CLTI Differential Equation

•

Please send corrections (or suggestions) to youngwlim@hotmail.com.

This document was produced by using OpenOffice and Octave.

Causal LTI Systems (1)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{d\, y(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{d\, x(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

$$M = N$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

Causal LTI Systems (2)

$$a_N \frac{d^N y(t)}{dt^N} + a_{N-1} \frac{d^{N-1} y(t)}{dt^N} + \cdots + a_1 \frac{dy(t)}{dt} + a_0 y(t) = b_M \frac{d^M x(t)}{dt^M} + b_{M-1} \frac{d^{M-1} x(t)}{dt^M} + \cdots + b_1 \frac{dx(t)}{dt} + b_0 x(t)$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$$Q(D) y(t) = P(D) x(t)$$

- **Zero Input Response**
- **Zero State Response (Convolution with h(t))**

- **Natural Response (Homogeneous Solution)**
- **Forced Response (Particular Solution)**

Zero Input Response $y_0(t)$ – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D)\, y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_0(t) = 0$$

Linear combination of $y_0(t)$ and its derivatives = 0

if and only if

$$y_0(t) = c e^{\lambda t}$$

$$\dot{y}_0(t) = c \lambda e^{\lambda t}$$

$$\ddot{y}_0(t) = c \lambda^2 e^{\lambda t}$$

$$\cdots$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{= 0} \; \underbrace{c e^{\lambda t}}_{\neq 0} = 0$$

Zero Input Response $y_0(t)$ – (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(D) y_0(t) = 0 \quad \Rightarrow \quad (D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_0(t) = 0$$

$$Q(\lambda) = 0 \quad \Leftrightarrow \quad \underbrace{(\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N)}_{= 0} \; \underbrace{c e^{\lambda t}}_{\neq 0} = 0$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N)$$

λ_i characteristic roots

$$c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t} = y_0(t)$$

$e^{\lambda_i t}$ characteristic modes

ZIR: a linear combination of the characteristic modes of the system

Zero State Response y(t) – (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

All initial conditions are zero

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau$$

Impulse response $h(t)$

$$y(t) = \int_{0^-}^{+t} x(\tau) y(t-\tau)\, d\tau \quad , \quad t \geq 0$$

Causality

causal system: Response cannot begin before the input

causal input: The input starts at t=0 $\quad h(\tau) = 0 \quad \tau < 0$

causal h(t): The causal system's response to a unit impulse cannot begin before t=0

$$h(t-\tau) = 0 \quad t-\tau < 0$$

Total Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$y(t) = \sum_{k=1}^{N} c_k e^{\lambda_k t} + x(t) * h(t)$$

Zero Input Response — Zero State Response

$$y(t) = y_n(t) + y_\Phi(t)$$

Natural Response — Forced Response

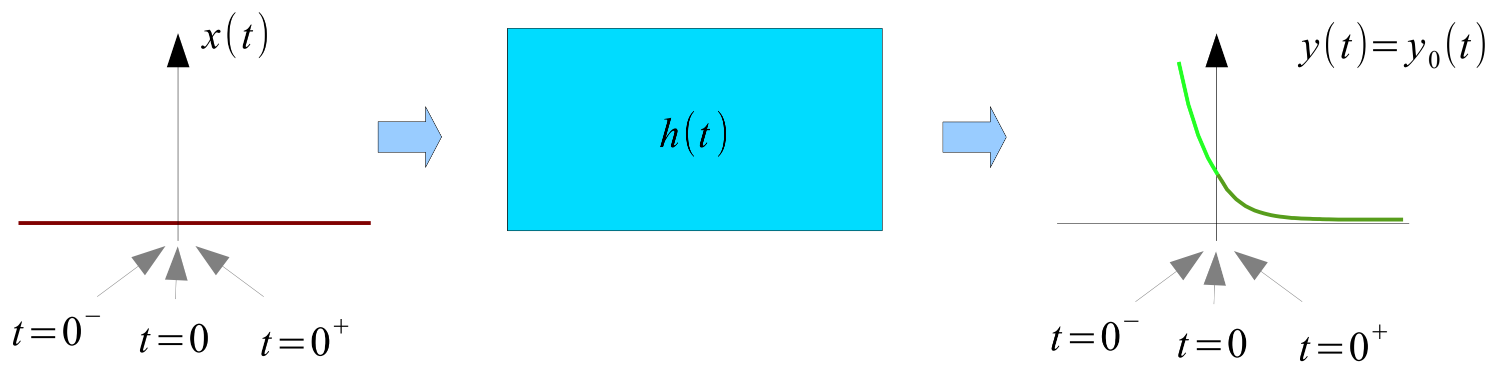
Young Won Lim
11/16/11

Zero Input Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



Input is zero

Only initial conditions drives the system

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

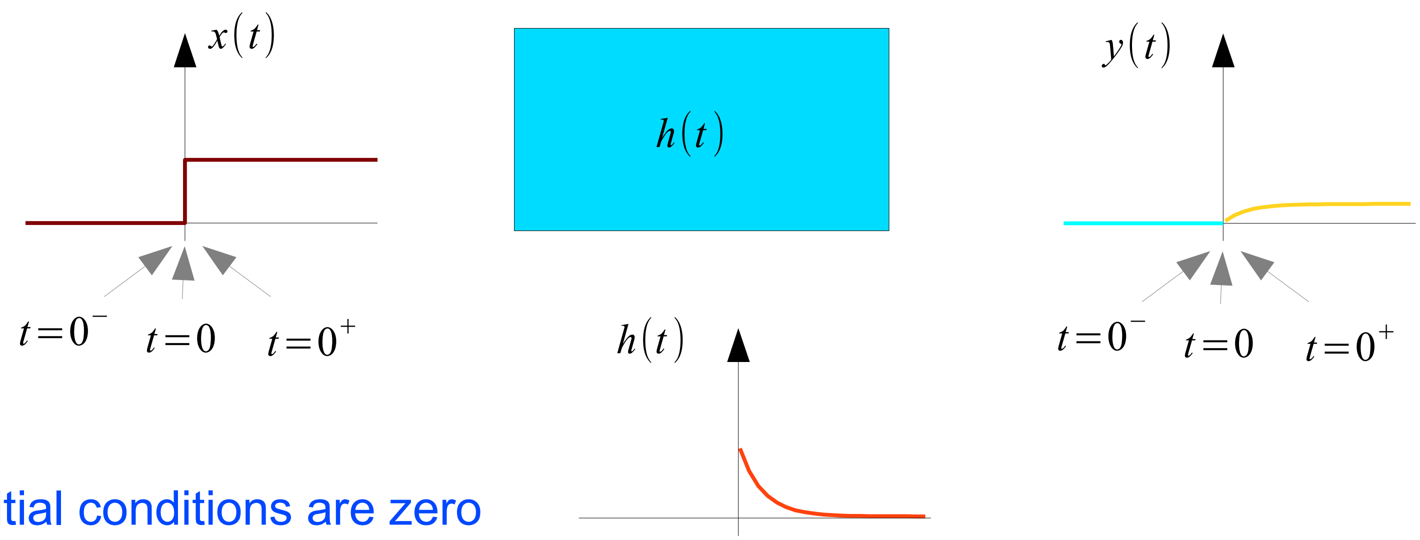
$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

Zero State Response

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



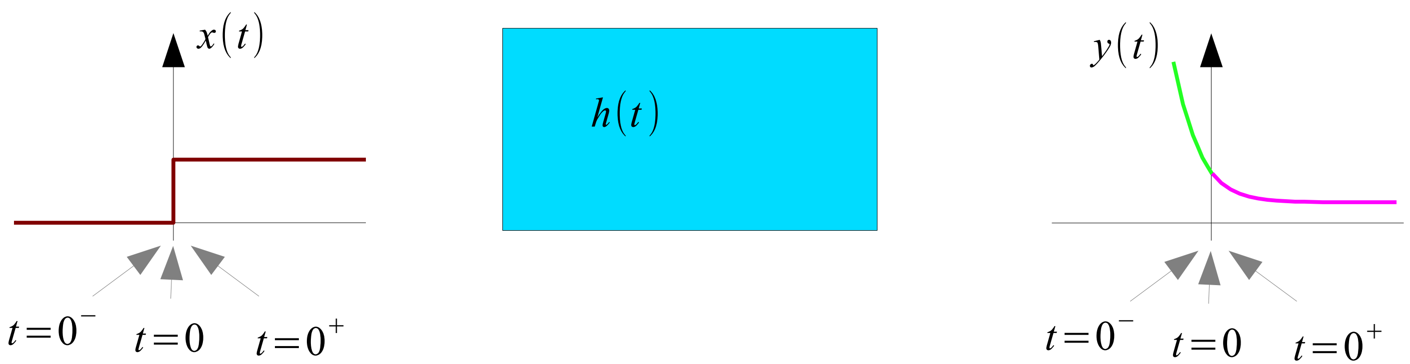
All initial conditions are zero

Total Response y(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$



$$y(t) = y_0(t) \quad t \le 0^-$$

zero input response

+

zero state response

because the input has not started yet

$$y(0^-) = y_0(0^-)$$

$$\dot{y}(0^-) = \dot{y}_0(0^-)$$

In general, the total response

$$y(0^-) \neq y(0^+)$$

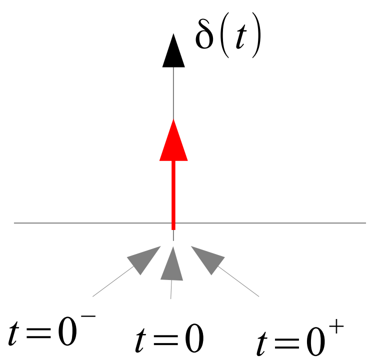
$$\dot{y}(0^-) \neq \dot{y}(0^+)$$

Impulse Response h(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

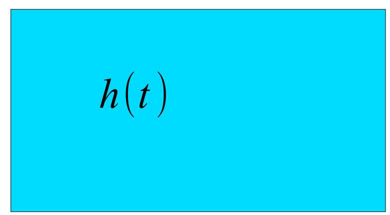
$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

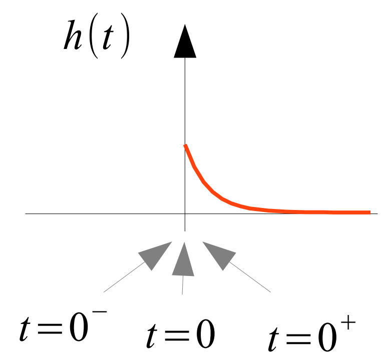


$\delta(0^-) = 0$ $\quad$ $\delta(0^+) = 0$

All init conditions are zero at $t=0^-$

Generates energy storage
Creates nonzero initial condition at $t=0^+$





$t \geq 0^+$ $(t \neq 0)$	h(t) = characteristic mode terms
$t=0$	h(t) can have at most an impulse $A_0 \delta(t)$

$$h(t) = A_0 \delta(t) + char\ mode\ terms \quad t \geq 0$$

h(t) can have at most a δ(t)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

If $\delta(t)$ is included in $h(t)$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) h(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) \delta(t) \qquad M = N$$

The highest order term $\delta^{(N+1)}(t)$ ⟷ $\delta^{(N)}(t)$ contradiction

h(t) cannot contain $\delta^{(i)}(t)$ at all

h(t) can contain at most δ(t)

Simplified Impulse Matching Method (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N)\, x(t)$$

$M = N$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$h(t) = b_0 \delta(t) + [P(D)\, y_n(t)]\, u(t)$$

$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \cdots = y_n^{(N-2)}(0) = 0 \qquad y_n^{(N-1)}(0) = 1$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N)\, y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$Q(D)\, w(t) = x(t)$$

$$Q(D)\, y_n(t) = \delta(t)$$

$$Q(D)\, w(t) = x(t)$$

$$Q(D)\, P(D)\, w(t) = P(D)\, x(t)$$

$$Q(D)\, y(t) = P(D)\, x(t)$$

$$y(t) = P(D)\, w(t)$$

Simplified Impulse Matching Method (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_0 \frac{d^M x(t)}{dt^M} + b_1 \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y(t) = (b_0 D^M + b_1 D^{M-1} + \cdots + b_{N-1} D + b_N) x(t)$$

$M = N$

$$Q(D) y(t) = P(D) x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) y_n(t) = \delta(t)$$

$$y_n^{(N)}(t) + a_1 y_n^{(N-1)}(t) + \cdots + a_{N-1} y_n^{(1)}(t) + y_n(t) = \delta(t)$$

$$h(t) = P(D)[y_n(t) u(t)]$$

$$h(t) = b_o \delta(t) + P(D) y_n(t), \quad t \geq 0$$

$$h(t) = b_o \delta(t) + [P(D) y_n(t)] u(t)$$

$$Q(D) w(t) = x(t)$$

$$Q(D) P(D) w(t) = P(D) x(t)$$

$$y(t) = P(D) w(t)$$

$$Q(D) y_n(t) = \delta(t)$$

$$Q(D) P(D) y_n(t) = P(D) \delta(t)$$

$$h(t) = P(D) y_n(t)$$

causal $y_n(t) u(t)$

$$h(t) = P(D)[y_n(t) u(t)]$$

Classical Solution (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

*When all the **characteristic mode terms** of the total system response together, they form the system's **natural response** (homogeneous, complementary solution)* $y_n(t)$

$$y(t) = y_n(t) + y_\Phi(t)$$

$$Q(D)\,[y_n(t) + y_\Phi(t)] = P(D)\,x(t)$$

*The remaining portion of noncharacteristic mode terms form the system's **forced response** (particular solution)* $y_\Phi(t)$

$$Q(D)\,y_n(t) = 0$$

$$Q(D)\,y_\Phi(t) = P(D)\,x(t)$$

Classical Solution (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) = \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

- *linear combination of the* ***characteristic modes****.* $y_n(t)$
- *the same form as that of the* ***zero input response***
- *only its constants are different*
- *these constants are determined from the* ***auxiliary conditions***
- *initial conditions at* $t = 0^+$
- *at* $t = 0^-$ *only the* ***zero input response***
- *initial condition at* $t = 0^-$ $\rightarrow$ *applied to* ***zero input response***
- ***zir*** *and* ***zsr*** *cannot be separated*

Young Won Lim
11/16/11

IC – Zero Input Response (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{dy(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{dx(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1}\lambda + a_N) = 0$$

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

ZIR: a linear combination of the characteristic modes of the system

$$y_0(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \cdots + c_N e^{\lambda_N t}$$

In practice, these initial conditions are known

$$y_0(0^-), \ \dot{y}_0(0^-), \ \ddot{y}_0(0^-), \ \cdots$$

But ZIR is not affected by the input.
Therefore, the following conditions are met

$$y_0(0^-) = y_0(0) = y_0(0^+)$$

$$\dot{y}_0(0^-) = \dot{y}_0(0) = \dot{y}_0(0^+)$$

$$\ddot{y}_0(0^-) = \ddot{y}_0(0) = \ddot{y}_0(0^+)$$

$$\cdots = \cdots = \cdots$$

IC – Impulse Response (1)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$t \geq 0^+$ h(t) = characteristic mode terms

$t \geq 0$ h(t) = $A_0 \delta(t)$ + characteristic mode terms

Simplified Impulse Matching Method ⇒ $h(t) = b_0 \delta(t) + [P(D) y_n(t)] u(t)$

$y_n(t)$ *linear combination of characteristic modes*
with the following initial conditions

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) = 0 \qquad y_n^{(N-1)}(0) = 1$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N y(t) = \delta(t)$$

$\frac{d^N y(t)}{dt^N}$ ⇓ $\delta(t)$

$\frac{d^{N-1} y(t)}{dt^{N-1}}$ ⇓ $u(t)$

remaining terms ⇓ *no jump discontinuity is allowed at t = 0*

IC – Impulse Response (2)

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\,x(t)}{dt} + b_N\, x(t)$$

$$\left(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N\right) \cdot y(t) \;=\; \left(b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N\right) \cdot x(t)$$

$$Q(D) \cdot y(t) \;=\; P(D) \cdot x(t)$$

$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) \;=\; 0 \qquad y_n^{(N-1)}(0) \;=\; 1$$

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + a_2 \frac{d^{N-2} y(t)}{dt^{N-2}} + \cdots + a_{N-1} \frac{d\,y(t)}{dt} + a_N\, y(t) \;=\; \delta(t)$$

$\delta(t)$ → (*integration*) → $u(t)$

no jump discontinuity is allowed at t = 0

$$y_n(0) = \dot{y}_n(0) = \ddot{y}_n(0) \;\cdots\; = y_n^{(N-2)}(0) \;=\; 0$$

unit jump discontinuity at t = 0

$$y_n^{(N-1)}(0) \;=\; 1$$

$$y_n^{(N)}(t) \;=\; \delta(t)$$

IC – Classical Solution

$$\frac{d^N y(t)}{dt^N} + a_1 \frac{d^{N-1} y(t)}{dt^{N-1}} + \cdots + a_{N-1} \frac{d\, y(t)}{dt} + a_N y(t) = b_{N-M} \frac{d^M x(t)}{dt^M} + b_{N-M+1} \frac{d^M x(t)}{dt^{M-1}} + \cdots + b_{N-1} \frac{d\, x(t)}{dt} + b_N x(t)$$

$$(D^N + a_1 D^{N-1} + \cdots + a_{N-1} D + a_N) \cdot y(t) = (b_{N-M} D^M + b_{N-M+1} D^{M-1} + \cdots + b_{N-1} D + b_N) \cdot x(t)$$

$$Q(D) \cdot y(t) = P(D) \cdot x(t)$$

$$Q(\lambda) = (\lambda^N + a_1 \lambda^{N-1} + \cdots + a_{N-1} \lambda + a_N) = 0$$

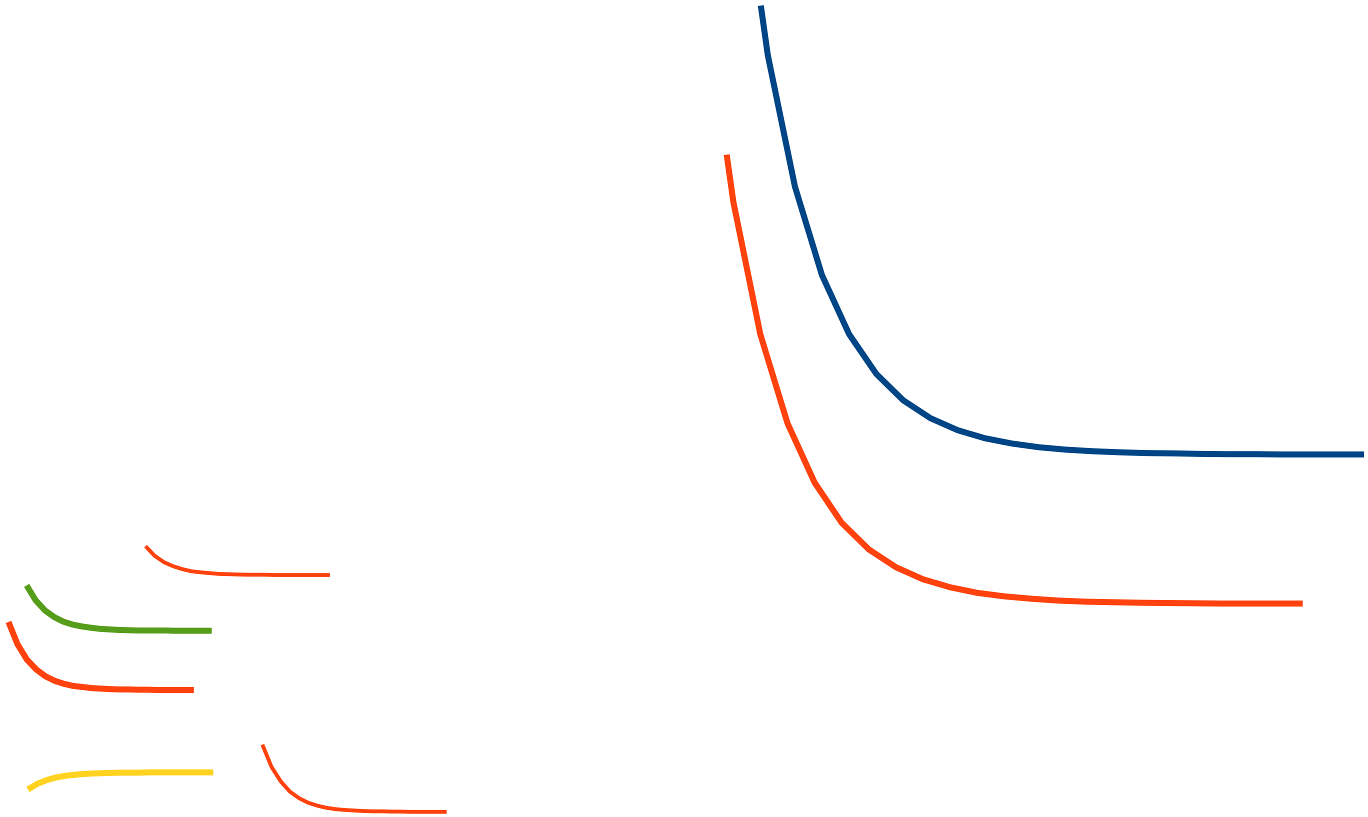
$y_n(t)$ *linear combination of characteristic modes with the following initial conditions*

$$Q(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_N) = 0$$

$y_0(0^+),\ \dot{y}_0(0^+),\ \ddot{y}_0(0^+),\ \cdots$

- *at* $t = 0^-$ *only the* ***zero input response***
- *initial condition at* $t = 0^-$ *only can be applied to* ***zero input response***
- ***zir*** *and* ***zsr*** *cannot be separated in the classical solution*
- *Therefore initial condition at* $t = 0^+$ *must be used.*

Impulse Response h(t)



References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] B.P. Lathi, Linear Systems and Signals (2nd Ed)