Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 14

Warm-up-exercises

Exercise 14.1. Let x be a real number, $x \neq 1$. Prove that for $n \in \mathbb{N}$ the relation

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1}$$

holds.

Exercise 14.2. Compute the series

$$\sum_{n=0}^{\infty} \left(\frac{1}{5}\right)^n = 1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \dots .$$

Exercise 14.3. Prove that the two series

$$\sum_{k=0}^{\infty} \frac{1}{2k+1} \text{ and } \sum_{k=0}^{\infty} \frac{1}{2k+2}$$

are divergent.

Exercise 14.4. Let $a, b \in \mathbb{R}_+$. Prove that the series

$$\sum_{k=0}^{\infty} \frac{1}{ak+b}$$

diverges.

Exercise 14.5. Prove that the series

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

diverges.

Exercise 14.6. Prove the Cauchy criterion for series of real numbers.

EXERCISE 14.7. Show that in a real sequence if you change finitely many sequence elements then neither the convergence nor the limit changes, and that in a series if you change a finite number of series terms then the convergence does not change, but the sum changes.

EXERCISE 14.8. Let

$$\sum_{k=0}^{\infty} a_k \text{ and } \sum_{k=0}^{\infty} b_k$$

be convergent series of real numbers with sums s and t. Prove the following statements.

(1) The series $\sum_{k=0}^{\infty} c_k$ with $c_k = a_k + b_k$ is convergent with sum equal to $s + t$.
(2) For $r \in \mathbb{R}$ the series $\sum_{k=0}^{\infty} d_k$ with $d_k = ra_k$ is also convergent with sum equal to rs.

EXERCISE 14.9. In a shared flat for students Student 1 prepares coffee and he puts the amount x_1 of coffee in the coffee filter. Then Student 2 looks horrified and says: "Do you want us all to be already completely awake?" and he takes the amount of coffee $x_2 < x_1$ back out of the filter. Then Student 3 comes and says: "Am I in a flat of sissies?" and he puts back an amount of coffee $x_3 < x_2$ in it. So it goes on indefinitely, alternating between putting in and taking out smaller amounts of coffee from the coffee filter. How can one characterize if the amount of coffee in the filter converges?

EXERCISE 14.10. Since on the previous day the coffee has become too weak for the supporters of a strong coffee, they develop a new strategy: they want to get up early, so that at the beginning of the day and between every two supporters of a weak coffee (who take out coffee from the coffee machine) there are always two supporters of a strong coffee putting in coffee. The amount of coffee that each person puts in or takes out does not change, and also the order in both camps does not change. Can they make the coffee stronger with this strategy?

EXERCISE 14.11. Two people, A and B, are in a pub. A wants to go home, but B still wants to drink a beer. "Well, we just drink another beer, but this is the very last" says A. Then B wants another beer, but since the previous beer was definitely the last one, they agree to drink a last half beer. After that they drink a last quarter of a beer, and then the last eighth of beer, and so on. How many "very last beer" do they drink overall?

Exercise 14.12. Let $k \geq 2$. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

is convergent.

Exercise 14.13. Prove the following *Comparison test.*

Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be two series of non-negative real numbers. The series $\sum_{k=0}^{\infty} b_k$ is divergent and moreover we have $a_k \geq b_k$ for all $k \in \mathbb{N}$. Then the series $\sum_{k=0}^{\infty} a_k$ is also divergent.

Hand-in-exercises

Exercise 14.14. (2 points)

Let $z \in \mathbb{R}$, $|z| < 1$. Determine and prove a formula for the series

$$\sum_{k=0}^{\infty} (-1)^k z^k .$$

Exercise 14.15. (3 points)

Compute the sum

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n .$$

Exercise 14.16. (4 points)

Let $g \in \mathbb{N}$, $g \geq 2$. A *sequence of digits*, given by

$$z_i \in \{0, 1, \ldots, g-1\} \text{ for } i \in \mathbb{Z},\ i \leq k ,$$

(where $k \in \mathbb{N}$) defines a real series[1]

$$\sum_{i=k}^{-\infty} z_i g^i .$$

Prove that such a series converges to a unique non-negative real number.

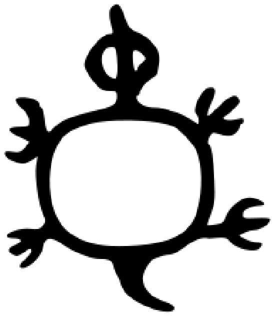
Exercise 14.17. (6 points)

Prove that the series

$$\sum_{n=0}^{\infty} \frac{n^2}{2^n}$$

converges.

[1] So here the index runs in the opposite direction.



Exercise 14.18. (5 points)

The situation in the turtle paradox of Zenon of Elea is the following: a slow turtle (with speed $v > 0$) has a starting point $s > 0$ compared with the faster Achilles (with speed $w > v$ and starting point 0). They start at the same time. Achilles can not catch the tortoise: when he arrives at the starting point of the tortoise $s_0 = s$, the turtle is not there anymore, but a little further, let's say at the point $s_1 > s_0$. When Achilles arrives at the point s_1, the turtle is once again a bit further at the point $s_2 > s_1$, and so on.

Calculate the elements of the sequence s_n, the associated time points t_n, and the respective limits. Compare these limits with the distance point where Achilles catches the turtle (you can calculate it directly from the given data).

Abbildungsverzeichnis