Geometric vector bundles

We have seen that the forcing algebra has locally the form $R_{f_i}[T_1, \ldots, T_{i-1}, T_{i+1}, \ldots, T_n]$ and its spectrum Spec (B) has locally the form $D(f_i) \times \mathbb{A}_K^{n-1}$. This description holds on the union $U = \bigcup_{i=1}^n D(f_i)$. Moreover, in the homogeneous case (f=0) the transition mappings are linear. Hence $V|_U$ is a geometric vector bundle according to the following definition.

Definition 2.1. Let X denote a scheme. A scheme

$$p : V \longrightarrow X$$

is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i \in I} U_i$ and U_i-isomorphisms

$$\psi_i : U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \longrightarrow V|_{U_i} = p^{-1}(U_i)$$

such that for every open affine subset $U \subseteq U_i \cap U_j$ the transition mappings

$$\psi_j^{-1} \circ \psi_i : \mathbb{A}^r_{U_i}|_U \longrightarrow \mathbb{A}^r_{U_j}|_U$$

are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \ldots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

Here we can restrict always to affine open coverings. If X is separated then the intersection of two affine open subschemes is again affine and then it is enough to check the condition on the intersection. The trivial bundle of rank r is the r-dimensional affine space $\mathbb{A}^r_X$ over X, and locally every vector bundle looks like this. Many properties of an affine space are enjoyed by general vector bundles. For example, in the affine space we have the natural addition

$$+ : (\mathbb{A}^r_U) \times_U (\mathbb{A}^r_U) \longrightarrow \mathbb{A}^r_U,\ (v_1, \ldots, v_r, w_1, \ldots, w_r) \longmapsto (v_1 + w_1, \ldots, v_r + w_r),$$

and this carries over to a vector bundle. The reason for this is that the isomorphisms occurring in the definition of a geometric vector bundle are linear, hence the addition on V coming from an isomorphism with some affine space is independent of the choosen isomorphism. For the same reason there is a unique closed subscheme of V called the *zero-section* which is locally defined to be $0 \times U \subseteq \mathbb{A}^r_U$. Also, the multiplication by a scalar, i.e. the mapping

$$\cdot : \mathbb{A}_U \times_U (\mathbb{A}^r_U) \longrightarrow \mathbb{A}^r_U,\ (s, v_1, \ldots, v_r) \longmapsto (sv_1, \ldots, sv_r),$$

carries over to a scalar multiplication

$$\cdot : \mathbb{A}_X \times_X V \longrightarrow V.$$

In particular, for every point $x \in X$ the fiber $V_x = V \times_X x$ is an affine space over $\kappa(x)$.

For a geometric vector bundle $p : V \to X$ and an open subset $U \subseteq X$ one sets

$$\Gamma(U, V) = \{s : U \to V|_U \,|\, p \circ s = \mathrm{id}_U\},$$

so this is the set of sections in V over U. This gives in fact for every scheme over X a set-valued sheaf. Because of the observations just mentioned, these sections can also be added and multiplied by elements in the structure sheaf, and so we get for every vector bundle a locally free sheaf, which is free on the open subsets where the vector bundle is trivial.

DEFINITION 2.2. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong \mathcal{O}^r|_{U_i}$ for every $i \in I$.

Vector bundles and locally free sheaves are essentially the same objects.

THEOREM 2.3. *Let X denote a scheme. Then the category of locally free sheaves on X and the category of geometric vector bundles on X are equivalent. A geometric vector bundle $V \to X$ corresponds to the sheaf of its sections, and a locally free sheaf $\mathcal{F}$ corresponds to the (relative) Spectrum of the symmetric algebra of the dual module $\mathcal{F}^*$.*

The free sheaf of rank r corresponds to the affine space $\mathbb{A}^r_X$ over X.

Torsors of vector bundles

We have seen that $V = \operatorname{Spec}(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n))$ acts on the spectrum of a forcing algebra $T = \operatorname{Spec}(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n + f))$ by addition. The restriction of V to U is a vector bundle, and T restricted to U becomes a V-torsor.

DEFINITION 2.4. Let V denote a geometric vector bundle over a scheme X. A scheme $T \to X$ together with an action

$$\beta : V \times_X T \longrightarrow T$$

is called a geometric (Zariski)-*torsor* for V (or a V-*principal fiber bundle* or a *principal homogeneous space*) if there exists an open covering $X = \bigcup_{i \in I} U_i$ and isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

such that the diagrams (we set $U = U_i$ and $\varphi = \varphi_i$)

$$\begin{array}{ccc} V|_U \times_U T|_U & \xrightarrow{\beta} & T|_U \\ \downarrow & & \downarrow \\ V|_U \times_U V|_U & \xrightarrow{\beta} & V|_U \end{array}$$

commute.

The torsors of vector bundles can be classified in the following way.

Proposition 2.5. *Let X denote a Noetherian separated scheme and let*

$$p : V \longrightarrow X$$

denote a geometric vector bundle on X with sheaf of sections $\mathcal{S}$. Then there exists a correspondence between first cohomology classes $c \in H^1(X, \mathcal{S})$ and geometric V-torsors.

Proof. We will describe this correspondence. Let T denote a V-torsor. Then there exists by definition an open covering $X = \bigcup_{i \in I} U_i$ such that there exists isomorphisms

$$\varphi_i : T|_{U_i} \longrightarrow V|_{U_i}$$

which are compatible with the action of $V|_{U_i}$ on itself. The isomorphisms φ_i induce automorphisms

$$\psi_{ij} = \varphi_j \circ \varphi_i^{-1} : V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j}.$$

These automorphisms are compatible with the action of V on itself, and this means that they are of the form

$$\psi_{ij} = \mathrm{Id}_V|_{U_i \cap U_j} + s_{ij}$$

with suitable sections $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$. This family defines a Cech-cocycle for the covering and gives therefore a cohomology class in $H^1(X, \mathcal{S})$. For the reverse direction, suppose that the cohomology class $c \in H^1(X, \mathcal{S})$ is represented by a Cech-cocycle $s_{ij} \in \Gamma(U_i \cap U_j, \mathcal{S})$ for an open covering $X = \bigcup_{i \in I} U_i$. Set $T_i := V|_{U_i}$. We take the morphisms

$$\psi_{ij} : T_i|_{U_i \cap U_j} = V|_{U_i \cap U_j} \longrightarrow V|_{U_i \cap U_j} = T_j|_{U_i \cap U_j}$$

given by $\psi_{ij} := \mathrm{Id}_V|_{U_i \cap U_j} + s_{ij}$ to glue the T_i together to a scheme T over X. This is possible since the cocycle condition guarantees the glueing condition for schemes (EGA I, 0, 4.1.7). The action of V_i on T_i by itself glues also together to give an action on T. □

It follows immediately that for an affine scheme (i.e. a scheme of type Spec (R)) there are no non-trivial torsor for any vector bundle. There will however be in general many non-trivial torsors on the punctured spectrum (and on a projective variety).

Forcing algebras and induced torsors

As T_U is a V_U-torsor, and as every V-torsor is represented by a unique cohomology class, there should be a natural cohomology class coming from the forcing data. To see this, let R be a noetherian ring and $I = (f_1, \ldots, f_n)$ be an ideal. Then on $U = D(I)$ we have the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

An element $f \in R$ defines an element $f \in \Gamma(U, \mathcal{O}_U)$ and hence a cohomology class $\delta(f) \in H^1(U, \mathrm{Syz}\,(f_1, \ldots, f_n))$. Hence f defines in fact a $\mathrm{Syz}\,(f_1, \ldots, f_n)$-torsor over U. We will see that this torsor is induced by the forcing algebra given by $f_1, \ldots, f_n$ and f.

THEOREM 2.6. *Let R denote a noetherian ring, let $I = (f_1, \ldots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c \in H^1(D(I), \mathrm{Syz}\,(f_1, \ldots, f_n))$ be the corresponding cohomology class and let $B = R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)$ denote the forcing algebra for these data. Then the scheme $\mathrm{Spec}\,(B)|_{D(I)}$ together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by c.*

Proof. We compute the cohomology class $\delta(f) \in \mathrm{Syz}\,(f_1, \ldots, f_n)$ and the cohomology class given by the forcing algebra. For the first computation we look at the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow \mathcal{O}_U^n \longrightarrow \mathcal{O}_U \longrightarrow 0\,.$$

On $D(f_i)$, the element f is the image of $(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0)$ (the non-zero entry is at the ith place). The cohomology class is therefore represented by the family of differences

$$(0, \ldots, 0, \frac{f}{f_i}, 0, \ldots, 0, -\frac{f}{f_j}, 0, \ldots, 0) \in \Gamma(D(f_i) \cap D(f_j), \mathrm{Syz}\,(f_1, \ldots, f_n))\,.$$

On the other hand, there are isomorphisms

$$V|_{D(f_i)} \longrightarrow T|_{D(f_i)},\ (s_1, \ldots, s_n) \longmapsto (s_1, \ldots, s_{i-1}, s_i + \frac{f}{f_i}, s_{i+1}, \ldots, s_n).$$

The difference of two such isomorphisms on $D(f_if_j)$ is the same as before. $\square$

EXAMPLE 2.7. Let $(R, \mathfrak{m})$ denote a two-dimensional normal local noetherian domain and let f and g be two parameters in R. On $D(\mathfrak{m})$ we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_U \cong \mathrm{Syz}\,(f, g) \longrightarrow \mathcal{O}_U^2 \stackrel{f,g}{\longrightarrow} \mathcal{O}_U \longrightarrow 0$$

and its corresponding long exact sequence of cohomology,

$$0 \longrightarrow R \longrightarrow R^2 \stackrel{f,g}{\longrightarrow} R \stackrel{\delta}{\longrightarrow} H^1(U, \mathcal{O}) \longrightarrow \ldots\,.$$

The connecting homomorphisms δ sends an element $h \in R$ to $\frac{h}{fg}$. The torsor given by such a cohomology class $c = \frac{h}{fg} \in H^1(U, \mathcal{O}_X)$ can be realized by the forcing algebra

$$R[T_1, T_2]/(fT_1 + gT_2 - h)\,.$$

Note that different forcing algebras may give the same torsor, because the torsor depends only on the spectrum of the forcing algebra restricted to the punctured spectrum of R. For example, the cohomology class $\frac{1}{fg} = \frac{fg}{f^2g^2}$ defines one torsor, but the two quotients yield the two forcing algebras

$R[T_1, T_2]/(fT_1 + gT_2 + 1)$ and $R[T_1, T_2]/(f^2T_1 + g^2T_2 + fg)$, which are quite different. The fiber over the maximal ideal of the first one is empty, whereas the fiber over the maximal ideal of the second one is a plane.

If R is regular, say $R = K[X, Y]$ (or the localization of this at (X, Y) or the corresponding power series ring) then the first cohomology classes are linear combinations of $\frac{1}{x^i y^j}$, $i, j \geq 1$. They are realized by the forcing algebras $K[X, Y]/(X^iT_1 + Y^jT_2 - 1)$. Since the fiber over the maximal ideal is empty, the spectrum of the forcing algebra equals the torsor. Or, the other way round, the torsor is itself an affine scheme.

In the next lectures we will deal with global properties of torsors and forcing algebras and how these properties are related to closure operations of ideals.

Exercise for Saturday: Show that f belongs to the radical of the ideal $(f_1, \ldots, f_n)$ if and only if the spectrum morphism

$$\text{Spec}\,(R[T_1, \ldots, T_n]/(f_1T_1 + \ldots + f_nT_n - f)) \longrightarrow \text{Spec}\,(R)$$

is surjective.