

CLTI Correlation (2B)

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Correlation

How signals move
relative to each other

Positively correlated the same direction

Average of product > product of averages

Negatively correlated the opposite direction

Average of product < product of averages

Uncorrelated

CrossCorrelation for Power Signals

Energy Signal

$$\begin{aligned} R_{xy}(\tau) &= \int_{-\infty}^{+\infty} x(t) y^*(t+\tau) dt \\ &= \int_{-\infty}^{+\infty} x(t-\tau) y^*(t) dt \end{aligned}$$

Energy Signal real $x(t), y(t)$

$$\begin{aligned} R_{xy}(\tau) &= \int_{-\infty}^{+\infty} x(t) y(t+\tau) dt \\ &= \int_{-\infty}^{+\infty} x(t-\tau) y(t) dt \end{aligned}$$

Power Signal

$$\begin{aligned} R_{xy}(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t) y^*(t+\tau) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t-\tau) y^*(t) dt \end{aligned}$$

Power Signal real $x(t), y(t)$

$$\begin{aligned} R_{xy}(\tau) &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t) y(t+\tau) dt \\ &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t-\tau) y(t) dt \end{aligned}$$

Periodic Power Signal

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t) y(t+\tau) dt$$

Correlation and Convolution

real $x(t)$, $y(t)$

Correlation
$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y(t+\tau) dt = \int_{-\infty}^{+\infty} x(t-\tau)y(t) dt$$

Convolution
$$x(t)*y(t) = \int_{-\infty}^{+\infty} x(t-\tau)y(\tau) d\tau$$

$$R_{xy}(\tau) = x(-\tau)*y(\tau)$$

$$x(-t) \iff X^*(f)$$

$$R_{xy}(\tau) \iff X^*(f)Y(f)$$

Correlation for Periodic Power Signals

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t)y(t+\tau) dt$$

Periodic Power Signal

$$R_{xy}(\tau) = \frac{1}{T} [x(-\tau) \circledast y(\tau)]$$

$$R_{xy}(\tau) \xleftrightarrow{\text{CTFS}} X^*[k]Y[k]$$

Circular Convolution

$$x(t) * y(t) \xleftrightarrow{\text{CTFS}} T X[k]Y[k]$$

$$x[n] * y[n] \xleftrightarrow{\text{CTFS}} N_0 Y[k]X[k]$$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y(t+\tau) dt$$

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t)y(t+\tau) dt$$

Correlation for Power & Energy Signals

One signal – a power signal
The other – an energy signal

Use the Energy Signal Version

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)y(t+\tau) dt$$

Autocorrelation

Energy Signal

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau) dt$$

total signal energy

$$R_{xx}(0) = \int_{-\infty}^{+\infty} x^2(t) dt$$

$$R_{xx}(0) \geq R_{xx}(\tau)$$

max at zero shift

$$R_{xx}(-\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau) dt$$

$$R_{xx}(+\tau) = \int_{-\infty}^{+\infty} x(s+\tau)x(s) ds$$

$$s = t - \tau$$

$$ds = dt$$

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} x(t-t_0)x(t-t_0+\tau) dt$$

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(s)x(s+\tau) ds$$

$$y(t) = x(t-t_0)$$

Power Signal

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t)x(t+\tau) dt$$

average signal power

$$R_{xx}(0) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x^2(t) dt$$

$$R_{xx}(0) \geq R_{xx}(\tau)$$

$$R_{xx}(-\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(t)x(t-\tau) dt$$

$$R_{xx}(+\tau) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_T x(s+\tau)x(s) ds$$

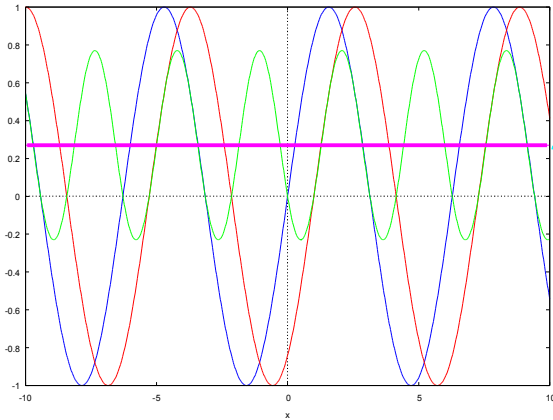
$$R_{yy}(\tau) = \lim_{T \rightarrow \infty} \int_T^T x(t-t_0)x(t-t_0+\tau) dt$$

$$R_{xx}(\tau) = \lim_{T \rightarrow \infty} \int_T^T x(s)x(s+\tau) ds$$

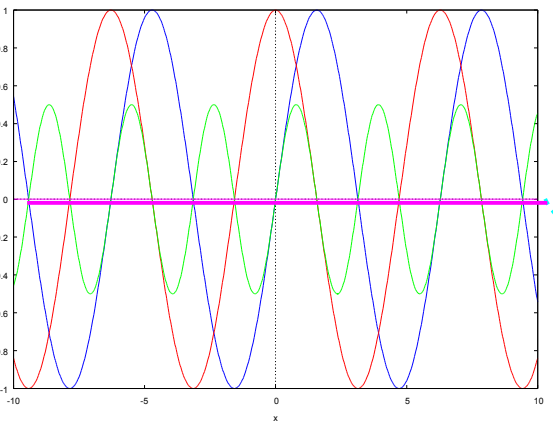
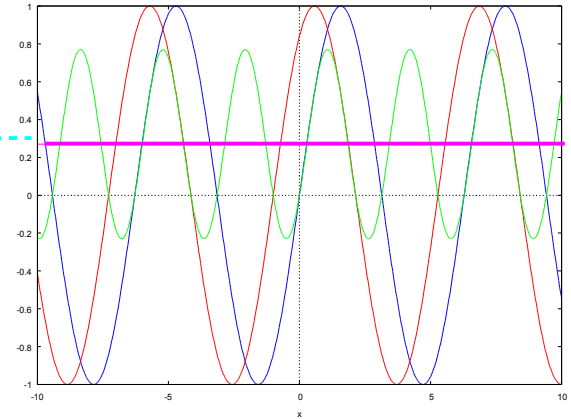
AutoCorrelation for Power Signals

$$R_{xx}(\tau) = \frac{1}{2\pi} \int_{2\pi} \sin(t) \sin(t+\tau) dt$$

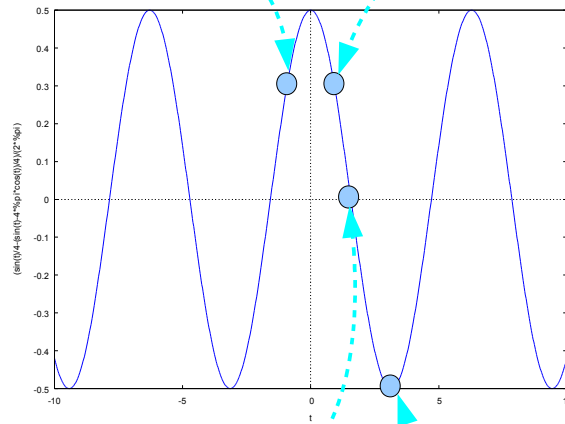
Positively correlated



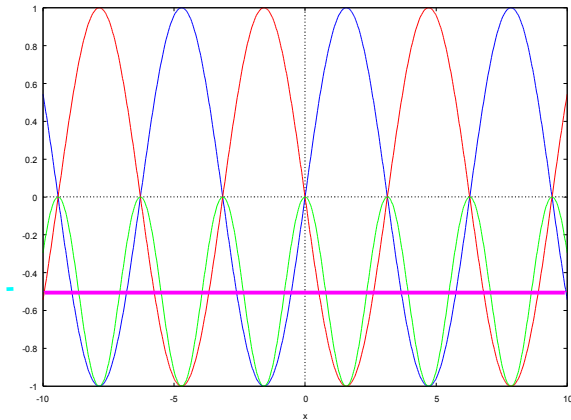
Positively correlated



Uncorrelated



Negatively correlated



Autocorrelation of Sinusoids

$$x(t) = A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2) = x_1(t) + x_2(t)$$

$$\begin{aligned} x(t)x(t+\tau) &= \{A_1 \cos(\omega_1 t + \theta_1) + A_2 \cos(\omega_2 t + \theta_2)\} \{A_1 \cos(\omega_1(t+\tau) + \theta_1) + A_2 \cos(\omega_2(t+\tau) + \theta_2)\} \\ &= A_1 \cos(\omega_1 t + \theta_1) A_1 \cos(\omega_1(t+\tau) + \theta_1) + A_2 \cos(\omega_2 t + \theta_2) A_2 \cos(\omega_2(t+\tau) + \theta_2) \\ &\quad + \underbrace{A_1 \cos(\omega_1 t + \theta_1) A_2 \cos(\omega_2(t+\tau) + \theta_2)} + \underbrace{A_2 \cos(\omega_2 t + \theta_2) A_1 \cos(\omega_1(t+\tau) + \theta_1)} \end{aligned}$$

$$\int_T A_1 \cos(\omega_1 t + \theta_1) A_2 \cos(\omega_2(t+\tau) + \theta_2) dt = 0$$

$$\int_T A_2 \cos(\omega_2 t + \theta_2) A_1 \cos(\omega_1(t+\tau) + \theta_1) dt = 0$$

$$R_x(\tau) = R_{x_1}(\tau) + R_{x_2}(\tau) \quad x_k(t) = A_k \cos(2\pi f_k t + \theta_k)$$

Autocorrelation of Random Signals

$$x(t) = \sum_{k=1}^N A_k \cos(\omega_k t + \theta_k)$$

$$R_x(\tau) = \sum_{k=1}^N R_k(\tau)$$

autocorrelation of $a_k \cos(\omega_k t + \theta_k)$

independent of choice of θ_k

random phase shift θ_k
the same amplitudes a
the same frequencies ω

} $x_k(t)$ different look
 $R_k(\tau)$ similar look

the amplitudes a
the frequencies ω

} can be observed
in the autocorrelation $R_k(\tau)$

similar look but not exactly the same

describes a signal generally, but not exactly
– suitable for a random signal

Autocorrelation Examples

AWGN signal

changes rapidly with time

current value has no correlation with past or future values

even at very short time period

random fluctuation except large peak at $\tau = 0$

ASK signal : sinusoid multiplied with rectangular pulse

regardless of sin or cos, the autocorrelation is always even function

cos wave multiplied by a rhombus pulse

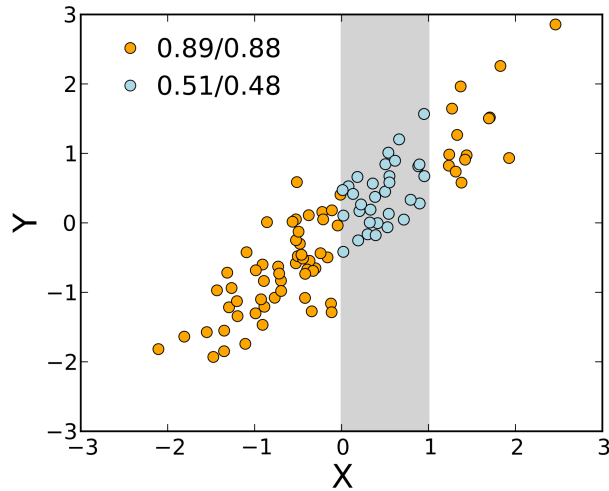
CrossCorrelation

$$R_{xy}(\tau) = R_{xy}(-\tau)$$

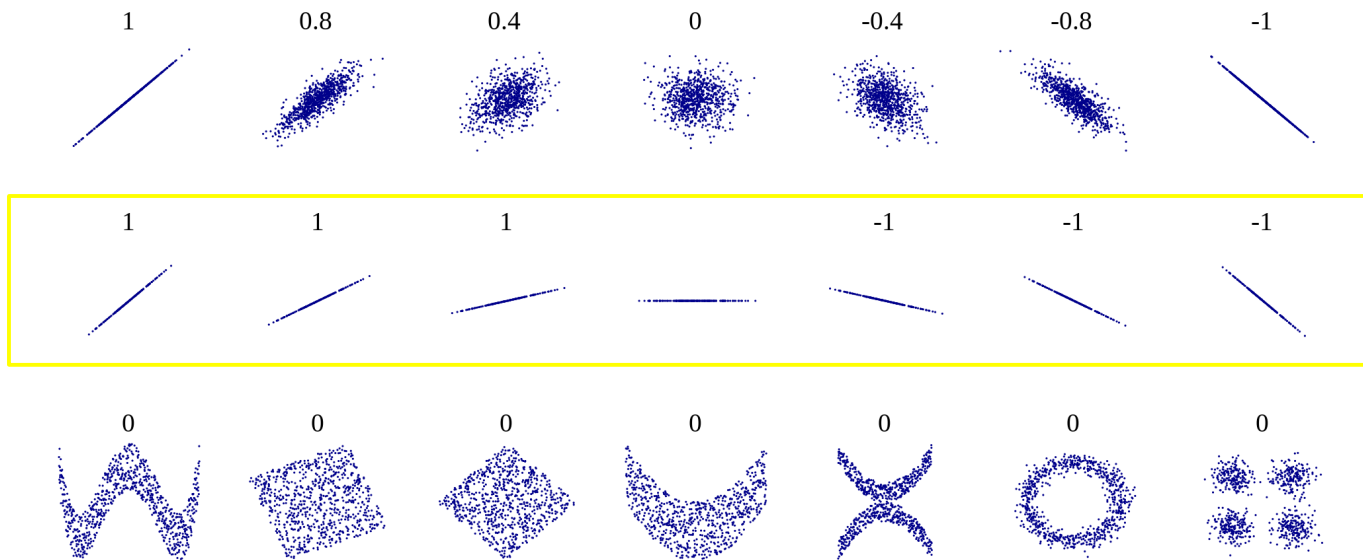
The largest peak occurs at a shift which is exactly the amount of shift
Between $x(t)$ and $y(t)$

The signal power of the sum depends strongly on whether two signals are correlated
Positively correlated vs. uncorrelated

Pearson's product-moment coefficient



$$\rho_{XY} = \frac{E[(X - m_x)(Y - m_y)]}{\sigma_X \sigma_Y}$$



regardless of slopes

Correlation Example – Sum (1)

$$x_1(t) = \sin(\omega t)$$

$$x_2(t) = \sin(\omega t + \frac{\pi}{2}) = \cos(\omega t)$$

$$x_3(t) = \sin(\omega t + \frac{\pi}{4})$$

$$x_4(t) = \sin(\omega t + \pi)$$

$$R_{12}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \cos(\omega t) dt = 0$$

$$R_{13}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi/4) dt = 0.354$$

$$R_{14}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi) dt = -0.5$$

$$\begin{aligned} f(t) &= x_1(t) + x_2(t) = \sin(\omega t) + \sin(\omega t + \frac{\pi}{2}) \\ &= 2 \sin(\frac{2\omega t + \pi/2}{2}) \cos(-\pi/4) \\ &= 2 \sin(\omega t + \frac{\pi}{4}) \cos(\frac{-\pi}{4}) = 0.707 \cdot 2 \sin(\omega t + \frac{\pi}{4}) \end{aligned}$$

sum of uncorrelated signals

The signal power of the sum depends strongly on whether two signals are correlated

positively correlated vs. uncorrelated

$$\begin{aligned} g(t) &= x_1(t) + x_3(t) = \sin(\omega t) + \sin(\omega t + \frac{\pi}{4}) \\ &= 2 \sin(\frac{2\omega t + \pi/4}{2}) \cos(-\pi/8) \\ &= 2 \sin(\omega t + \frac{\pi}{8}) \cos(\frac{-\pi}{8}) = 0.924 \cdot 2 \sin(\omega t + \frac{\pi}{4}) \end{aligned}$$

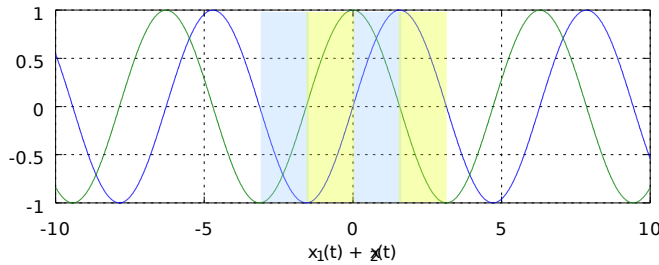
sum of positively correlated signals

$$\begin{aligned} h(t) &= x_1(t) + x_4(t) = \sin(\omega t) + \sin(\omega t + \pi) \\ &= \sin(\omega t) - \sin(\omega t) \\ &= 0 \end{aligned}$$

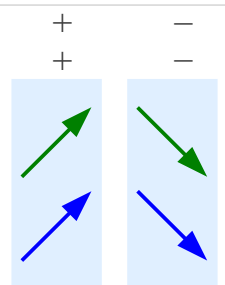
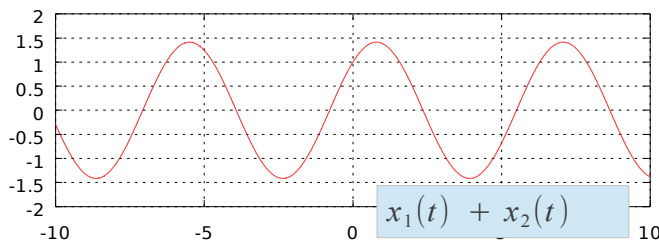
sum of negatively correlated signals

Correlation Example – Sum (2)

$$R_{12}(0) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sin(\omega t) \cos(\omega t) dt = 0$$

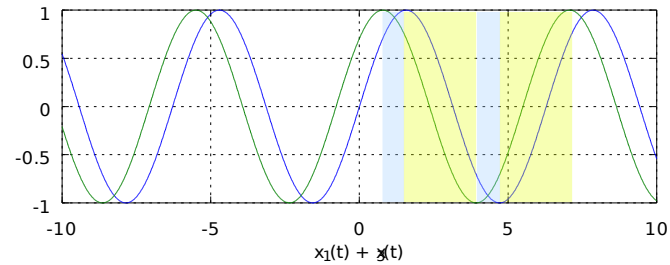


0.707 · 2

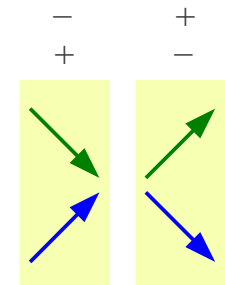
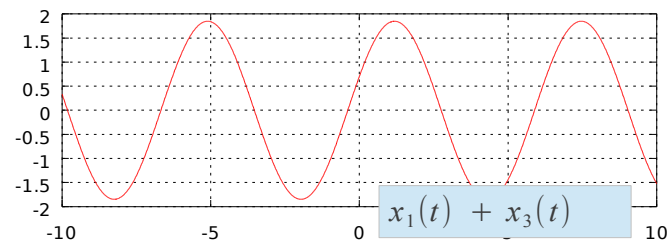


both signals increase or decrease

$$R_{13}(0) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sin(\omega t) \sin(\omega t + \pi/4) dt = 0.354$$

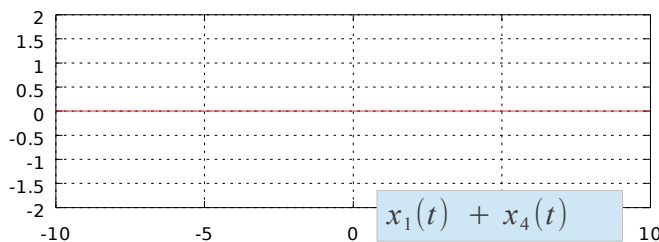
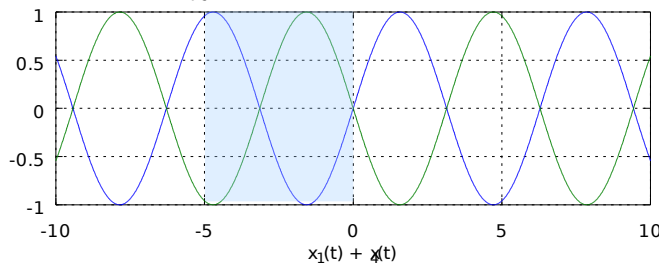


0.924 · 2

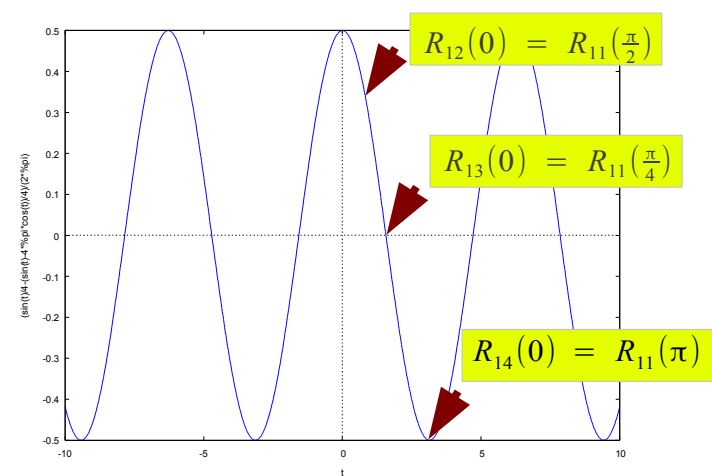


one increases, the other decreases

$$R_{14}(0) = \frac{1}{2\pi} \int_{-2\pi}^{2\pi} \sin(\omega t) \sin(\omega t + \pi) dt = -0.5$$

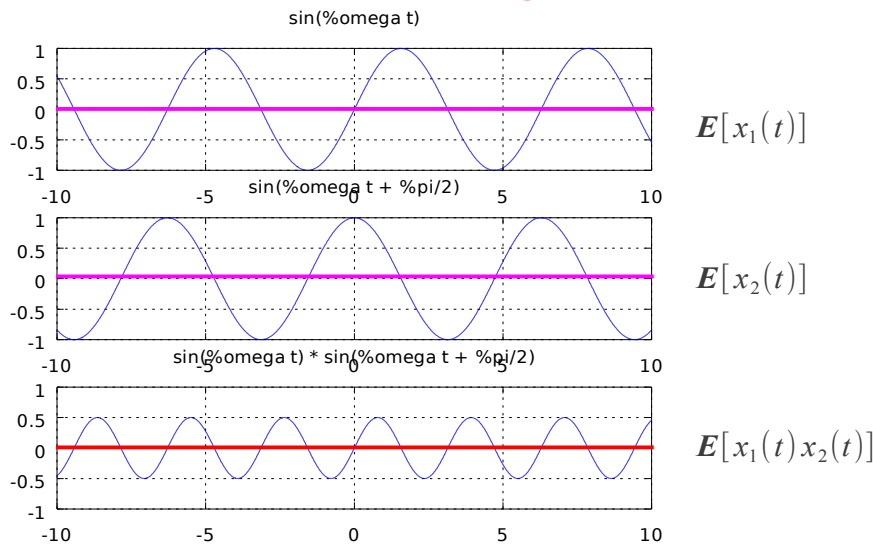


autocorrelation $R_{11}(\tau)$

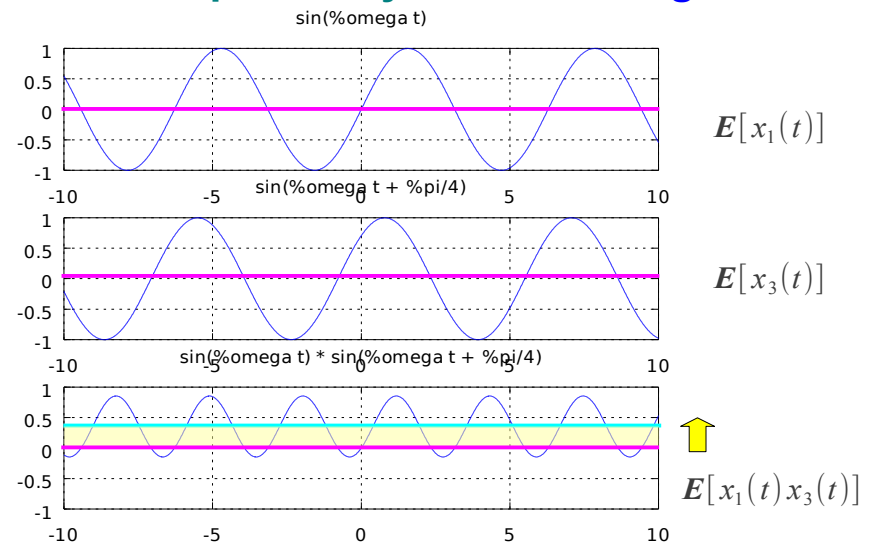


Correlation Example – Product

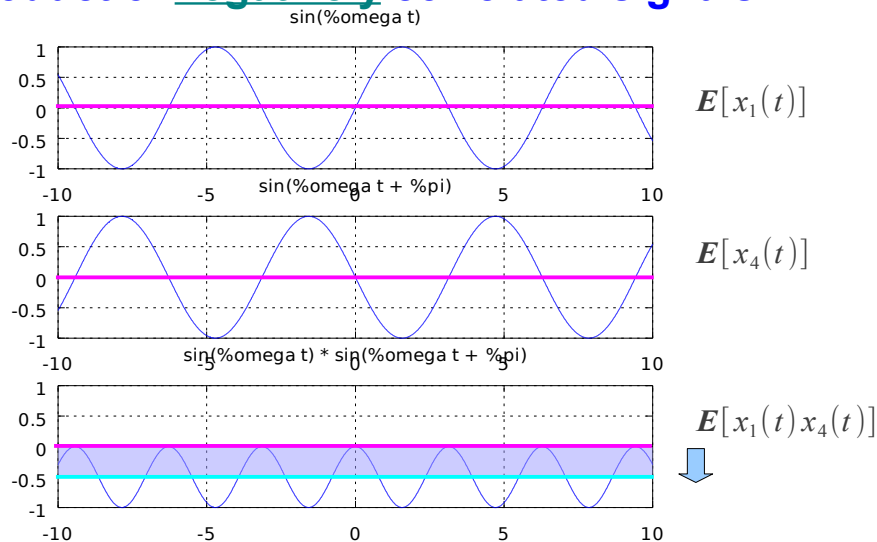
product of uncorrelated signals



product of positively correlated signals



product of negatively correlated signals



Positively correlated

the same direction

↑ **Average of product > Product of averages**

Negatively correlated

the opposite direction

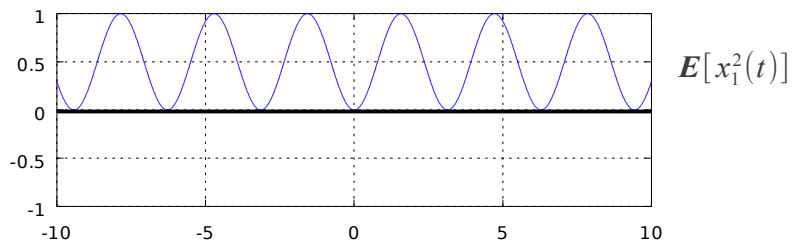
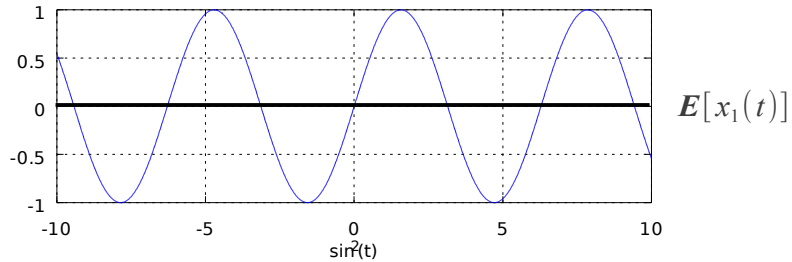
↓ **Average of product < Product of averages**

Uncorrelated

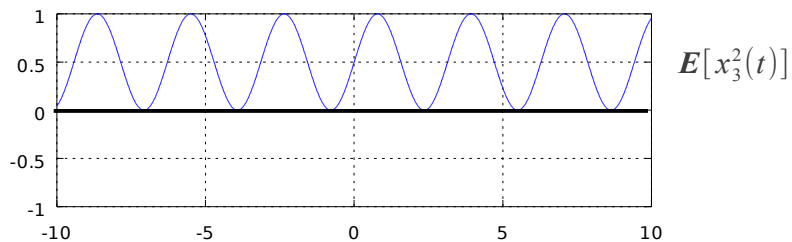
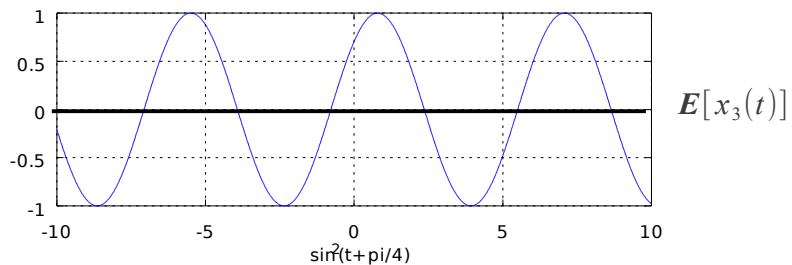
Average of product \cong Product of averages

Correlation Example – Mean, Variance

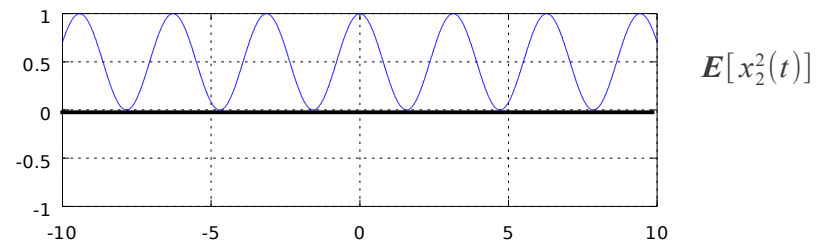
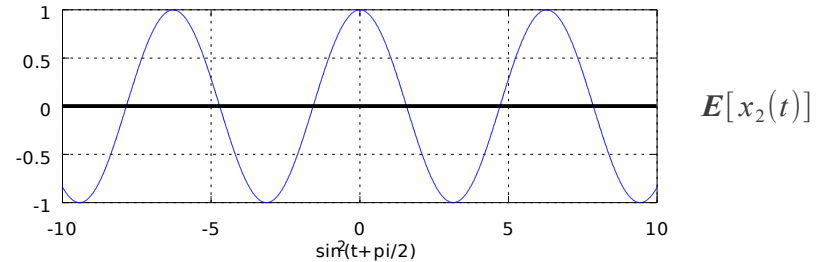
$$\sigma_1^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t) dt = 0.5 \quad m_1 = 0$$



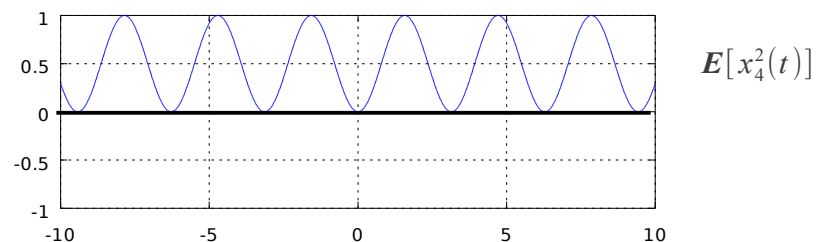
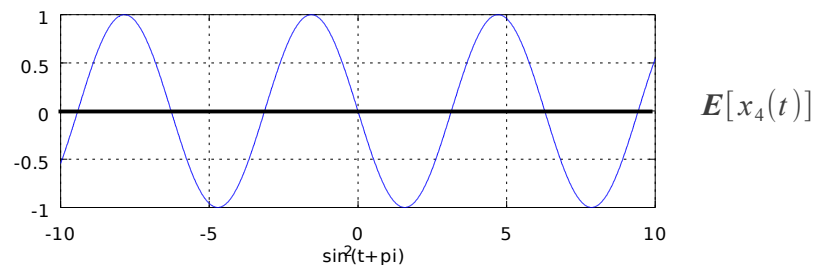
$$\sigma_3^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \frac{\pi}{4}) dt = 0.5 \quad m_3 = 0$$



$$\sigma_2^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \frac{\pi}{2}) dt = 0.5 \quad m_1 = 0$$



$$\sigma_4^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \pi) dt = 0.5 \quad m_4 = 0$$



Correlation Example – Correlation Coefficients

$$x_1(t) = \sin(\omega t) \quad x_2(t) = \sin(\omega t + \frac{\pi}{2}) \quad x_3(t) = \sin(\omega t + \frac{\pi}{4}) \quad x_4(t) = \sin(\omega t + \pi)$$

$$\sigma_1^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t) dt = 0.5 \quad m_1 = 0$$

$$\sigma_2^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \frac{\pi}{2}) dt = 0.5 \quad m_2 = 0$$

$$\sigma_3^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \frac{\pi}{4}) dt = 0.5 \quad m_3 = 0$$

$$\sigma_4^2 = \frac{1}{2\pi} \int_{2\pi} \sin^2(\omega t + \pi) dt = 0.5 \quad m_4 = 0$$

$$R_{12}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \cos(\omega t) dt = 0$$

$$R_{13}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi/4) dt = 0.354$$

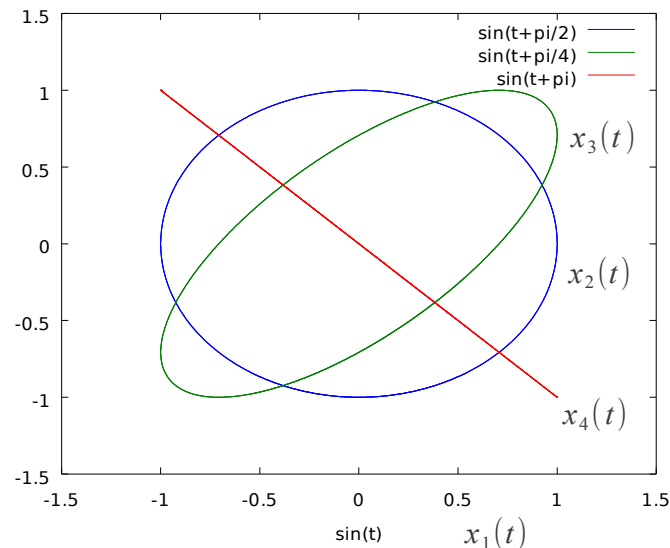
$$R_{14}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi) dt = -0.5$$

$$\rho_{XY} = \frac{E[(X - m_x)(Y - m_y)]}{\sigma_X \sigma_Y}$$

$$\rho_{12} = \frac{E[x_1(t)x_2(t)]}{\sigma_1 \sigma_2} = \frac{R_{12}(0)}{0.5} = 0$$

$$\rho_{13} = \frac{E[x_1(t)x_3(t)]}{\sigma_1 \sigma_3} = \frac{R_{13}(0)}{0.5} = 0.177$$

$$\rho_{14} = \frac{E[x_1(t)x_4(t)]}{\sigma_1 \sigma_4} = \frac{R_{14}(0)}{0.5} = -1$$



Correlation Example – Auto & Cross Correlation

special case: sinusoidal signals

$$x_1(t) = \sin(\omega t)$$

$$x_2(t) = \sin(\omega t + \frac{\pi}{2})$$

$$x_3(t) = \sin(\omega t + \frac{\pi}{4})$$

$$x_4(t) = \sin(\omega t + \pi)$$

$$R_{12}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \cos(\omega t) dt = 0$$

$$R_{13}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi/4) dt = 0.354$$

$$R_{14}(0) = \frac{1}{2\pi} \int_{2\pi} \sin(\omega t) \sin(\omega t + \pi) dt = -0.5$$

CrossCorrelation



$$x_1(t + \frac{\pi}{2\omega}) = \sin(\omega(t + \frac{\pi}{2\omega}))$$



$$x_1(t + \frac{\pi}{4\omega}) = \sin(\omega(t + \frac{\pi}{4\omega}))$$



$$x_1(t + \frac{\pi}{\omega}) = \sin(\omega(t + \frac{\pi}{\omega}))$$



$$R_{11}(\frac{\pi}{2\omega})$$



$$R_{11}(\frac{\pi}{4\omega})$$



$$R_{11}(\frac{\pi}{\omega})$$

AutoCorrelation

Random Signal

Random Signal

No exact description of the signal

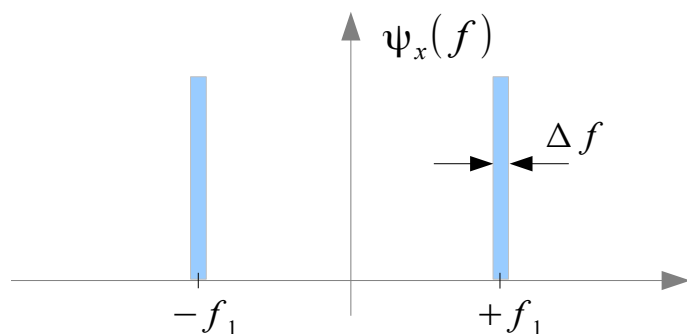
But we can estimate

- Autocorrelation
- Energy spectral densities (ESD)
- Power spectral densities (PSD)

Total Energy

$$E_x = \int_{-\infty}^{+\infty} \Psi_x(f) df$$

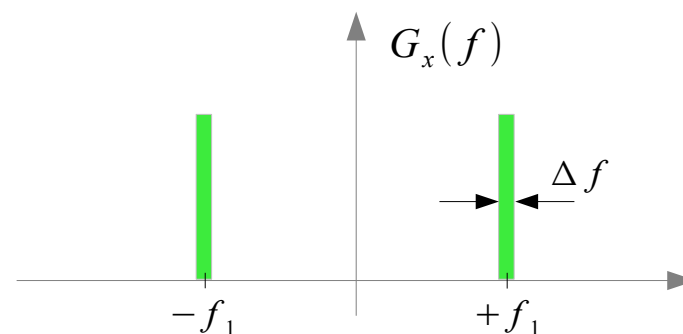
energy spectral densities



Average Power

$$P_x = \int_{-\infty}^{+\infty} G_x(f) df$$

power spectral densities



Energy Spectral Density (ESD)

Parseval's Theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

Energy Spectral Density $|X(f)|^2 = \Psi_x(f)$

Real $x(t)$ \rightarrow even, non-negative, real $\Psi_x(f)$

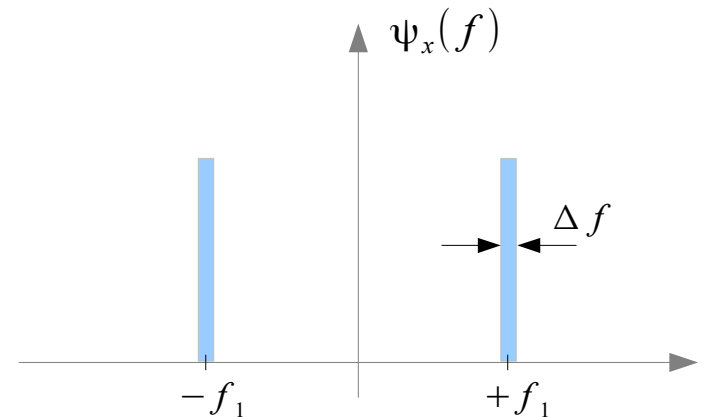
$$E_x = 2 \int_0^{+\infty} \Psi_x(f) df$$

$$E_y = 2 \int_0^{+\infty} \Psi_y(f) df = 2 \int_0^{+\infty} |Y(f)|^2 df$$

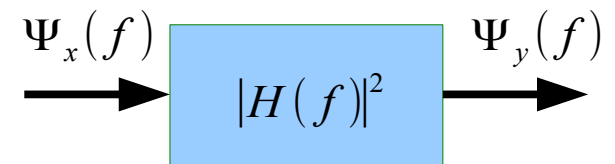
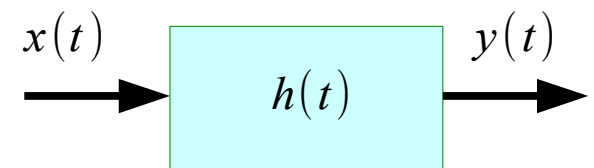
$$= 2 \int_0^{+\infty} |H(f) X(f)|^2 df = 2 \int_0^{+\infty} |H(f)| |X(f)|^2 df$$

$$= 2 \int_0^{+\infty} |H(f)|^2 \Psi_x(f) df$$

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f)$$



The distribution of signal energy versus frequency

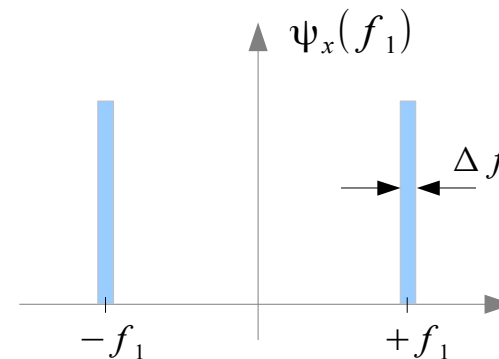
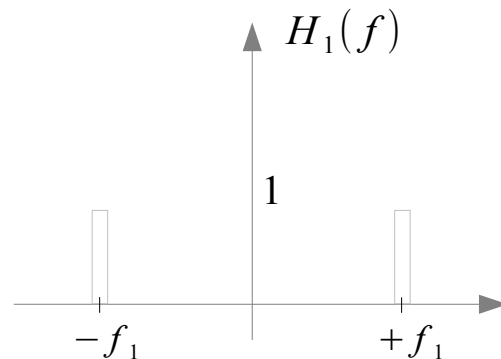
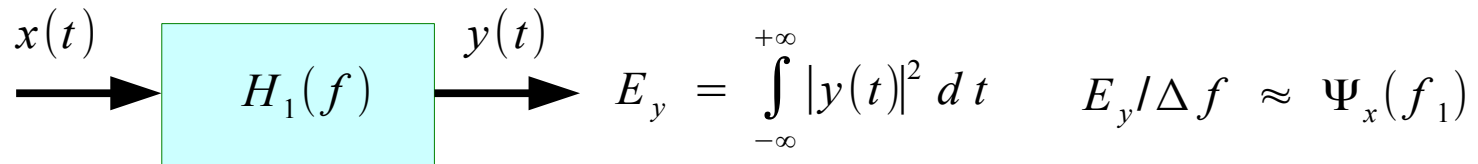


Conceptual ESD Estimation

Parseval's Theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

Energy Spectral Density $|X(f)|^2 = \Psi_x(f)$



ESD and Autocorrelation

$$R_x(t) \quad \longleftrightarrow \quad \Psi_x(f) \quad (= |X(f)|^2)$$

$$R_x(t) \quad \longleftrightarrow \quad X^*(f)X(f)$$

$$R_x(t) = x(-t)*x(t) = \int_{-\infty}^{+\infty} x(-\tau)x(t-\tau) d\tau \quad \longrightarrow \quad R_x(t) = \int_{-\infty}^{+\infty} x(\tau)x(\tau+t) d\tau$$

Power Spectral Density (PSD)

Many signals are considered as a **power signal**

The **steady state** signals activated for a long time ago will be expected to continue

$$x_T(t) = \text{rect}\left(\frac{t}{T}\right)x(t) = \begin{cases} x(t) & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

The **truncated version** of $x(t)$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(\tau) e^{-2\pi f t} d\tau = \int_{-T/2}^{+T/2} x_T(\tau) e^{-2\pi f t} d\tau$$

$$\Psi_{x_T}(f) = |X_T(f)|^2$$

The **ESD** of a **truncated version** of $x(t)$

$$G_{X_T}(f) = \frac{\Psi_{X_T}}{T} = \frac{1}{T} |X_T(f)|^2$$

The **PSD** of a **truncated version** of $x(t)$

$$G_x(f) = \lim_{T \rightarrow \infty} G_{X_T}(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X_T(f)|^2$$

The estimated **PSD**

$$2 \int_{f_L}^{f_H} G(f) df$$

The power of a finite signal power signal in a bandwidth f_L to f_H

Conceptual PSD Estimation

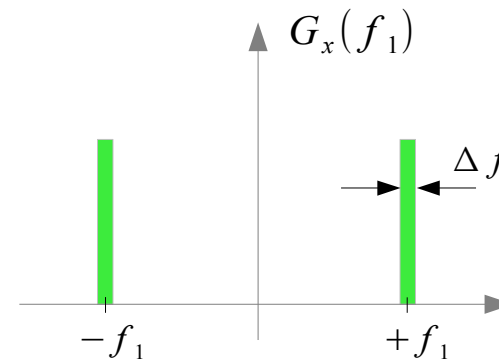
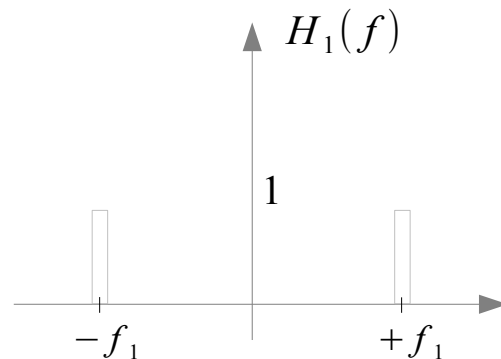
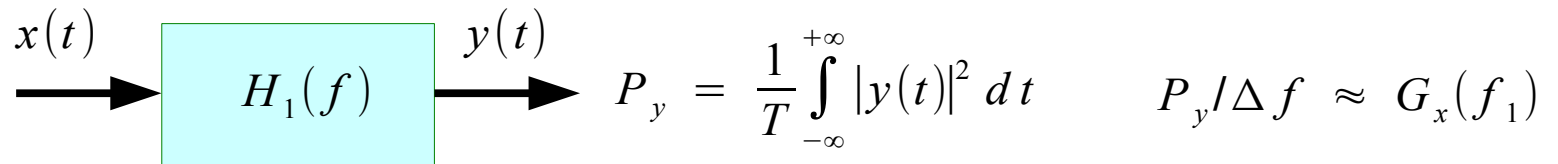
Parseval's Theorem

$$E_{x_T} = \int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 df$$

Energy Spectral Density

$$|X_T(f)|^2 = \Psi_{x_T}(f)$$

$$G_{X_T}(f) = \frac{\Psi_{X_T}}{T} = \frac{1}{T} |X_T(f)|^2$$



ESD and Band-pass Filtering

$$E_y = 2 \int_0^{+\infty} \Psi_y(f) df = 2 \int_0^{+\infty} |Y(f)|^2 df = 2 \int_0^{+\infty} |H(f)X(f)|^2 df$$

$$E_y = 2 \int_0^{+\infty} |H(f)|^2 \Psi_x(f) df = 2 \int_{f_L}^{f_H} \Psi_x(f) df$$

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f)H^*(f)\Psi_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

PSD and Band-pass Filtering

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

References

- [1] <http://en.wikipedia.org/>
- [2] M.J. Roberts, Signals and Systems,