Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 8

Warm-up-exercises

EXERCISE 8.1. Let K be a field and let V be a K-vector space of dimension $n = \dim(V)$. Suppose that n vectors $v_1, \ldots, v_n$ in V are given. Prove that the following facts are equivalent.

(1) $v_1, \ldots, v_n$ form a basis for V.
(2) $v_1, \ldots, v_n$ form a system of generators for V.
(3) $v_1, \ldots, v_n$ are linearly independent.

EXERCISE 8.2. Let K be a field and let $K[X]$ denote the polynomial ring over K. Let $d \in \mathbb{N}$. Show that the set of all polynomials of degree $\leq d$ is a finite dimensional subspace of $K[X]$. What is its dimension?

EXERCISE 8.3. Show that the set of real polynomials of degree ≤ 4 which have a zero at -2 and a zero at 3 is a finite dimensional subspace of $\mathbb{R}[X]$. Determine the dimension of this vector space.

EXERCISE 8.4. Let K be a field and let V and W be two finite-dimensional K-vector spaces with $\dim(V) = n$ and $\dim(W) = m$. What is the dimension of the Cartesian product $V \times W$?

EXERCISE 8.5. Let V be a finite-dimensional vector space over the complex numbers, and let $v_1, \ldots, v_n$ be a basis of V. Prove that the family of vectors

$$v_1, \ldots, v_n \text{ and } iv_1, \ldots, iv_n$$

form a basis for V, considered as a real vector space.

EXERCISE 8.6. Consider the standard basis e_1, e_2, e_3, e_4 in $\mathbb{R}^4$ and the three vectors

$$\begin{pmatrix} 1 \\ 3 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 5 \\ 7 \end{pmatrix} \text{ and } \begin{pmatrix} -4 \\ 9 \\ -5 \\ 1 \end{pmatrix}.$$

Prove that these vectors are linearly independent and extend them to a basis by adding an appropriate standard vector as shown in Theorem 8.2. Can one take any standard vector?

EXERCISE 8.7. Determine the transformation matrices $M^{\mathfrak{u}}_{\mathfrak{v}}$ and $M^{\mathfrak{v}}_{\mathfrak{u}}$ for the standard basis $\mathfrak{u}$ and the basis $\mathfrak{v}$ in $\mathbb{R}^4$ which is given by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

EXERCISE 8.8. Determine the transformation matrices $M^{\mathfrak{u}}_{\mathfrak{v}}$ and $M^{\mathfrak{v}}_{\mathfrak{u}}$ for the standard basis $\mathfrak{u}$ and the basis $\mathfrak{v}$ of $\mathbb{C}^2$ which is given by the vectors

$$v_1 = \begin{pmatrix} 3+5i \\ 1-i \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 2+3i \\ 4+i \end{pmatrix},$$

EXERCISE 8.9. We consider the families of vectors

$$\mathfrak{v} = \begin{pmatrix} 7 \\ -4 \end{pmatrix}, \begin{pmatrix} 8 \\ 1 \end{pmatrix} \text{ and } \mathfrak{u} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

in $\mathbb{R}^2$.

a) Show that $\mathfrak{v}$ and $\mathfrak{u}$ are both a basis of $\mathbb{R}^2$.

b) Let $P \in \mathbb{R}^2$ denote the point which has the coordinates $(-2, 5)$ with respect to the basis $\mathfrak{v}$. What are the coordinates of this point with respect to the basis $\mathfrak{u}$?

c) Determine the transformation matrix which describes the change of basis from $\mathfrak{v}$ to $\mathfrak{u}$.

Hand-in-exercises

EXERCISE 8.10. (4 points)

Show that the set of all real polynomials of degree ≤ 6 which have a zero at -1, at 0 and at 1 is a finite dimensional subspace of $\mathbb{R}[X]$. Determine the dimension of this vector space.

EXERCISE 8.11. (3 points)

Let K be a field and let V be a K-vector space. Let $v_1, \ldots, v_m$ be a family of vectors in V and let

$$U = \langle v_i,\ i = 1, \ldots, m \rangle$$

be the subspace they span. Prove that the family is linearly independent if and only if the dimension of U is exactly m.

Exercise 8.12. (4 points)

Determine the transformation matrices $M^{\mathfrak{u}}_{\mathfrak{v}}$ and $M^{\mathfrak{v}}_{\mathfrak{u}}$ for the standard basis $\mathfrak{u}$ and the basis $\mathfrak{v}$ of $\mathbb{R}^3$ which is given by the vectors

$$v_1 = \begin{pmatrix} 4 \\ 5 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 3 \\ -8 \end{pmatrix} \text{ and } v_3 = \begin{pmatrix} 5 \\ 7 \\ -3 \end{pmatrix}$$

Exercise 8.13. (6 points)

We consider the families of vectors

$$\mathfrak{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix} \text{ and } \mathfrak{u} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ -2 \end{pmatrix}$$

in $\mathbb{R}^3$.

a) Show that $\mathfrak{v}$ and $\mathfrak{u}$ are both a basis of $\mathbb{R}^3$.

b) Let $P \in \mathbb{R}^3$ denote the point which has the coordinates $(2, 5, 4)$ with respect to the basis $\mathfrak{v}$. What are the coordinates of this point with respect to the basis $\mathfrak{u}$?

c) Determine the transformation matrix which describes the change of basis from $\mathfrak{v}$ to $\mathfrak{u}$.