Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 22

Warm-up-exercises

Exercise 22.1. Determine the Taylor polynomial of degree 4 of the function

$$\mathbb{R} \longrightarrow \mathbb{R},\, x \longmapsto \sin x \ \cos x\,,$$

at the zero point.

Exercise 22.2. Determine all the Taylor polynomials of the function

$$f(x) = x^4 - 2x^3 + 2x^2 - 3x + 5$$

at the point $a = 3$.

Exercise 22.3. Let $\sum_{n=0}^{\infty} c_n(x-a)^n$ be a convergent power series. Determine the derivative $f^{(k)}(a)$.

Exercise 22.4. Let $p \in \mathbb{R}[Y]$ be a polynomial and

$$g : \mathbb{R}_+ \longrightarrow \mathbb{R},\, x \longmapsto g(x) = p(\frac{1}{x})e^{-\frac{1}{x}}.$$

Prove that the derivative $g'(x)$ has also the shape

$$g'(x) = q(\frac{1}{x})e^{-\frac{1}{x}}\,,$$

where q is a polynomial.

Exercise 22.5. We consider the function

$$f : \mathbb{R}_+ \longrightarrow \mathbb{R},\, x \longmapsto f(x) = e^{-\frac{1}{x}}.$$

Prove that for all $n \in \mathbb{N}$ the n-th derivative $f^{(n)}$ satisfies the following property

$$\lim_{x \in \mathbb{R}_+,\, x \to 0} f^{(n)}(x) = 0\,.$$

Exercise 22.6. Determine the Taylor series of the function $f(x) = \frac{1}{x}$ at point $a = 2$ up to order 4 (Give also the Taylor polynomial of degree 4 at point 2, where the coefficients must be stated in the most simple form).

Exercise 22.7. Determine the Taylor polynomial of degree 3 of the function

$$f(x) = x \cdot \sin x$$

at point $a = \frac{\pi}{2}$.

Exercise 22.8. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto f(x),$$

be a differentiable function with the property

$$f' = f \text{ and } f(0) = 1\,.$$

Prove that $f(x) = \exp x$ for all $x \in \mathbb{R}$.

Exercise 22.9. Determine the Taylor polynomial up to fourth order of the inverse of the sine function at the point 0 with the power series approach described in Remark 22.8.

Hand-in-exercises

Exercise 22.10. (4 points)

Find the Taylor polynomials in 0 up to degree 4 of the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto \sin(\cos x) + x^3 \exp(x^2).$$

Exercise 22.11. (4 points)

Discuss the behavior of the function

$$f : [0, 2\pi] \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \sin x \ \cos x\,,$$

concerning zeros, growth behavior, (local) extrema. Sketch the graph of the function.

Exercise 22.12. (4 points)

Discuss the behavior of the function

$$f : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow \mathbb{R},\ x \longmapsto f(x) = \sin^3 x - \frac{1}{4} \sin x\,,$$

concerning zeros, growth behavior, (local) extrema. Sketch the graph of the function.

Exercise 22.13. (4 points)

Determine the Taylor polynomial up to fourth order of the natural logarithm at point 1 with the power series approach described in Remark 22.8 from the power series of the exponential function.

Exercise 22.14. (8 points)

For $n \geq 3$ let A_n be the area of a circle inscribed in the unit regular n-gon. Prove that $A_n \leq A_{n+1}$.