

Forcing algebras and closure operations

Let R denote a commutative ring and let $I = (f_1, \dots, f_n)$ be an ideal. Let $f \in R$ and let

$$B = R[T_1, \dots, T_n]/(f_1T_1 + \dots + f_nT_n - f)$$

be the corresponding forcing algebra and

$$\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(R)$$

the corresponding spectrum morphism. How are properties of φ (or of the R -algebra B) related to certain ideal closure operations?

We start with some examples. The element f belongs to the ideal I if and only if we can write $f = r_1f_1 + \dots + r_nf_n$. By the universal property of the forcing algebra this means that there exists an R -algebra-homomorphism

$$B \longrightarrow R,$$

hence $f \in I$ holds if and only if φ admits a scheme section. This is also equivalent to

$$R \longrightarrow B$$

admitting an R -module section or B being a pure R -algebra (so for forcing algebras properties might be equivalent which are not equivalent for arbitrary algebras).

The radical of an ideal

Now we look at the radical of the ideal I ,

$$\text{rad}(I) = \{f \in R \mid f^k \in I \text{ for some } k\}.$$

The importance of the radical comes mainly from Hilbert's Nullstellensatz, saying that for algebras of finite type over an algebraically closed field there is a natural bijection between radical ideals and closed algebraic zero-sets. So geometrically one can see from an ideal only its radical. As this is quite a coarse closure operation we should expect that this corresponds to a quite coarse property of the morphism φ as well. Indeed, it is true that $f \in \text{rad}(I)$ if and only if φ is surjective. This is true since the radical of an ideal is the intersection of all prime ideals in which it is contained. Hence an element f belongs to the radical if and only if for all residue class homomorphisms

$$\varphi : R \longrightarrow \kappa(\mathfrak{p})$$

where I is sent to 0, also f is sent to 0. But this means for the forcing equation that whenever the equation degenerates to 0, then also the inhomogeneous part becomes zero, and so there will always be a solution to the inhomogeneous equation.

Exercise: Define the radical of a submodule inside a module.

Integral closure of an ideal

Another closure operation is integral closure. It is defined by

$$\bar{I} = \{f \in R \mid f^k + a_1 f^{k-1} + \dots + a_{k-1} f + a_k = 0 \text{ for some } k \text{ and } a_i \in I^i\}.$$

This notion is important for describing the normalization of the blow up of the ideal I . Another characterization is that there exists a $z \in R$, not contained in any minimal prime ideal of R , such that $z f^n \in I^n$ holds for all n . Another equivalent property - the valuative criterion - is that for all ring homomorphisms

$$\theta : R \longrightarrow D$$

to a discrete valuation domain D (assume that R is noetherian) the containment $\theta(f) \in \theta(I)D$ holds.

The characterization of the integral closure in terms of forcing algebras requires some notions from topology. A continuous map

$$\varphi : X \longrightarrow Y$$

between topological spaces X and Y is called a *submersion*, if it is surjective and if Y carries the image topology (quotient topology) under this map. This means that a subset $W \subseteq Y$ is open if and only if its preimage $\varphi^{-1}(W)$ is open. Since the spectrum of a ring endowed with the Zariski topology is a topological space, this notion can be applied to the spectrum morphism of a ring homomorphism. With this notion we can state that $f \in \bar{I}$ if and only if the forcing morphism

$$\varphi : \text{Spec}(B) \longrightarrow \text{Spec}(R)$$

is a universal submersion (universal means here that for any ring change $R \rightarrow R'$ to a noetherian ring R' , the resulting homomorphism $R' \rightarrow B'$ still has this property). The relation between these two notions stem from the fact that also for universal submersions there exists a criterion in terms of discrete valuation domains. For a morphism

$$Z \longrightarrow \text{Spec}(D)$$

(D a discrete valuation domain) to be a submersion means that above the only chain of prime ideals in $\text{Spec}(D)$, namely $(0) \subset \mathfrak{m}_D$, there exists a chain of prime ideals $\mathfrak{p}' \subseteq \mathfrak{q}'$ in Z lying over this chain.

So this property is stronger than lying over (which means surjective) but weaker than the going-down or going-up property (in the presence of surjectivity).

If we are dealing only with algebras of finite type over the complex numbers \mathbb{C} , then we may also consider the corresponding complex spaces with their natural topology induced from the euclidean topology of \mathbb{C}^n . Then universal submersive with respect to the Zariski topology is the same as submersive in the complex topology (the target space needs to be normal).

EXAMPLE 3.1. Let K be a field and consider $R = K[X]$. Since this is a principal ideal domain, the only interesting forcing algebras (if we are only interested in the local behavior around (X)) are of the form $K[X, T]/(X^n T - X^m)$. For $m \geq n$ this $K[X]$ -algebra admits a section (corresponding to the fact that $X^m \in (X^n)$), and if $n \geq 1$ there exists an affine line over the maximal ideal (X) . So now assume $m < n$. If $m = 0$, then we have a hyperbola mapping to an affine line, with the fiber over (X) being empty, corresponding to the fact that 1 does not belong to the radical of (X^n) for $n \geq 1$. So assume finally $1 \leq m < n$. Then X^m belongs to the radical of (X^n) , but not to its integral closure (which is the identical closure on a one-dimensional regular ring). We can write the forcing equation as $X^n T - X^m = X^m(X^{n-m} T - 1)$. So the spectrum of the forcing algebra consists of a (thickend) line over (X) and of a hyperbola. The forcing morphism is surjective, but it is not a submersion. For example, the preimage of (X) is a connected component hence open, but this single point is not open.

EXAMPLE 3.2. Let K be a field and let $R = K[X, Y]$ be the polynomial ring in two variables. We consider the ideal $I = (X^2, Y)$ and the element X . This element belongs to the radical of this ideal, hence the forcing morphism

$$\text{Spec}(K[X, Y, T_1, T_2]/(X^2 T_1 + Y T_2 + X)) \longrightarrow \text{Spec}(K[X, Y])$$

is surjective. We claim that it is not a submersion. For this we look at the reduction modulo Y . In $K[X, Y]/(Y) \cong K[X]$ the ideal becomes (X^2) which does not contain X . Hence by the valuative criterion for integral closure, X does not belong to the integral closure of the ideal. One can also say that the chain $V(X, Y) \subset V(Y)$ in the affine plane does not have a lift (as a chain) to the spectrum of the forcing algebra.

For the ideal $I = (X^2, Y^2)$ and the element XY the situation looks different. Let

$$\theta : K[X, Y] \longrightarrow D$$

be a ring homomorphism to a discrete valuation domain D . If X or Y is mapped to 0, then also XY is mapped to 0 and hence belongs to the extendend ideal. So assume that $\theta(X) = u\pi^r$ and $\theta(Y) = v\pi^s$, where π is a local parameter of D and u and v are units. Then $\theta(XY) = uv\pi^{r+s}$ and the exponent is at least the minimum of $2r$ and $2s$, hence $\theta(XY) \in (\pi^{2r}, \pi^{2s}) = (\theta(X^2), \theta(Y^2))D$. Hence XY belongs to the integral closure of (X^2, Y^2) and the forcing morphism

$$\text{Spec}(K[X, Y, T_1, T_2]/(X^2 T_1 + Y^2 T_2 + XY)) \longrightarrow \text{Spec}(K[X, Y])$$

is a universal submersion.

Continuous closure

Suppose now that $R = \mathbb{C}[X_1, \dots, X_k]$. Then every polynomial $f \in R$ can be considered as a continuous function

$$f : \mathbb{C}^k \longrightarrow \mathbb{C}, (x_1, \dots, x_k) \longmapsto f(x_1, \dots, x_k)$$

in the complex topology. If $I = (f_1, \dots, f_n)$ is an ideal and $f \in R$ is an element, we say that f belongs to the *continuous closure* of I , if there exist continuous functions

$$g_1, \dots, g_n : \mathbb{C}^k \longrightarrow \mathbb{C}$$

such that

$$f = \sum_{i=1}^n g_i f_i$$

(identity of functions) (the same definition works for \mathbb{C} -algebras of finite type).

It is not at all clear at once that there may exist polynomials $f \notin I$ but inside the continuous closure of I . For $\mathbb{C}[X]$ it is easy to show that the continuous closure is (like the integral closure) just the ideal itself. We also remark that when we would only allow holomorphic functions g_1, \dots, g_n then we could not get something larger. However, with continuous functions we can for example write

$$X^2 Y^2 = g_1 X^3 + g_2 Y^3.$$

Continuous closure is always inside the integral closure and hence also inside the radical. The element XY does not belong to the continuous closure of (X^2, Y^2) , though it belongs to the integral closure of I . In terms of forcing algebras, an element f belongs to the continuous closure if and only if the complex forcing mapping

$$\varphi_{\mathbb{C}} : \text{Spec}(B)_{\mathbb{C}} \longrightarrow \text{Spec}(R)_{\mathbb{C}}$$

(between the corresponding complex spaces) admits a continuous section.

The closure operations we have considered so far can be characterized by some property of the forcing algebra. However, they can not be characterized by a property of the corresponding torsor alone. For example, for $R = K[X, Y]$, we may write

$$\frac{1}{XY} = \frac{X}{X^2 Y} = \frac{XY}{X^2 Y^2} = \frac{X^2 Y^2}{X^3 Y^3},$$

so the torsors given by the forcing algebras

$$R[T_1, T_2]/(XT_1 + YT_2 + 1), \quad R[T_1, T_2]/(X^2 T_1 + YT_2 + X),$$

$$R[T_1, T_2]/(X^2 T_1 + Y^2 T_2 + XY) \quad \text{and} \quad R[T_1, T_2]/(X^3 T_1 + Y^3 T_2 + X^2 Y^2)$$

are all the same (the restriction over $D(X, Y)$), but their global properties are quite different. We have a non-surjection, a surjective non submersion, a submersion which does not admit (for $K = \mathbb{C}$) a continuous section and a map which admits a continuous section.