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Vector bundles, forcing algebras and local cohomology

Lecture 8

In the remaining lectures we will continue with the question when are the torsors given by a forcing algebras over a two-dimensional ring affine? We will look at the graded situation to be able to work on the corresponding projective curve.

In particular we want to address the following questions

(1) Is there a procedure to decide whether the torsor is affine?
(2) Is it non-affine if and only if there exists a geometric reason for it not to be affine (because the superheight is too large)?
(3) How does the affineness vary in an arithmetic family, when we vary the prime characteristic?
(4) How does the affineness vary in a geometric family, when we vary the base ring?

In terms of tight closure, these questions are directly related to the tantalizing question of tight closure (is it the same as plus closure), the dependence of tight closure on the characteristic and the localization problem of tight closure.

Geometric interpretation in dimension two

We will restrict now to the two-dimensional homogeneous case in order to work on the corresponding projective curve. We want to find an object over the curve which corresponds to the forcing algebra or its induced torsor.

Let R be a two-dimensional standard-graded normal domain over an algebraically closed field K. Let $C = \text{Proj } R$ be the corresponding smooth projective curve and let

$$I = (f_1, \ldots, f_n)$$

be an R_+-primary homogeneous ideal with generators of degrees $d_1, \ldots, d_n$. Then we get on C the short exact sequence

$$0 \longrightarrow \text{Syz}\,(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0\,.$$

Here $\text{Syz}\,(f_1, \ldots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, of rank $n-1$ and of degree

$$((n-1)m - \sum_{i=1}^{n} d_i) \deg\,(C)\,.$$

Thus a homogeneous element f of degree m defines a cohomology class $\delta(f) \in H^1(C, \mathrm{Syz}\,(f_1, \ldots, f_n)(m))$, so this defines a torsor over the projective curve. We mention an alternative description of the torsor corresponding to a first cohomology class in a locally free sheaf which is better suited for the projective situation.

REMARK 8.1. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields projective subbundles

$$\mathbb{P}(\mathcal{S}^*) \subset \mathbb{P}(\mathcal{S}'^*)\,.$$

If V is the corresponding vector bundle, one may think of $\mathbb{P}(\mathcal{S}^*)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x running through the zero point. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement then is over every point then an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^*) - \mathbb{P}(\mathcal{S}^*)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

Semistability of vector bundles

In the situation of a forcing algebra for homogeneous elements, this torsor T can also be obtained as Proj B, where B is the (not necessarily positively) graded forcing algebra. In particular, it follows that the containment $f \in I^*$ is equivalent to the property that T is not an affine variety. For this properties, positivity (ampleness) properties of the syzygy bundle are crucial. We need the concept of (Mumford) semistability.

DEFINITION 8.2. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable*, if $\frac{\deg(\mathcal{T})}{\mathrm{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\mathrm{rk}(\mathcal{S})}$ for all subbundles $\mathcal{T}$.

Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

An important property of a semistable bundle of negative degree is that it can not have any global section $\neq 0$. Note that a semistable vector bundle need not be strongly semistable, the following is probably the simplest example.

EXAMPLE 8.3. Let C be the smooth Fermat quartic given by $x^4+y^4+z^4$ and consider on it the syzygy bundle $\mathrm{Syz}\,(x,y,z)$ (which is also the restricted cotangent bundle from the projective plane). This bundle is semistable. Suppose that the characteristic is 3. Then its Frobenius pull-back is $\mathrm{Syz}\,(x^3,y^3,z^3)$. The curve equation gives a global nontrivial section of this bundle of total degree 4. But the degree of $\mathrm{Syz}\,(x^3,y^3,z^3)(4)$ is negative, hence it can not be semistable anymore.

For a strongly semistable vector bundle $\mathcal{S}$ on C and a cohomology class $c \in H^1(C,\mathcal{S})$ with corresponding torsor we obtain the following affineness criterion.

THEOREM 8.4. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a strongly semistable vector bundle over C together with a cohomology class $c \in H^1(C,\mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if* $\deg(\mathcal{S}) < 0$ *and* $c \neq 0$ *($F^e(c) \neq 0$ in positive characteristic).*

This result rests on the ampleness of $\mathcal{S}'^{\vee}$ occuring in the dual exact sequence $0 \to \mathcal{O}_C \to \mathcal{S}'^{\vee} \to \mathcal{S}^{\vee} \to 0$ given by c (work of Hartshorne and Gieseker). It implies for a strongly semistable syzygy bundles the following *degree formula* for tight closure.

THEOREM 8.5. *Suppose that* $\mathrm{Syz}\,(f_1,\ldots,f_n)$ *is strongly semistable. Then*

$$R_m \subseteq I^* \text{ for } m \geq \frac{\sum d_i}{n-1} \text{ and (for almost all prime numbers)}$$
$$R_m \cap I^* \subseteq I \text{ for } m < \frac{\sum d_i}{n-1}.$$

We indicate the proof of the inclusion result. The degree condition implies that $c = \delta(f) \in H^1(C,\mathcal{S})$ is such that $\mathcal{S} = \mathrm{Syz}\,(f_1,\ldots,f_n)(m)$ has nonnegative degree. Then also all Frobenius pull-backs $F^*(\mathcal{S})$ have nonnegative degree. Let $\mathcal{L} = \mathcal{O}(k)$ be a twist of the tautological line bundle on C such that its degree is larger than the degree of ω_C^{-1}, the dual of the canonical sheaf. Let $z \in H^0(Y,\mathcal{L})$ be a non-zero element. Then $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L})$, and by Serre duality we have

$$H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L}) \cong H^0(F^{e*}(\mathcal{S}^*) \otimes \mathcal{L}^{-1} \otimes \omega_C)^{\vee}\,.$$

On the right hand side we have a semistable sheaf of negative degree, which can not have a nontrivial section. Hence $zF^{e*} = 0$ and therefore f belongs to the tight closure.

Harder-Narasimhan filtration

In general, there exists an exact criterion depending on c and the *strong Harder-Narasimhan filtration* of $\mathcal{S}$. For this we give the definition of the Harder-Narasimhan filtration.

Definition 8.6. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C over an algebraically closed field K. Then the (uniquely determined) filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that all quotient bundles $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing slopes $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$, is called the *Harder-Narasimhan filtration* of $\mathcal{S}$.

The Harder-Narasimhan filtration exists uniquely (by a Theorem of Harder and Narasimhan). A Harder-Narasimhan filtration is called strong if all the quotients $\mathcal{S}_i/\mathcal{S}_{i-1}$ are strongly semistable. A Harder-Narasimhan filtration is not strong in general, however, by a Theorem of A. Langer, there exists some Frobenius pull-back $F^{e*}(\mathcal{S})$ such that its Harder-Narasimhan filtration is strong.

Theorem 8.7. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

be a strong Harder-Narasimhan filteration. Then the torsor $T(c)$ is an affine scheme if and only if the following (inductively defined property starting with t) holds: there is an i such that $\deg(\mathcal{S}_i/\mathcal{S}_{i-1}) < 0$ and the image of c in this sheaf is $\neq 0$ (and also the Frobenius pull-backs of this class are $\neq 0$).