

Hermitian Inner Product Space (3B)

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Complex Vector Inner Product

Hermitian inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} = \sum x_i^* y_i$$

\mathbf{x}^H : conjugate transpose

Norm of Hermitian inner products

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^H \cdot \mathbf{x}} = \sqrt{\sum x_i^* x_i}$$

the length of a vector

$$\mathbf{x} = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \cdot \mathbf{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - jb_1 & a_2 - jb_2 & \cdots & a_n - jb_n \end{pmatrix} \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix} = \sum_{i=1}^n a_i^2 + b_i^2$$

Cauchy-Schwartz Inequality

For all vectors \mathbf{x} and \mathbf{y} of an inner product space

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent \Rightarrow maximum

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \quad \mathbf{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \quad \mathbf{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

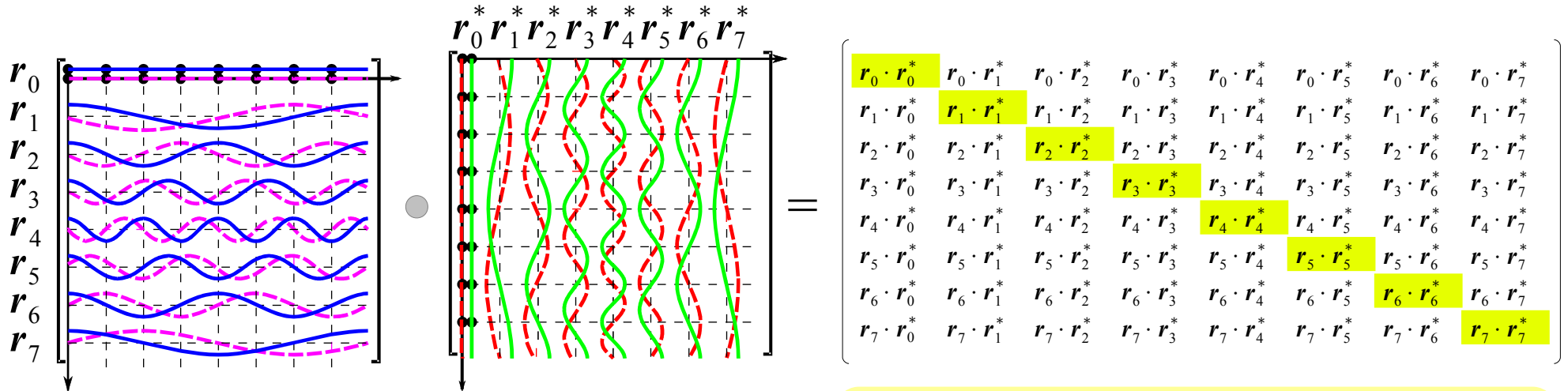
Inner product is maximum
when $\mathbf{y} = k \mathbf{x}$

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq k \left(\sum_{i=1}^n a_i^2 + b_i^2 \right)$$

Orthogonality

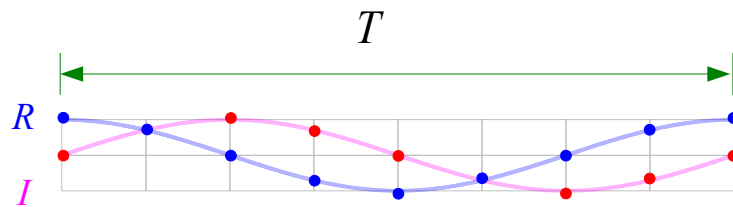
$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{cases} A^H = B \\ B^H = A \end{cases} \begin{cases} AB = NI \\ BA = NI \end{cases} \Rightarrow \begin{cases} A^H A = A A^H = NI \\ B^H B = B B^H = NI \end{cases}$$



$$\begin{aligned} \langle \mathbf{r}_i^H, \mathbf{r}_i^H \rangle &= \mathbf{r}_i \cdot \mathbf{r}_i^* = N \\ \langle \mathbf{r}_i^H, \mathbf{r}_j^H \rangle &= \mathbf{r}_i \cdot \mathbf{r}_j^* = 0 \quad (i \neq j) \end{aligned}$$

Inner Product Examples (1)

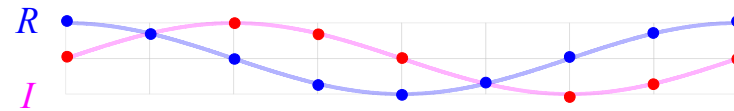
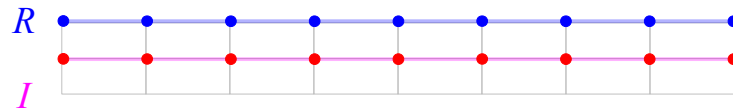


$$f_0 = 1/T$$

$$\omega_0 = 2\pi/T$$

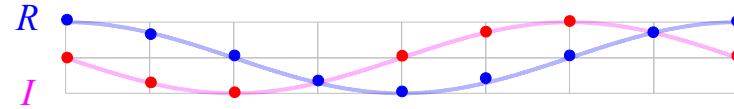
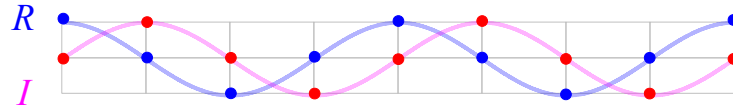
$$\leftarrow e^{j1\omega_0 t}$$

$$e^{+j(1-1)\omega_0 t} = e^{+j0\omega_0 t}$$



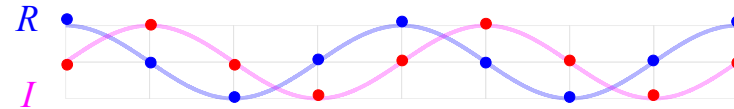
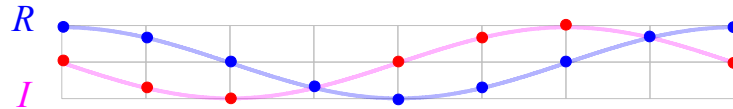
$$\leftarrow e^{j1\omega_0 t}$$

$$e^{+j(1+1)\omega_0 t} = e^{+j2\omega_0 t}$$



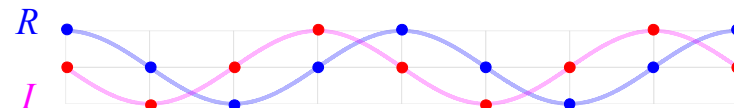
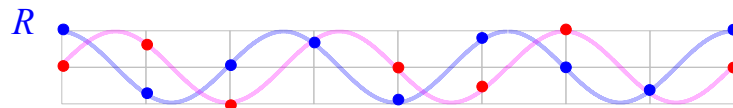
$$\leftarrow e^{j(-1)\omega_0 t}$$

$$e^{+j(1-2)\omega_0 t} = e^{+j(-1)\omega_0 t}$$



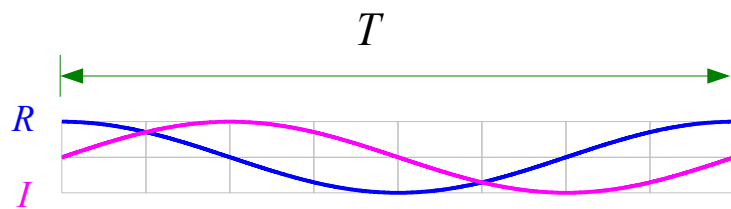
$$\leftarrow e^{j2\omega_0 t}$$

$$e^{+j(1+2)\omega_0 t} = e^{+j3\omega_0 t}$$



$$\leftarrow e^{j(-2)\omega_0 t}$$

Inner Product Examples (2)

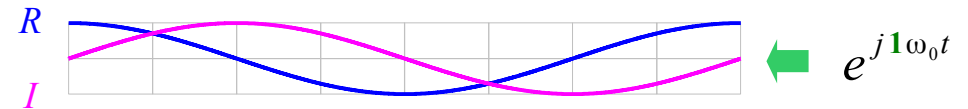


$$f_0 = 1/T$$

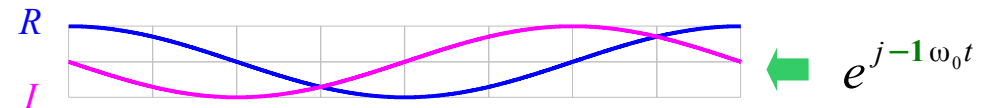
$$\omega_0 = 2\pi/T$$

$$\leftarrow e^{j1\omega_0 t}$$

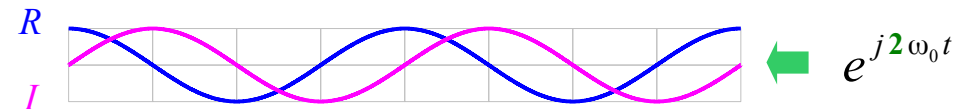
$$\langle e^{j1\omega_0 t}, e^{j1\omega_0 t} \rangle = \int_0^T e^{+j(1-1)\omega_0 t} dt = T \quad \leftarrow$$



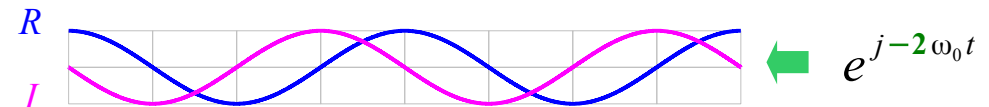
$$\langle e^{j1\omega_0 t}, e^{j-1\omega_0 t} \rangle = \int_0^T e^{+j(1+1)\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j1\omega_0 t}, e^{j2\omega_0 t} \rangle = \int_0^T e^{+j(1-2)\omega_0 t} dt = 0 \quad \leftarrow$$



$$\langle e^{j1\omega_0 t}, e^{j-2\omega_0 t} \rangle = \int_0^T e^{+j(1+2)\omega_0 t} dt = 0 \quad \leftarrow$$



N=8 DFT Matrix in Cosine and Sine Terms

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j \sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

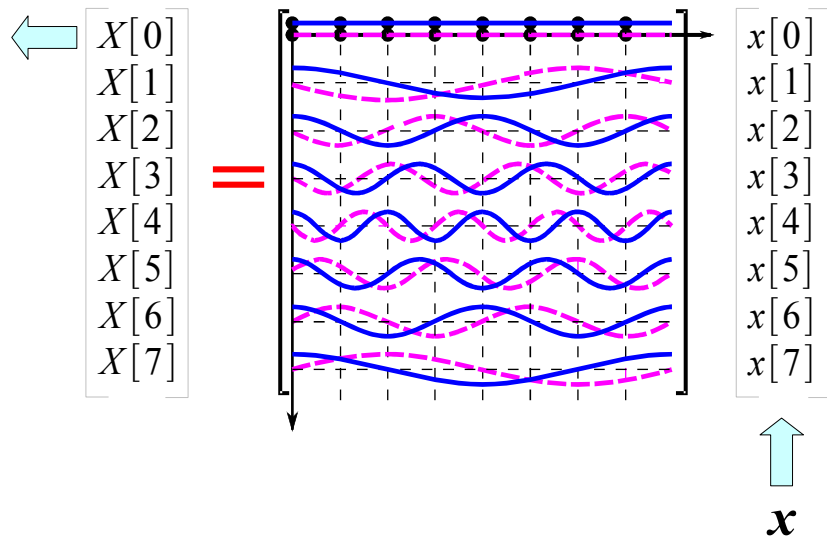
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 0$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 0$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 1$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 3$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 5$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 7$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 1$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 3$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 5$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 7$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 6$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 6$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 3$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 1$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 7$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 5$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 3$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 1$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 7$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 5$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 4$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 4$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 5$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 7$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 1$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 3$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 5$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 7$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 1$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 3$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 2$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 2$
$\cos(\pi/4)\cdot 0$	$\cos(\pi/4)\cdot 7$	$\cos(\pi/4)\cdot 6$	$\cos(\pi/4)\cdot 5$	$\cos(\pi/4)\cdot 4$	$\cos(\pi/4)\cdot 3$	$\cos(\pi/4)\cdot 2$	$\cos(\pi/4)\cdot 1$
$-j \sin(\pi/4)\cdot 0$	$-j \sin(\pi/4)\cdot 7$	$-j \sin(\pi/4)\cdot 6$	$-j \sin(\pi/4)\cdot 5$	$-j \sin(\pi/4)\cdot 4$	$-j \sin(\pi/4)\cdot 3$	$-j \sin(\pi/4)\cdot 2$	$-j \sin(\pi/4)\cdot 1$

N=8 DFT Matrix Real and Imaginary Terms

$$W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn} = \cos\left(\frac{\pi}{4}\cdot k\cdot n\right) - j \sin\left(\frac{\pi}{4}\cdot k\cdot n\right)$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\ 0 & -j & -1 & +j & 0 & -j & -1 & +j \\ 0 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & +j & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 \\ 0 & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \\ 0 & +j & -1 & -j & 0 & +j & -1 & -j \\ 0 & +\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & +j & -\frac{1}{\sqrt{2}}+j\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} & -j & +\frac{1}{\sqrt{2}}-j\frac{1}{\sqrt{2}} \end{bmatrix}$$

N=8 DFT : Inner Product X[0]

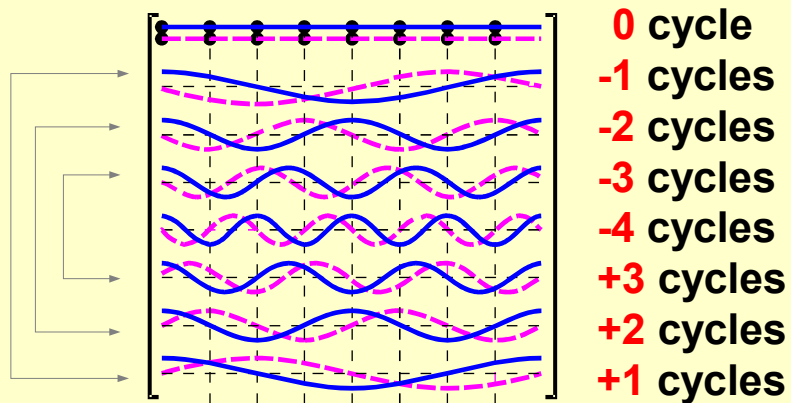


$X[0]$ measures "0 cycle" component in \mathbf{x}

$$\langle \mathbf{r}_0^H, \mathbf{x} \rangle = \mathbf{r}_0 \cdot \mathbf{x} \leq \|\mathbf{r}_0^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_0^H$

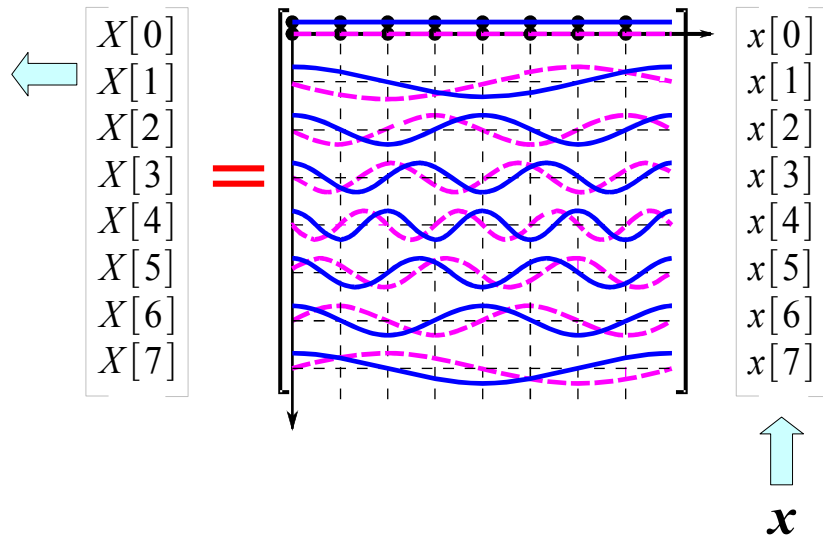
When \mathbf{x} looks like this, $X[0]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[1]

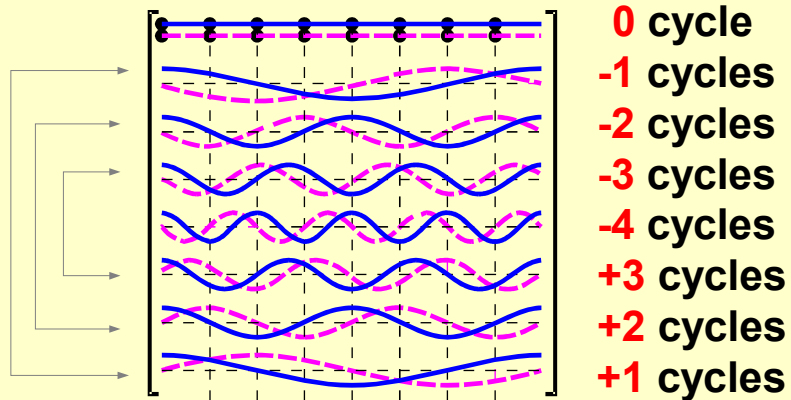


X[1] measures "+1 cycle" component in x

$$\langle \mathbf{r}_1^H, \mathbf{x} \rangle = \mathbf{r}_1 \cdot \mathbf{x} \leq \|\mathbf{r}_1^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_1^H$

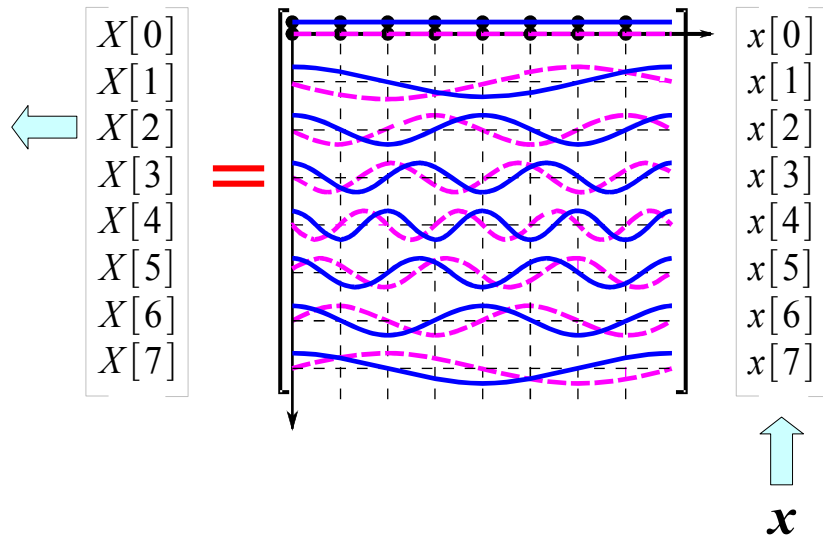
When x looks like this, $X[1]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[2]

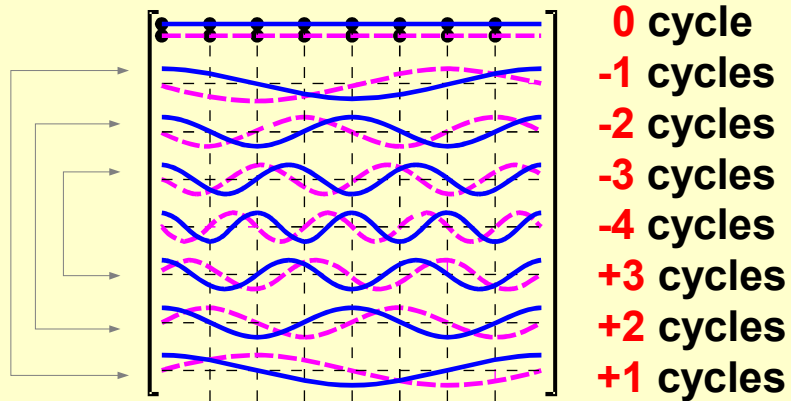


X[2] measures “+2 cycle” component in **x**

$$\langle \mathbf{r}_2^H, \mathbf{x} \rangle = \mathbf{r}_2 \cdot \mathbf{x} \leq \|\mathbf{r}_2^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_2^H$

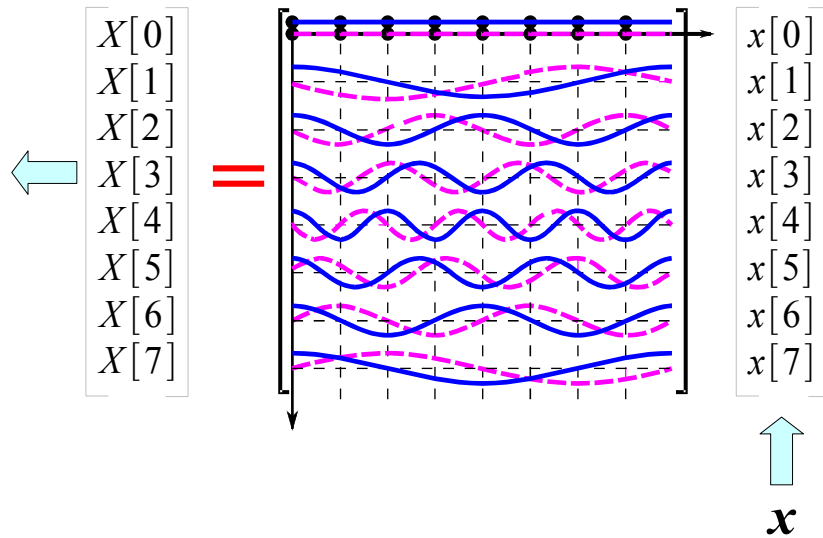
When \mathbf{x} looks like this, $X[2]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

..... $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[3]

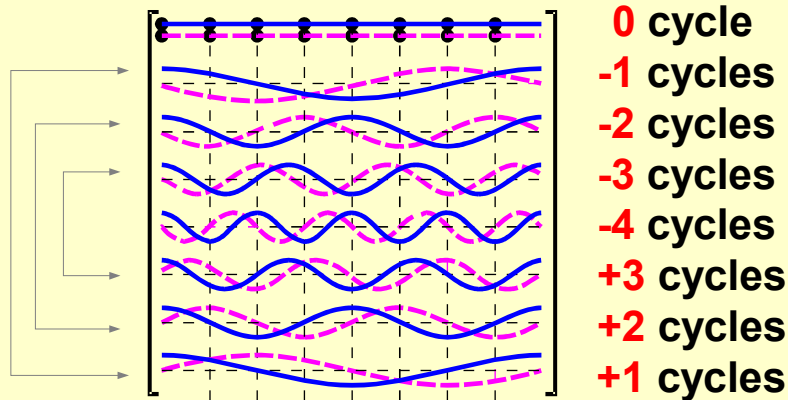


$X[3]$ measures “+3 cycle” component in \mathbf{x}

$$\langle \mathbf{r}_3^H, \mathbf{x} \rangle = \mathbf{r}_3 \cdot \mathbf{x} \leq \|\mathbf{r}_3^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_3^H$

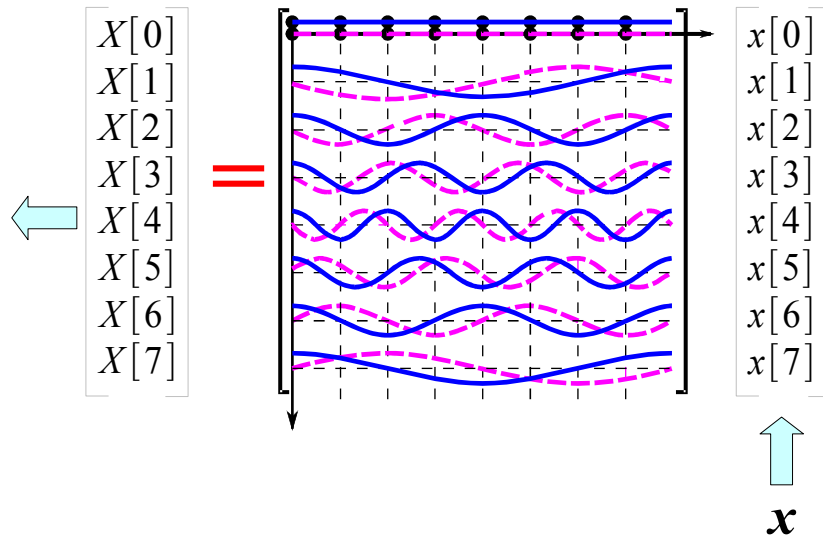
When \mathbf{x} looks like this, $X[3]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[4]

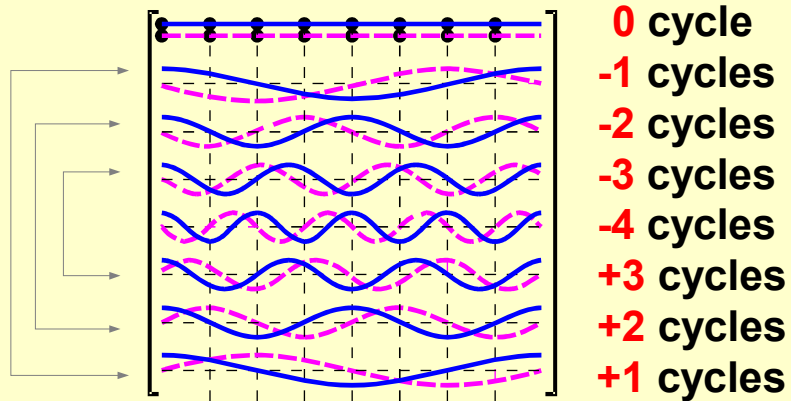


$X[4]$ measures “+4 cycle” component in \mathbf{x}

$$\langle \mathbf{r}_4^H, \mathbf{x} \rangle = \mathbf{r}_4 \cdot \mathbf{x} \leq \|\mathbf{r}_4^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_4^H$

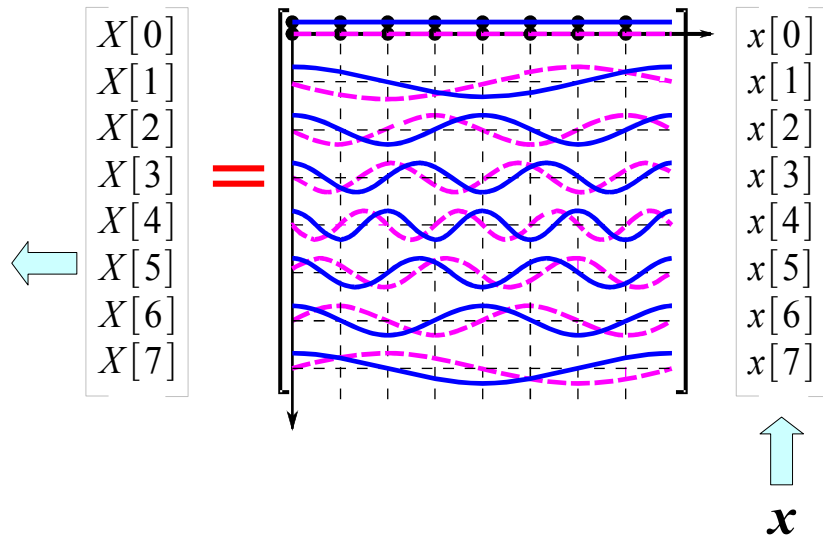
When \mathbf{x} looks like this, $X[4]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[5]

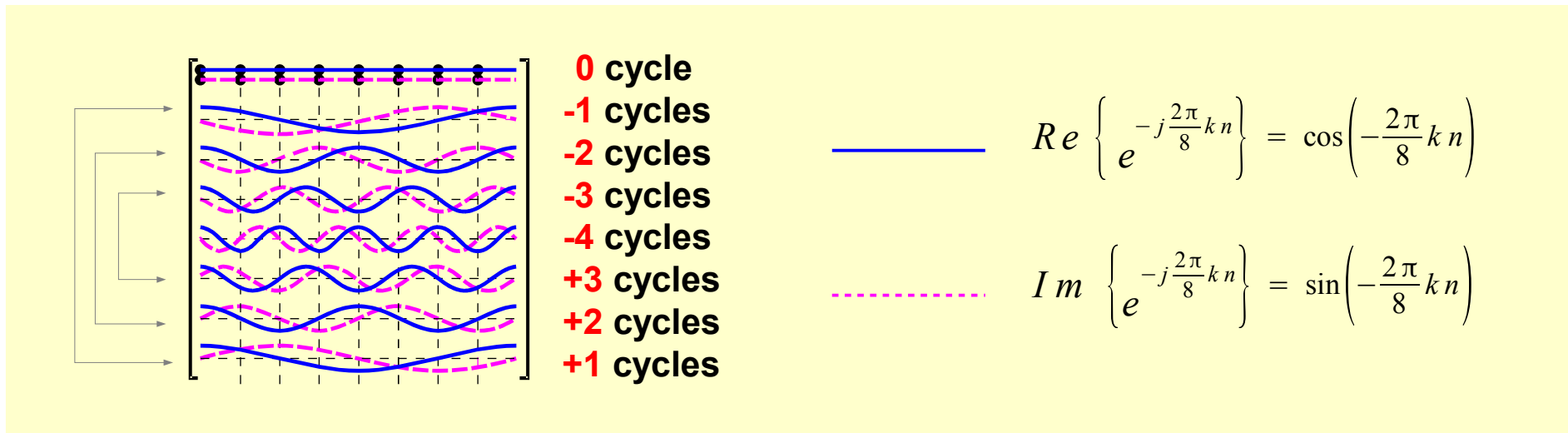


X[5] measures “-3 cycle” component in **x**

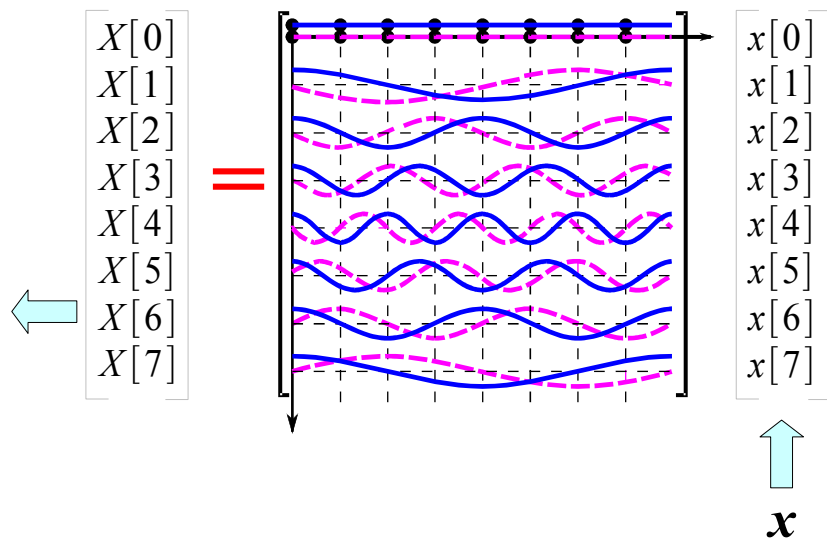
$$\langle \mathbf{r}_5^H, \mathbf{x} \rangle = \mathbf{r}_5 \cdot \mathbf{x} \leq \|\mathbf{r}_5^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_5^H$

When \mathbf{x} looks like this, $X[5]$ is max.



N=8 DFT : Inner Product X[6]

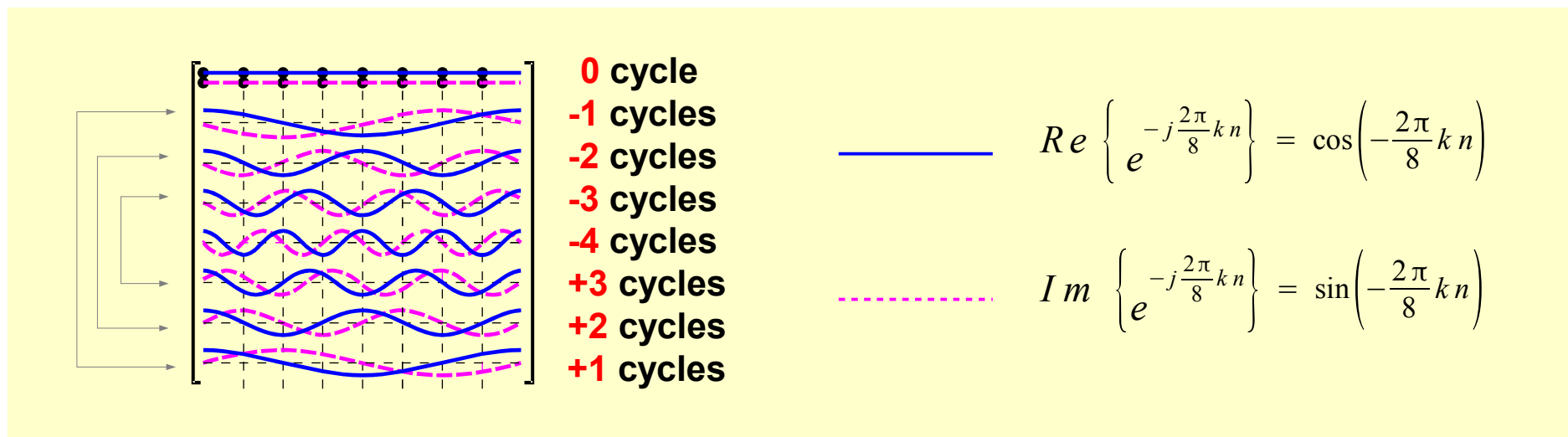


X[6] measures “-2 cycle” component in **x**

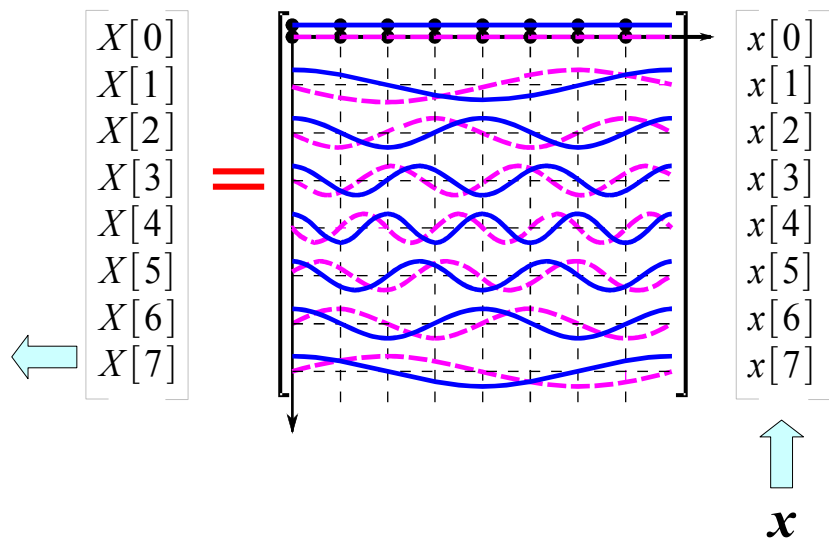
$$\langle \mathbf{r}_6^H, \mathbf{x} \rangle = \mathbf{r}_6 \cdot \mathbf{x} \leq \|\mathbf{r}_6^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_6^H$

When \mathbf{x} looks like this, $X[6]$ is max.



N=8 DFT : Inner Product X[7]

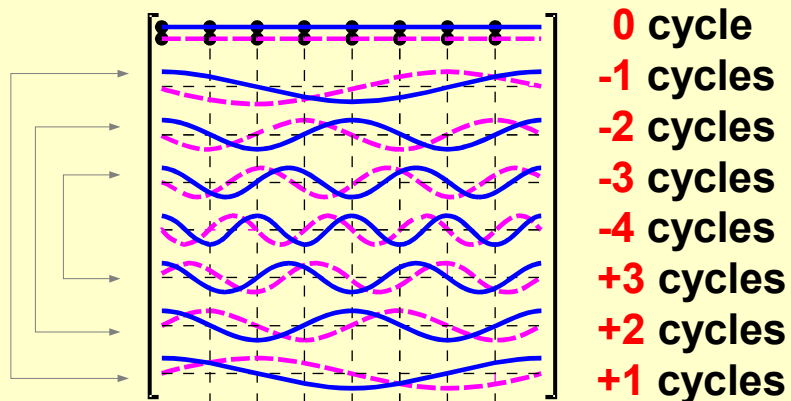


$X[7]$ measures “-1 cycle” component in x

$$\langle \mathbf{r}_7^H, \mathbf{x} \rangle = \mathbf{r}_7 \cdot \mathbf{x} \leq \|\mathbf{r}_7^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_7^H$

When x looks like this, $X[7]$ is max.

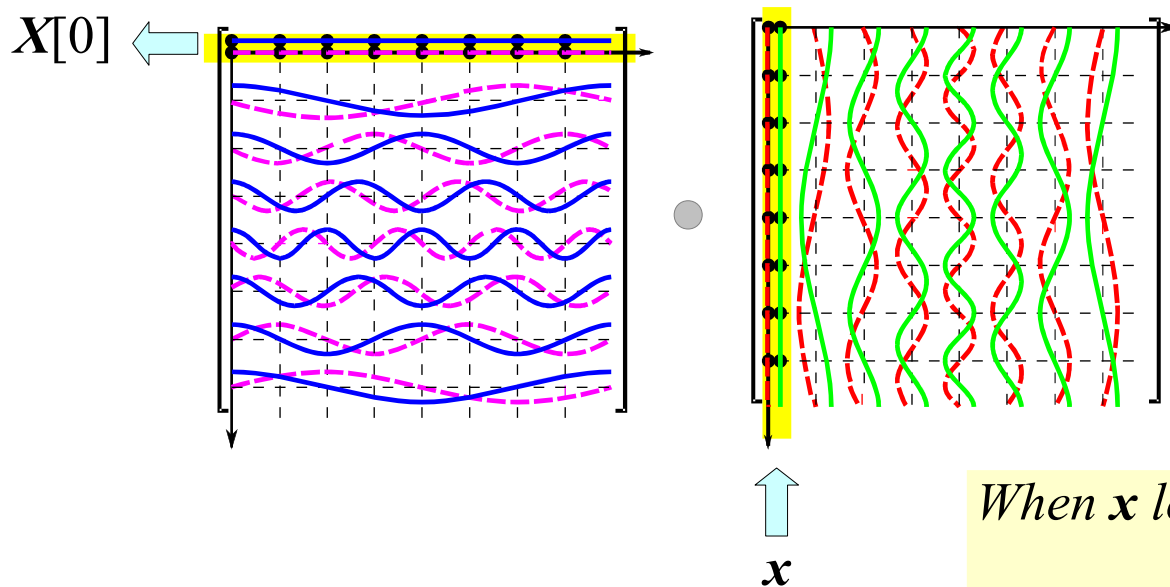


$$\text{Re} \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$$

$$\text{Im} \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$$

N=8 DFT : X[0] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

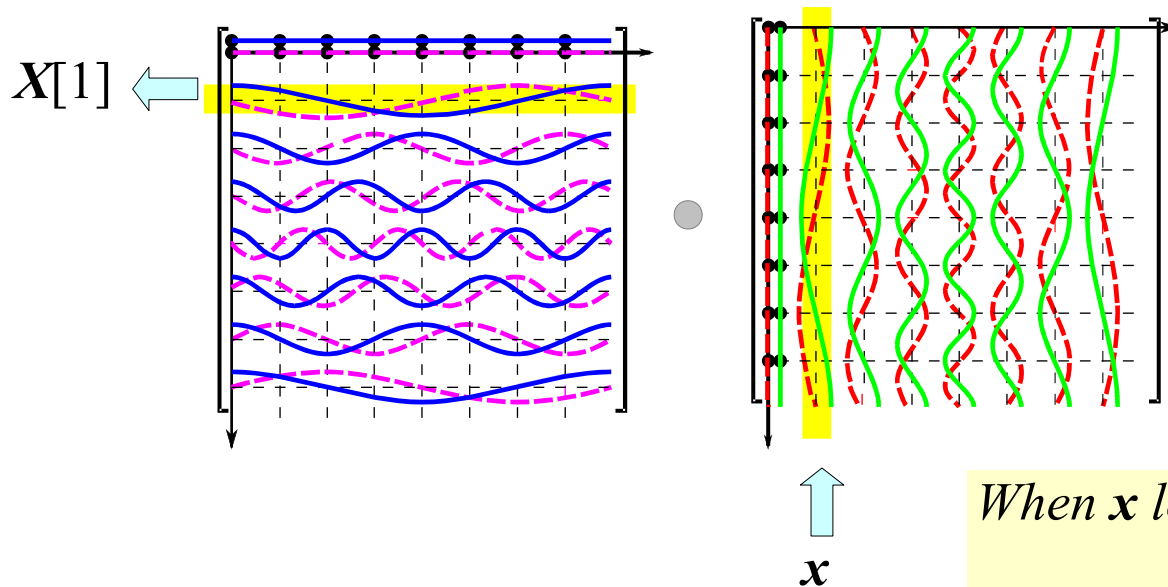
Unitary Matrix

When x looks like this, $X[0]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 0$

$$X[0] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix} \bullet \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}^T$$

N=8 DFT : X[1] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

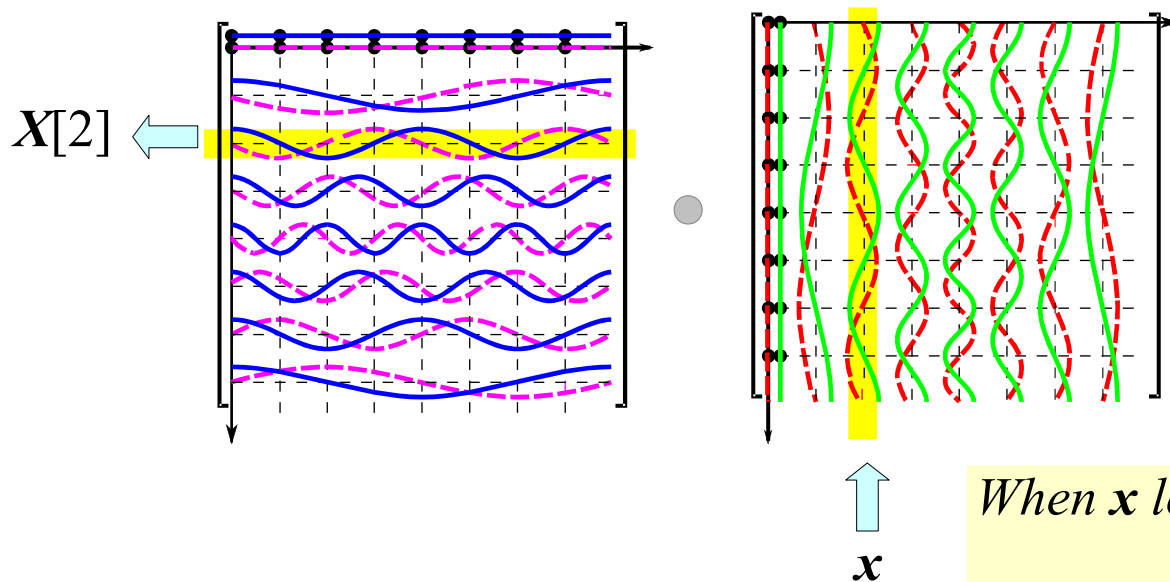
Unitary Matrix

When x looks like this, $X[1]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 1$

$$X[1] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 1} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 5} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 7} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[2] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

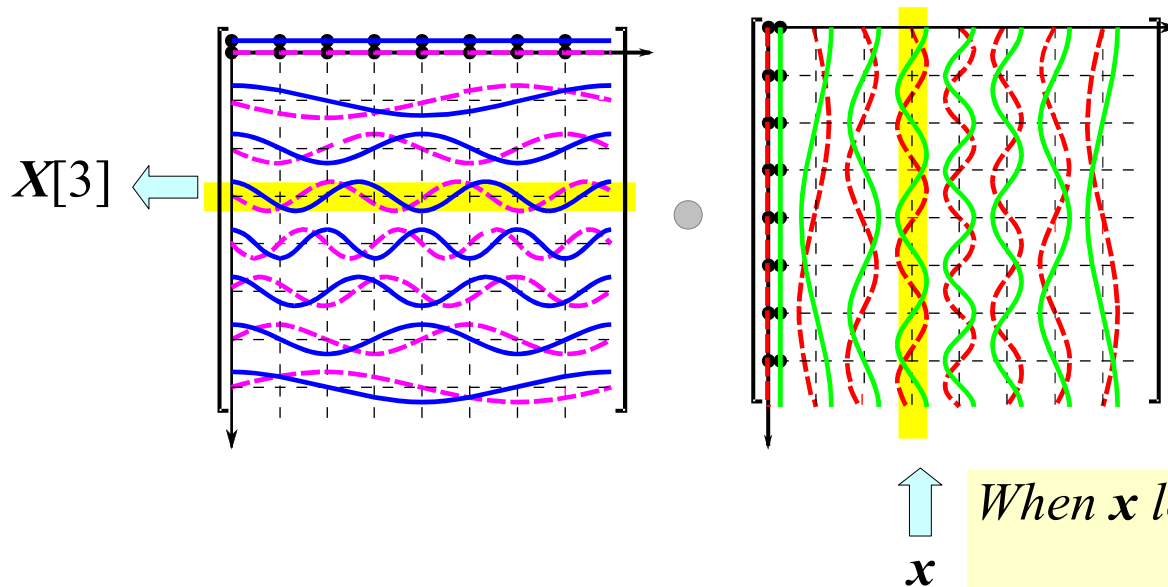
Unitary Matrix

When x looks like this, $X[2]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 2$

$$X[2] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 6} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[3] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

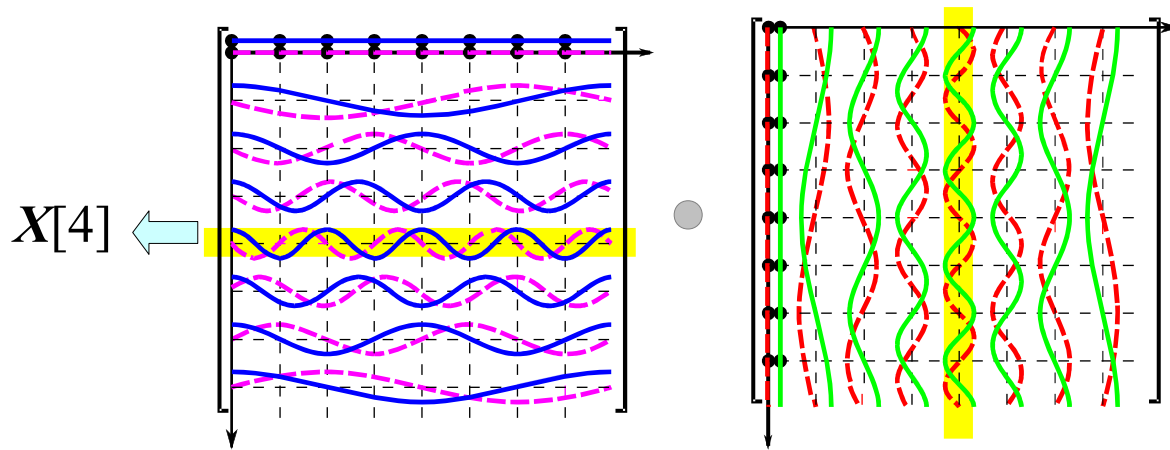
Unitary Matrix

When x looks like this, $X[3]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 3$

$$X[3] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 1} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 7} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 5} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix} \bullet$$

N=8 DFT : X[4] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

When x looks like this, $X[4]$ is max. (=N)

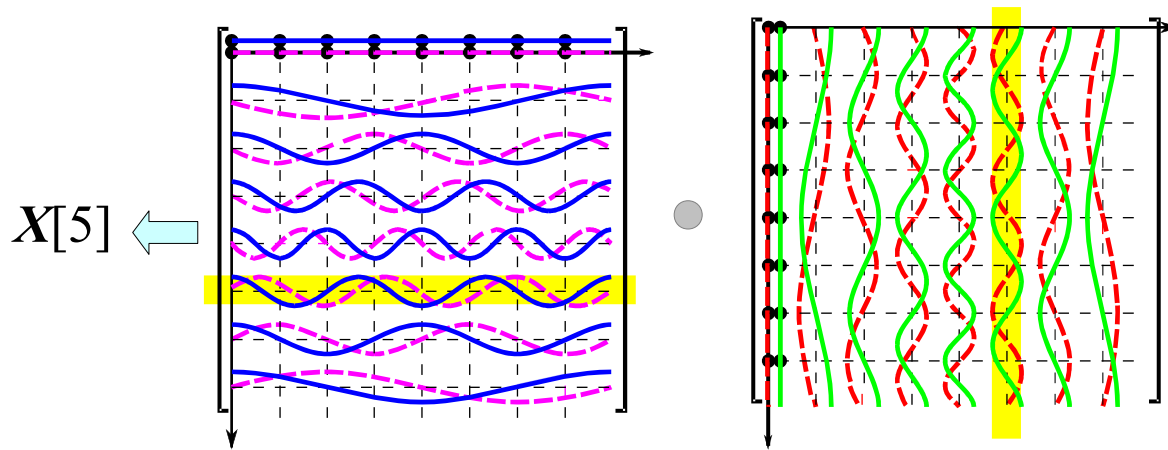
$$X[k] = 0 \text{ for } k \neq 4$$

x

$$X[4] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 4} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[5] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

When x looks like this, $X[5]$ is max. ($=N$)

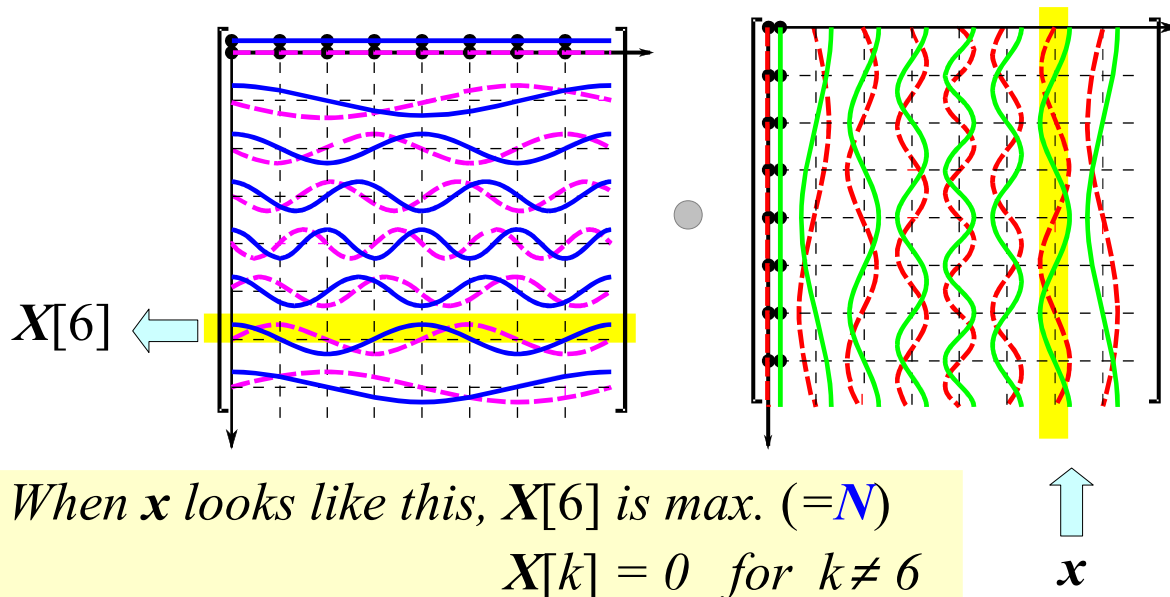
$$X[k] = 0 \text{ for } k \neq 5$$

x

$$X[5] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 5} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 7} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 1} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 3} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[6] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

When x looks like this, $X[6]$ is max. ($=N$)

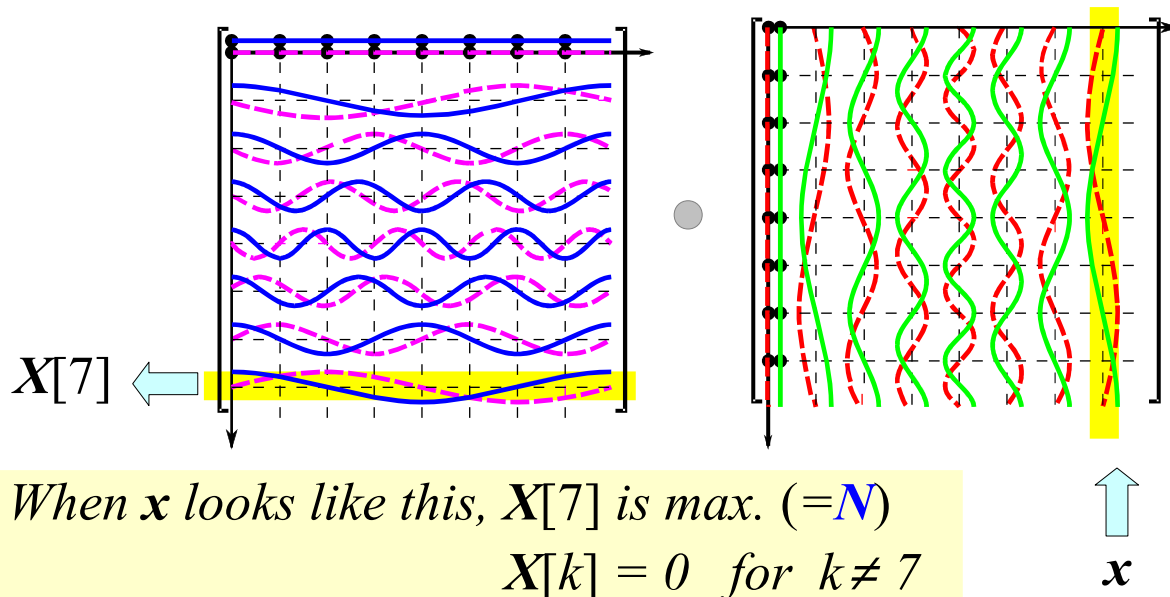
$$X[k] = 0 \text{ for } k \neq 6$$

\uparrow
 x

$$X[6] = \begin{bmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 2} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{bmatrix} \bullet$$

N=8 DFT : X[7] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

$$X[7] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 7} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 5} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 1} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix} \bullet$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann