Prof. Dr. H. Brenner **Osnabrück WS 2011/2012**

Mathematics for natural sciences I

Exercise sheet 24

Warm-up-exercises

Exercise 24.1. Compute the definite integral

$$\int_2^5 \frac{x^2+3x-6}{x-1}\, dx\,.$$

Exercise 24.2. Determine the second derivative of the function

$$F(x) = \int_0^x \sqrt{t^5 - t^3 + 2t}\, dt\,.$$

Exercise 24.3. An object is released at time 0 and it falls freely without air resistance from a certain height down to the earth thanks to the (constant) gravity force. Determine the velocity $v(t)$ and the distance $s(t)$ as a function of time t. After which time the object has traveled 100 meters?

Exercise 24.4. Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that the function

$$h(x) = \int_0^{g(x)} f(t)\, dt$$

is differentiable and determine its derivative.

Exercise 24.5. Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Consider the following sequence

$$a_n := \int_{\frac{1}{n+1}}^{\frac{1}{n}} f(t)\, dt\,.$$

Determine whether this sequence converges and, in case, determine its limit.

Exercise 24.6. Let $\sum_{n=1}^\infty a_n$ be a convergent series with $a_n \in [0,1]$ for all $n \in \mathbb{N}$ and let $f : [0,1] \to \mathbb{R}$ be a Riemann-integrable function. Prove that the series

$$\sum_{n=1}^\infty \int_0^{a_n} f(x) dx$$

is absolutely convergent.

EXERCISE 24.7. Let f be a Riemann-integrable function on $[a,b]$ with $f(x) \geq 0$ for all $x \in [a,b]$. Show that if f is continuous at a point $c \in [a,b]$ with $f(c) > 0$, then

$$\int_a^b f(x)dx > 0\,.$$

EXERCISE 24.8. Prove that the equation

$$\int_0^x e^{t^2} dt = 1$$

has exactly one solution $x \in [0,1]$.

EXERCISE 24.9. Let

$$f, g : [a,b] \longrightarrow \mathbb{R}$$

be two continuous functions such that

$$\int_a^b f(x)dx = \int_a^b g(x)dx\,.$$

Prove that there exists $c \in [a,b]$ such that $f(c) = g(c)$.

Hand-in-exercises

EXERCISE 24.10. (2 points)

Determine the area below[1] the graph of the sine function between 0 and π.

EXERCISE 24.11. (3 points)

Compute the definite integral

$$\int_1^7 \frac{x^3 - 2x^2 - x + 5}{x+1}\, dx\,.$$

EXERCISE 24.12. (3 points)

Determine an antiderivative for the function

$$\frac{1}{\sqrt{x} + \sqrt{x+1}}\,.$$

EXERCISE 24.13. (4 points)

Compute the area of the surface, which is enclosed by the graphs of the two functions f and g such that

$$f(x) = x^2 \text{ and } g(x) = -2x^2 + 3x + 4\,.$$

[1]Here we mean the area between the graph and the x-axis.

EXERCISE 24.14. (4 points)

We consider the function

$$f : \mathbb{R} \longrightarrow \mathbb{R},\ t \longmapsto f(t),$$

with

$$f(t) = \begin{cases} 0 \text{ for } t = 0, \\ \sin \frac{1}{t} \text{ for } t \neq 0\,. \end{cases}$$

Show, with reference to the function $g(x) = x^2 \cos \frac{1}{x}$, that f has an antiderivative.

EXERCISE 24.15. (3 points)

Let

$$f, g : [a, b] \longrightarrow \mathbb{R}$$

be two continuous functions and let $g(t) \geq 0$ for all $t \in [a, b]$. Prove that there exists $s \in [a, b]$ such that

$$\int_a^b f(t)g(t)\, dt = f(s) \int_a^b g(t)\, dt\,.$$