CORDIC in VHDL (1A)

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07/20/2011

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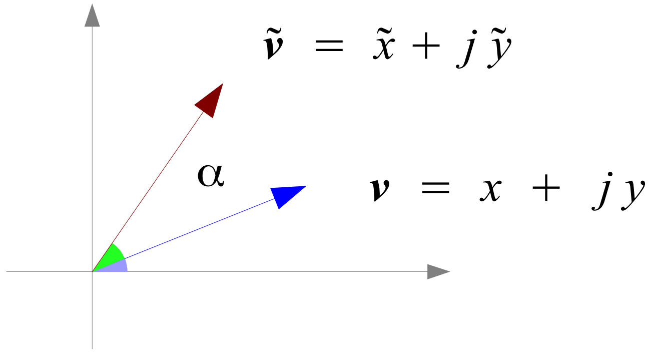
CORDIC Background

1. G Hampson,
A VHDL Implementation of a CORDIC Arithmetic Processor Chip
Monash University, Technical Report 94-9, 1994

Angle Expansion

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$



elementary angle

$$\theta_0 = \tan^{-1}(2^{0}) =$$
$$\theta_1 = \tan^{-1}(2^{-1}) =$$
$$\theta_2 = \tan^{-1}(2^{-2}) =$$
$$\theta_3 = \tan^{-1}(2^{-3}) =$$

α can be expanded by

a set of elementary angles α_i

pseudo-digits q_i

$$\alpha_i = \begin{cases} \pi/2 & i = -1 \\ \tan^{-1}(2^{-i}) & i = 0, 1, 2, \cdots, n-1 \end{cases}$$

$$q_i = \begin{cases} -1 \\ +1 \end{cases}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

angle expansion error z_n

$$|z_n| \le 2^{-(n-1)}$$

Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$

$$= \boldsymbol{v} \exp\left(j \left(\sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n \right)\right)$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=-1}^{n-1} e^{j q_i \alpha_i} \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} e^{j q_i \alpha_i} \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \cdot (1 + j q_i 2^{-i}) \right) \cdot e^{j z_n}$$

$$= \boldsymbol{v} \cdot (j q_{-1}) \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot \left(\prod_{i=0}^{n-1} (1 + j q_i 2^{-i}) \right) \cdot e^{j z_n}$$

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$e^{j\theta} = \cos\theta + j \sin\theta$$

$$e^{j q_{-1} \alpha_0} = e^{j q_{-1} \frac{\pi}{2}} = j q_{-1} \quad \left(e^{\pm j \frac{\pi}{2}} = \pm j \right)$$

$$e^{j q_i \alpha_i} = \cos(q_i \alpha_i) + j \sin(q_i \alpha_i)$$

$$= \cos(q_i \alpha_i) \cdot (1 + j \tan(q_i \alpha_i))$$

$$= \cos(q_i \alpha_i) \cdot (1 + j q_i 2^{-i})$$

$$= \cos(\alpha_i) \cdot (1 + j q_i 2^{-i})$$

$$\left(\cos(\pm\alpha_i) = \cos(\alpha_i) \right)$$

Rotating Vector

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$

rotated by α

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i) \right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i 2^{-i}) \right) \cdot e^{jz_n}$$

$$K_n = \prod_{i=0}^{n-1} \frac{1}{\sqrt{1 + 2^{-2i}}}$$

series rotations of α_i

n iterations

q_{-1}	q_0	q_1		$\cdots$	$\cdots$		q_{n-1}

$$((((\alpha - q_{-1}\alpha_{-1}) - q_0\alpha_0) - q_1\alpha_1) \quad \cdots \quad - q_{n-1}\alpha_{n-1})$$

$z_{-1}=\alpha$	z_0	z_1	z_2		z_n

$$\begin{aligned} &if\ (z_0 \geq 0)\ \ q_0 = +1 \\ &if\ (z_0 < 0)\ \ q_0 = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_1 \geq 0)\ \ q_1 = +1 \\ &if\ (z_1 < 0)\ \ q_1 = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_2 \geq 0)\ \ q_2 = +1 \\ &if\ (z_2 < 0)\ \ q_2 = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_{-1} \geq 0)\ \ q_{-1} = +1 \\ &if\ (z_{-1} < 0)\ \ q_{-1} = -1 \end{aligned}$$

$$\begin{aligned} &if\ (z_{n-1} \geq 0)\ \ q_{n-1} = +1 \\ &if\ (z_{n-1} < 0)\ \ q_{n-1} = -1 \end{aligned}$$

q_{-1}	q_0	q_1	q_2		q_{n-1}

$$\tilde{\boldsymbol{v}} = \boldsymbol{v}\, e^{j\alpha}$$

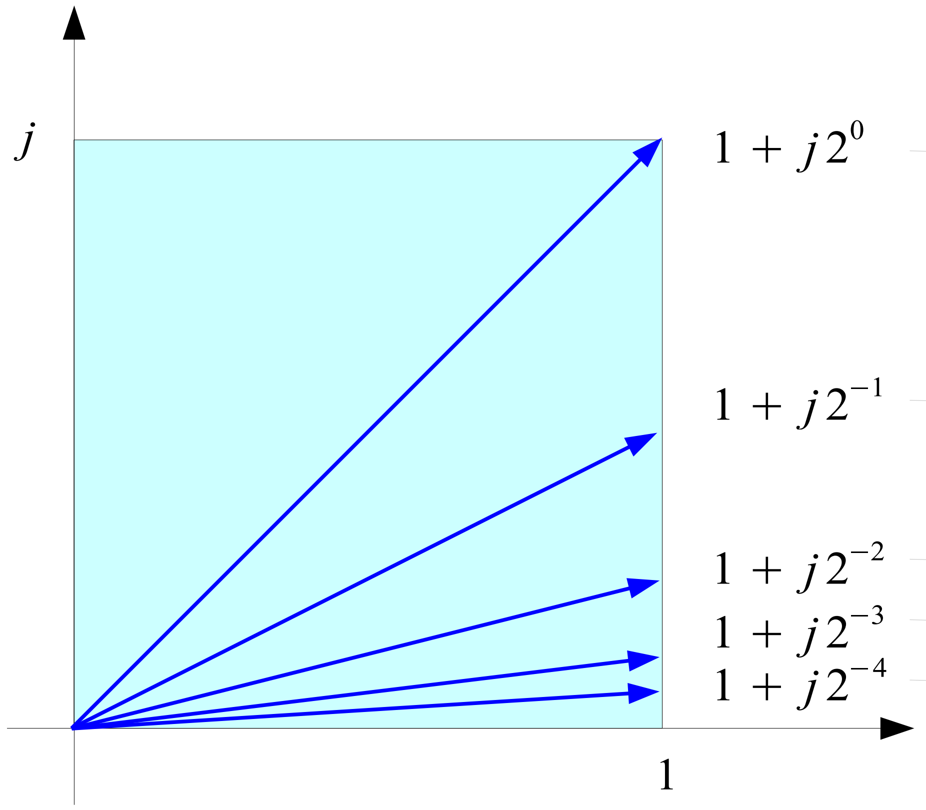
rotated by α

$$\alpha = \sum_{i=-1}^{n-1} q_i \cdot \alpha_i + z_n$$

$$= \boldsymbol{v} \cdot \left(\prod_{i=0}^{n-1} \cos(\alpha_i)\right) \cdot (jq_{-1}) \cdot \left(\prod_{i=0}^{n-1} (1 + j\, q_i 2^{-i})\right) \cdot e^{jz_n}$$

q_{-1}	q_0	q_1		$\cdots$	$\cdots$		q_{n-1}

Elementary Angle: $\tan^{-1}(K)$



$$\theta_L = \tan^{-1}(2^{-L}) = \tan^{-1}(K)$$

$$\theta_0 = \tan^{-1}(2^{0}) = 45.00000$$

$$\theta_1 = \tan^{-1}(2^{-1}) = 26.56505$$

$$\theta_2 = \tan^{-1}(2^{-2}) = 14.03624$$

$$\theta_3 = \tan^{-1}(2^{-3}) = 7.12502$$

$$\theta_4 = \tan^{-1}(2^{-4}) = 3.57633$$

Represent arbitrary angle θ

in terms of $\pm\theta_{0,}\ \pm\theta_{1,}\ \pm\theta_{2,}\ \pm\theta_{3,}\ \cdots,\ \pm\theta_L,\ \cdots$ $\left(K = \frac{1}{2^L},\quad L = 0, 1, 2, \cdots\right)$

Phase and Magnitude of 1 + jK (1)

Cumulative Magnitude

L	$K = \frac{1}{2^L}$	$R = 1 + jK$	*Phase of R*	*Magnitude of R*	*CORDIC Gain*
0	1.0	1 + j1.0	45°	1.41421356	1.414213562
1	0.5	1 + j0.5	26.56505°	1.11803399	1.581138830
2	0.25	1 + j0.25	14.03624°	1.03077641	1.629800601
3	0.125	1 + j0.125	7.12502°	1.00778222	1.642484066
4	0.0625	1 + j0.0625	3.57633°	1.00195122	1.645688916
5	0.03125	1 + j0.03125	1.78991°	1.00048816	1.646492279
6	0.015625	1 + j0.015625	0.89517°	1.00012206	1.646693254
7	0.007813	1 + j0.007813	0.44761°	1.00003052	1.646743507
•••	•••	•••	•••	•••	•••
					1.647 ←

$$R = 1 + jK \xrightarrow[L \;=\; 0, 1, 2, \cdots]{K \;=\; 1/2^L} \sqrt{1^2 + K^2} > 1.0$$

References

[1] http://en.wikipedia.org/
[2] G Hampson, A VHDL Implementation of a CORDIC Arithmetic Processor Chip
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