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Vector bundles, forcing algebras and local cohomology

Lecture 6

If R is a normal local domain of dimension 2 and $I = (f_1, \ldots, f_n)$ an $\mathfrak{m}$-primary ideal, then $f \in I^*$ (or inside the solid closure) if and only if $D(\mathfrak{m}) \subseteq \operatorname{Spec}(B)$ is an affine scheme, where B denotes the forcing algebra. Here we will discuss in general, with this application in mind, when a scheme is affine.

Affine schemes

A scheme U is called *affine* if it is isomorphic to the spectrum of some commutative ring R. If the scheme is of finite type over a field (or a ring) K (if we have a variety), then this is equivalent to saying that there exist global functions

$$g_1, \ldots, g_m \in \Gamma(U, \mathcal{O}_U)$$

such that the mapping

$$U \longrightarrow \mathbb{A}_K^m,\ x \longmapsto (g_1(x), \ldots, g_m(x)),$$

is a closed embedding. The relation to cohomology is given by the following well-known theorem of Serre.

Theorem 6.1. *Let U denote a noetherian scheme. Then the following properties are equivalent.*

1. *U is an affine scheme.*
2. *For every quasicoherent sheaf $\mathcal{F}$ on U and all $i \geq 1$ we have $H^i(U, \mathcal{F}) = 0$.*
3. *For every coherent ideal sheaf $\mathcal{I}$ on U we have $H^1(U, \mathcal{I}) = 0$.*

It is in general a difficult question whether a given scheme U is affine. For example, suppose that $X = \operatorname{Spec}(R)$ is an affine scheme and

$$U = D(\mathfrak{a}) \subseteq X$$

is an open subset (such schemes are called quasiaffine) defined by an ideal $\mathfrak{a} \subseteq R$. When is U itself affine? The cohomological criterion above simplifies to the condition that $H^i(U, \mathcal{O}_X) = 0$ for $i \geq 1$.

Of course, if $\mathfrak{a} = (f)$ is a principal ideal (or up to radical a principal ideal), then $U = D(f) \cong \operatorname{Spec}(R_f)$ is affine. On the other hand, if $(R, \mathfrak{m})$ is a local ring of dimension ≥ 2, then

$$D(\mathfrak{m}) \subset \operatorname{Spec}(R)$$

is not affine, since

$$H^{d-1}(U, \mathcal{O}_X) \; = \; H^d_{\mathfrak{m}}(R) \; \neq \; 0$$

by the relation between sheaf cohomology and local cohomology and a theorem of Grothendieck.

Codimension condition

One can show that for an open affine subset $U \subseteq X$ the closed complement $Y = X \setminus U$ must be of pure codimension one (U must be the complement of the support of an effective divisor). In a regular or (locally $\mathbb{Q}$)- factorial domain the complement of every divisor is affine, since the divisor can be described (at least locally geometrically) by one equation. But it is easy to give examples to show that this is not true for normal threedimensional domains. The following example is a standardexample for this phenomenon and is in fact given by a forcing algebra.

EXAMPLE 6.2. Let K be a field and consider the ring

$$R = K[x, y, u, v]/(xu - yv)\,.$$

The ideal $\mathfrak{p} = (x, y)$ is a prime ideal in R of height one. Hence the open subset $U = D(x, y)$ is the complement of an irreducible hypersurface. However, U is not affine. For this we consider the closed subscheme

$$\mathbb{A}^2_K \cong Z = V(u, v) \subseteq \operatorname{Spec}(R)$$

and

$$Z \cap U \subseteq U\,.$$

If U were affine, then also the closed subscheme $Z \cap U \cong \mathbb{A}^2_K \setminus \{(0, 0)\}$ would be affine, but this is not true, since the complement of the punctured plane has codimension 2.

Ring of global sections of affine schemes

LEMMA 6.3. *Let R be a noetherian ring and $U = D(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$ an open subset. Then the following hold.*

1. *U is an affine scheme if and only if $\mathfrak{a}\Gamma(U, \mathcal{O}_X) = (1)$.*
2. *If this holds, and $q_1 f_1 + \ldots + q_n f_n = 1$ with $\mathfrak{a} = (f_1, \ldots, f_n)$ and $q_i \in \Gamma(U, \mathcal{O}_X)$, then $\Gamma(U, \mathcal{O}_X) = R[q_1, \ldots, q_n]$. In particular, the ring of global sections over U is finitely generated over R.*

Proof. We only give a sketch. (1). There always exists a natural scheme morphism

$$U \longrightarrow \operatorname{Spec}(\Gamma(U, \mathcal{O}_X)),$$

and U is affine if and only if this morphism is an isomorphism. It is always an open embedding (because it is an isomorphism on the $D(f)$, $f \in \mathfrak{a}$), and the image is $D(\mathfrak{a}\Gamma(U, \mathcal{O}_X))$. This is everything if and only if the extended ideal is the unit ideal.

(2). We write $1 = q_1 f_1 + \ldots + q_n f_n$ and consider the natural morphism

$$U \longrightarrow \text{Spec}\,(R[q_1, \ldots, q_n])$$

corresponding to the ring inclusion $R[q_1, \ldots, q_n] \subseteq \Gamma(U, \mathcal{O}_X)$. This morphism is again an open embedding and its image is everything. $\square$

An application of this is the follwoing computation.

EXAMPLE 6.4. We consider the Fermat cubic $R = K[X, Y, Z]/(X^3+Y^3+Z^3)$, the ideal $I = (X, Y)$ and the element Z. We claim that for characteristic $\neq 3$ the element Z does not belong to the solid closure of I. Equivalently, the open subset

$$D(X, Y) \subseteq \text{Spec}\,(R[S, T]/(XS + YT + Z))$$

is affine. For this we show that the extended ideal inside the ring of global sections is the unit ideal. First of all we get the equation

$$X^3 + Y^3 \;=\; (XS + YT)^3 \;=\; X^3S^3 + 3X^2S^2YT + 3XSY^2T^2 + Y^3T^3$$

or, equivalently,

$$X^3\left(S^3 - 1\right) + 3X^2YS^2T + 3XY^2ST^2 + Y^3\left(T^3 - 1\right) \;=\; 0.$$

We write this as

$$\begin{aligned} X^3(S^3 - 1) &= -3X^2YS^2T - 3XY^2ST^2 - Y^3(T^3 - 1) \\ &= Y\left(-3X^2YS^2T - 3XY^2ST^2 - Y^3\left(T^3 - 1\right)\right), \end{aligned}$$

which yields on $D(X, Y)$ the rational function

$$Q \;=\; \frac{S^3 - 1}{Y} \;=\; \frac{-3X^2YS^2T - 3XY^2ST^2 - Y^3\left(T^3 - 1\right)}{X^3}.$$

This shows that $S^3 - 1 = QY$ belongs to the extended ideal. Similarly, one can show that also the other coefficients $3S^2T, 3ST^2, T^3 - 1$ belong to the extended ideal. Therefore in characteristic different from 3, the extended ideal is the unit ideal.

We will see later also examples where the ring of global sections is not finitely generated.

EXAMPLE 6.5. We consider the Fermat cubic $R = K[X, Y, Z]/(X^3 + Y^3 + Z^3)$, the ideal $I = (X, Y)$ and the element Z^2. We claim that in positive characteristic $\neq 3$ the element Z^2 does belong to the tight closure of I. Equivalently, the open subset

$$D(X, Y) \subseteq \text{Spec}\,(R[S, T]/(XS + YT + Z^2))$$

is not affine. The element Z^2 defines the cohomology class

$$c = \frac{Z^2}{XY} \in H^1(D(X, Y), \mathcal{O}_X)$$

and its Frobenius pull-backs are $F^{e*}(c) = \frac{Z^{2q}}{X^q Y^q} \in H^1(D(X,Y), \mathcal{O}_X)$. This cohomology module has a $\mathbb{Z}$-graded structure (the degree is given by the difference of the degree of the numerator and the degree of the denominator) and, moreover, it is 0 in positive degree (this is related to the fact that the corresponding projective curve is elliptic). Therefore for any homogeneous element $t \in R$ of positive degree we have $tF^{e*}(c) = 0$ and so Z^2 belongs to the tight closure.

From this it follows also that in characteristic 0 the element Z^2 belongs to the solid closure, because affineness is an open property in an arithmetic family.