overleftarrow{\gamma}$ is homotopic to the constant path, as shown by the homotopy

$$H(s,t) = \begin{cases} \gamma(2st) \text{ for } t \leq 1/2 \\ = \gamma(2s(1-t)) \text{ for } t \geq 1/2 \end{cases} .$$

So the inverse path is really the inverse in the group structure. □

**Remark 4.1.7.** For a path-connected space $X$ the fundamental group does not depend on the point $P$, as follows from exercise 7.6.

**Example 4.1.8.** Let $X = \mathbb{R}^2$. Consider the unit circle $[0,1] \longrightarrow \mathbb{R}^2$ given by $t \mapsto (\cos(2\pi t), \sin(2\pi t))$. Its start and end point is $(1,0)$. By shrinking the circles but keeping the start and end point we see that this circle path is homotop to the constant path:

$$H(s,t) := (s + (1-s)\cos(2\pi t), (1-s)\sin(2\pi t)) .$$

It is clear that for this homotopy we need the area inside the circle. If instead of $\mathbb{R}^2$ we would just deal with the unit circle or with $\mathbb{R}^2 - \{(0,0)\}$, then - as we will see more precisely below - it is not possible to perform such a homotopy, and the circle path is not homotopic to the constant path.

**Definition 4.1.9.** A metric space $X$ is called *contractible* if there exists a point $P \in X$ and a **continuous** (“*contracting*”) *mapping* $H : I \times X \longrightarrow X$ with $H(0,-) = \mathrm{id}$ and $H(1,-) = P$ (the constant mapping) and with $H(s,P) = P$ for all $s \in I$.

**Theorem 4.1.10.** *Suppose that $G$ is a* **contractible space***. Then its* **fundamental group is trivial***. This is in particular true for a* **star-shaped subspace** *of a vector space.*

*Proof.* Suppose that $G$ is **contractible** with respect to $P$ and consider $\pi(P,X)$. Let $H : I \times X \longrightarrow X$ be a **contracting homotopy** to $P$ and let a closed path (starting and ending in $P$ be given). Then

$$I \times I \xrightarrow{\mathrm{id}\times\gamma} I \times X \xrightarrow{H} X ,$$

which sends $(s,t) \longmapsto H(s,\gamma(t))$, is a **homotopy** between $\gamma$ and the **constant path** $\equiv P$. □

**Definition 4.1.11.** A topological space is called *simply connected* if it is **path-connected** and if its **fundamental group is trivial**.

So Theorem 4.1.10 says that **contractible spaces are simply connected**. This is true for (open and closed) balls and for every finite dimensional vector space as a whole.

**Proposition 4.1.12.** *Let $\varphi : X \longrightarrow Y$ be a* **continuous mapping** *and let $P \in X$ be a point which is mapped to $Q = \varphi(P)$. Then we get a* **natural group homomorphism**

$$\pi(\varphi) : \pi(X,P) \longrightarrow \pi(Y,P),\ [\gamma] \longmapsto [\varphi \circ \gamma] .$$

*Proof.* One first has to show that this is **well-defined**. So let $\gamma \sim \beta$ with a homotopy $H$. Then $\varphi \circ H$ gives a homotopy between $\varphi \circ \gamma$ and $\varphi \circ \beta$. That it is a group homomorphism is clear from the definition of the concatenation of two paths. $\square$

In particular, the image of a **zero-homotopic** (= **homotopic to constant path**) is again **zero-homotopic**.

4.2. **Coverings.**

In this section we will deal with topological spaces, not only with metric spaces. We introduce coverings, which we will use for the computation of fundamental groups and in the proof of the monodromy theorem 5.1.2.

**Definition 4.2.1.** Let $Y$ and $X$ be **topological spaces** and let $p : Y \longrightarrow X$ be a **continuous mapping**. The mapping is called a *covering* (and $Y$ is called a *covering space* of $X$) if the following condition holds:

For every **point** $P \in X$ there exists an **open neighborhood** $P \in U$ such that $p^{-1}(U)$ is the **disjoint union of open subsets** $V_i$, $i \in I$, (where $I$ is some index set), such that the **restrictions** $V_i \longrightarrow U$ are **homeomorphisms for every** $i$.

A covering of the form $\biguplus_{i\in I} V_i \longrightarrow U$, where each $V_i$ is mapped homeomorphically to $U$, is called *trivial*. So *locally*, every covering is trivial, but the *global* properties can be quite involved. The word couple **local/global** is very important in modern mathematics.

The standard examples are the following.

**Example 4.2.2.** Let $p : \mathbb{R} \longrightarrow S^1$ be given by $t \mapsto (\cos(t), \sin(t))$. For a point $P_0 \in S^1$, $P_0 = (\cos(t_0), \sin(t_0))$, the preimage consists of $t_0 + 2n\pi$, $n \in \mathbb{Z}$. The preimage of the set

$$\{(\cos(t), \sin(t))\colon t_0 - \delta < t < t_0 + \delta\}$$

for $\delta$ small enough ($\delta < \pi$) is the disjoint union of open intervals

$$\biguplus_{n\in\mathbb{Z}} (t_0 + 2n\pi - \delta, t_0 + 2n\pi + \delta)\,.$$

On each of these mappings we have a homeomorphism. To see this extend the mapping to the closed interval,

$$[t_0 + 2n\pi - \delta, t_0 + 2n\pi\delta] \longrightarrow \{(\cos(t), \sin(t))\colon t_0 - \delta \le t \le t_0 + \delta\}\,.$$

Then closed subsets are compact, so their image is compact and hence again closed, and we have a homeomorphism.

**Example 4.2.3.** Let $p : \mathbb{C} \longrightarrow \mathbb{C}^\times = \mathbb{C} - \{0\}$ given by $z \mapsto \exp(z)$. That this is in fact a covering can be best seen using Example 4.2.2. To see this relation consider the mappings

$$\mathbb{C} \longrightarrow \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}_+ \times S^1 \longrightarrow \mathbb{C}^\times$$

given by

$$(x, y) \longmapsto (\exp(x), y),\ (u, v) \longmapsto (u, \cos(v), \sin(v)),\ (r, s_1, s_2) \longmapsto (rs_1, rs_2)\,.$$

The composed mapping is

$$(x, y) \mapsto (\exp(x)\cos(y), \exp(x)\sin(y)) = \exp(x + iy)$$

(by **Euler formula**, see Example 6.3.4), so this is our original mapping. The mappings on the left and on the right are homeomorphisms, and the mapping in the middle is in the first component the identity and in the second component the covering from Example 4.2.2. So by Exercise 8.6 this is a covering.

**Example 4.2.4.** Let $k$ be a natural number. Then $p : S^1 \longrightarrow S^1$ given by

$$(\cos u, \sin u) \longmapsto (\cos nu, \sin nu),\ 0 \le u \le 2\pi\,,$$

is a covering. The preimage of a point $(\cos(u_0), \sin(u_0))$ (and similarly, of a neighborhood), consists of the $n$ points $(\cos(\frac{u_0}{n} + \frac{2\pi k}{n}), \sin(\frac{u_0}{n} + \frac{2\pi k}{n}))$, $k = 0, \ldots, n-1$.

**Example 4.2.5.** Let $n$ be a natural number. Then $p : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ given by $z \mapsto z^n$ is a covering (the corresponding mapping $\mathbb{C} \longrightarrow \mathbb{C}$ is not). Using the identification $\mathbb{C}^\times \cong \mathbb{R}_+ \times S^1$ (given by $(r, u) \mapsto ru$) taking $n$th powers is the mappings

$$(r, u) \longmapsto (r^n, u^n)\,.$$

In the first component this is a homeomorphism, and in the second this is just the covering from Exercise 4.2.4, as $(\cos nt, \sin nt) = (\cos t, \sin t)^n$ (recall that multiplying of numbers on the unit circle is adding of the angles).

**Definition 4.2.6.** Let $p : Y \longrightarrow X$ be a **continuous mapping** between topological **spaces**, and let $g : T \longrightarrow X$ be another **continuous map**, where $T$ is also a topological space. Then a **continuous mapping** $\tilde{g} : T \longrightarrow Y$ is called a *lifting* of $g$ if $p \circ \tilde{g} = g$.

$$\begin{tikzcd} & Y \arrow[d, "p"] \\ T \arrow[ur, "\tilde{g}"] \arrow[r, "g"'] & X \end{tikzcd}$$

**Lemma 4.2.7.** *Let $p : Y \longrightarrow X$ be a* **covering** *and let $g : T \longrightarrow X$ be a* **continuous map**. *Let $g_1, g_2 : T \longrightarrow Y$ be* **two continuous liftings**. *Then the set $S = \{t \in T : g_1(t) = g_2(t)\}$ is* **open and closed**. *In particular: If $T$ is* **connected** *and two* **liftings coincide in one point**, *then they are* **identical**.

*Proof.* Let $t$ be a point in this set $S$. There exists an **open neighborhood** of $P = g(t) \in U$ such that the **restriction** $p^{-1}(U) \longrightarrow U$ is a **trivial covering**. The point $Q = g_1(t) = g_2(t)$ lies in one $V_i$ which is **mapped homeomorphically** to $U$. As the liftings are continuous there exists a (common) **open neighborhood** $t \in W$ such that $g_1(W), g_2(W) \subseteq V_i$. But then $g_1$ and $g_2$ are **determined** to be $(p|V)^{-1} \circ g$, so they are the same on $W$. Hence $P \in W \subseteq S$ and $S$ is open.

Assume now that $t$ is not in the set. This means that $g_1(t) \neq g_2(t)$ and that $g_1(t) \in V_i$ and $g_2(t) \in V_j$ for $i \neq j$. Then there exists again a (common) **open neighborhood** $t \in W$ such that $g_1(W) \subseteq V_i$ and $g_2(W) \subseteq V_j$. Hence on $W$ the **two liftings do not have a point in common**, and so $W$ is an **open neighborhood** of $t$ outside of the given set. Hence also the complement is open. $\square$

**Theorem 4.2.8.** *Let $p : Y \longrightarrow X$ be a* **covering** *and let $\gamma : I = [a, b] \longrightarrow X$ be a* **path**. *Fix a* **point** *$Q \in Y$ mapping to $P = \gamma(a) \in X$. Then there exists a* **uniquely determined lifting** *$\tilde{\gamma} : I \longrightarrow Y$ with $\tilde{\gamma}(a) = Q$.*

*Proof.* By the **compactness** of a closed interval there **exist** $a = a_0 < a_1 < \ldots < a_k < b$ such that $\gamma([a_\ell, a_{\ell+1}]) \subseteq U_\ell$, where the **covering** on $U_\ell$ is **trivial** (Exercise 8.1). We denote these paths by $\gamma_\ell$, $\ell = 0, \ldots, k$. We **define a lifting** for these paths **successively**. First we have $Q \in V_{0,i}$ for some $i$ and so on $[a, a_1]$ we must have $\tilde{\gamma}_0 = (p|_{V_{0,i}})^{-1} \circ \gamma_0$. This determines a point $\tilde{\gamma}(a_1) \in Y$ over $\gamma(a_1)$, which we use to construct $\tilde{\gamma}_1$. At the end we get a **complete lifting**. The **uniqueness** was already proven in Lemma 4.2.7. □

**Theorem 4.2.9.** *Let $p : Y \longrightarrow X$ be a* **covering** *and let $H : [0, 1] \times [0, 1] \longrightarrow X$ be* **continuous**. *Fix a point $Q \in Y$ mapping to $P = H(0, 0) \in X$. Then there exists a* **uniquely determined lifting** *$\tilde{H} : [0, 1] \times [0, 1] \longrightarrow Y$ with $\tilde{H}(0, 0) = Q$. If $H(s, 0)$ is constant, then also $\tilde{H}(s, 0)$ is constant.*

*Proof.* We skip this proof, which is similar to the proof of 4.2.8 (but more complicated). The last sentence follows since $\tilde{H}(s, 0)$ is a lifting of the path $H(s, 0)$, and this is uniquely determined by one point. □

**Corollary 4.2.10.** *Let $p : Y \longrightarrow X$ be a* **covering** *and let $\gamma_1, \gamma_2 : I = [a, b] \longrightarrow X$ be* **two paths** *which are* **homotopic** *in $X$ (so in particular $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$). Let $\tilde{\gamma}_1$ be a* **lifting** *of $\gamma_1$ and let $\tilde{\gamma}_2$ be a* **lifting** *of $\gamma_2$ with $\tilde{\gamma}_1(a) = \tilde{\gamma}_2(a)$. Then also $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are* **homotopic** *and in particular $\tilde{\gamma}_1(b) = \tilde{\gamma}_2(b)$.*

*Proof.* Write $I$ for the interval and let $H : [0, 1] \times I \longrightarrow X$ be a **homotopy** between the **two paths**. By Theorem 4.2.9 there **exists** $\tilde{H} : [0, 1] \times I \longrightarrow Y$ such that $\tilde{H}(0, a) = \tilde{\gamma}_1(a) = \tilde{\gamma}_2(a)$. Then $\tilde{H}(0, -) = \tilde{\gamma}_1$ and $\tilde{H}(1, -) = \tilde{\gamma}_2$. □

## 4.3. **The circle and the punctured plane.**

We compute now some fundamental groups with the help of coverings.

**Lemma 4.3.1.** *Let $\gamma : [0, 1] \longrightarrow S^1$ be a* **closed path**, *let $\mathbb{R} \longrightarrow S^1$, $t \mapsto (\cos t, \sin t)$ be the covering described in Example 4.2.2. Let $\tilde{\gamma}$ be a* **lifting**. *Then $\gamma$ is* **homotopic to the constant path** *(so $[\gamma] = 0$ in $\pi(S^1)$) if and only if the* **lifting** *$\tilde{\gamma}$ is* **closed** *(so $\tilde{\gamma}(0) = \tilde{\gamma}(1)$).*

*Proof.* One direction follows from Corollary 4.2.10. On the other hand, if $\tilde{\gamma}(0) = \tilde{\gamma}(1)$, then $\tilde{\gamma}$ is (as a path in $\mathbb{R}$, which is contractible) **homotopic to the constant path**. So its **image path**, which is $\gamma$, must also be **homotopic to the constant path**. □

**Theorem 4.3.2.** *The* **fundamental group** *of the* **circle** *$S^1$ is $\mathbb{Z}$. Every* **closed path** *$\gamma : [0, 1] \longrightarrow S^1$ is* **homotopic** *to one* **path of the form**

$$t \longmapsto (\cos 2n\pi t, \sin 2n\pi t) = e^{2\pi int}$$

*for one $n \in \mathbb{Z}$. The* **assignment** *$\gamma \mapsto n$ gives this* **correspondence**.

*Proof.* We may assume that the path has **start** and **end point** $(1, 0)$. Let $\mathbb{R} \longrightarrow S^1$ be the covering, and let $\tilde{\gamma} : [0, 1] \longrightarrow \mathbb{R}$ be a **lifting**. The preimage of $(1, 0)$ is exactly $\mathbb{Z}2\pi$. Hence we can write $\tilde{\gamma}(1) - \tilde{\gamma}(0) = 2n\pi$ with a **uniquely determined** $n \in \mathbb{Z}$. Let $\gamma_n$ be the path $t \longmapsto (\cos 2n\pi t, \sin 2n\pi t)$. A **lifting** to $\mathbb{R}$ of this path is of the form $t \mapsto 2\pi k + 2n\pi t$ for some $k$. Now look at the path $\gamma\overleftarrow{\gamma_n} = \gamma\gamma_{-n}$. Any **lifting** of this path has the same start and end point. Hence $\gamma\gamma_{-n}$ is homotopic to the constant path by Lemma 4.3.1 and so $\gamma$ is homotopic to $\gamma_n$. □

**Corollary 4.3.3.** *The* **fundamental group** *of the* **punctured plane** $\mathbb{C}^\times = \mathbb{R}^2 - \{0\}$ *is* $\mathbb{Z}$*,*

$$\pi(\mathbb{C}^\times) \cong \mathbb{Z}\,.$$

*Every* **closed path** *(starting and ending) in* $(1, 0)$ *is* **homotopic to a path of type**

$$t \longmapsto (\cos 2n\pi t, \sin 2n\pi t)\,.$$

*Proof.* This follows from the **homeomorphism**

$$S^1 \times \mathbb{R}_+ \cong \mathbb{C}^\times$$

given by $(\theta, r) \mapsto r\theta$ (polar coordinates), Exercise 7.7 and the fact that $\mathbb{R}_+$ is **contractible**. □

## 5. Exact and closed differential forms

We are still dealing with the question when is a **closed continuous differential form** $\omega \in \Omega(V, W)$ **exact** and how the **path-integrals** $\int_\gamma \omega$ depend on the **paths** $\omega$. This section gives the answers in the form of the **monodromy theorem** and its corollaries.

### 5.1. The monodromy theorem.

For a **closed differential form** $\omega : G \longrightarrow \mathrm{Hom}(V, W)$ we construct a **covering space** $G_\omega \longrightarrow G$, the so-called "**integral covering**". This will be the main technical tool to proof the **monodromy theorem**.

**Construction 5.1.1.** Let $\omega : G \longrightarrow \mathrm{Hom}(G, W)$ be a **closed differential form**. For each point $P \in G$ there exists an **open neighborhood** $P \in U \subseteq G$ (a ball for example) such that the differential form **restricted** to $U$ is **exact** (Theorem 3.5.7), so **locally** on $U$ there **exists a primitive function**.

Let $G_\omega := G \times W$ as a set together with the **projection** $p : G \times W \longrightarrow G$. We will define a **topology** on $G_\omega$ such that this projection is a **covering**. We consider the following **subsets** of $G_\omega$:

$$\Gamma(U, F) = \{(P, F(P)),\ P \in U\} \subseteq G \times W\,,$$

where $U$ runs through all **open subsets** of $G$ such that $\omega|_U$ is **exact** and $F$ runs through all **primitive functions** for $\omega$ on $U$ (they differ by an element of $W$, if $U$ is connected) ($\Gamma$ stands for **graph**).

We define a **topology** on $G_\omega$ by **declaring** these subsets and **all unions of such subsets to be open**. Note that **the intersection of two such sets** is the union of such sets. To see this let $(P, y) \in \Gamma(U, F) \cap \Gamma(V, G)$. Then $P \in U \cap V$, and let $P \in B \subseteq U \cap V$ be an open ball neighborhood. Since on a connected subset a primitive function is **determined by one value** (Theorem 3.1.8) and since $y = F(P) = G(P)$, it follows that $F|_B = G|_B$, and so

$$(P, y) \in \Gamma(B, F|_B) \subseteq \Gamma(U, F) \cap \Gamma(V, G)\,.$$

It follows that **all unions of such sets form a topology** and make $G_\omega$ into a **topological space** (we do not claim that it is metric). For an open subset $U \subseteq G$ with the property that $\omega|_U$ is exact, we have that $p^{-1}(U) \cong U_\omega \longrightarrow U$ is a **trivial covering**, and consists of **disjoint copies** of $U$, one for each $y \in W$. Therefore $G_\omega$ it is a **covering space** which we call the *covering space for the differential form* $\omega$ or just the *integral covering*.

The following theorem is called the *monodromy theorem*:

**Theorem 5.1.2.** *Let* $\omega$ *be a* **continuous closed differential form** *defined on* $G \subseteq V$ *with values in* $W$*. Let* $\gamma_1, \gamma_2 : [a, b] \longrightarrow G$ *be* **two piecewise differentiable paths** *which are* **homotopic**. *Then the* **path integrals are the same**,

$$\int_{\gamma_1} \omega = \int_{\gamma_2} \omega\,.$$

*Proof.* First, let $\gamma : [a, b] \longrightarrow G$ be a **piecewise differentiable path** and let $\tilde{\gamma}$ be a **lifting** (which exists by Theorem 4.2.8) in the **integral covering** $G_\omega$ for $\omega$ as constructed in 5.1.1. We **claim** that $\int_\gamma \omega = \tilde{\gamma}(b) - \tilde{\gamma}(a)$. For this let $a < a_1 < \ldots < a_k < b$ be such that the images of every restricted path $\gamma_\ell : [a_\ell, a_{\ell+1}] \longrightarrow G$ lie **completely** in an open subset $U_\ell$ on which $\omega$ is **exact** (exists by the compactness of the interval, see Exercise 8.1) and such that $\gamma_\ell$ is differentiable. Let $F_\ell$ be the **primitive function** on $U_\ell$ such that $F_\ell(\gamma(a_\ell)) = \tilde{\gamma}(a_\ell)$. Note that $F_\ell$ is a **section** $U_\ell \longrightarrow G_\omega$ and that then $\tilde{\gamma} = F_\ell \circ \gamma_\ell$ holds on $[a_\ell, a_{\ell+1}]$ by Theorem 4.2.8.

The path integral is the sum over the path integrals for the paths $\gamma_\ell$, and these are

$$\int_{\gamma_\ell} \omega = F_\ell(\gamma(a_{\ell+1})) - F_\ell(\gamma(a_\ell)) = \tilde{\gamma}(a_{\ell+1}) - \tilde{\gamma}(a_\ell)\,.$$

Hence **summing up** for $\ell = 0, \ldots, k$ gives $\int_\gamma \omega = \tilde{\gamma}(b) - \tilde{\gamma}(a)$.

Now let $\gamma_1$ and $\gamma_2$ be **two homotopic paths** in $G$. Then by Corollary 4.2.10 also **two liftings** $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ with same start point are **homotopic** (and have also the same end point). Therefore

$$\int_{\gamma_1} \omega = \tilde{\gamma}_1(b) - \tilde{\gamma}_1(a) = \tilde{\gamma}_2(b) - \tilde{\gamma}_2(a) = \int_{\gamma_2} \omega\,.$$

∎

**Corollary 5.1.3.** *Let $\omega$ be a* **closed continuous differential form** *defined on $G \subseteq V$ with values in $W$. Let $\gamma : [a,b] \longrightarrow G$ be a* **closed path** *(with start and end point $P$) and suppose that it is* **homotopic to the constant path***. Then*

$$\int_\gamma \omega = 0\,.$$

*Proof.* This follows directly from Theorem 5.1.2, since the **path integral** over a **constant path** is 0. ☐

The following theorem generalizes the same fact which was given for star-shaped open sets in Theorem 3.5.7.

**Theorem 5.1.4.** *Let $G$ be* **simply connected***. Let $\omega$ be a* **continuous closed differential form***. Then $\omega$ is* **exact***. In particular, path integrals over paths in a simply connected subset depend only on the start point and the end point, and path integrals over closed paths are 0. A primitive function for $\omega$ can be found by choosing a point $P$ and then defining $\varphi(Q) := \int_\gamma \omega$, where $\gamma$ is an arbitrary path connecting $P$ and $Q$.*

*Proof.* This follows directly from Theorem 5.1.2 and Theorem 3.4.1. ☐

**Example 5.1.5.** These results give also a method to show that certain paths are not homotopic to the constant path by showing that the path integral for a certain differential form is not 0. For example, since $\int_\gamma \frac{1}{z}dz = 2\pi i$ over the unit circle $\gamma$ in $\mathbb{C}^\times$, the circle is not homotopic to the constant path. In particular, $\pi(\mathbb{C}^\times) \neq 0$.

An integration version of Corollary 4.3.3 is the following.

**Corollary 5.1.6.** *The mapping*

$$\pi(\mathbb{C}^\times) \longrightarrow \mathbb{Z},\ \gamma \longmapsto \frac{1}{2\pi i}\int_\gamma \frac{dz}{z}$$

*is an isomorphism.*

*Proof.* We may assume that our paths start and end in $(1,0)$. This mapping is **well defined** (as a mapping to $\mathbb{C}$) by the **monodromy theorem** 5.1.2. By Corollary 4.3.3, **every path in $\mathbb{C}^\times$ is homotopic** to $\gamma_n : t \mapsto (\cos 2\pi nt, \sin 2\pi nt)$. Then

$$\int_{\gamma_n} = \int_0^1 \frac{1}{e^{2\pi int}} d(e^{2\pi int}) = 2\pi in \int_0^1 e^{-2\pi int} e^{2\pi int} dt = 2\pi in\,.$$

Hence the image is $\mathbb{Z}$. ☐

## 5.2. **The period mapping.**

We look again at the monodromy theorem from several perspectives (this section can be skipped). We define $\Omega^{cl}(G,W)$ to be the continuous closed differential form on

$G$ with values in $W$ and by $\Omega^{ex}(G,W)$ the exact forms. So $\Omega^{ex}(G,W) \subseteq \Omega^{cl}(G,W)$ is a subvector space. By Theorem 5.1.2, the mapping

$$\pi(G,P) \times \Omega^{cl}(G,W) \longrightarrow W,\ (\gamma,\omega) \longmapsto \int_\gamma \omega$$

is well defined. Any class $[\gamma] \in \pi(G,P)$ is **represented** by a **piecewise differentiable** (even piecewise linear) path, along which we can evaluate the integral, and the result will be the same for every homotopic path.

A similar thing happens with respect to the second argument: for an **exact differential form**, the path integral over every closed path is zero (Theorem 3.4.1). It follows that for a **closed form** $\omega$ and a **second exact form** $\alpha$ the path integral over a closed path of $\omega$ and of $\omega + \alpha$ are the same. Hence we have a mapping

$$\pi(G,P) \times \Big(\Omega^{cl}(G,W)/\Omega^{ex}(G,W)\Big) \longrightarrow W,\ (\gamma,\omega) \longmapsto \int_\gamma \omega\,.$$

Here $\Omega^{cl}(G,W)/\Omega^{ex}(G,W)$ is the residue class vector space (in particular a residue class group).

One can look at this mapping (the **pairing between closed paths and closed differential forms**) in several ways. One may look at it as giving a group homomorphism

$$\pi(G,P) \longrightarrow \mathrm{Hom}_{\mathbb{K}}(\Omega^{cl}(G,W),W),\ \gamma \longmapsto (\omega \longmapsto \int_\gamma \omega)\,.$$

The homomorphism property follows from the fact that the path integral over the concatenation of two paths is just the sum of the two path integrals.

Note here the following: the group on the right is (as it is a vector space) **commutative**, but the **fundamental group is in general not commutative**. This has an important consequence in understanding the kernel of this group homomorphism. If $\gamma$ and $\beta$ are two paths, then in general $\gamma\beta \neq \beta\gamma$, or equivalently, $\beta^{-1}\gamma\beta\gamma^{-1} \neq 0$. The image of this combination of $\gamma$ and $\beta$ must however be always 0. In general, for a **group** $G$ we denote the **subgroup generated** by $h^{-1}ghg^{-1}$ by $[G,G]$. This is a **normal subgroup**, and the **residue class group** $G/[G,G]$ is **commutative**.

**Definition 5.2.1.** Let $G$ be a **pathwise connected topological space** with **fundamental group** $\pi = \pi(G,P)$. Then we set $H_1(G) := \pi/[\pi,\pi]$ and we call this (commutative) group the *first homology group* of $G$.

So we have in fact a pairing

$$H_1(G) \times (\Omega^{cl}(G,W)/\Omega^{ex}(G,W)) \longrightarrow W\,.$$

One can look at this mapping as a vector space homomorphism

$$\Omega^{cl}(G,W) \longrightarrow \mathrm{Hom}(\pi(G,P) \longrightarrow W),\ \omega \longmapsto (\gamma \longmapsto \int_\gamma \omega)\,.$$

So a **fixed differential form** $\omega$ gives a **group homomorphism** $\pi(G,P) \longrightarrow W$. Taking into account Definition 5.2.1, this can also be seen as a group homomorphism

$$\psi_\omega : H_1(G) \longrightarrow W\,,$$

which is called the *period mapping* of $\omega$. One important case for this setting is for $G = \mathbb{C} - \{0\}$ (or a ball without a point), which we describe in the next section.

**Corollary 5.2.2.** *A* **continuous closed differential form** $\omega$ *on* $G \subseteq V$ *is* **exact** *if and only if the* **period mapping** $\psi_\omega : H_1(G) \longrightarrow W$ *is trivial.*

*Proof.* This is a reformulation of Theorem 3.4.1. □

## 6. The main theorems of complex function theory

From now on we will consider open subsets $G \subseteq \mathbb{C} \cong \mathbb{R}^2$and complex valued functions $f : G \longrightarrow \mathbb{C}$and the corresponding **complex-valued differential forms** $f(z)dz$. The relation between **complex-differentiable functions** and **closed (real-symmetric) differential forms** (Theorem 3.6.1) provides a bridge between the **theory of differential forms** and **complex analysis**. With this relationship we will prove the **basic results of function theory in one complex variable**, among them the **residue calculus**, **Cauchy integral theorem** and **Cauchy integral formula**, the **differentiability of arbitrary order** and the **analyticity of complex-differentiable functions**, the **theorem of Liouville**, the **identity principle** and **openness**.

### 6.1. Residue calculus.

For a **complex-differentiable function** $f$ on the disc $B$ the function $f/(z-P)$ is defined on the punctured disc $B - \{P\}$ and is complex-differentiable there. This function might be or might not be **extendable** to a **continuous function** on the whole $B$. Anyway, via the correspondence of Theorem 3.6.1 we have a **closed form** $\frac{f(z)}{z-P}dz$ (which is also a closed real-differential form) on $B - \{P\}$. It will turn out that the complex differentiability of $f$ is such a strong property that the **value** of $f(P)$ can be **computed by computing the path integrals** of this **differential form** for a **path which runs one time counter-clockwise around the center**. This is the **residue calculus**.

**Definition 6.1.1.** For a **continuous closed** ($\mathbb{C}$-valued) **differential form** $\omega$ defined on a **punctured disc** $B(P,r) - \{P\}$ we set

$$\operatorname{Res}(\omega, P) = \frac{1}{2\pi i}\int_{\gamma_1} \omega$$

and call it **the residue** of $\omega$ in $P$. Here $\gamma_1$ is a **closed path** inside $B$ (with arbitrary start and end point) which runs around $P$ one time **counter-clockwise**.

**Remark 6.1.2.** The residue is a complex number. By the **monodromy theorem** 5.1.2 it does not matter which (one round, counter-clockwise) path we choose to compute it. It is also independent of the start and end point.

More generally, if $G$ is any **open subset** in $\mathbb{C}$, $P \in \mathbb{C}$ and if $\omega$ is a **continuous closed differential form** defined on $G - \{P\}$, then $\mathrm{Res}(\omega, P)$ is defined as before by taking an open disc $B(P,r) \subseteq G$ (so that $\gamma_1$ runs only around $P$, not around any other missing points).

We gather together some properties of the residue.

**Proposition 6.1.3.** *Let $G \subseteq \mathbb{C}$ be open, $P \in G$. Let $\omega$ be a* **real symmetric** *(in particular closed)* **complex-valued differential form** *on $G - \{P\}$. Then the following hold.*

(i) *The mapping* $\mathrm{Res}(-, P) : \Omega^{cl}(G - \{P\}) \longrightarrow \mathbb{C}$ *is* $\mathbb{C}$-**linear**.
(ii) *If $\omega$ is* **exact** *on $G - \{P\}$ (or in a* **punctured ball neighborhood** *$B - \{P\} \subseteq G - \{P\}$), then* $\mathrm{Res}(\omega, P) = 0$.
(iii) *If $\omega$* **extends** *to a* **continuous closed differential form** *$\tilde{\omega}$ on $G$, then* $\mathrm{Res}(\omega, P) = 0$.
(iv) $\mathrm{Res}(\frac{dz}{z-P}, P) = 1$.
(v) $\mathrm{Res}((z-P)^i dz, P) = 0$ *for $i \geq 0$ and for $i \leq -2$.*
(vi) *If $f(z) = \sum_{n=a}^{\infty} c_n (z-P)^n$ (where $a \in \mathbb{Z}$) such that the* **power series** *(see Section 6.3) $\sum_{n=0}^{\infty} c_n (z-P)^n$ converges (in $G$ or in a neighborhood of $P$), then* $\mathrm{Res}(f(z)dz, P) = c_{-1}$.

*Proof.* (i) The **linearity** follows from the **linearity of path integrals** (Theorem 3.3.11). (ii) follows from Theorem 3.4.1.

(iii). The **extended closed form** $\tilde{\omega}$ is **exact** on a ball neighborhood of $P$ (by Theorem 3.5.7), hence $\omega$ itself must be **exact**, and so (ii) gives (iii).

(iv). This was computed in Corollary 5.1.6.

(v). For $i \geq 0$ this follows from (ii) (or (iii)) and for $i \leq -2$ this follows from (ii).

(vi) follows from the previous results. □

**Example 6.1.4.** Compute the residue of the differential form $f(z)g(z)dz$, where

$$f(z) = z^{-3} - 2z^{-1} + 1 + 4z \text{ and } g(z) = 4z^{-4} - z^{-2} + z^{-1} + 6 - 3z^2 + 2z^3 .$$

We are only interested at the coefficient for $-1$. So we have to add $-3 - 12 + 1 - 4 = -18$.

## 6.2. **Cauchy integral theorem and Cauchy integral formula.**

We continue to **reconstruct complex-differentiable functions** by knowing their **effect** on **path integrals**.

The following theorem, called the **Cauchy integral theorem**, is a reformulation of 5.1.4

**Theorem 6.2.1.** *Let $f : G \longrightarrow \mathbb{C}$ be a* **continuously complex-differentiable function** *in a* **simply connected domain** *$G \subseteq \mathbb{C}$. Then for* **every closed path** *$\gamma$ we have $\int_\gamma f(z)dz = 0$.*

*Proof.* By 3.6.1 the form $f(z)dz$ is **closed** and by Theorem 5.1.4 the form is **exact** and all path integrals over closed paths are 0. □

The following theorem is the **Cauchy integral formula**.

**Theorem 6.2.2.** *Let $f : U \longrightarrow \mathbb{C}$ be a* **continuously complex-differentiable function**. *Then for $P \in U$ we have*

$$f(P) = \mathrm{Res}(\frac{f(z)}{z-P}dz; P)\,.$$

*Proof.* The form $\frac{f(z)}{z-P}dz$ is a (continuous and) **closed differential form** on $G = U - \{P\}$. We write

$$\frac{f(z)}{z-P}dz = \frac{f(z)-f(P)}{z-P}dz + \frac{f(P)}{z-P}dz$$

and by the **residue calculus** we have that $f(P) = \mathrm{Res}(\frac{f(P)}{z-P}dz; P)$ (Proposition 6.1.3 (iv) and Corollary 5.1.6). So we only have to show that $\mathrm{Res}(\frac{f(z)-f(P)}{z-P}dz, P) = 0$.Since $f$ is **complex-differentiable** and since **the expression inside the differential form** is the **differential quotient**, we know that there exist constants $C$ and $\delta$ such that $\left|\frac{f(z)-f(P)}{z-P}\right| \leq C$ for $z \in B(P,\delta)$ ($\subseteq U$). For $\varepsilon < \delta$ we have by 3.3.11 (v) the estimate

$$\left|\mathrm{Res}(\frac{f(z)-f(P)}{z-P}, P)\right| = \left|\frac{1}{2\pi i}\right| \left|\int_{\gamma_{1,\varepsilon}} \frac{f(z)-f(P)}{z-P}dz\right|$$

$$= \frac{1}{2\pi}\left|\int_0^{2\pi} \frac{f(\gamma_{1,\varepsilon}(t))-f(P)}{\gamma_{1,\varepsilon}(t)-P}\cdot\gamma'_{1,\varepsilon}(t)dt\right| \leq \frac{1}{2\pi}\int_0^{2\pi} C\cdot\left|\gamma'_{1,\varepsilon}(t)\right| dt \leq C\varepsilon\,.$$

As $\varepsilon$ can be chosen arbitrarily small, this expression must be 0. □

**Remark 6.2.3.** There are several ways to express the Cauchy integral formula. One can find the formulations/notations

$$f(P) = \mathrm{Res}(\frac{f(z)}{z-P}dz; P) = \frac{1}{2\pi i}\int_{\gamma_1}\frac{f(z)}{z-P}dz = \frac{1}{2\pi i}\int_{\partial B}\frac{f(z)}{z-P}dz\,.$$

Here $\gamma_1$ is a counter-clockwise path around the point $P$ (so that the path is entirely inside a disc inside $U$) and $\partial B$ denotes the border of such a disc (with the same meaning).

**Remark 6.2.4.** With the help of the Cauchy integral theorem and the Cauchy integral formula one can **compute** a lot of **path integrals**. Suppose that $U \subseteq \mathbb{C}$ is open, $P \in U$ and set $G = U - \{P\}$. Let $f : U \longrightarrow \mathbb{C}$ be a continuously complex-differentiable function, let $\omega = \frac{f(z)}{z-P}dz$ be the differential form on $G$ and let $\gamma : I \longrightarrow G$ be a piecewise differentiable path. Then one can apply either the Cauchy integral theorem or the Cauchy integral formula under the following conditions.

Suppose first that $P$ is "**outside of** $\gamma$" and inside a simply connected subset $H \subseteq G$. Then $\omega|_H$ is exact and so by the **Cauchy integral formula** we have $\int_\gamma \omega = 0$.

Suppose next that $P$ is "**inside of** $\gamma$", and that $\gamma$ runs once counter-clockwise around $P$ inside $B(P,r) \subseteq U$. Then $\int_\gamma \omega = 2\pi i f(P)$ by the **Cauchy integral formula**. See exercise 9.1 for examples.

**Remark 6.2.5.** Theorems 6.2.1 and 6.2.2 do hold also without the assumption that the complex differential is continuous. The same is true for 3.5.7 and 3.6.1. This generalization is known under the name **Lemma of Goursat**, and is more difficult to prove.

**Corollary 6.2.6.** *Suppose the situation of Theorem 6.2.2 and suppose that* $\overline{B(P,r)} \subseteq U$. *Then*

$$f(P) = \frac{1}{2\pi} \int_0^{2\pi} f(P + re^{it}) dt\,.$$

*Proof.* We apply Theorem 6.2.2 with the path $\gamma(t) = P + re^{it} = P + r\cos t + ri\sin t$. This gives

$$\begin{aligned} f(P) &= \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-P} dz = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(P+re^{it})}{re^{it}} dre^{it} \\ &= \frac{1}{2\pi i} \int_0^{2\pi} f(P+re^{it}) i dt = \frac{1}{2\pi} \int_0^{2\pi} f(P+re^{it}) dt\,. \end{aligned}$$

□

**Corollary 6.2.7.** *Suppose the situation of Theorem 6.2.2 and suppose that* $\overline{B(P,r)} \subseteq U$. *Let* $S = S(P,r)$. *Then*

$$|f(P)| \le |f|_{S(P,r)} = \max_{w\in S} |f(w)|\,.$$

*Proof.* This follows immediately from Corollary 6.2.6 □

**Theorem 6.2.8.** *Suppose that $f$ is a* **continuously complex-differentiable function** *on $B(Q,R)$. Then $f$ is* **infinitely often complex-differentiable** *on $B(Q,R)$. For a point $P \in B(Q,r)$, $r < R$, the $n$-th* **derivative** *is*

$$f^{(n)}(P) = \frac{n!}{2\pi i} \int_{\gamma_{Q,r}} \frac{f(z)}{(z-P)^{n+1}} dz\,.$$

*Proof.* For $n = 0$ this is Theorem 6.2.2, namely that

$$f(P) = \frac{1}{2\pi i} \int_{\gamma_{Q,r}} \frac{f(z)}{(z-P)} dz$$

for all $P \in B(Q,r)$. Suppose first more generally that we **have** an **equation**

$$h(P) = \int_\gamma g(z,P) dz = \int_a^b g(\gamma(t),P) \cdot \gamma'(t) dt\,,$$

where in the integral $P$ is a **parameter**, on which the integral depends. Suppose that $g(z,P)$ is **complex-differentiable** considered as a function in $P$ (on a certain open subset), and write $\frac{\partial g}{\partial P}$ for this **derivative**. Then also $h$ is **complex-differentiable** and

$$h'(P) = \int_\gamma \frac{\partial g}{\partial P}(z,P) dz = \int_a^b \frac{\partial g}{\partial P}(\gamma(t),P) \cdot \gamma'(t) dt\,.$$

This is a theorem of **one dimensional integration theory**. Now we do induction on $n$ and **apply** this **differentiation rule** on $h = f^{(n)}$. This gives

$$f^{(n+1)}(P) = h'(P) = \left(\frac{n!}{2\pi i}\int_{\gamma_{Q,r}} \frac{f(z)}{(z-P)^{n+1}}dz\right)'$$

$$= \frac{n!}{2\pi i}\int_{\gamma_{Q,r}} \frac{\partial}{\partial P}\frac{f(z)}{(z-P)^{n+1}}dz = \frac{n!}{2\pi i}(n+1)\int_{\gamma_{Q,r}} \frac{f(z)}{(z-P)^{n+2}}dz\,.$$

□

The next theorem is called the **Cauchy inequality**.

**Theorem 6.2.9.** *Suppose that $f$ is a* **continuously complex-differentiable function** *on $B(Q,R)$ and let $r < R$. Let $M = \max\{|f(z)| \colon |z-Q| = r\}$ (the* **maximum** *of $|f|$ on the* **circle** *$S(Q,r)$). Then for $P \in B(Q,r)$ and $n \in \mathbb{N}$ we get the* **inequality**

$$\left|f^{(n)}(P)\right| \le \frac{Mr(n!)}{(r-|P-Q|)^{n+1}}\,.$$

*Proof.* By 3.3.11 (v) we have

$$\begin{aligned}
\left|\int_{\gamma_{Q,r}} \frac{f(z)}{(z-P)^{n+1}}dz\right| &= \left|\int_0^{2\pi} \frac{f(\gamma_{Q,r}(t))}{(\gamma_{Q,r}(t)-P)^{n+1}}\gamma'_{Q,r}(t)dt\right| \\
&\le \int_0^{2\pi} \left|\frac{f(\gamma_{Q,r}(t))}{(\gamma_{Q,r}(t)-P)^{n+1}}\right| \left|\gamma'_{Q,r}(t)\right| dt \\
&\le \int_0^{2\pi} \frac{M}{(r-|P-Q|)^{n+1}} r dt \\
&= \frac{2\pi M r}{(r-|P-Q|)^{n+1}}\,.
\end{aligned}$$

In the last estimate we have used $|f(\gamma_{Q,r}(t))| \le M$, $|\gamma_{Q,r}(t) - P| \ge r - |P-Q|$ (triangle inequality) and $\left|\gamma'_{Q,r}(t)\right| = r$ (for a standard circle path of radius $r$). Then Theorem 6.2.8 gives the result. □

## 6.3. **Review on power series.**

We recall briefly (without proofs) some results about **convergent power series** and **analytic functions**.

Recall that a **power series** is an expression of the form

$$\sum_{n=0}^{\infty} c_n (z-P)^n\,.$$

Here $c_n$ are **coefficients** in $\mathbb{C}$, $P \in \mathbb{C}$ (the **center of the power series**) and $z$ should be thought of as a variable. If we **plug in** for $z$ a complex number, we get a **series of complex numbers**, which **might converge or not**.

**Proposition 6.3.1.** *Let $\sum_{n=0}^{\infty} c_n(z-P)^n$ be a power series. Then the following holds.*

(i) *There exists a number $R$ (which might be $0$ in the non convergent case, and which might also be $\infty$) such that for $z \in B(P,R)$ we have that the power series $\sum_{n=0}^{\infty} c_n(z-P)^n$* **converges**.
(ii) *If $R > 0$, then for $z \in B(P,R)$ we get a function $f : B(P,R) \longrightarrow \mathbb{C}$ by setting $f(z) = \sum_{n=0}^{\infty} c_n(z-P)^n$.*
(iii) *The sequence of polynomials $f_k = \sum_{n=0}^{k} c_n(z-P)^n$***converges uniformly** *to the* **function** *$f$ on $B(P,r)$ for every $r < R$. This means that for every $\delta$ there exists $n_0$ such that for all $m \geq n_0$ and all $z \in B(P,r)$ we have $\|\sum_{n=m}^{\infty}(z-P)^n\| \leq \delta$.*
(iv) *The function $f(z)$ is continuous.*

**Example 6.3.2.** The **geometric series** $\sum_{n=0}^{\infty} z^n$ has center 0 and all coefficients are 1. It converges for $|z| < 1$, and the identity

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

holds.

**Example 6.3.3.** The **exponential series** $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$ has center 0 and the coefficients are $c_n = \frac{1}{n!}$. This series converges for every $z$ and the resulting function is (by definition, well, this is one way to define it) the **exponential function**.

**Example 6.3.4.** The **cos series** and the **cos series** are

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \text{and} \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}\,.$$

From these power series one gets immediately the relationship

$$\exp(x+iy) = \exp(x)(\cos y + i \sin y)\,.$$

For $x = 0$ this is the **Euler formula** $\exp(iy) = \cos y + i \sin y$.

**Definition 6.3.5.** A function $f : G \longrightarrow \mathbb{K}$, $G \subseteq \mathbb{K}$, is called **analytic** if for every point $P \in G$ there exists an $r > 0$ and a **power series** $\sum_{n=0}^{\infty} c_n(z-P)^n$ which converges on $B(P,r)$ and such that $f(z) = \sum_{n=0}^{\infty} c_n(z-P)^n$for $z \in B(P,r)$.

**Theorem 6.3.6.** *An* **analytic function** *$f$ given on $B(P,r)$ as the power series $f(z) = \sum_{n=0}^{\infty} c_n(z-P)^n$is* **differentiable**, *and its* **derivative is again analytic**, *namely*

$$f'(z) = \sum_{n=0}^{\infty} (n+1)c_{n+1}(z-P)^n\,.$$

*In particular, an* **analytic function is differentiable of arbitrary order**.

A similar statement holds for integration. If $f(z) = \sum_{n=0}^{\infty} c_n(z-P)^n$ converges on $B(P,R)$, then a primitive function exists and is given as $\sum_{n=0}^{\infty} \frac{1}{n+1} c_n z^{n+1}$ (in particular, this power series converges on the same disc). More generally, integration and uniformly convergent sequence of functions are interchangeable, that is, $\int(\lim_{n\longrightarrow\infty} f_n) = \lim_{n\longrightarrow\infty}(\int f_n)$.

## 6.4. Complex differentiable functions are analytic.

We show now that **continuously complex-dif ferentiable functions are analytic**. This is also called **Taylor's theorem**.

**Theorem 6.4.1.** *Let $f : G \longrightarrow \mathbb{C}$ be a* **continuously complex-differentiable function**, *and $z_0 \in \overline{B(z_0, r)} \subseteq G$. Then for $z \in B(z_0, r)$ we have the* **power series**

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n ,$$

*where*

$$c_n = \frac{1}{2\pi i} \int_{\gamma_{z_0 r}} \frac{f(u)}{(u - z_0)^{n+1}} du .$$

*Proof.* We write

$$\frac{f(u)}{u - z} = \frac{f(u)}{u - z_0 - (z - z_0)} = \frac{f(u)}{(u - z_0)(1 - \frac{z - z_0}{u - z_0})} = \sum_{n=0}^{\infty} \frac{f(u)}{(u - z_0)^{n+1}} (z - z_0)^n ,$$

where we have used the **geometric series** $\frac{1}{1 - \frac{a}{b}} = \sum_{n=0}^{\infty} (\frac{a}{b})^n$, which is allowed since $\left| \frac{z - z_0}{u - z_0} \right| < 1$ (as $u \in S(z_0, r)$). Also, for fixed $z$ and **variable** $u \in S(z_0, r)$, this series is **uniformly convergent**. From Theorem 6.2.2 we have

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_{z_0 r}} \frac{f(u)}{u - z} du = \frac{1}{2\pi i} \int_{\gamma_{z_0 r}} (\sum_{n=0}^{\infty} \frac{f(u)}{(u - z_0)^{n+1}} (z - z_0)^n) du .$$

By a theorem on integration, **integration** and **uniformly convergent series** are **interchangeable**, hence (as $z$ and $z_0$ are constant in the integral) this is

$$= \sum_{n=0}^{\infty} (\frac{1}{2\pi i} \int_{\gamma_{z_0 r}} \frac{f(u)}{(u - z_0)^{n+1}} du)(z - z_0)^n .$$

□

## 6.5. The theorem of Liouville.

**Definition 6.5.1.** We call a **continuously complex-differentiable function** $f : G \longrightarrow \mathbb{C}$ **holomorphic**. For $G = \mathbb{C}$, such a function is called **entire**.

We know already that **holomorphic** is the same as **analytic**.

**Definition 6.5.2.** Two open subsets $G, D \subseteq \mathbb{C}$ are called **biholomorphic** if there exists a **bijective holomorphic mapping** $f : G \longrightarrow D$ such that the **inverse** $f^{-1} : D \longrightarrow G$ is also **holomorphic**.

The following theorem is the **Theorem of Liouville**.

**Theorem 6.5.3.** *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be an* **entire-holomorphic** *function which is* **bounded**. *Then $f$ is* **constant** *(so every non-constant entire function is unbounded: for every real number $M$ there exists $z \in \mathbb{C}$ such that $|f(z)| \geq M$).*

*Proof.* Assume that $f$ is **bounded**, say $|f(z)| \leq M$ for all $z \in \mathbb{C}$. We can apply the **Cauchy inequality** (Theorem 6.2.9) for $z = P = Q$ and an **arbitrary radius** $r$ to infer

$$|f'(z)| \leq \frac{Mr}{r^2} = \frac{M}{r}\,.$$

As $r$ might be **arbitrarily large**, it follows that $f'(z) = 0$ for all points $z \in \mathbb{C}$. Hence $f$ is **constant**. □

**Corollary 6.5.4.** *The* **complex plane** *and the* **open disc** $B(0,1)$ *are* **not biholomorphic equivalent**.

*Proof.* This follows immediately from Theorem 6.5.3, since every holomorphic function $f : \mathbb{C} \longrightarrow B(0,1)$ is bounded, hence constant, and cannot be a bijection. □

**Remark 6.5.5.** It can be easily shown that **all discs** in $\mathbb{C}$ are **biholomorphic** to each other. A **disc** is also **biholomorphic** to the **upper complex half plane** $\{z \in \mathbb{C}\colon Re(z) > 0\}$. In fact, a deep theorem, the so-called **Riemann mapping theorem** (**Riemannscher Abbildungssatz**), tells that **every simply-connected open subset** of $\mathbb{C}$, which is **not** $\mathbb{C}$, is **biholomorphic to the disc**.

From the **Theorem of Liouville** we can also deduce the **Fundamental Theorem of Algebra**

**Theorem 6.5.6.** *Let* $p : \mathbb{C} \longrightarrow \mathbb{C}$ *be a* **non-constant polynomial**,

$$p(z) = c_n z^n + c_{n-1} z^{n-1} + \ldots + c_1 z + c_0$$

*($c_n \neq 0$, $n \geq 1$). Then* $p$ **has a zero**, *that is, there* **exists** $z \in \mathbb{C}$ *such that* $p(z) = 0$.

*Proof.* Suppose $p$ has **no zero**. Then $p$ is **invertible**, that is, **the function**

$$\frac{1}{p} : \mathbb{C} \longrightarrow \mathbb{C}\,, z \longmapsto \frac{1}{p(z)}$$

is an **entire function**. As the **polynomial is not constant**, its inverse is also **not constant**, hence it is **unbounded** by the **Theorem of Liouville**.

On the other hand,

$$|p(z)| \geq |c_n z^n| - \left|c_{n-1} z^{n-1} + \ldots + c_1 z + c_0\right|\,.$$

Let $c$ be the maximum $\max\{|c_\nu| : \nu = 0, \ldots, n\}$. Then for $|z| \geq 1$ we have

$$\left|c_{n-1} z^{n-1} + \ldots + c_1 z + c_0\right| \leq nc\,|z|^{n-1}\,.$$

This together gives for $z$ with $|z| \geq \frac{1+nc}{|c_n|} =: M$ that

$$|p(z)| \geq |c_n z^n| - nc\,|z|^{n-1} = (|c_n| \cdot |z| - nc)\,|z|^{n-1} \geq |z|^{n-1}\,.$$

Hence $\left|\frac{1}{p(z)}\right| \leq \left|\frac{1}{z^{n-1}}\right| \leq \frac{1}{M}$ for $|z| \geq M$ (say $n \geq 2$, for $n = 1$ the result is clear anyway). Hence **this inverse function** is all in all **bounded**, since $\{z\colon |z| \leq M\}$ is **compact** and the **function is bounded** on this set anyway. So we get a **contradiction**. □

## 6.6. **Identity theorem.**

The **analyticity** of **holomorphic functions** (Theorem 6.4.1) implies that a rather **small amount of information determines already a holomorphic function completely**. This is the content of the **identity theorem** or **identity principle**.

**Definition 6.6.1.** Let $Z \subseteq X$ be an (arbitrary) subset of a metric space. A point $P \in X$ is called an **accumulation point** of $Z$ if for **every** $r > 0$ there **exists** a **point** $Q \in Z \cap B(P, r)$, $Q \neq P$.

**Theorem 6.6.2.** *Let $G \subseteq \mathbb{C}$ be* **open and connected** *and let $f : G \longrightarrow \mathbb{C}$ be a* **holomorphic function**. *Then the following are equivalent.*

(i) $f = 0$.
(ii) *The zero set $Z = \{P \in G\colon f(P) = 0\}$ has an* **accumulation point** *in $G$.*
(iii) *There* **exists a point** *$Q \in G$ such that $f^{(k)}(Q) = 0$ holds for all $k \in \mathbb{N}$.*

*Proof.* The implication (i) ⇒ (ii) is trivial. (ii) ⇒ (iii). Let $Q$ be an **accumulation point** of the set

$$Z = \{P \in G\colon f(P) = 0\}$$

We want to show that **all derivatives** of $f$ at $Q$ are 0. **Assume** that this is not true and let $k \in \mathbb{N}$ be the **minimal order** such that $f^{(k)}(Q) \neq 0$. As $f$ is **analytic** (Theorem 6.4.1), we can write

$$f(z - Q) = (z - Q)^k \tilde{f}(z)\,,$$

where $\tilde{f}(Q) \neq 0$. Then also $\tilde{f}(z) \neq 0$ in a **certain open neighborhood** around $Q$, and therefore also $f(z) \neq 0$ in such a neighborhood. But then $Q$ cannot be an accumulation point of $Z$.

(iii) ⇒ (i). The sets

$$W_k = \{P \in G\colon f^{(k)}(P) = 0\}$$

are **closed**. Hence also their **intersection** $W = \bigcap_{k=0}^{\infty} W_k$is **closed**. Let $P \in W$ be a point in this intersection, that is, **all derivatives** at $P$ **are zero**. This means that the **power series with center** $P$ which describes the function in a **disc neighborhood** $B$ around $P$ must be **zero**. But then for all points in this **disc neighborhood** all **derivatives** are again **zero**. So $B \subseteq W$ and $W$ is also open. As $G$ is **connected**, it follows that $W = G$ and so $f = 0$ on $G$. □

The following theorem is called the **identity theorem**.

**Theorem 6.6.3.** *Let $G \subseteq \mathbb{C}$ be* **open and connected** *and let $g, h : G \longrightarrow \mathbb{C}$ be* **two holomorphic functions**. *Then the following are equivalent.*

(i) $g = h$.
(ii) *The set $\{P \in G\colon g(P) = h(P)\}$ has an* **accumulation point** *in $G$.*
(iii) *There* **exists a point** *$Q \in G$ such that $g^{(k)}(Q) = h^{(k)}(Q)$ holds for all $k \in \mathbb{N}$.*

*Proof.* This statement follows from and is in fact equivalent to Theorem 6.6.3 by looking at the difference $f = g - h$. □

**Corollary 6.6.4.** *Every* **holomorphic function** *is* **determined** *on an* **arbitrarily small open disc**. *This means that if $g, h : G \longrightarrow \mathbb{C}$ are given on a connected open subset $G \subseteq \mathbb{C}$ and $g|_B = h|_B$ for some disc $B = B(P, r) \subseteq G$, then $g = h$ on $G$.*

*Proof.* This follows immediately from Theorem 6.6.3 (ii). □

**Corollary 6.6.5.** *For a* **function** $f : \mathbb{R} \longrightarrow \mathbb{R}$ *(or defined on an interval) there exists* **at most one holomorphic extension** $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$ *(that is, a function $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{C}$ such that $\tilde{f}|_{\mathbb{R}} = f$).*

*Proof.* Let $g$ and $h$ be **two extensions** of $f$. Then $g$ and $h$ are **identical on the real numbers** (or on any real interval, which is not just a point), and such a subset has clearly an **accumulation point**. So $g = h$ by Theorem 6.6.3. □

**Remark 6.6.6.** Functions like the **exponential function** or the **trigonometric function** are often first defined for **real numbers** (with real values). There **exist holomorphic extensions** (using the power series) to the **complex numbers**, and by Corollary 6.6.5 these are the only possibilities.

By the **identity principle** also certain **rules** which hold for the real numbers pass over to the complex numbers. For example, suppose we know the rule

$$\exp(x + y) = \exp(x)\exp(y)$$

only for real arguments. As soon as the exponential mapping is established for complex arguments, **both sides** of this identity **are holomorphic functions** on $\mathbb{C}$. Since they are identical on $\mathbb{R}$, they must be identical also on $\mathbb{C}$.

**Definition 6.6.7.** A subset $Z \subseteq X$ in a metric space $X$ is called **discrete** if for every point $P \in Z$ there exists $r > 0$ such that $B(P, r) \cap Z = \{P\}$.

**Corollary 6.6.8.** *Let $f : G \longrightarrow \mathbb{C}$ be a* **holomorphic non-constant function**. *Then for every $w \in \mathbb{C}$ the set $f^{-1}(w) = \{P \in G : f(P) = w\}$ is* **discrete**, **closed** *and in particular* **countable**. *In particular, the zero set of a holomorphic function is discrete, closed and countable.*

*Proof.* The closedness holds for every continuous map. The discreteness follows again from Theorem 6.6.3. □

**Example 6.6.9.** For a **non-constant polynomial** the **zero set** is always **finite**. The **trigonometric functions** sin and cos are **holomorphic functions** where the **zero set** is **not finite**, but still **countable**.

The following corollary is used in the proof of the openness theorem 6.7.4.

**Corollary 6.6.10.** *Let $f : G \longrightarrow \mathbb{C}$ be a holomorphic non-constant function, $Q \in G$, $G$ connected. Then there exists a $\delta > 0$ such that $f(Q) \neq f(u)$ for all $u \in S(Q, \delta)$.*

*Proof.* Suppose to the **contrary** that **for all radiuses** $\delta$ such that $B(Q,\delta) \subseteq G$ there **exists a point** $P_\delta \in S(Q,\delta)$ such that $f(Q) = f(P_\delta)$. Then the set $\{P \in G : f(P) = f(Q)\}$ has an **accumulation point** and the map would be **constant** $f(Q)$ by Theorem 6.6.3. □

We close this section with an **algebraic property** of the **ring of holomorphic functions**

$$\mathcal{O}(G) = \{f : G \longrightarrow \mathbb{C} \colon f \text{ holomorphic}\}\,.$$

**Corollary 6.6.11.** *Let $G \subseteq \mathbb{C}$ be open. Then the* **ring of holomorphic functions** *$\mathcal{O}(G)$ is an* **integral domain** *if and only if $G$ is* **connected**.

*Proof.* Assume that $G$ is **connected**. Let $z_n$ be a **sequence** in $G$, **convergent** to $z \in G$ and such that $z_n \neq z_k$ for $n \neq k$. Suppose that $g \cdot h = 0$, both **holomorphic functions**. Then $(g \cdot h)(z_n) = g(z_n) \cdot h(z_n) = 0$ and so $g(z_n) = 0$ or $h(z_n) = 0$ for every $n$. Then for at least one of these functions, say for $g$, we must have $g(z_n) = 0$ for **infinitely many** $n$. Hence $z$ is an **accumulation point** for this **subsequence** and so $g = 0$ by Theorem 6.6.3.

If $G$ is **not connected**, then let $U \subset G$, $U \neq \emptyset$, be a subset which is **open and closed** with (also open and closed) complement $V$. Then the function

$$e(P) = \begin{cases} 1 \text{ for } P \in U \\ 0 \text{ for } P \in V \end{cases}$$

is (**continuous** and) **holomorphic**, since this is a **local property**. The same is true for $1 - e$, hence $e(1 - e) = 0$. □

## 6.7. **Openness of holomorphic functions.**

**Definition 6.7.1.** Let $\varphi : X \longrightarrow Y$ be a map between topological spaces. Then $\varphi$ is called **open** if for **every open subset** $U \subseteq X$ also **the image** $\varphi(U)$ is **open**.

We want to show that **non-constant holomorphic mappings are open**. For this we need two lemmas.

**Lemma 6.7.2.** *Let $B$ be an* **open disc** *with* **center** *$Q$, $\overline{B} \subseteq G$ and $f : G \longrightarrow \mathbb{C}$ holomorphic. Let $S$ be the* **sphere** *of the disc. Suppose that $|f(Q)| < \min_{u \in S} |f(u)|$. Then $f$* **has** *a* **zero inside** *$B$.*

*Proof.* Suppose that $f$ has no zero in $B$. Then $f$ has no zero in $\overline{B} \subseteq U \subseteq G$, $U$ open (as it has no zero on $S$, and since the non-zero locus is open). So the function $z \mapsto \frac{1}{f(z)}$ is **holomorphic** on $U$. By Corollary 6.2.7 we have

$$|f(Q)|^{-1} \leq \max_{u \in S} |f(u)|^{-1} = (\min_{u \in S} |f(u)|)^{-1}\,.$$

This gives $|f(Q)| \geq \min_{u \in S} |f(u)|$ in **contradiction** to the assumption. □

**Lemma 6.7.3.** *Let $B$ be an* **open disc** *with center $Q$, $\overline{B} \subseteq G$ and $f : G \longrightarrow \mathbb{C}$* **holomorphic***. Let $S$ be the* **sphere** *of the disc. Set $\delta := \frac{1}{2}\min_{u\in S}|f(u) - f(Q)|$ and suppose that this is $> 0$. Then $B(f(Q),\delta) \subseteq f(B)$.*

*Proof.* We have to show that **every point** in $B(f(Q),\delta)$ **is in** the **image** of $f$, so let $P \in B(f(Q),\delta)$ be given. For $u \in S$ we have

$$|f(u) - P| \geq |f(u) - f(Q)| - |P - f(Q)| > 2\delta - \delta = \delta\,.$$

Hence $\min_{u\in S}|f(u) - P| > |f(Q) - P|$. By Lemma 6.7.2 (applied to $f(u) - P$) we infer the **existence** of a point $z \in B$ with $f(z) = P$. □

**Theorem 6.7.4.** *Let $f : G \longrightarrow \mathbb{C}$ be* **holomorphic** *and* **not constant***, $G$ a* **connected** *subset in $\mathbb{C}$. Then $f$ is* **open***.*

*Proof.* Let $U \subseteq G$ be **open** and let $Q \in U$. By Corollary 6.6.10 there exists a disc $Q \in B(Q,r) \subset \overline{B} \subseteq U$ such that $f(Q) \neq f(u)$ for all $u \in S(Q,r) = S$. Hence $\delta = \frac{1}{2}\min_{u\in S}|f(u) - f(Q)| > 0$ and by Lemma 6.7.3 we deduce that $B(f(Q),\delta) \subseteq f(U)$. Therefore $f(U)$ is open. □

## 7. Exercises

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Holger Brenner

## PMA444 Real and complex analysis 2007: Exercise sheet 1

Please hand in on Monday, the 12th of February your solutions to the following questions.

**Exercise 1.1.** Show that the following subsets of the complex numbers are open; draw a picture of the situation (it is difficult to draw the openness property). Give also a description of the (closed) complement.

a) The upper (open) half $\{z \in \mathbb{C} : Im(z) > 0\}$ (2 marks).

b) The quadrant $\{z \in \mathbb{C} : Im(z) > 0 \text{ and } Re(z) > 0\}$ (2 marks).

c) $\{z \in \mathbb{C} : 1 < |z| < 2\}$ (2 marks).

d) $\mathbb{C} - \{P\}$, where $P \in \mathbb{C}$ is a point (2 marks).

e) $\mathbb{C} - \mathbb{R}$ (2 marks).

f) $\mathbb{C} - \mathbb{R}_{\geq 0}$, where $\mathbb{R}_{\geq 0}$ denotes the nonnegative real numbers (2 marks).

g) $\mathbb{C} - \mathbb{N}$ (3 marks).

h) $\mathbb{C} - \{(x, y) : y = x^2\}$ (3 marks).

**Exercise 1.2.** Give an example in $\mathbb{R}$ showing that the intersection of infinitely many open subsets is not open in general (3 marks).

**Exercise 1.3.** Let $X$ and $Y$ be two metric spaces.

a) Show that their product

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

is also a metric space by setting

$$d((x_1, y_1), (x_2, y_2)) := \max(d(x_1, x_2), d(y_1, y_2)) \text{ (3 marks)}.$$

b) In the special case $X = Y = \mathbb{R}$, draw an open ball in the product space $X \times Y = \mathbb{R} \times \mathbb{R}$ (2 marks).

c) Let $Z$ be another metric space. Let $g : Z \longrightarrow X$ and $h : Z \longrightarrow Y$ be two mappings. Show that the (product) mapping $f : Z \longrightarrow X \times Y$, given by $f(z) = (g(z), h(z))$, is continuous if and only if $g$ and $h$ are continuous (6 marks).

**Exercise 1.4.** a) Prove Proposition 1.1.8 (4 marks).

b) Formulate Proposition 1.1.8 for closed subsets (3 marks).

**Exercise 1.5.** Fix $k \in \mathbb{N}$. Show that the sequence $1/n^k$, $n \in \mathbb{N}$, converges to 0, using the function $x \mapsto x^k$. What happens for $k = 1/2$, what for $k = -1$ (6 marks)?

**Exercise 1.6.** Let $X$ be a metric space and let $f : X \longrightarrow \mathbb{R}$ be a continuous function. Show that the subset

$$\{P \in X : f(P) \geq 0\}$$

is closed. Show also for a given real number $\alpha$ that the subsets

$$\{P \in X : f(P) = \alpha\}$$

are closed (4 marks).

**Exercise 1.7.** Let $X$ be a metric space, $P \in X$, $r > 0$. Then the set

$$\overline{B(P,r)} = \{Q \in X : d(P,Q) \leq r\}$$

is called the *closed ball* (with center $P$ of radius $r$). The set

$$S(P,r) = \{Q \in X : d(P,Q) = r\}$$

is called the *sphere* (with center $P$ of radius $r$).

Show that the closed ball and that the sphere is closed (2 marks).

**THE UNIVERSITY OF SHEFFIELD**
**DEPARTMENT OF PURE MATHEMATICS**
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 2**

Please hand in on Monday, the 19th of February your solutions to the following questions.

**Exercise 2.1.** Let $V$ be a normed $\mathbb{K}$-vector space. Show that $d(v, w) := \|w - v\|$ is a distance (so that $V$ is a metric space) (3 marks).

**Exercise 2.2.** Let $(V, \| \|)$ be a normed $\mathbb{K}$-vector space.

a) Show that $| \|v\| - \|w\| | \leq \|v - w\|$ for $v, w \in V$ (2 marks).

b) Show that the norm is a continuous mapping (2 marks).

**Exercise 2.3.** a) Prove that the Euclidean norm, the sumnorm and the maximum norm on $\mathbb{K}^n$ are in fact norms (13 marks). If you want you can consider only the case $\mathbb{R}^2$.

b) Draw pictures of the open balls with center 0 in $\mathbb{R}^2$ of radius 1 of these three norms (3 marks).

c) Show for $v \in \mathbb{K}^n$ that

$$\|v\|_{\max} \leq \|v\|_{\mathrm{Euc}} \leq \|v\|_{\mathrm{sum}} \leq n \|v\|_{\max} \quad (6 \text{ marks}) .$$

**Exercise 2.4.** a) Let $V = \mathbb{R}^3$. Compute the distance of the two vectors $v = (5, 2 - 3)$ and $w = (-1, 0, 4)$ is the three norms (3 marks).

b) Let $V = \mathbb{C}^2$. Compute the three norms of the vector $P = (3 - 2i, 2 + 5i)$ (3 marks).

**Exercise 2.5.** a) Show that the rational numbers $\mathbb{Q} \subseteq \mathbb{R}$ form neither a closed nor an open subset (4 marks).

b) Show that $\mathbb{Q}$ is not connected (3 marks).

**Exercise 2.6.** Let $f$ and $g$ be continuous mappings $\mathbb{K}^n \longrightarrow \mathbb{K}$. On which subset of $\mathbb{K}^n$ is the function $f/g$ defined (2 marks)? Is this subset open (2 marks)?

**Exercise 2.7.** Let $[0, 2\pi) = \{t \in \mathbb{R} : 0 \leq t < 2\pi\}$ be the interval which is closed on the left but open on the right, together with the induced metric and topology. Let $S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$ be the unit circle (again with the induced metric). Consider the map

$$\psi : [0, 2\pi) \longrightarrow S^1,\ t \longmapsto (\cos t, \sin t) .$$

a) Show that $\psi$ is bijective and continuous (use what you know about cos and sin) (4 marks).

b) Show that the inverse mapping is not continuous. E.g. describe a sequence $t_n \in [0, 2\pi)$ which does not converge in $[0, 2\pi)$, but such that $\psi(t_n) \in S^1$ does converge (4 marks).

**Exercise 2.8.** In this exercise we look at the addition and the multiplication, that is at the mappings

$$+ : \mathbb{K}^2 \longrightarrow \mathbb{K}, (x, y) \longmapsto x + y \quad \text{and} \quad \times : \mathbb{K}^2 \longrightarrow \mathbb{K}, (x, y) \longmapsto x \times y\,.$$

a) Show that the addition is continuous. Is it linear? (2 marks)

b) Show that the multiplication is continuous. Is it linear? (6 marks)

c) Prove in a) and b) whether it is linear or not, and write down a matrix if yes (5 marks).

**Exercise 2.9.** Let $V$ be a normed $\mathbb{K}$-vector space, and let $P_1, P_2, P_3, P_3$ be four points (think at the corners of a tetragon).

a) Prove the following inequality

$\|P_1 - P_3\| + \|P_2 - P_4\| \leq \|P_1 - P_2\| + \|P_2 - P_3\| + \|P_3 - P_4\| + \|P_4 - P_1\|$ (4 marks).

b) Give an interpretation in words of this inequality (2 marks).

**Exercise 2.10.** Let $X$ be a metric space. Consider the relation where $P$ is related to $Q$ if there exists a continuous mapping $\gamma : [0, 1] \longrightarrow X$ with $\gamma(0) = P$ and $\gamma(1) = Q$. Show that this relation is an equivalence relation (4 marks).

**Exercise 2.11.** Have a look at Topologist's sine curve under Wikipedia for an example of a connected, but not-path connected metric space.

**THE UNIVERSITY OF SHEFFIELD**
**DEPARTMENT OF PURE MATHEMATICS**
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 3**

Please hand in on Monday, the 26th of February your solutions to the following questions.

**Exercise 3.1.** a) Compute the total differential of the mapping

$$\varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^2,\ (x,y) \longmapsto (xy - 2y^3 + 5, x^3 - xy^2 + y)$$

for every point (3 marks).

b) What is the total differential in the point $P = (1,2)$ (2 marks).

c) What is the directional derivative in this point in direction $(4,-3)$ (2 marks).

d) What is the value of $\varphi$ in this point (1 mark).

**Exercise 3.2.** a) Compute the total differential of the mapping

$$\varphi : \mathbb{K}^3 \longrightarrow \mathbb{K}^2,\ (x,y,z) \longmapsto (xy - zy + 2z^2, \sin(x^2yz))$$

for every point (4 marks).

b) What is the total differential in the point $P = (1,-1,\pi)$ (3 marks).

c) What is the directional derivative in this point in direction $(2,0,5)$ (3 marks).

d) What is the value of $\varphi$ in this point (1 mark).

**Exercise 3.3.** a) Compute the total differential of the mapping

$$\varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^3,\ (x,y) \longmapsto (x + y^2, xy, \exp(x))$$

for every point (4 marks).

b) What is the total differential in the point $P = (3,2)$ (3 marks).

c) What is the directional derivative in this point in direction $(-1,-7)$ (3 marks).

d) What is the value of $\varphi$ in this point (1 mark).

**Exercise 3.4.** Study Example 2.2.9.

**Exercise 3.5.** We want to illustrate the chain rule in considering the mappings

$$\varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^3,\ (u,v) \longmapsto (uv, u - v, v^2)$$

and

$$\psi : \mathbb{K}^3 \longrightarrow \mathbb{K}^2,\ (x,y,z) \longmapsto (xyz^2, y\exp(xz))$$

and their composition $\psi \circ \varphi$.

a) Compute the differential $D\varphi$ for an arbitrary point $P \in \mathbb{K}^2$ with the help of partial derivatives (3 marks).

b) Compute the differential $D\psi$ for an arbitrary point $Q \in \mathbb{K}^3$ with the help of partial derivatives (3 marks).

c) Compute explicitly the composed mapping $\psi \circ \varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^2$ (3 marks).

d) Compute the differential of $\psi \circ \varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^2$ in a point $P \in \mathbb{K}^2$ directly with partial derivatives (3 marks).

e) Compute the differential of $\psi \circ \varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^2$ in a point $P \in \mathbb{K}^2$ directly with the help of the chain rule and part a) and b) (3 marks).

**Exercise 3.6.** a) Show the converse of Theorem 1.3.7, namely that a connected subset of $\mathbb{R}$ is an interval (3 marks).

b) Deduce the intermediate value theorem from part a) and from Theorem 1.3.3 (3 marks).

**Exercise 3.7.** Let $f : X \longrightarrow Y$ be a continuous map between metric spaces. Assume that $X$ is path-connected. Show that also the image $f(X)$ is path-connected (3 marks).

**Exercise 3.8.** Let $V$, $W_1$ and $W_2$ be three normed $\mathbb{K}$-vector spaces of finite dimension. Recall that $W_1 \times W_2$ is also a normed vector space.

a) Let $L_1 : V \longrightarrow W_1$ and $L_2 : V \longrightarrow W_2$ be two $\mathbb{K}$-linear mappings. Show that also the mapping

$$L_1 \times L_2 : V \longrightarrow W_1 \times W_2,\ v \longmapsto (L_1(v), L_2(v))$$

is $\mathbb{K}$-linear (3 marks).

b) Let $f_1 : V \longrightarrow W_1$ and $f_2 : V \longrightarrow W_2$ be two mappings which are differentiable in a point $P \in V$. Show that also the map

$$f = (f_1 \times f_2) : V \longrightarrow W_1 \times W_2,\ P \longmapsto (f_1(P), f_2(P))$$

is differentiable in $P$ with total differential given by

$$(Df)_P = (Df_1)_P \times (Df_1)_P \ \text{(5 marks)}.$$

**Exercise 3.9.** Determine for addition and for multiplication

$$+ : \mathbb{K}^2 \longrightarrow \mathbb{K}, (x, y) \longmapsto x + y \quad \text{and} \quad \times : \mathbb{K}^2 \longrightarrow \mathbb{K}, (x, y) \longmapsto x \times y\,.$$

the total differential (4 marks).

THE UNIVERSITY OF SHEFFIELD
DEPARTMENT OF PURE MATHEMATICS
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 4**

Please hand in on Monday, the 5th of March your solutions to the following questions.

**Exercise 4.1.** Express the mapping

$$\mathbb{C} \longrightarrow \mathbb{C},\ z \longmapsto 2z^3 - z^2 + 3z + 2 - i$$

in terms of real coordinates. Write down the real differential of this mapping (5 marks).

**Exercise 4.2.** a) Express the mappings

$$\mathbb{C} \longrightarrow \mathbb{C},\ z \longmapsto z^k$$

in terms of real coordinates for $k = 0, \ldots, 5$ (6 marks).

b) Express the mappings

$$\mathbb{C} - \{0\} \longrightarrow \mathbb{C},\ z \longmapsto z^k$$

in terms of real coordinates for $k = -1, -2, -3$ (6 marks).

**Exercise 4.3.** Consider the mapping

$$\mathbb{C} = \mathbb{R}^2 \longrightarrow \mathbb{C} = \mathbb{R}^2$$

given in real coordinates as $(x, y) \mapsto (x^3 - xy^2, 5x^2y^2 - y) = (g, h)$. Compute the real differential and check whether the Cauchy-Riemann differential equations hold (3 marks).

**Exercise 4.4.** Let $V$ and $W$ be two finite-dimensional $\mathbb{K}$-vector spaces. Consider the evaluation mapping

$$\mathrm{Ev} : \mathrm{Hom}_{\mathbb{K}}(V, W) \times V \longrightarrow W,\ (L, v) \longmapsto L(v)\,.$$

Note that the Hom-space on the left is also a finite-dimensional $\mathbb{K}$-vector space.

a) Is the evaluation map linear (2 marks)?

b) Determine the directional derivative of this evaluation map in a point $(L, v)$ in direction $(M, u)$ using the definition of total differentiability (6 marks).

**Exercise 4.5.** Let $G \subseteq V$ be open and let $f, g : G \longrightarrow \mathbb{K}$ be two differentiable functions. Use the chain rule applied to the diagram

$$G \xrightarrow{f,g} \mathbb{K} \times \mathbb{K} \xrightarrow{\text{mult}} \mathbb{K}\,.$$

and Exercise 3.9 to show that $(D(f \cdot g))_P = g(P)(Df)_P + f(P)(Dg)_P$ (4 marks).

**Exercise 4.6.** Let $f_1, \dots, f_n$ be functions in one variable which are continuously differentiable. Find the differential of the mapping $\mathbb{K}^n \longrightarrow \mathbb{K}^n$ given by $(x_1, \dots, x_n) \mapsto (f_1(x_1), \dots, f_n(x_n))$ (3 marks).

**Exercise 4.7.** Show that the mappings $Re : \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto Re(z)$ and $Im : \mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto Im(z)$ are real-linear, but not complex-linear. (3 marks).

**Exercise 4.8.** Is the complex conjugation $\mathbb{C} \longrightarrow \mathbb{C}$, $z \mapsto \overline{z}$ complex-differentiable? Is it real-differentiable (3 marks)?

**Exercise 4.9.** Consider the complex-linear mapping $L : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by the $2 \times 2$-complex matrix

$$\begin{pmatrix} 5 & 3+4i \\ 2-2i & i \end{pmatrix} .$$

Write the corresponding matrix which describes the same linear mapping $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ (4 marks).

**Exercise 4.10.** Let $V$ and $W$ be $\mathbb{Q}$-vector spaces and let $L : V \longrightarrow W$ be an additive map (that is, $L(v + w) = L(v) + L(w)$) for all $v, w \in V$. Show that $L$ is already $\mathbb{Q}$-linear, that is, $L(qv) = qL(v)$ (5 marks).

**Exercise 4.11.** Study the proof of Theorem 2.2.6.

**THE UNIVERSITY OF SHEFFIELD**
**DEPARTMENT OF PURE MATHEMATICS**
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 5**

Please hand in on Monday, the 12th of March your solutions to the following questions.

**Exercise 5.1.** The pull-back of differential forms is linear.

a) What is meant by this statement? Give an explicit formulation (3 marks).

b) Prove your statement in a) (3 marks).

c) Can you describe and prove a statement about the pull-back of $f\omega$, where $f$ is a function with values in $\mathbb{K}$ (4 marks).

**Exercise 5.2.** Let $\omega \in \Omega(G, W)$ be a differential form. Let $\psi : G' \longrightarrow G$ and $\theta : G'' \longrightarrow G'$ be differential maps, where $G' \subseteq V'$ and $G'' \subseteq V''$ are open subsets in further vector spaces $V'$ and $V''$. Show that

$$(\psi \circ \theta)^* \omega = \theta^*(\psi^*(\omega)) \quad \text{(4 marks)}\,.$$

**Exercise 5.3.** Let $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$, $t \mapsto (\cos t, \sin t)$. Compute the path integral for the following differential forms on $\mathbb{R}^2$.

a) $\omega(x, y) = xdx + ydy$ (3 marks).

b) $\omega(x, y) = xdx - ydy$ (3 marks).

c) $\omega(x, y) = ydx + xdy$ (3 marks).

d) $\omega(x, y) = ydx - xdy$ (3 marks).

**Exercise 5.4.** Let $\gamma : [-1, 1] \longrightarrow \mathbb{R}^2$, $t \mapsto (t, t^2)$ be given. Compute the path integral for the differential form given by $\omega(x, y) = (x + y)dx + xydy$ (3 marks).

**Exercise 5.5.** Let $a, b, c, d, r, s \geq 1$ be natural numbers. Consider the path $\gamma : [0, 1] \longrightarrow \mathbb{R}^2$ given by $t \mapsto (t^r, t^s)$. Compute the path integral for the differential form given by $\omega(x, y) = x^a y^b dx + x^c y^d dy$ (5 marks).

**Exercise 5.6.** Let $\gamma : [a, b] \longrightarrow G$ be a constant (or stationary) path, that is, $\gamma(t) = P$ for all $t$. Then show that $\int_\gamma \omega = 0$ for every $\omega \in \Omega(G, W)$ (2 marks).

**Exercise 5.7.** Consider the exact differential form given by

$$(x, y, z) \longmapsto 2xydx + (x^2 + z)dy + (y - 2z)dz\,.$$

Construct a primitive mapping $\varphi$ using the method of Theorem 3.4.1 taking $P_0 = 0$ in two different ways.

a) Use the direct linear path from 0 to a given point $P = (x, y, z)$ (2 marks).

b) Use the piecewise linear path from 0 to a given point $P = (x, y, z)$ parallel to the axes (2 marks).

**Exercise 5.8.** Let on $\mathbb{K}^3$ be the differential form

$$(x, y, z) \longmapsto x^2 dx + 3z^2 dy - 5xydz$$

be given and let $\psi : \mathbb{K}^2 \longrightarrow \mathbb{K}^3$ be the mapping given by

$$(u, v) \longmapsto (u \exp(uv^2), u^2 - v, 3) .$$

Compute the pull-back $\psi^*(\omega)$ (5 marks).

**Exercise 5.9.** Show that the differential form

$$(x, y) \longmapsto (x^2 + y)dx + 2xydy$$

is not exact by giving two different paths (say from $(0, 0)$ to $1, 1)$ with different path integrals. Give also a closed path such that its path integral is not 0 (5 marks).

**Exercise 5.10.** Let $\omega \in \Omega(G, \mathbb{K})$ be a differential form, let $\gamma : I \longrightarrow G$ be a piecewise differentiable path and let $\theta : I' = [c, d] \longrightarrow I = [a, b]$ be a differential mapping with $\theta(c) = a$ and $\theta(d) = b$. Consider the composed path $\gamma \circ \theta : I' \longrightarrow G$. Then show that $\int_\gamma \omega = \int_{\gamma\circ\theta} \omega$ (4 marks).

**Exercise 5.11.** Show that the mapping $\mathbb{C} \longrightarrow \mathrm{Mat}_2(\mathbb{R})$ given by

$$a + bi \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is a ring homomorphism (4 marks).

**Exercise 5.12.** Let $V$ and $W$ be two normed finite-dimensional $\mathbb{K}$-vector spaces. Consider the $\mathbb{K}$-vector space $\mathrm{Hom}(V, W)$ (which is also finite dimensional).

a) Show that defining

$$\|L\| := \max_{v \in V,\, \|v\|=1} \|L(v)\|$$

for $L : V \longrightarrow W$ makes $\mathrm{Hom}(V, W)$ into a normed $\mathbb{K}$-vector space (the maximum exists, since $\{v \in V \colon \|v\| = 1\}$ is compact) (6 marks).

b) Show that

$$\|L(v)\| \leq \|L\| \cdot \|v\| \quad \text{for} \quad v \in V$$

(this means that $\|L\|$ gives a bound for a linear mapping) (2 marks).

c) Let $V = W$ and let $\lambda \in \mathbb{K}$ be an eigenvalue for $L : V \longrightarrow V$. Show that $|\lambda| \leq \|L\|$ (2 marks).

THE UNIVERSITY OF SHEFFIELD
DEPARTMENT OF PURE MATHEMATICS
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 6**

Please hand in on Monday, the 19th of March your solutions to the following questions.

**Exercise 6.1.** Let $f : \mathbb{C} \longrightarrow \mathbb{C}$, $f(z) = -2+iz^2+3z^4$. Determine the differential form $f(z)dz$ in real coordinates and show by explicit computation that it is symmetric (4 marks).

**Exercise 6.2.** Express the differential form $dz/z$, which is defined on $\mathbb{C}^\times$ with values in $\mathbb{C}$, in real coordinates. Show explicitly that the real part and the imaginary part of this form are both symmetric. Show that the real part is exact and that the imaginary part is not exact (8 marks).

**Exercise 6.3.** Consider the mapping

$$\psi : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{C}^\times,\ (r, \theta) \longmapsto (r \cos \theta, r \sin \theta)$$

(the restriction of this mapping to $\mathbb{R}_+ \times [0, 2\pi]$ is called polar coordinates).

a) Compute the differential of this mapping (2 marks).

b) Compute the pull-back of the differential form $\omega = dz/z$ under this mapping (6 marks).

c) Show that the pull-back is exact (though $\omega$ is not exact) (2 marks).

d) Consider for fixed $r_0 \in \mathbb{R}_+$ the path $\overline{\gamma} : [0, 2\pi] \longrightarrow \mathbb{R}_+ \times \mathbb{R}$, $t \mapsto (r_0, t)$. Compute the composed path $\gamma = \psi \circ \overline{\gamma}$. Say in words what this composed path is. Compute $\int_{\overline{\gamma}} \psi^*(\omega)$ and $\int_\gamma \omega$ (6 marks).

**Exercise 6.4.** Consider the complex exponential mapping

$$\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times,\ u \longmapsto \exp(u)\,.$$

a) Write down this mapping in real coordinates (2 marks).

b) Compute the pull-back of the differential form $\omega = dz/z$ under this mapping (2 marks).

c) Show that the pull-back is exact (though $\omega$ is not exact) (2 marks).

d) Consider for fixed $a_0 \in \mathbb{R}_+$ the path $\tilde{\gamma} : [0, 2\pi] \longrightarrow \mathbb{C}$, $t \mapsto (a_0, t)$. Compute $\int_{\exp \circ \tilde{\gamma}} \omega$ and $\int_{\tilde{\gamma}} \exp^*(\omega)$ (4 marks).

e) Find a relationship between the complex exponential mapping and the mapping considered in Exercise 6.3 (3 marks).

**Exercise 6.5.** Compute the path integral for the differential form $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by
$$(x,y) \longmapsto \ln(\sqrt{x+y})dx + y^2 dy$$
along the linear path from $(1,5)$ to $(5,1)$ (4 marks).

**Exercise 6.6.** Compute the path integral for the differential form $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}$ given by
$$(x,y) \longmapsto \frac{1}{x+y}dx + xydy$$
along the linear path from $(1,1)$ to $(4,4)$ (4 marks).

**Exercise 6.7.** Let $u = (0,1)$, $v = (1,2)$ and $w = (3,0)$. Compute for the mapping
$$\varphi : \mathbb{K}^2 \longrightarrow \mathbb{K}^2,\ (x,y) \longmapsto (xy^3 + y^2, x^2 - xy)$$
the directional derivatives $D_u\varphi$, $D_v\varphi$, $D_w\varphi$ and the higher derivatives $D_uD_u\varphi$, $D_vD_v\varphi$, $D_wD_w\varphi$, $D_uD_v\varphi$, $D_uD_w\varphi$, $D_vD_u\varphi$, $D_vD_w\varphi$, $D_wD_u\varphi$, $D_wD_v\varphi$ (12 marks).

**Exercise 6.8.** Check which of the following differential forms are symmetric on $\mathbb{R}^2$. Give also, if possible, a primitive function.

a) $\omega = adx + bdy$ (where $a, b \in \mathbb{R}$).

b) $\omega = \sin xydx + \sin xydy$.

c) $\omega = y \sin xydx + x \sin xydy$.

d) $\omega = 2xy^3dx + 3x^2y^2dy$ (8 marks).

**Exercise 6.9.** Let $X$ and $Y$ be two metric spaces and let $[a,b]$ and $[b,c]$ be two intervals in $\mathbb{R}$ $(a < b < c)$. Let
$$f_1 : [a,b] \times Y \longrightarrow X \text{ and } f_2 : [b,c] \times Y \longrightarrow X$$
be two continuous mappings and suppose that on $b \times Y$ both mappings coincide. Then the mapping $f : [a,c] \times Y \longrightarrow X$ (to be defined in an obvious manner) is also continuous (use the sequence criterion for continuity) (7 marks).

**THE UNIVERSITY OF SHEFFIELD**
**DEPARTMENT OF PURE MATHEMATICS**
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 7**

Please hand in on Monday, the 16th of April your solutions to the following questions.

**Exercise 7.1.** Show that $\mathbb{C}^\times = \mathbb{C} - \{0\}$ is connected, but not star-shaped (3 marks).

**Exercise 7.2.** Let $B = B(P, r) \subseteq V$ be a ball in a finite dimensional $\mathbb{K}$-vector space $V$. Let $\gamma_1, \gamma_2 : [a, b] \longrightarrow B$ be two paths with $\gamma_1(a) = \gamma_2(a)$ and $\gamma_1(b) = \gamma_2(b)$. Show that $\gamma_1$ and $\gamma_2$ are homotopic. In particular, a path $\gamma$ in $B$ is homotopic to the line segment which connects $\gamma(a)$ and $\gamma(b)$ (6 marks).

**Exercise 7.3.** Show that for every path $\gamma : [a, b] \longrightarrow G$ ($G$ open in a vector space $V$) there exists a piecewise differentiable path which is homotopic to $\gamma$ (use Exercise 7.2 and the compactness of $[a, b]$) (4 marks).

**Exercise 7.4.** Give a piecewise linear path which is homotopic to the unit circle inside $\mathbb{C}^\times$ (3 marks).

**Exercise 7.5.** Show that a star-shaped open subset $G$ in a vector space $V$ is contractible by giving an explicit contraction (3 marks).

**Exercise 7.6.** Let $X$ be a metric path-connected space. Show that for every two points $P$ and $Q$ there exists a group isomorphism

$$\pi(X, P) \longrightarrow \pi(X, Q)\,.$$

This means essentially that the fundamental group is independent of the chosen point (4 marks).

**Exercise 7.7.** Let $X$ and $Y$ be two path-connected topological (or metric) spaces. Show that there exists an isomorphism

$$\pi(X \times Y) \cong \pi(X) \times \pi(Y) \text{ (6 marks).}$$

**Exercise 7.8.** Let $g_j$, $j = 1, \ldots, n$ be differentiable functions in one variable, $g_j : \mathbb{K} \longrightarrow W$. Show that the differential form

$$\omega : \mathbb{K}^n \longrightarrow W, (x_1, \ldots, x_n) \longmapsto g_1(x_1)dx_1 + \ldots + g_n(x_n)dx_n$$

is symmetric. Show also that it is exact by giving a primitive function (3 marks).

**Exercise 7.9.** Let $f : \mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ be a differentiable function and consider the differential form $\omega : \mathbb{R}^3 - \{0\} \longrightarrow \mathrm{Hom}(\mathbb{R}^3, \mathbb{R})$ given by

$$P = (x, y, z) \longmapsto f(\|P\|)xdx + f(\|P\|)ydy + f(\|P\|)zdz$$

(where $\| \ \|$ is the Euclidean norm). Such a differential form (or the corresponding vector field) describes a central force, that is, a force which is directed to a central object and depends only on the distance to the central object (sun, atom).

a) Show that $\omega$ is symmetric (3 marks).

b) Write $f(t) = \frac{h(t)}{t}$ and suppose that $H$ is a primitive function for $h$ on $\mathbb{R}_+$. Show that

$$\varphi : P \longmapsto H(\|P\|)$$

is a primitive mapping for $\omega$ (4 marks).

c) Let $\gamma : [a, b] \longrightarrow \mathbb{R}^3 - \{0\}$ be a path such that $\gamma(a) = P$ and $\gamma(b) = Q$. Show that $\int_\gamma \omega = H(\|Q\|) - H(\|P\|)$ (1 mark).

d) Describe the central differential forms given by

$$f(t) = \frac{1}{t}, \quad f(t) = -\frac{1}{t^2}, \quad f(t) = \frac{1}{t^3} \ .$$

explicitly and find primitive mappings for them (6 marks).

THE UNIVERSITY OF SHEFFIELD
DEPARTMENT OF PURE MATHEMATICS
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 8**

Please hand in on Monday, the 23th of April your solutions to the following questions.

**Exercise 8.1.** Let $\gamma : [a, b] \longrightarrow G$ be a path and let $p : Y \longrightarrow G$ be a covering. Show that there exist $a < a_1 < \ldots < a_k < b$ such that the images of every restricted path $[a_\ell, a_{\ell+1}] \longrightarrow G$ lies completely in an open subset $U_\ell$ such that $p^{-1}(U_\ell) \longrightarrow U_\ell$ is trivial (use the compactness of the closed interval) (6 marks).

**Exercise 8.2.** Compute the preimage of the following points $P \in \mathbb{C}^\times$ under the following coverings $p$.

a) $P = 1, 2, -1, i$ and $p = \exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ (8 marks).

b) $P = 1, 2, -1, i, -i, 5 + 12i$ and $p = z^2 : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$ (8 marks).

c) $P = 1$ for $p = z^n$, where $n \in \mathbb{N}$ (4 marks).

**Exercise 8.3.** Consider the covering $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \mapsto z^k$ ($k \in \mathbb{Z}$). Describe the corresponding mapping on the fundamental groups (4 marks).

**Exercise 8.4.** Consider the covering $\mathbb{C}^\times \longrightarrow \mathbb{C}^\times$, $z \mapsto z^k$. Can you describe a lifting for $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ with respect to this covering (3 marks).

**Exercise 8.5.** Now consider the two coverings

$$\varphi_k, \varphi_\ell : \mathbb{C}^\times \longrightarrow \mathbb{C}^\times,\ z \longmapsto z^k \text{ and } z \longmapsto z^\ell\,.$$

Prove that $\varphi_\ell$ has a lifting with respect to $\varphi_k$ if and only if $k$ divides $\ell$ (4 marks).

**Exercise 8.6.** Let $p : Y \longrightarrow X$ be a covering of metric spaces, and let $Z$ be another metric space. Show that also the product mapping

$$p \times \mathrm{id} : Y \times Z \longrightarrow X \times Z\,, (y, z) \longmapsto (p(y), z)$$

is a covering (the products are metric spaces by Exercise 1.3) (5 marks).

**Exercise 8.7.** Show that the projection $\mathbb{R}^2 \longrightarrow \mathbb{R}$, $(x, y) \mapsto x$ is not a covering in showing that Lemma 4.2.7 does not hold (take $T = \mathbb{R}$ and $g = \mathrm{id}$ and describe two different liftings with a common point) (4 marks).

**Exercise 8.8.** Let $P_1, \ldots, P_k$ be different points in $\mathbb{C}$. Show that there is a surjective group homomorphism
$$\pi(\mathbb{C} - \{P_1, \ldots, P_k\}) \longrightarrow \mathbb{Z}^k$$
by looking at the mappings $\pi(\mathbb{C} - \{P_1, \ldots, P_k\}) \longrightarrow \pi(\mathbb{C} - \{P_i\})$ for each $i$. (6 marks).

**Exercise 8.9.** Let the points $P = (0,0)$, $P_1 = (3,1)$, $P_2 = (-2,2)$ and $P_3 = (-3,-3)$ be given. Construct explicitly a piecewise differentiable closed path starting and ending in $P$ which represents $(1,3,-1)$ in $\pi(\mathbb{C} - \{P_1\}) \times \pi(\mathbb{C} - \{P_2\}) \times \pi(\mathbb{C} - \{P_3\})$. Make sure that your path is defined on $[0,1]$ (7 marks).

**Exercise 8.10.** Study examples 4.2.2 - 4.2.5 in detail. Study 4.3.

**THE UNIVERSITY OF SHEFFIELD**
**DEPARTMENT OF PURE MATHEMATICS**
Holger Brenner

## PMA444 Real and complex analysis 2007: Exercise sheet 9

Please hand in on Monday, the 30th of April your solutions to the following questions.

**Exercise 9.1.** Let $\gamma$ be the path which runs counter-clockwise along the sides of the triangle given by $(0,0),(3,-2i),(3,2i)$. Compute the following integrals using either the Cauchy integral formula or the Cauchy integral theorem.

$$\int_\gamma \frac{e^z}{z-2}dz, \quad \int_\gamma \frac{e^z}{z+1}dz, \quad \int_\gamma \frac{e^{3z}}{z-2+i}dz, \quad \int_\gamma \frac{z\cos(z)}{z^2-4}dz \quad \text{(8 marks)} .$$

**Exercise 9.2.** Compute the residue of the differential form $f(z)g(z)dz$, where $f(z)=z^{-4}-3z^{-2}+2z^{-1}+3+2z$ and $g(z)=3z^{-3}-z^{-2}+z^{-1}+6-2z-3z^2+z^3$ (3 marks).

**Exercise 9.3.** Let $P_1,\ldots,P_k$ be different points in $\mathbb{C}$. Show that there is a surjective group homomorphism

$$\pi(\mathbb{C}-\{P_1,\ldots,P_k\}) \longrightarrow \mathbb{Z}^k$$

using differential forms and path integrals (compare Exercise 8.8). (6 marks).

**Exercise 9.4.** Consider the function $f:\mathbb{R}\longrightarrow\mathbb{R}$ defined by

$$f(x) := \begin{cases} \exp(-x^{-1}) \text{ for } x>0 \\ 0 \text{ for } x\leq 0 \end{cases} .$$

Show that for $x>0$ all derivatives of $f(x)$ exist and are of the form $g(x^{-1})\exp(-x^{-1})$ with some polynomial $g$. Deduce that $f$ is differentiable of arbitrary order and that $f^{(k)}(0)=0$ for all $k$. Conclude that this is not an analytic function at 0 (this also shows that the identity theorem does not hold for real-differentiable functions) (6 marks).

**Exercise 9.5.** Show that there exists a bijective real-differentiable mapping $\mathbb{R}^2\longrightarrow B=B(0,1)$ from the real plane to the unit disc. (The Theorem of Liouville will tell us that there does not exist a bijective complex-differentiable map; you may use Exercise 9.4) (8 marks).

THE UNIVERSITY OF SHEFFIELD
DEPARTMENT OF PURE MATHEMATICS
Holger Brenner

**PMA444 Real and complex analysis 2007: Exercise sheet 10**

Please hand in on Monday, the 7th of May your solutions to the following questions.

**Exercise 10.1.** Give a holomorphic bijection (with holomorphic inverse function) between the two open discs $B(P, r)$ and $B(Q, s)$ (4 marks).

**Exercise 10.2.** a) Give an expression for $\cos(x+iy)$ where only the real exponential and the real trigonometric functions occur (use $\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$ and Euler formula) (5 marks).

b) Check that the Cauchy-Riemann differential equations hold for your expression (4 marks).

c) Deduce from the identity principle that this is the only way to extend cos as a complex-differential function to $\mathbb{C}$ (3 marks).

**Exercise 10.3.** Compute the coefficients of the power series around 1 for the complex-differentiable function $z \mapsto \frac{1}{z}$ using Taylor's theorem 6.4.1. Recall that an expression of the form $\frac{1}{(u-a_1)^{k_1} \cdot (u-a_1)^{k_1}}$ can be expressed as a sum of fractions with easier denominators (this method is called "partial fractions") (10 marks).

**Exercise 10.4.** Compute the coefficients of the power series around 1 for the complex-differentiable function $z \mapsto \frac{1}{z}$ again using the geometric series (3 marks).

**Exercise 10.5.** Show that a non-discrete subset of a metric space has an accumulation point (3 marks).

**Exercise 10.6.** Let $f : G \longrightarrow \mathbb{C}$ be holomorphic, $G \subseteq \mathbb{C}$ open and connected. Assume that $|f(z)|$ is constant. Then $f$ itself is constant (3 marks).

**Exercise 10.7.** Let $p : Y \longrightarrow X$ be a covering. Show that $p$ is an open mapping (4 marks).

**Exercise 10.8.** Show that the exponential mapping $\exp : \mathbb{C} \longrightarrow \mathbb{C}^\times$ is surjective. Use that this mapping is a group homomorphism and the openness principle (4 marks).

**Exercise 10.9.** Deduce from the fundamental theorem of algebra that every polynomial $p \in \mathbb{C}[z]$ factors in a product of linear factors (4 marks).