Locally free sheaves

We start this lecture series by asking what are the easiest modules M over a commutative ring R. There are several possible answers to this question, but the answer should definitely include the ring R itself and also the 0-module. Another easy module is the direct product $R^n = R \times \cdots \times R$ of the ring with itself. These modules are called free modules of rank r. Ideals might look easy at first sight, but in fact they are not, with the exception of a principal ideal domain, where all non-zero ideals are isomorphic as a module to R. Instead we consider here R-modules which have the property that their localizations are free. For this we look at a typical example, the so-called syzygy modules. Let R be a commutative ring and let I be an ideal generated by finitely many elements $I = (f_1, \ldots, f_n)$. The free resolution of the residue class ring R/I is the exact complex

$$\ldots \longrightarrow R^{n_2} \longrightarrow R^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} R \longrightarrow R/I \longrightarrow 0\,.$$

This resolution goes (unless I has finite projective dimension) on forever, but we can break it up to obtain the exact complex

$$0 \longrightarrow M = \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow R^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} R \longrightarrow R/I \longrightarrow 0\,,$$

where the module M is just defined to be the kernel of the R-module-homomorphism

$$R^n \longrightarrow R,\ (s_1, \ldots, s_n) \longmapsto \sum_{i=1}^{n} s_i f_i.$$

This kernel consists exactly of the syzygies for these elements, hence it is called (the first) syzygy module. This module can be already quite complicated, however, we can make the following observation. Let us fix one i, say $i = 1$, and look at the induced sequence over the localization R_{f_1}. As localization is an exact functor, we still get an exact sequence, and since $f_1 \in I$, the ideal I_{f_1} contains now a unit and therefore we have $(R/I)_{f_1} = 0$, so we can rewrite the induced sequence as

$$0 \longrightarrow M_{f_1} \longrightarrow (R_{f_1})^n \stackrel{f_1,\ldots,f_n}{\longrightarrow} R_{f_1} \longrightarrow 0\,.$$

We claim that we have an R_{f_1}-module isomorphism

$$(R_{f_1})^{n-1} \longrightarrow M_{f_1} \cong (\operatorname{Syz}(f_1, \ldots, f_n))_{f_1}$$

by sending the j-th standard vector e_j ($j = 2, \ldots, n$) to

$$v_j = (-\frac{f_j}{f_1}, \ldots, 0, 1, 0, \ldots, 0)$$

(the 1 stands at the jth position). This is obviously well-defined, since f_1 is a unit in R_{f_1}, and evidently the given tuple is a syzygy. If $s = (s_1, \ldots, s_n)$

is a syzygy, then $\sum_{j=2}^{n} s_j e_j$ is a preimage, since it is mapped under this homomorphism to

$$\sum_{j=2}^{n} s_j v_j = (-\sum_{j=2}^{n} s_j \frac{f_j}{f_1}, s_2, \ldots, s_n) = (s_1, s_2, \ldots, s_n).$$

Hence we have a surjection. The injectivity follows immediately by looking at the components 2 to n in the syzygy. This means that the syzygy module when restricted to the open subset $D(f_1)$ (viewed as an R_{f_1}-module) is free of rank $n-1$, and the same holds for all $D(f_j)$. Hence the syzygy module restricted to the open subset

$$U = \bigcup_{j=1}^{n} D(f_j)$$

has the property that there exists a covering by open subsets such that the restrictions to these open subsets are free modules. In general, the syzygy module is not free as an R-module nor as an $\mathcal{O}_U$-module on U. The above given explicit isomorphism on $D(f_1)$ (such an isomorphism is called a local trivialization of M on $D(f_1)$) uses that f_1 is a unit, hence this can not be extended to give an isomorphism on U. On the intersection $D(f_1) \cap D(f_2) = D(f_1 f_2)$ f_1 as well as f_2 are units, hence the above isomorphisms (let's call them ψ_1 on $D(f_1)$ and ψ_2 on $D(f_2)$) induce two different isomorphisms on $D(f_1 f_2)$ between $(R_{f_1 f_2})^{n-1}$ and $M_{f_1 f_2}$ We can connect them to get an isomorphism

$$\psi_2^{-1} \circ \psi_1 : (R_{f_1 f_2})^{n-1} \longrightarrow (R_{f_1 f_2})^{n-1}$$

which is given by the (over $R_{f_1 f_2}$) invertible $(n-1) \times (n-1)$-matrix

$$\begin{pmatrix} -\frac{f_2}{f_1} & -\frac{f_3}{f_1} & -\frac{f_4}{f_1} & \ldots & -\frac{f_n}{f_1} \\ 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & \ldots & 0 & -1 \end{pmatrix}.$$

We have seen that a syzygy module as above considered on $U = \bigcup_{i=1}^{n} D(f_i)$ has the following two properties: On the $D(f_i)$, which cover U, there are isomorphisms with a free module, and if we connect two such isomorphisms then the transition map is linear. These two properties give rise to what is called a locally free sheaf (the second condition is somehow hidden in the coherence. It will be explicit in the equivalent definition of a geometric vector bundle below). We will now give the precise definition. For this we will work in the context of schemes. If you are not familiar with the theory of schemes, it is enough to think of X as the spectrum $\operatorname{Spec}(R)$ of a ring or an open subset $D(\mathfrak{a}) \subseteq \operatorname{Spec}(R)$ of it defined by an ideal $\mathfrak{a} \subseteq R$ (such schemes are called quasiaffine). Recall that $\operatorname{Spec}(R)$ consists of all prime ideals of R

together with the Zariski topology where a basis is given by

$$D(f) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) |\, f \not\in \mathfrak{p}\} \cong \mathrm{Spec}\,(R_f).$$

If $R = K[X_1, \ldots, X_n]$, where K is a field, then one should consider Spec (R) as the usual affine space K^n, where the points $(a_1, \ldots, a_n)$ correspond to the maximal ideals of the form $(X_1 - a_1, \ldots, X_n - a_n)$ (and all maximal ideals with residue class field K are of this form). If $R = K[X_1, \ldots, X_n]/(H)$, then one should think of Spec (R) as the closed subset of affine space K^n consisting of the points $(a_1, \ldots, a_n)$ such that $H(a_1, \ldots, a_n) = 0$. For some also the word sheaf might be scary. As a first good approximation, one may think of a quasicoherent sheaf as an R-module M together with the family of localizations M_f which are associated to the open subsets $D(f)$ $(f \in R)$.

Definition 1.1. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong \mathcal{O}^r|_{U_i}$ for every $i \in I$.

The easist locally free sheaves are $\mathcal{O}_X^r$ $(r \in \mathbb{N})$, these are called free. The definition says exactly that locally a locally free sheaf is such a free sheaf. Over a local ring, any locally free sheaf is free, so there is not much to say. However, if we consider over a local ring R the modules which are locally free outside the unique closed point $\mathfrak{m}$ of Spec (R), i.e. on $D(\mathfrak{m})$ (which is called the punctured spectrum), then this is already a very important class of modules. Examples of this type will be the first syzygy module for an $\mathfrak{m}$-primary ideal. The following two theorems give equivalent characterizations of locally free sheaves on an affine scheme Spec (R). Basically it says that the term locally can be understood in any meaningful sense.

Theorem 1.2. *Let R denote a commutative noetherian ring and let M denote a finitely generated R-module. Let $r \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) *The localizations $M_\mathfrak{p}$ are free of rank r for every prime ideal $\mathfrak{p} \in$* Spec (R).
(2) *The localizations $M_\mathfrak{m}$ are free of rank r for every maximal ideal $\mathfrak{m}$ of R.*
(3) *There exists elements $f_1, \ldots, f_k \in R$ which generate the unit ideal and such that the localizations M_{f_j} are free of rank r for every $j = 1, \ldots, k$.*
(4) *The coherent sheaf $\tilde{M}$ on* Spec (R) *associated to M is locally free.*

Theorem 1.3. *Let R denote a commutative noetherian ring and let M denote a finitely generated R-module. Let $r \in \mathbb{N}$. Then the following conditions are equivalent.*

(1) *$\tilde{M}$ is locally free.*
(2) *M is a projective module.*
(3) *M is a (faithfully) flat module.*

The following theorem provides many locally free sheaves. The syzygy sheaves discussed above are a special case of this construction, since they are the kernel of $\mathcal{O}_X^n \to \mathcal{O}_X$, which is surjective on $U = \bigcup_{i=1}^n D(f_i)$.

THEOREM 1.4. *Let X denote a scheme and let $\mathcal{F}$ and $\mathcal{G}$ denote two locally free sheaves (of rank r and s) together with a surjective sheaf homomorphism*

$$\psi : \mathcal{F} \longrightarrow \mathcal{G}.$$

Then the kernel sheaf $\ker(\psi)$ *is also locally free (of rank $r - s$).*

Geometric vector bundles

We develop an equivalent but more geometric notion for a locally free sheaf. Both concepts are equally important, and it is good to switch from one perspective to the other.

DEFINITION 1.5. Let X denote a scheme. A scheme

$$p : V \longrightarrow X$$

is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i \in I} U_i$ and U_i-isomorphisms

$$\psi_i : U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \longrightarrow V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset $U \subseteq U_i \cap U_j$ the transition mappings

$$\psi_j^{-1} \circ \psi_i : \mathbb{A}^r_{U_i}|_U \longrightarrow \mathbb{A}^r_{U_j}|_U$$

are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \ldots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

Here we can restrict always to affine open coverings. If X is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersection. The trivial bundle of rank r is the r-dimensional affine space $\mathbb{A}^r_X$ over X, and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+ : (\mathbb{A}^r_U) \times_U (\mathbb{A}^r_U) \longrightarrow \mathbb{A}^r_U,\ (v_1, \ldots, v_r, w_1, \ldots, w_r) \longmapsto (v_1 + w_1, \ldots, v_r + w_r),$$

and this carries over to a vector bundle. The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on V coming from an isomorphism with some affine space is independent of the chosen isomorphism. For the same reason there is a unique closed subscheme of V called the *zero-section* which is locally defined to be $0 \times U \subseteq \mathbb{A}^r_U$. Also, the multiplication by a scalar, i.e. the mapping

$$\cdot : \mathbb{A}_U \times_U (\mathbb{A}^r_U) \longrightarrow \mathbb{A}^r_U,\ (s, v_1, \ldots, v_r) \longmapsto (sv_1, \ldots, sv_r),$$

carries over to a scalar multiplication

$$\cdot : \mathbb{A}_X \times_X V \longrightarrow V.$$

In particular, for every point $x \in X$ the fiber $V_x = V \times_X x$ is an affine vector space over $\kappa(x)$. This given we can say that a vector bundle V is in particular a commutative group scheme (but one which is defined over an arbitrary base X, not over the spectrum of a field), meaning that we have morphismus

$$+ : V \times V \longrightarrow V,\ 0 : X \longrightarrow V \text{ and } - : V \longrightarrow V$$

fulfilling certain natural arrow-conditions expressing associativity, that 0 is the neutral element and that $-$ gives the negative. This viewpoint will be later important when we have a look at the torsors of this group scheme. For a geometric vector bundle $p : V \to X$ and an open subset $U \subseteq X$ one sets

$$\Gamma(U, V) = \{s : U \to V|_U \,|\, p \circ s = \mathrm{id}_U\},$$

so this is the set of sections in V over U. This gives in fact for every scheme over X a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

Theorem 1.6. *Let X denote a scheme. Then the category of locally free sheaves on X and the category of geometric vector bundles on X are equivalent. A geometric vector bundle $V \to X$ corresponds to the sheaf of its sections, and a locally free sheaf $\mathcal{F}$ corresponds to the (relative) Spectrum of the symmetric algebra of the dual module $\mathcal{F}^*$.*

The free sheaf of rank r corresponds to the affine space $\mathbb{A}_X^r$ over X.

Remark 1.7. For a surjective morphism

$$\varphi : \mathcal{O}_X^n \longrightarrow \mathcal{O}_X$$

on a scheme X given by elements $f_1, \ldots, f_n \in \Gamma(X, \mathcal{O}_X)$ (the surjectivity means that these elements generate locally the unit ideal) we can realize the corresponding locally free kernel sheaf in the following natural way. We can directly look at the corresponding surjection of geometric vector bundles

$$\varphi : \mathbb{A}_X^n \longrightarrow \mathbb{A}_X,\ (v_1, \ldots, v_n) \longmapsto \sum_{i=1}^n f_i v_i,$$

and the kernel consists for every base point $x \in X$ in the solution set

$$\{(v_1, \ldots, v_n) \in (\kappa(x))^n | \sum_{i=1}^n f_i(x) v_i = 0\}$$

to this linear equation over the residue class field $\kappa(x)$. So fiberwise this syzygy bundle is a very simple object, but of course the solution space varies with the basis. If $X = \mathrm{Spec}\,(R)$ is affine, then one can also describe the syzygy bundle as the spectrum of the R-algebra

$$R[T_1, \ldots , T_n]/(f_1T_1 + \ldots + f_nT_n)\,.$$

If the elements $f_1, \ldots , f_n \in R$ do not generate the unit ideal in R, then the syzygy module yields only a vector bundle on the open subset $D(f_1, \ldots , f_n)$. However, the algebra just mentioned,

$$A = R[T_1, \ldots , T_n]/(f_1T_1 + \ldots + f_nT_n)\,,$$

always gives rise to a commutative group scheme $\mathrm{Spec}\,(A)$ over $\mathrm{Spec}\,(R)$. Note that

$$A \otimes_R A \cong R[T_1, \ldots , T_n, S_1, \ldots , S_n]/(f_1T_1 + \ldots + f_nT_n, f_1S_1 + \ldots + f_nS_n)\,.$$

The coadditon is given by

$$A \longrightarrow A \otimes_R A,\ T_i \longmapsto T_i + S_i,$$

and the addition is given by

$$\mathrm{Spec}\,(A{\otimes}_RA) \longrightarrow \mathrm{Spec}\,(A),\ (t_1, \ldots , t_n, s_1, \ldots , s_n) \longmapsto (s_1 + t_1, \ldots , s_n + t_n).$$

The zero element and the negatives are also defined in an obvious way. Also, the fibers of $\mathrm{Spec}\,(A)$ over a point $x \in \mathrm{Spec}\,(R)$ is always a vector space over the residue class field. However, the dimension may vary. If $x \in X$ is a point where all the functions $f_1, \ldots , f_n$ vanish (and such points exist if these elements do not generate the unit ideal), then the equation which defines A degenerates to 0 and then the dimension of the fiber is n instead of $n - 1$. This corresponds to the property that the linear equation degenerates and hence the dimension of the solution space goes up. In the next lecture we will study torsors of vector bundles and forcing algebras, which correspond to inhomogeneous linear equations varying with a basis.