Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 1

Warm-up-exercises

EXERCISE 1.1. Let A, B and C denote three sets. Prove the following identities.

(1) $A \cap \emptyset = \emptyset$,
(2) $A \cup \emptyset = A$,
(3) $A \cap B = B \cap A$,
(4) $A \cup B = B \cup A$,
(5) $A \cap (B \cap C) = (A \cap B) \cap C$,
(6) $A \cup (B \cup C) = (A \cup B) \cup C$,
(7) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(8) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
(9) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

EXERCISE 1.2. Prove the following (settheoretical versions of) syllogisms of Aristotle. Let A, B, C denote sets.

(1) Modus Barbara: $B \subseteq A$ and $C \subseteq B$ imply $C \subseteq A$.
(2) Modus Celarent: $B \cap A = \emptyset$ and $C \subseteq B$ imply $C \cap A = \emptyset$.
(3) Modus Darii: $B \subseteq A$ and $C \cap B \neq \emptyset$ imply $C \cap A \neq \emptyset$.
(4) Modus Ferio: $B \cap A = \emptyset$ and $C \cap B \neq \emptyset$ imply $C \not\subseteq A$.
(5) Modus Baroco: $B \subseteq A$ and $B \not\subseteq C$ imply $A \not\subseteq C$.

EXERCISE 1.3. Prove the following formulas by induction.

(1)
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2},$$
(2)
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6},$$
(3)
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

EXERCISE 1.4. Show that (with $n = 3$ being the only exception) the relation

$$2^n \geq n^2$$

holds.

EXERCISE 1.5. Show by induction that for every $n \in \mathbb{N}$ the number

$$6^{n+2} + 7^{2n+1}$$

is a multiple of 43.

EXERCISE 1.6. Prove by induction that the following inequality holds

$$1 \cdot 2^2 \cdot 3^3 \cdots n^n \leq n^{\frac{n(n+1)}{2}} .$$

EXERCISE 1.7. Prove by induction that the following formula holds for all $n \in \mathbb{N}_+$

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n+1} \frac{n(n+1)}{2} .$$

EXERCISE 1.8. The cities $S_1, \ldots, S_n$ are connected by roads and there is exactly one road between each couple of cities. Due to construction works at the moment all roads are drivable only in one direction. Show that nevertheless there exists one city from which you can reach all the others.

Hand-in-exercises

EXERCISE 1.9. (4 points)

Let A and B be two sets. Show that the following facts are equivalent.

(1) $A \subseteq B$,
(2) $A \cap B = A$,
(3) $A \cup B = B$,
(4) $A \setminus B = \emptyset$,
(5) There exist a set C such that $B = A \cup C$,
(6) There exist a set D such that $A = B \cap D$.

EXERCISE 1.10. (3 points)

Prove by induction that the sum of consecutive odd numbers (starting from 1) is always a square number.

EXERCISE 1.11. (3 points)

Fix $m \in \mathbb{N}$. Show by induction that the following identity holds.

$$(2m+1)\prod_{i=1}^{m}(2i-1)^2 = \prod_{k=1}^{m}(4k^2-1)\,.$$

EXERCISE 1.12. (4 points)

An n-chocolate is a rectangular grid, which is divided by $a-1$ longitudinal grooves and by $b-1$ transverse grooves into $n = a \cdot b$ ($a, b \in \mathbb{N}_+$) smaller bite-sized rectangles. A dividing step of a chocolate is the complete severing of a chocolate along a longitudinal or a transverse groove. A complete breakdown of a chocolate is a consequence of division steps (each one applied to a previously obtained intermediate chocolate), whose final product consists of all the small bite-sized pieces, more handy to be eaten. Show by induction that each breakdown of an n-chocolate consists of exactly $n-1$ division steps.