DFT Analysis (5B)

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7/29/11

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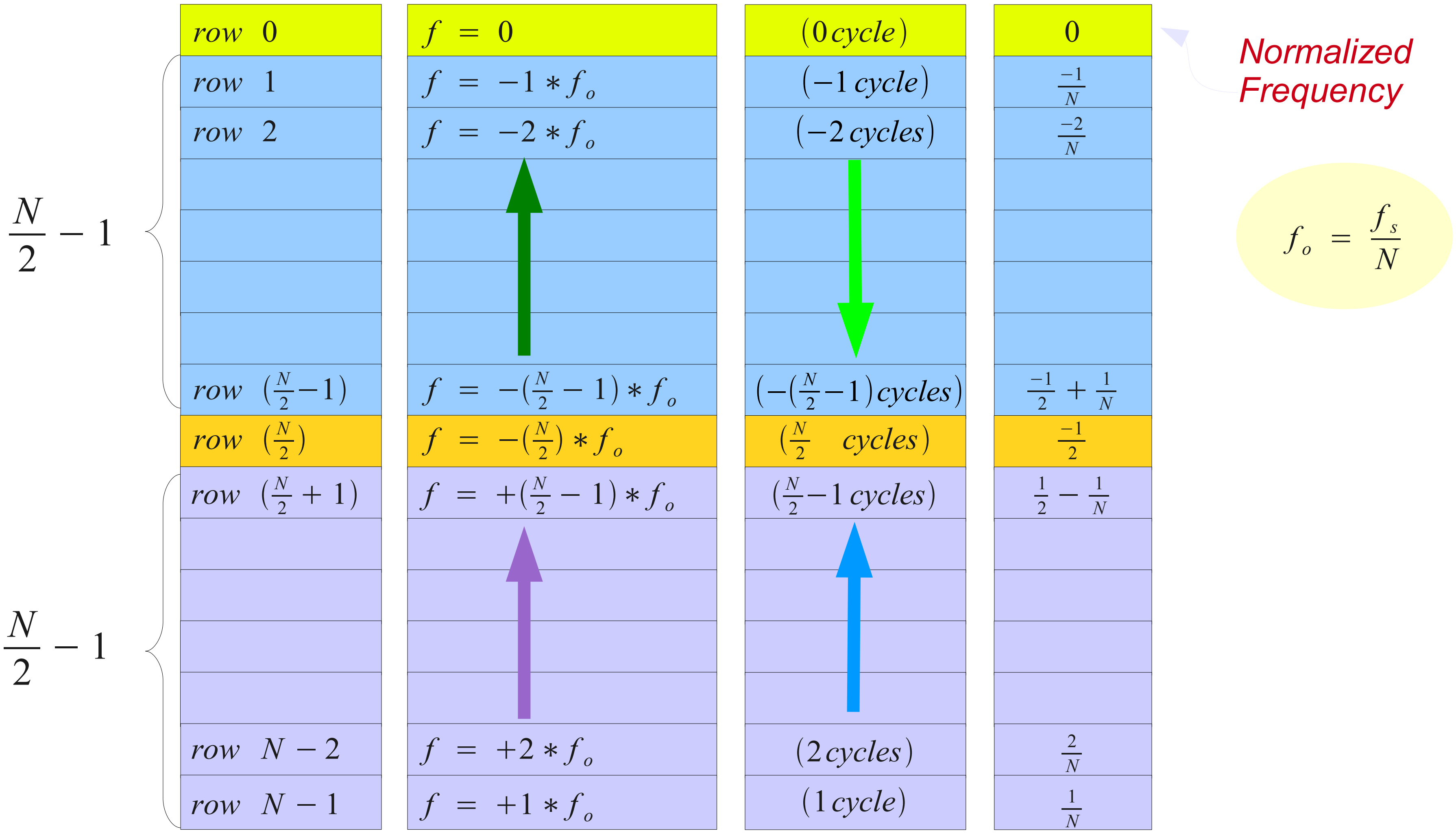
Frequency View of a DFT Matrix

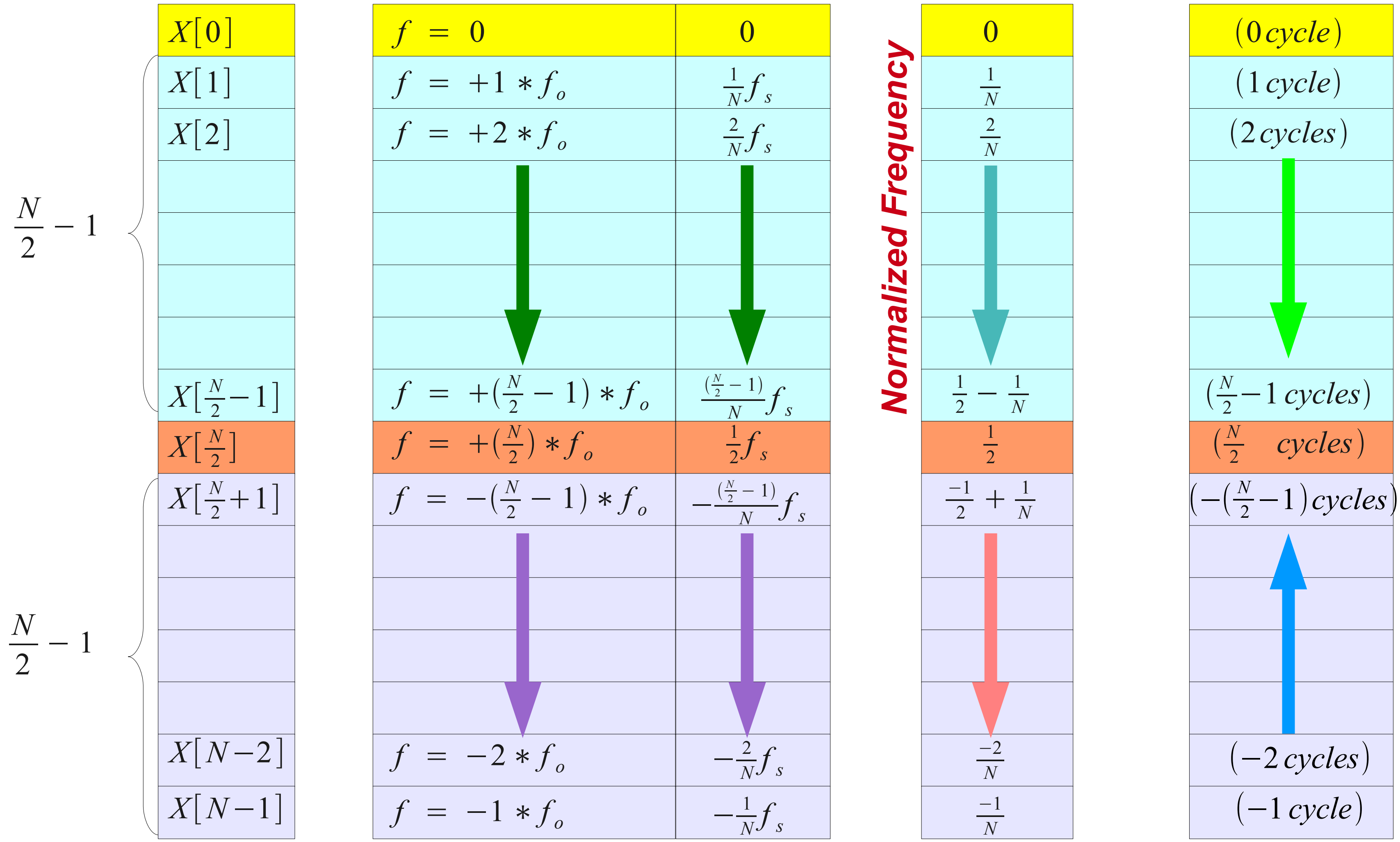


	$row\ \ 0$	$f\ =\ 0$	$(0\,cycle)$	0
	$row\ \ 1$	$f\ =\ -1 * f_o$	$(-1\,cycle)$	$\frac{-1}{N}$
	$row\ \ 2$	$f\ =\ -2 * f_o$	$(-2\,cycles)$	$\frac{-2}{N}$
$\frac{N}{2}-1$				
	$row\ \ (\frac{N}{2}-1)$	$f\ =\ -(\frac{N}{2}-1) * f_o$	$(-(\frac{N}{2}-1)\,cycles)$	$\frac{-1}{2}+\frac{1}{N}$
	$row\ \ (\frac{N}{2})$	$f\ =\ -(\frac{N}{2}) * f_o$	$(\frac{N}{2}\ \ cycles)$	$\frac{-1}{2}$
	$row\ \ (\frac{N}{2}+1)$	$f\ =\ +(\frac{N}{2}-1) * f_o$	$(\frac{N}{2}-1\,cycles)$	$\frac{1}{2}-\frac{1}{N}$
$\frac{N}{2}-1$				
	$row\ \ N-2$	$f\ =\ +2 * f_o$	$(2\,cycles)$	$\frac{2}{N}$
	$row\ \ N-1$	$f\ =\ +1 * f_o$	$(1\,cycle)$	$\frac{1}{N}$

Normalized Frequency

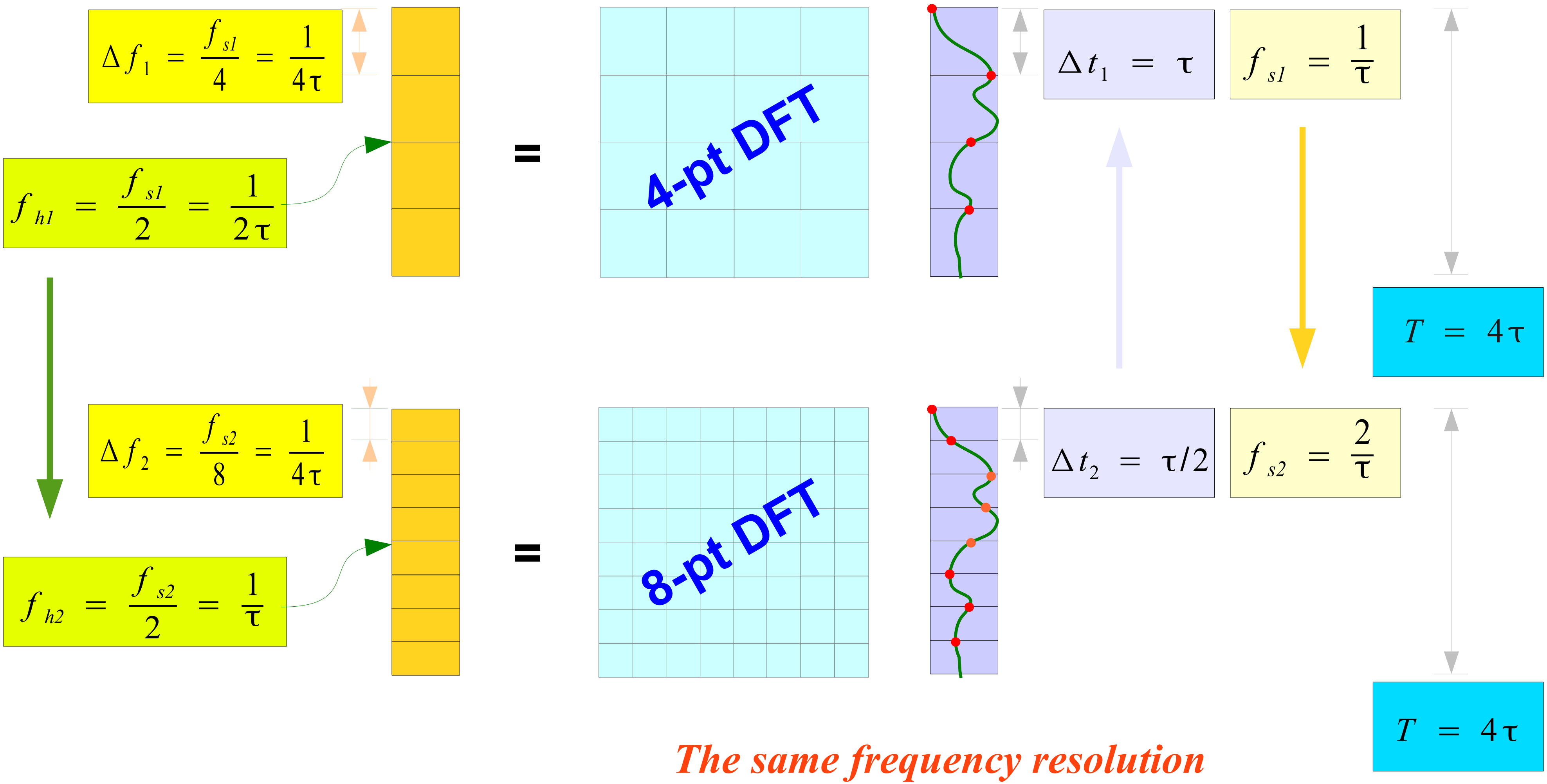
$$f_o\ =\ \frac{f_s}{N}$$

Frequency View of a X[i] Vector



			Normalized Frequency	
$X[0]$	$f = 0$	0	0	$(0\, cycle)$
$X[1]$	$f = +1 * f_o$	$\frac{1}{N} f_s$	$\frac{1}{N}$	$(1\, cycle)$
$X[2]$	$f = +2 * f_o$	$\frac{2}{N} f_s$	$\frac{2}{N}$	$(2\, cycles)$
$X[\frac{N}{2}-1]$	$f = +(\frac{N}{2}-1) * f_o$	$\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{1}{2} - \frac{1}{N}$	$(\frac{N}{2}-1\, cycles)$
$X[\frac{N}{2}]$	$f = +(\frac{N}{2}) * f_o$	$\frac{1}{2} f_s$	$\frac{1}{2}$	$(\frac{N}{2}\quad cycles)$
$X[\frac{N}{2}+1]$	$f = -(\frac{N}{2}-1) * f_o$	$-\frac{(\frac{N}{2}-1)}{N} f_s$	$\frac{-1}{2} + \frac{1}{N}$	$(-(\frac{N}{2}-1)\, cycles)$
$X[N-2]$	$f = -2 * f_o$	$-\frac{2}{N} f_s$	$\frac{-2}{N}$	$(-2\, cycles)$
$X[N-1]$	$f = -1 * f_o$	$-\frac{1}{N} f_s$	$\frac{-1}{N}$	$(-1\, cycle)$

Frequency and Time Interval (1)



Freq Domain

Time Domain

$$\Delta f_1 = \frac{f_{s1}}{4} = \frac{1}{4\tau}$$

$$f_{h1} = \frac{f_{s1}}{2} = \frac{1}{2\tau}$$

=

$$\Delta t_1 = \tau$$

$$f_{s1} = \frac{1}{\tau}$$

$$T = 4\tau$$

$$\Delta f_2 = \frac{f_{s2}}{8} = \frac{1}{4\tau}$$

$$f_{h2} = \frac{f_{s2}}{2} = \frac{1}{\tau}$$

=

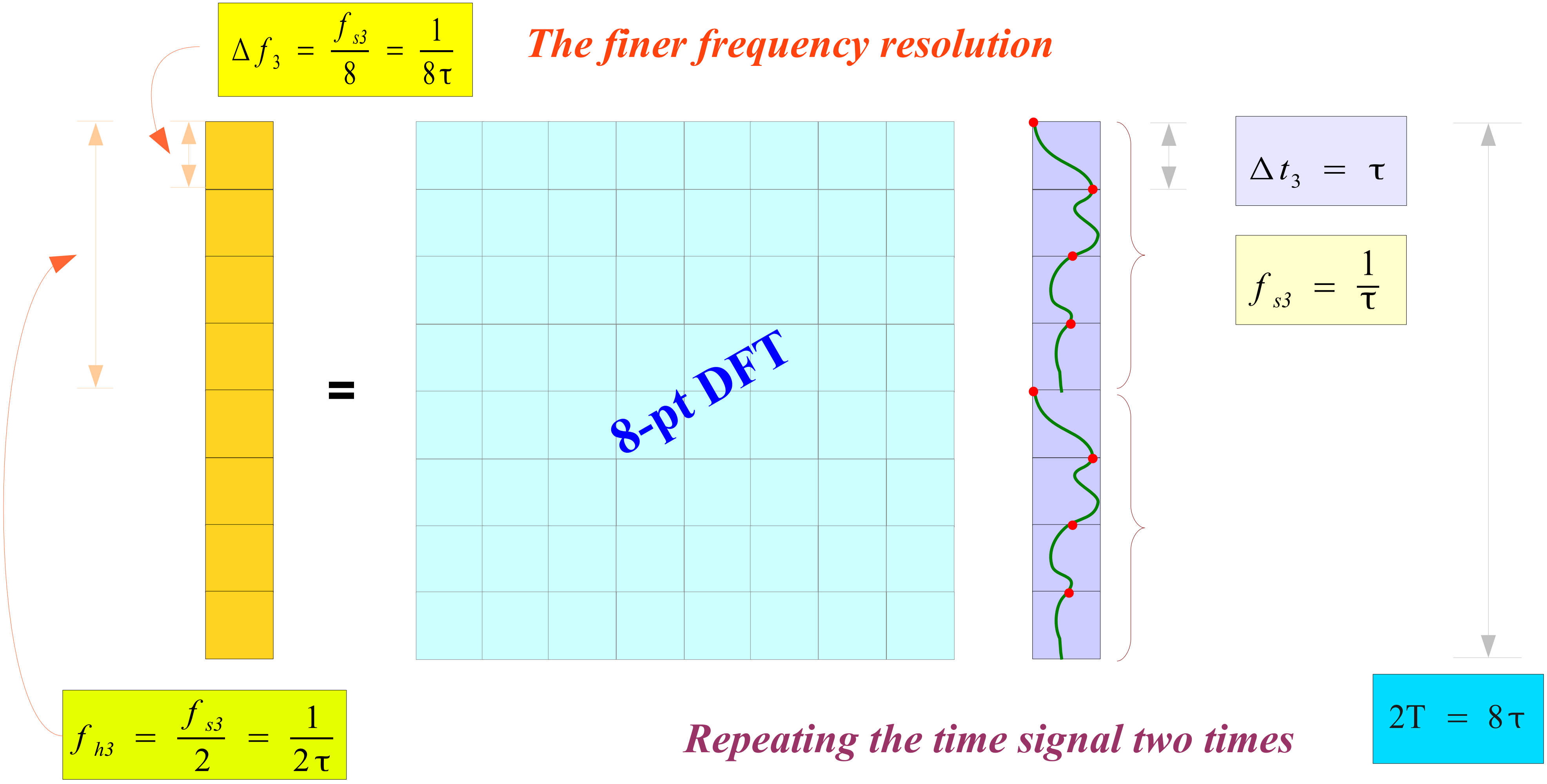
$$\Delta t_2 = \tau/2$$

$$f_{s2} = \frac{2}{\tau}$$

$$T = 4\tau$$

The same frequency resolution

Frequency and Time Interval (2)



$$\Delta f_3 = \frac{f_{s3}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution

$$\Delta t_3 = \tau$$

$$f_{s3} = \frac{1}{\tau}$$

=

$$f_{h3} = \frac{f_{s3}}{2} = \frac{1}{2\tau}$$

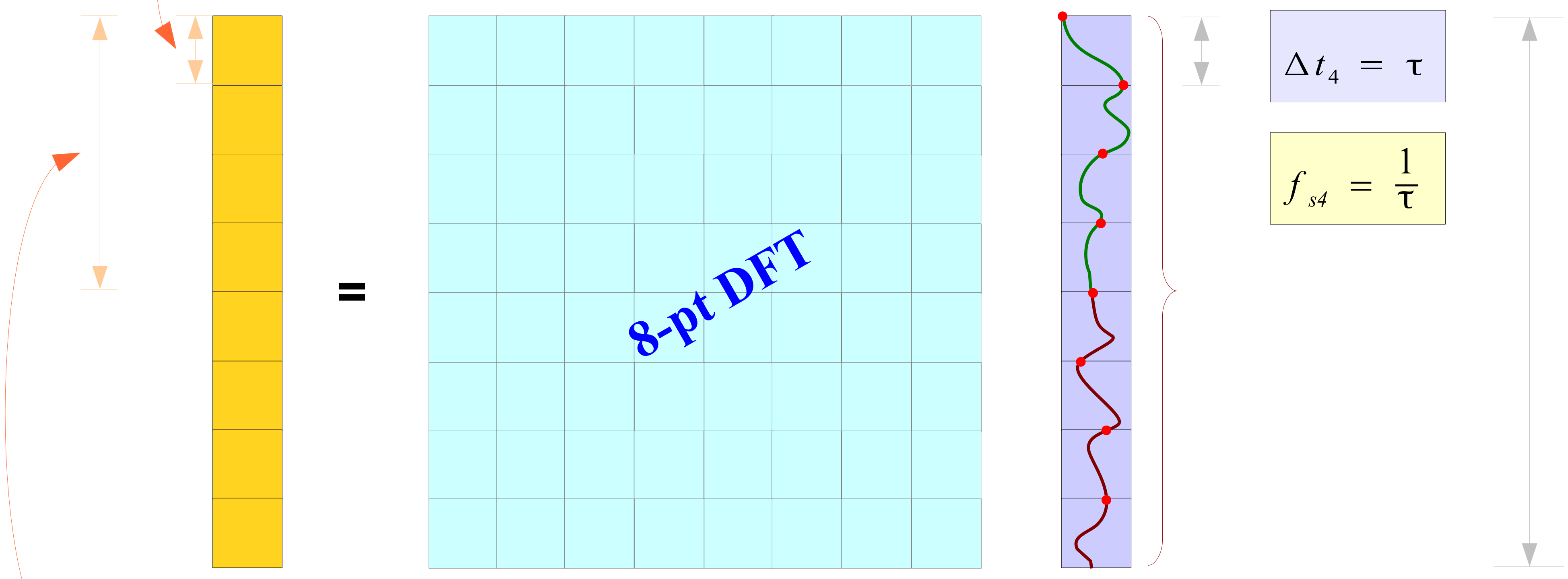
Repeating the time signal two times

$$2T = 8\tau$$

Frequency and Time Interval (3)

$$\Delta f_4 = \frac{f_{s4}}{8} = \frac{1}{8\tau}$$

The finer frequency resolution



$$\Delta t_4 = \tau$$

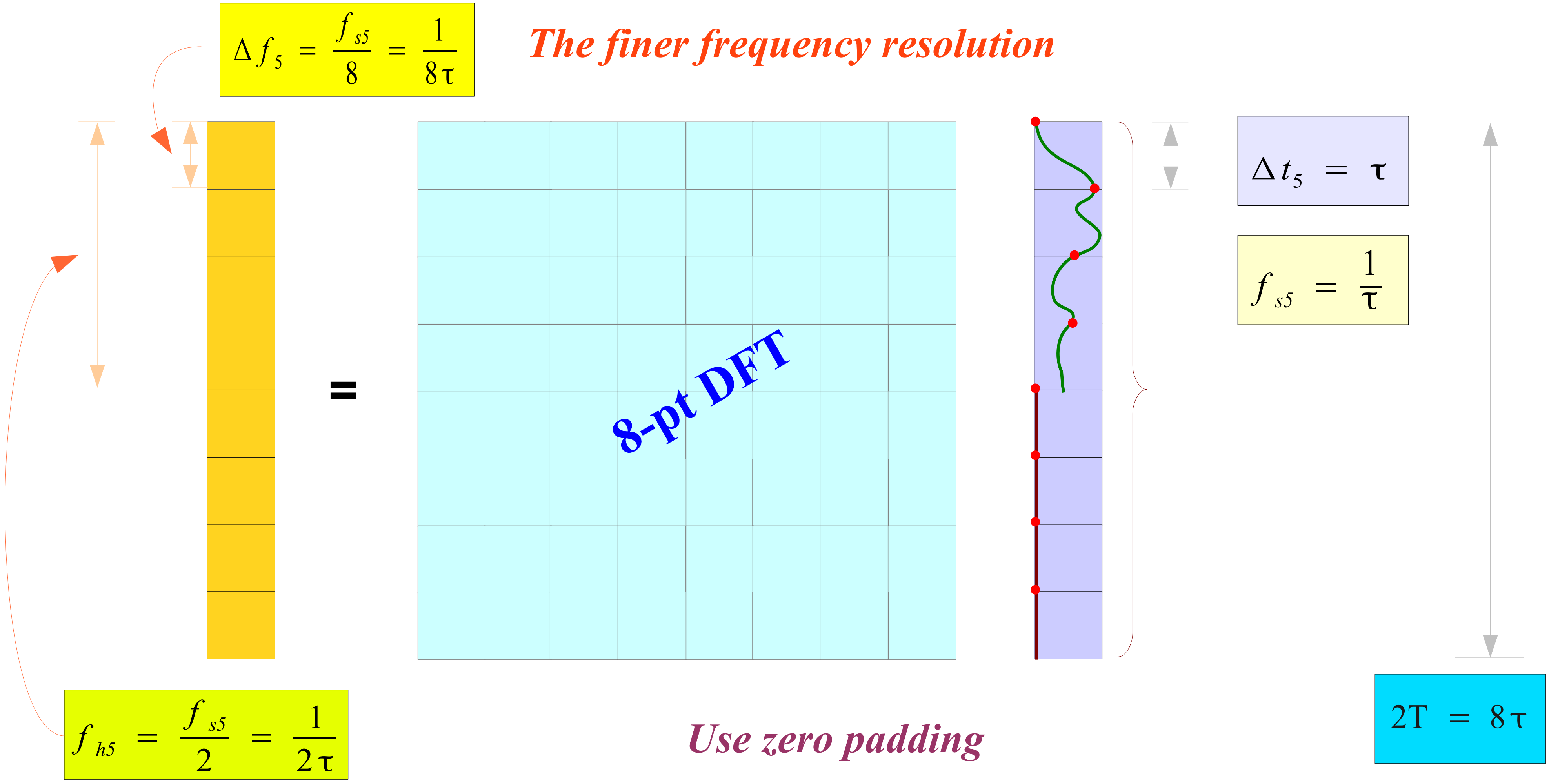
$$f_{s4} = \frac{1}{\tau}$$

$$f_{h4} = \frac{f_{s4}}{2} = \frac{1}{2\tau}$$

Use two times more samples (2T)

$$2T = 8\tau$$

Frequency and Time Interval (4)



$\Delta f_5 = \frac{f_{s5}}{8} = \frac{1}{8\tau}$

The finer frequency resolution

$=$

$\Delta t_5 = \tau$

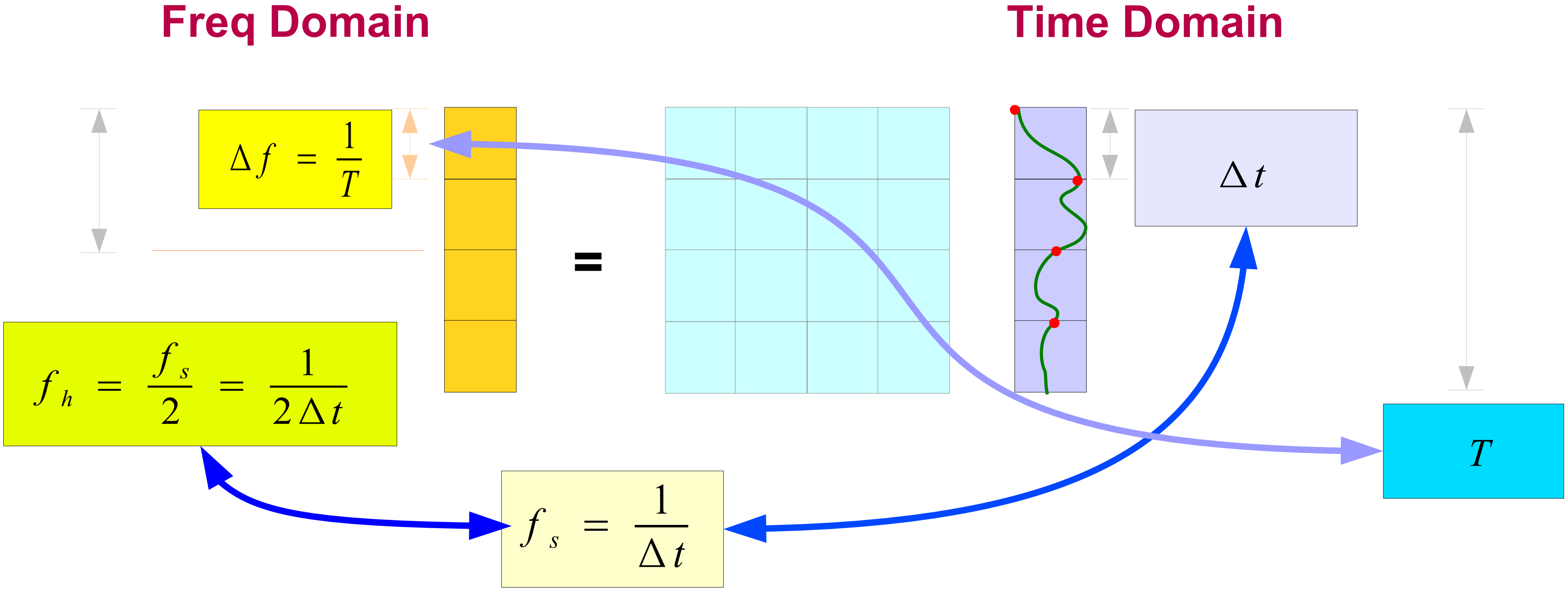
$f_{s5} = \frac{1}{\tau}$

$f_{h5} = \frac{f_{s5}}{2} = \frac{1}{2\tau}$

Use zero padding

$2T = 8\tau$

Frequency and Time Interval (5)



Freq Domain

Time Domain

$\Delta f = \frac{1}{T}$

$f_h = \frac{f_s}{2} = \frac{1}{2\Delta t}$

$f_s = \frac{1}{\Delta t}$

Δt

T

	Periodic Signals	Aperiodic Signals	Random Signals	
Frequency Spacing	$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$	$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$	$\frac{1}{N \Delta t} \sum x^2 \Delta t$
Two Sided	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S(k) = \frac{\Delta t}{N} \lvert X(k) \rvert^2$	$P = \sum_{k=0}^{N-1} S(k) \Delta f$
One Sided				
$k=0,\ \frac{N}{2}$	$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$	$S_1(k) = 2S(k)$	$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$
$k=1, \cdots, \frac{N}{2}-1$	$\frac{2}{N} X(k)$	$\frac{2 \Delta t}{N} X(k)$	$S_1(k) = S(k)$	
Frequency Scale	$k \Delta f$	$k \Delta f$	$k \Delta f$	

Periodic Signals	Aperiodic Signals
Frequency Spacing	
$\Delta f = \frac{1}{N \Delta t}$	$\Delta f = \frac{1}{N \Delta t}$
Two Sided Fourier Series Coefficient	Two Sided Fourier Series Coefficient
$\frac{1}{N} X(k)$	$\frac{\Delta t}{N} X(k)$
One Sided Fourier Series Coefficient	One Sided Fourier Series Coefficient
$\frac{1}{N} X(k) \qquad k = 0, \ \frac{N}{2}$	$\frac{\Delta t}{N} X(k) \qquad k = 0, \ \frac{N}{2}$
$\frac{2}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$	$\frac{2 \Delta t}{N} X(k) \qquad k = 1, \cdots, \frac{N}{2} - 1$
Frequency Scale	
$k \Delta f$	$k \Delta f$

Random Signals

One-sided Power Spectral Density

$$P=\sum_{k=0}^{N-1} S(k)\Delta f$$

One-sided Power Spectral Density

$$P=\sum_{k=0}^{N/2} S_1(k)\Delta f$$

$$S_1(k)=2\mathrm{S}(k) \qquad k=1,\ldots,\tfrac{N}{2}-1$$

$$S_1(k)=S(k) \qquad k=0,\tfrac{N}{2}$$

Two Sided Fourier Series Coefficient

$$\frac{1}{N\Delta t}\sum x^2\Delta t$$

$$\sum S\Delta f=\frac{1}{N\Delta t}\sum S$$

$$S(k)=\frac{\Delta t}{N}|X(k)|^2$$

$$k\Delta f$$

Amplitude Spectrum

$$A_k = \tfrac{1}{N}\left|X(k)\right| = \tfrac{1}{N}\sqrt{\Re^2(X(k))+\Im^2(X(k))}$$

$$k = 0, 1, 2, \cdots, N-1$$

One Sided Amplitude Spectrum

$$\bar{A}_k = \tfrac{1}{N}\left|X(0)\right| \qquad k=0$$

$$\bar{A}_k = \tfrac{2}{N}\left|X(0)\right| \qquad k=1, 2, \cdots, N/2$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im(X(k))}{\Re(X(k))}\right) \qquad k=0, 1, 2, \cdots, N-1$$

Power Spectrum

$$P_k = \tfrac{1}{N^2}\left|X(k)\right|^2 = \tfrac{1}{N^2}\left\{\Re^2(X(k))+\Im^2(X(k))\right\}$$

$$k = 0, 1, 2, \cdots, N-1$$

One Sided Power Spectrum

$$\bar{P}_k = \tfrac{1}{N^2}\left|X(0)\right|^2 \qquad k=0$$

$$\bar{P}_k = \tfrac{2}{N^2}\left|X(0)\right|^2 \qquad k=1, 2, \cdots, N/2$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Data Truncation
Frequency Resolution
Zero Padding
Periodogram
Spectral Plot

Amplitude spectrum in quantity peak
Phase spectrum in radians
Amplitude spectrum in volts rms
Phase spectrum in degrees
Power spectrum

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple of the FFT length

Leakage

$[0, \quad \frac{f_s}{2}]$

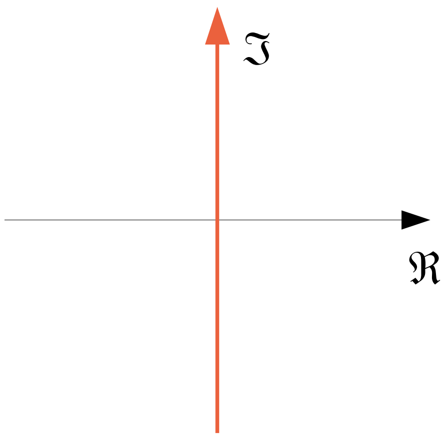
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$$f(t)\, e^{-j\omega t}$$

$$-\infty < \omega < +\infty$$

Not so useful in transient analysis

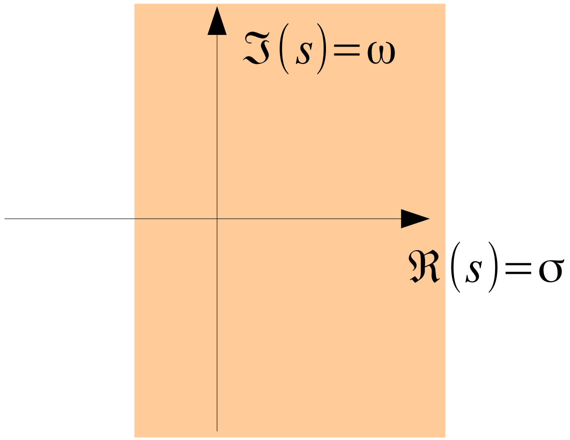


Laplace Transform

$$f(t)\, e^{-st} = f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



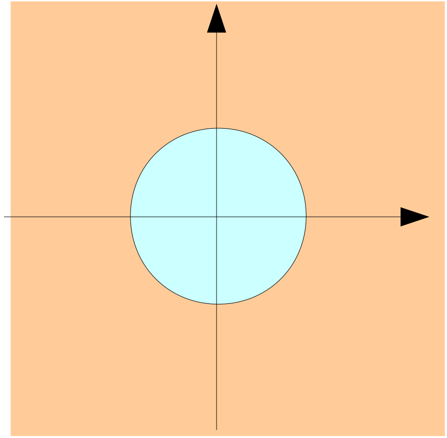
z Transform

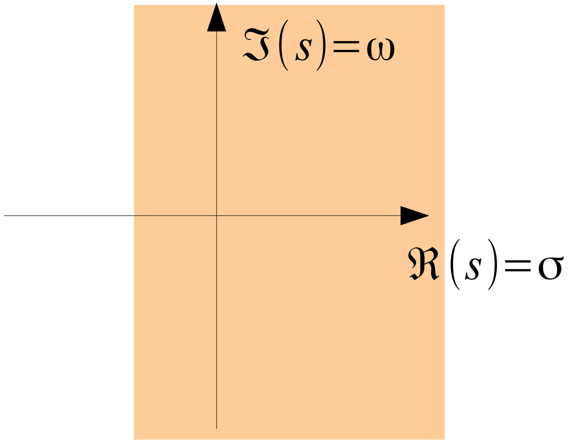
$$f[n]\, z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T}\, e^{j\omega T}$$





References

[1] http://en.wikipedia.org/
[2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
[3] A “graphical interpretation” of the DFT and FFT, by Steve Mann