

## Vector bundles, forcing algebras and local cohomology

### Lecture 3

#### Geometric vector bundles

We have seen that the fibers of the spectrum of a forcing algebra are (empty or) affine spaces. However, this is not only fiberwise true, but more general: If we localize the forcing algebra at  $f_i$  we get

$$(R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n - f))_{f_i} \cong R_{f_i}[T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n],$$

since we can write

$$T_i = - \sum_{j \neq i} \frac{f_j}{f_i} T_j + \frac{f}{f_i}.$$

So over every  $D(f_i)$  the spectrum of the forcing algebra is an  $(n - 1)$ -dimensional affine space over the base. So locally, restricted to  $D(f_i)$ , we have isomorphisms

$$T|_{D(f_i)} \cong D(f_i) \times \mathbb{A}^{n-1}.$$

On the intersections  $D(f_i) \cap D(f_j)$  we get two identifications with affine space, and the transition morphisms are linear if  $f = 0$ , but only affine-linear in general (because of the translation with  $\frac{f}{f_i}$ ).

So the forcing algebra has locally the form  $R_{f_i}[T_1, \dots, T_{i-1}, T_{i+1}, \dots, T_n]$  and its spectrum  $\text{Spec}(B)$  has locally the form  $D(f_i) \times \mathbb{A}_K^{n-1}$ . This description holds on the union  $U = \bigcup_{i=1}^n D(f_i)$ . Moreover, in the homogeneous case ( $f=0$ ) the transition mappings are linear. Hence  $V|_U$  is a geometric vector bundle according to the following definition.

DEFINITION 3.1. Let  $X$  denote a scheme. A scheme

$$p : V \longrightarrow X$$

is called a *geometric vector bundle* of rank  $r$  over  $X$  if there exists an open covering  $X = \bigcup_{i \in I} U_i$  and  $U_i$ -isomorphisms

$$\psi_i : U_i \times \mathbb{A}^r = \mathbb{A}_{U_i}^r \longrightarrow V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset  $U \subseteq U_i \cap U_j$  the transition mappings

$$\psi_j^{-1} \circ \psi_i : \mathbb{A}_{U_i}^r|_U \longrightarrow \mathbb{A}_{U_j}^r|_U$$

are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring  $\Gamma(U, \mathcal{O}_X)[T_1, \dots, T_r]$  given by  $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$ .

Here we can restrict always to affine open coverings. If  $X$  is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersection. The trivial bundle of rank  $r$  is the  $r$ -dimensional affine space  $\mathbb{A}_X^r$  over  $X$ , and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+ : (\mathbb{A}_U^r) \times_U (\mathbb{A}_U^r) \longrightarrow \mathbb{A}_U^r, (v_1, \dots, v_r, w_1, \dots, w_r) \longmapsto (v_1 + w_1, \dots, v_r + w_r),$$

and this carries over to a vector bundle, that is, we have an addition

$$\alpha : V \times_X V \longrightarrow V.$$

The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on  $V|_U$  coming from an isomorphism with some affine space over  $U$  is independent of the chosen isomorphism. For the same reason there is a unique closed subscheme of  $V$  called the *zero-section* which is locally defined to be  $0 \times U \subseteq \mathbb{A}_U^r$ . Also, the multiplication by a scalar, i.e. the mapping

$$\cdot : \mathbb{A}_U \times_U (\mathbb{A}_U^r) \longrightarrow \mathbb{A}_U^r, (s, v_1, \dots, v_r) \longmapsto (sv_1, \dots, sv_r),$$

carries over to a scalar multiplication

$$\cdot : \mathbb{A}_X \times_X V \longrightarrow V.$$

In particular, for every point  $P \in X$  the fiber  $V_P = V \times_X P$  is an affine space over  $\kappa(P)$ .

For a geometric vector bundle  $p : V \rightarrow X$  and an open subset  $U \subseteq X$  one sets

$$\Gamma(U, V) = \{s : U \rightarrow V|_U \mid p \circ s = \text{id}_U\},$$

so this is the set of sections in  $V$  over  $U$ . This gives in fact for every scheme over  $X$  a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

**DEFINITION 3.2.** A coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$  on a scheme  $X$  is called *locally free* of rank  $r$ , if there exists an open covering  $X = \bigcup_{i \in I} U_i$  and  $\mathcal{O}_{U_i}$ -module-isomorphisms  $\mathcal{F}|_{U_i} \cong \mathcal{O}^r|_{U_i}$  for every  $i \in I$ .

Vector bundles and locally free sheaves are essentially the same objects.

**THEOREM 3.3.** *Let  $X$  denote a scheme. Then the category of locally free sheaves on  $X$  and the category of geometric vector bundles on  $X$  are equivalent. A geometric vector bundle  $V \rightarrow X$  corresponds to the sheaf of its sections, and a locally free sheaf  $\mathcal{F}$  corresponds to the (relative) Spectrum of the symmetric algebra of the dual module  $\mathcal{F}^*$ .*

The free sheaf of rank  $r$  corresponds to the affine space  $\mathbb{A}_X^r$  over  $X$ .

### Torsors of vector bundles

We have seen that  $V = \text{Spec}(R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n))$  acts on the spectrum of a forcing algebra  $T = \text{Spec}(R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n + f))$  by addition. The restriction of  $V$  to  $U$  is a vector bundle, and  $T$  restricted to  $U$  becomes a  $V$ -torsor.

**DEFINITION 3.4.** Let  $V$  denote a geometric vector bundle over a scheme  $X$ . A scheme  $T \rightarrow X$  together with an action

$$\beta : V \times_X T \longrightarrow T$$

is called a geometric (Zariski)-*torsor* for  $V$  (or a  $V$ -*principal fiber bundle* or a *principal homogeneous space*) if there exists an open covering  $X = \bigcup_{i \in I} U_i$  and isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

such that the diagrams (we set  $U = U_i$  and  $\varphi = \varphi_i$ )

$$\begin{array}{ccc} V|_U \times_U T|_U & \xrightarrow{\beta} & T|_U \\ \downarrow & & \downarrow \\ V|_U \times_U V|_U & \xrightarrow{\alpha} & V|_U \end{array}$$

commute.

The torsors of vector bundles can be classified in the following way.

**PROPOSITION 3.5.** *Let  $X$  denote a Noetherian separated scheme and let*

$$p : V \longrightarrow X$$

*denote a geometric vector bundle on  $X$  with sheaf of sections  $\mathcal{S}$ . Then there exists a correspondence between first cohomology classes  $c \in H^1(X, \mathcal{S})$  and geometric  $V$ -torsors.*

*Proof.* We will describe this correspondence. Let  $T$  denote a  $V$ -torsor. Then there exists by definition an open covering  $X = \bigcup_{i \in I} U_i$  such that there exists isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

which are compatible with the action of  $V|_{U_i}$  on itself. The isomorphisms  $\varphi_i$  induce automorphisms

$$\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j}.$$

These automorphisms are compatible with the action of  $V$  on itself, and this means that they are of the form

$$\psi_{ij} = \text{Id}_V|_{U_i \cap U_j} + s_{ij}$$

with suitable sections  $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$ . This family defines a Čech-cocycle for the covering and gives therefore a cohomology class in  $H^1(X, \mathcal{S})$ . For the reverse direction, suppose that the cohomology class  $c \in H^1(X, \mathcal{S})$  is represented by a Čech-cocycle  $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$  for an open covering  $X = \bigcup_{i \in I} U_i$ . Set  $T_i := V|_{U_i}$ . We take the morphisms

$$\psi_{ij} : T_i|_{U_i \cap U_j} = V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$$

given by  $\psi_{ij} := \text{Id}_V|_{U_i \cap U_j} + s_{ij}$  to glue the  $T_i$  together to a scheme  $T$  over  $X$ . This is possible since the cocycle condition guarantees the glueing condition for schemes (EGA I, 0, 4.1.7). The action of  $T_i = V|_{U_i}$  on itself glues also together to give an action on  $T$ .  $\square$

It follows immediately that for an affine scheme (i.e. a scheme of type  $\text{Spec}(R)$ ) there are no non-trivial torsors for any vector bundle. There will however be in general many non-trivial torsors on the punctured spectrum (and on a projective variety).