Prof. Dr. H. Brenner — Osnabrück WS 2011/2012

Mathematics for natural sciences I

Exercise sheet 15

Warm-up-exercises

Exercise 15.1. Show that a linear function

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto ax,$$

is continuous.

Exercise 15.2. Prove that the function

$$\mathbb{R} \longrightarrow \mathbb{R},\ x \longmapsto |x|,$$

is continuous.

Exercise 15.3. Prove that the function

$$\mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0},\ x \longmapsto \sqrt{x},$$

is continuous.

Exercise 15.4. Let $T \subseteq \mathbb{R}$ be a subset and let

$$f : T \longrightarrow \mathbb{R}$$

be a continuous function. Let $x \in T$ be a point such that $f(x) > 0$. Prove that $f(y) > 0$ for all y in a non-empty open interval $]x - a, x + a[$.

Exercise 15.5. Let $a < b < c$ be real numbers and let

$$f : [a, b] \longrightarrow \mathbb{R}$$

and

$$g : [b, c] \longrightarrow \mathbb{R}$$

be continuous functions such that $f(b) = g(b)$. Prove that the function

$$h : [a, c] \longrightarrow \mathbb{R}$$

such that

$$h(t) = f(t) \text{ for } t \leq b \text{ and } h(t) = g(t) \text{ for } t > b$$

is also continuous.

EXERCISE 15.6. Compute the limit of the sequence

$$x_n = 5\left(\frac{2n+1}{n}\right)^3 - 4\left(\frac{2n+1}{n}\right)^2 + 2\left(\frac{2n+1}{n}\right) - 3$$

for $n \to \infty$.

EXERCISE 15.7. Let

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

be a continuous function which takes only finitely many values. Prove that f is constant.

EXERCISE 15.8. Give an example of a continuuos function

$$f : \mathbb{Q} \longrightarrow \mathbb{R},$$

which takes exactly two values.

EXERCISE 15.9. Prove that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

defined by

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}\,, \\ 0, & \text{otherwise}\,, \end{cases}$$

is only at the zero point 0 continuous.

EXERCISE 15.10. Let $T \subseteq \mathbb{R}$ be a subset and let $a \in \mathbb{R}$ be a point. Let $f : T \to \mathbb{R}$ be a function and $b \in \mathbb{R}$. Prove that the following statements are equivalent.

(1) We have
$$\lim_{x \to a} f(x) = b\,.$$
(2) For all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in T$ with $d(x, a) \leq \delta$ the inequality $d(f(x), b) \leq \epsilon$ holds.

Hand-in-exercises

EXERCISE 15.11. (4 points)

We consider the function

$$f(x) = \begin{cases} 1 \text{ for } x \leq -1 \\ x^2 \text{ for } -1 < x < 2 \\ -2x + 7 \text{ for } x \geq 2\,. \end{cases}$$

Determine the points $x \in \mathbb{R}$ where f is continuous.

Exercise 15.12. (4 points)

Compute the limit of the sequence

$$b_n = 2a_n^4 - 6a_n^3 + a_n^2 - 5a_n + 3\,,$$

where

$$a_n = \frac{3n^3 - 5n^2 + 7}{4n^3 + 2n - 1}\,.$$

Exercise 15.13. (3 points)

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

is for no point $x \in \mathbb{R}$ continuous.

Exercise 15.14. (3 points)

Decide whether the sequence

$$a_n = \sqrt{n+1} - \sqrt{n}$$

converges and in case determine the limit.

Exercise 15.15. (4 points)

Determine the limit of the rational function

$$\frac{2x^3 + 3x^2 - 1}{x^3 - x^2 + x + 3}$$

at the point $a = -1$.