Local cohomology and ideal closure operations I

In these two talks I want to discuss three topics related to local cohomology: the affineness of (quasi-affine) schemes, the relation of local cohomology to closure operations, in particular tight closure, and the behaviour of local cohomology (and cohomological dimension) in an (arithmetic or geometric) deformation.

Affine schemes

A scheme U is called *affine* if it is isomorphic to the spectrum of some commutative ring R. If the scheme is of finite type (if we have a variety), then this is equivalent to saying that there exists global functions

$$g_1, \ldots, g_k \in \Gamma(U, \mathcal{O}_U)$$

such that the mapping

$$U \longrightarrow \mathbb{A}^k,\ x \longmapsto (g_1(x), \ldots, g_k(x)),$$

is a closed embedding. The relation to cohomology is given by the following well-known theorem of Serre.

THEOREM 1.1. *Let U denote a noetherian scheme. Then the following properties are equivalent.*

(1) *U is an affine scheme.*
(2) *For every quasicoherent sheaf $\mathcal{F}$ on U and all $i \geq 1$ we have $H^i(U, \mathcal{F}) = 0$.*
(3) *For every coherent ideal sheaf $\mathcal{I}$ on U we have $H^1(U, \mathcal{I}) = 0$.*

It is in general a difficult question whether a given scheme U is affine. For example, suppose that $X = \operatorname{Spec}(R)$ is an affine scheme and

$$U = D(\mathfrak{a}) \subseteq X$$

is an open subset (such schemes are called quasiaffine) defined by an ideal $\mathfrak{a} \subseteq R$. When is U itself affine? The cohomological criterion above simplifies to the condition that $H^1(U, \mathcal{O}_X) = 0$.

Of course, if $\mathfrak{a} = (f)$ is a principal ideal (or up to radical a principal ideal), then $U = D(f) \cong \operatorname{Spec}(R_f)$ is affine. On the other hand, if $(R, \mathfrak{m})$ is a local ring of dimension ≥ 2, then

$$D(\mathfrak{m}) \subset \operatorname{Spec}(R)$$

is not affine, since

$$H^{d-1}(U, \mathcal{O}_X) = H^d_{\mathfrak{m}}(R)$$

by the relation between sheaf cohomology and local cohomology and a theorem of Grothendieck. A variant of this observation shows that for an open affine subset $U \subseteq X$ the closed complement $Y = X \setminus U$ must be of pure codimension one (U must be the complement of the support of an effective divisor). In a regular or (locally $\mathbb{Q}$)- factorial domain the complement of every divisor is affine, since the divisor can be described (at least locally geometrically) by one equation. But it is easy to give examples to show that this is not true for normal threedimensional domains.

Example 1.2. Let K be a field and consider the ring

$$R = K[x, y, u, v]/(xu - yv)\,.$$

The ideal $\mathfrak{p} = (x, y)$ is a prime ideal in R of height one. Hence the open subset $U = D(x, y)$ is the complement of an irreducible hypersurface. However, U is not affine. For this we consider the closed subscheme

$$\mathbb{A}^2_K \cong Z = V(u, v) \subseteq \operatorname{Spec}(R)$$

and

$$Z \cap U \subseteq U\,.$$

If U were affine, then also the closed subscheme $Z \cap U \cong \mathbb{A}^2_K \setminus \{(0, 0)\}$ would be affine, but this is not true, since the complement of the punctured plane has codimension 2.

The argument employed in this example rests on the following definition and the next theorem.

Definition 1.3. Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal. The (noetherian) *superheight* is the supremum

$$\sup\,(\operatorname{ht}(IS) : \, S \text{ is a notherian } R-\text{algebra})\,.$$

Theorem 1.4. *Let R be a noetherian commutative ring and let $I \subseteq R$ be an ideal and $U = D(I) \subseteq X = \operatorname{Spec}(R)$. Then the following are equivalent.*

(1) *U is an affine scheme.*
(2) *I has superheight ≤ 1 and $\Gamma(U, \mathcal{O}_X)$ is a finitely generated R-algebra.*

It is not true at all that the ring of global sections of an open subset U of the spectrum X of a noetherian ring is of finite type over this ring. This is not even true if X is an affine variety. This problem is directly related to Hilbert's fourteenth problem, which has a negative answer. We will later present examples where U has superheight one, yet is not affine, hence its ring of global sections is not finitely generated.

Forcing algebras and their torsors

We want to deal now with a very special class of open subsets and ask whether they are affine or not and what their cohomological dimension is. Though it is in some sense a very special class it exhibits already a very rich behaviour. These open subsets are given by so-called forcing equations and forcing algebras.

Definition 1.5. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

This algebra was introduced by Hochster. The forcing algebra forces that f belongs to the extended ideal $(f_1, \ldots, f_n)B$. It yields a scheme morphism

$$\varphi : \operatorname{Spec} B \longrightarrow X = \operatorname{Spec} R.$$

We are interested in the relationship:

How is f related to I?

Does f belong to certain closure operations of I? $\Longleftrightarrow$
Properties of φ.

Examples

$$f \in I \Longleftrightarrow \varphi \text{ has a scheme-section}\,.$$

$$f \in \operatorname{rad} I \Longleftrightarrow \varphi \text{ is surjective}\,.$$

$$f \in \overline{I} \text{ (integral closure)} \Longleftrightarrow \varphi \text{ is a universal submersion}\,.$$

Tight closure

We want to deal with tight closure, a closure operation introduced by Hochster and Huneke.

Let R be a noetherian domain of positive characteristic, let

$$F : R \longrightarrow R,\ f \longmapsto f^p,$$

be the *Frobenius homomorphism*, and

$$F^e : R \longrightarrow R,\ f \longmapsto f^q,\ q = p^e\,,$$

its eth iteration. Let I be an ideal and set

$$I^{[q]} = \text{ extended ideal of } I \text{ under } F^e$$

Then define the *tight closure* of I to be the ideal

$$I^* := \{f \in R : \text{ there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e\}\,.$$

The relation between tight closure and forcing algebras is given in the following theorem.

THEOREM 1.6. *Let R be a normal excellent local domain with maximal ideal $\mathfrak{m}$ over a field of positive characteristic. Let $f_1, \ldots, f_n$ generate an $\mathfrak{m}$-primary ideal I and let f be another element in R. Then $f \in I^*$ if and only if*

$$H^{\dim(R)}_{\mathfrak{m}}(B) \neq 0\,,$$

where $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ denotes the forcing algebra of these elements.

If the dimension d is at least two, then

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}B}(B) \cong H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B)\,.$$

This means that we have to look at the cohomological properties of the complement of the exceptional fiber over the closed point. Such an open subset is called a torsor. If the dimension is two, then we have to look whether the first cohomology of the structure sheaf vanishes. This is true if and only if the open subset $D(\mathfrak{m}B)$ is an *affine scheme* (the spectrum of a ring).

The right hand side of this equivalence - the non-vanishing of the top-dimensional local cohomology - is independent of any characteristic assumption, and the basis for solid closure.

It is a fact that tight closure is difficult to compute. Since tight closure can be formulated with local cohomology, it follows that it must be quite difficult to give a general criterion for vanishing of local cohomology.

An important property of tight closure is that it is trivial for regular rings, i.e. $I^* = I$ for every ideal I. This implies the following cohomological property.

COROLLARY 1.7. Let $(R, \mathfrak{m})$ denote a regular local ring of dimension d and of positive characteristic, let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$. Let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ be the corresponding forcing algebra. Then for the extended ideal $\mathfrak{m}B$ we have

$$H^d_{\mathfrak{m}B}(B) \;=\; H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) \;=\; 0.$$

Proof. This follows from Fakt and $f \notin I^*$. □

In dimension two this is true in every (even mixed) characteristic.

THEOREM 1.8. *Let $(R, \mathfrak{m})$ denote a two-dimensional regular local ring, let $I = (f_1, \ldots, f_n)$ be an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$. Let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f)$ be the corresponding forcing algebra. Then for the extended ideal $\mathfrak{m}B$ we have*

$$H^2_{\mathfrak{m}B}(B) \;=\; H^1(D(\mathfrak{m}B), \mathcal{O}_B) \;=\; 0.$$

In particular, the open subset zusatz$1D(\mathfrak{m}B)$ is an affine scheme if and only if $f \notin I$.

The example above, the equation $xu+vy=0$ can be considered as the forcing algebra for the ideal (x,y) and the element $0 \in (x,y)$. The non-affineness of $D(x,y)$ corresponds to this containment.

We will continue in the next lecture with a detailed study of the situation of a two-dimensional graded base ring.

In higher dimension in characteristic zero it is not true that a regular ring is solidly closed, as was shown by the following example of Paul Roberts.

EXAMPLE 1.9. Let K be a field of characteristic 0 and let

$$B = K[X,Y,Z][U,V,W]/(X^2U + Y^2V + Z^2W - X^3Y^3Z^3)\,.$$

Then the ideal $\mathfrak{a} = (X,Y,Z)B$ has the property that $H^3_{\mathfrak{a}}(B) \neq 0$. This means that in $R = K[X,Y,Z]$ the element $X^3Y^3Z^3$ belongs to the solid closure of the ideal (X^2,Y^2,Z^2), and hence the threedimensional polynomial ring is not solidly closed.

This example uses a forcing equation of a special type: For parameters $x_1,\ldots,x_d$ in a d-dimensional local ring R and some $s \in \mathbb{N}$ one considers the forcing algebra given by

$$(x_1 \cdots x_d)^s = x_1^{s+1}T_1 + \ldots + x_d^{s+1}T_{d+1}\,.$$

The monomial conjecture states that this equation does not have a solution in R. It is open only in mixed characteristic. The equation expresses that the Cech cohomology class $\frac{1}{x_1\cdots x_n}$ is mapped to 0 in the forcing algebra. Robert's computation shows that this does not imply that the complete local cohomology module vanishes. Therefore solid closure is not a characteristic-free replacement for tight closure. There is a variant, called parasolid closure, which is characteristic free and has all the properties of tight closure (over a field). A detailed understanding of the top-dimensional local cohomology of the torsors given by the forcing algebras for these special equations could solve the monomial conjecture.

Plus closure

The above mentioned (finite) superheight condition is also related to another closure operation, the plus closure.

For an ideal $I \subseteq R$ in a domain R define

$I^+ = \{f \in R : \text{ there exists a finite domain extension } R \subseteq T \text{ such that } f \in IT\}\,.$

Equivalent: let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered by Artin). Then

$$f \in I^+ \text{ if and only if } f \in IR^+\,.$$

The plus closure commutes with localization.

We also have the inclusion $I^+ \subseteq I^*$. Here the question arises:

Question: Is $I^+ = I^*$?

This question is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure means that there exists a d-dimensional closed subscheme inside the torsor which meet the exceptional fiber (the fiber over the maximal ideal) in one point, and this means that the superheight of the extended ideal is d. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension.

Remark 1.10. In characteristic zero, the plus closure behaves very differently compared with positive characteristic. If R is a normal domain of characteristic 0, then the trace map shows that the plus closure is trivial, $I^+ = I$ for every ideal I. This implies also that if R is a twodimensional normal local ring of characteristic 0 and I an $\mathfrak{m}$-primary ideal and $f \in R$ an element with $f \notin I$, then the extendend ideal $\mathfrak{m}B$ inside the forcing algebra B has superheight 1. If moreover f belongs to the solid closure of I, then $D(\mathfrak{m}B)$ is not affine and so by Fakt its ring of global sections is not finitely generated.

Example 1.11. Let K be a field and consider the Fermat ring

$$R = K[X,Y,Z]/(X^d + Y^d + Z^d)$$

together with the ideal $I = (X,Y)$ and $f = Z^2$. For $d \geq 3$ we have $Z^2 \notin (X,Y)$. This element is however in the tight closure $(X,Y)^*$ of the ideal in positive characteristic (assume that the characteristic p does not divide d) and is therefore also in characteristic 0 inside the tight closure and inside the solid closure. Hence the open subset

$$D(X,Y) \subseteq \operatorname{Spec}\left(K[X,Y,Z,S,T]/(X^d + Y^d + Z^d, SX + TY - Z^2)\right)$$

is not an affine scheme. In positive characteristic, Z^2 is also contained in the plus closure $(X,Y)^+$ and therefore this open subset contains punctured surfaces (the spectrum of the forcing algebra contains two-dimensional closed subschemes which meet the exceptional fiber $V(X,Y)$ in only one point; the ideal (X,Y) has superheight 2 in the forcing algebra). In characteristic zero however, due to Fakt the superheight is one and therefore by Fakt the algebra $\Gamma(D(X,Y), \mathcal{O}_B)$ is not finitely generated. For $K = \mathbb{C}$ and $d = 3$ one can also show that $D(X,Y)_{\mathbb{C}}$ is, considered as a complex space, a Stein space.