Correlation (1A)

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11/6/12

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Probability Density Function

Probability Density Function $\quad f(x;\ t) = \dfrac{\partial}{\partial x} F(x;\ t)$

$$P[\, a \le X(t) \le b \,] = \int_a^b f(x;\ t)\,dx$$

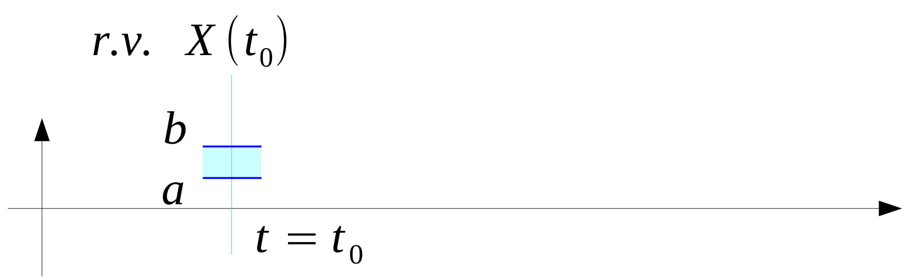
Cumulative Distribution Function $\quad F(x;\ t) = \displaystyle\int_{-\infty}^{x} f(x;\ t)\,dx$

$$P[\, X(t) \le b \,] = \int_{-\infty}^{b} f(x;\ t)\,dx = F(b;\ t)$$

$X(t)$	*random variable X(t) at a given time t*
$f(x;\ t)$	*The probability density function of random variable X(t)*

$$P[\, x \le X(t) \le x + dt \,] = f(x;\ t)\, dx$$

$$P[\, X(t) = x \,] \Rightarrow f(x;\ t)$$



Probability Density Function

$X(t)$ *random variable $X(t)$ at a given time t*

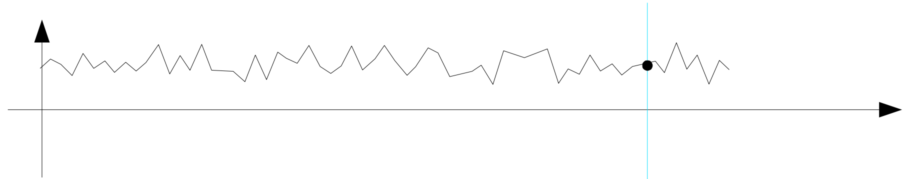
$f(x;\, t)$ *The probability density function of random variable $X(t)$*

$$P[\, x \le X(t) \le x + dt \,] \;=\; f(x;\; t)\, dx$$

$$P[\, X(t) = x \,] \;\Rightarrow\; f(x;\; t)$$

$X(t_0)$ *random variable at a given time t_0*

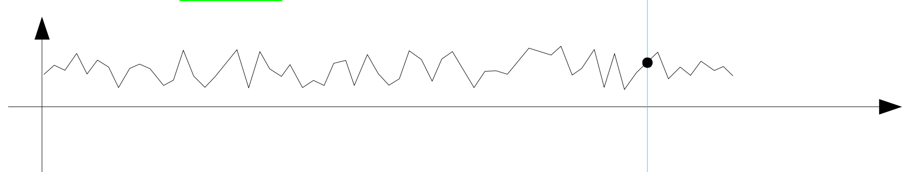
1st trial $x_1(t)$



$X(t_0) \;=\; x_1(t_0)$

r.v. $X(t_0)$ *has a value* $x_1(t_0)$ *at time* t_0 *in the* ***1st trial***

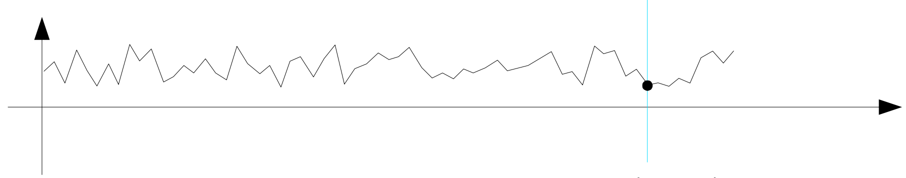
2nd trial $x_2(t)$



$X(t_0) \;=\; x_2(t_0)$

r.v. $X(t_0)$ *has a value* $x_2(t_0)$ *at time* t_0 *in the* ***2nd trial***

3rd trial $x_3(t)$



$t = t_0$

$X(t_0) \;=\; x_3(t_0)$

r.v. $X(t_0)$ *has a value* $x_3(t_0)$ *at time* t_0 *in the* ***3rd trial***

Joint Probability Density Function

Joint Probability Density Function $f(x_1, x_2;\ t_1, t_2) = \dfrac{\partial^2}{\partial x_1 \partial x_2} F(x_1, x_2;\ t_1, t_2)$

$$P[\{a \le X(t_1) \le b\} \cap \{c \le X(t_2) \le d\}] = \int_a^b \int_c^d f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$$

Joint Cumulative Distribution Function $F(x_1, x_2;\ t_1, t_2) = \displaystyle\int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2$

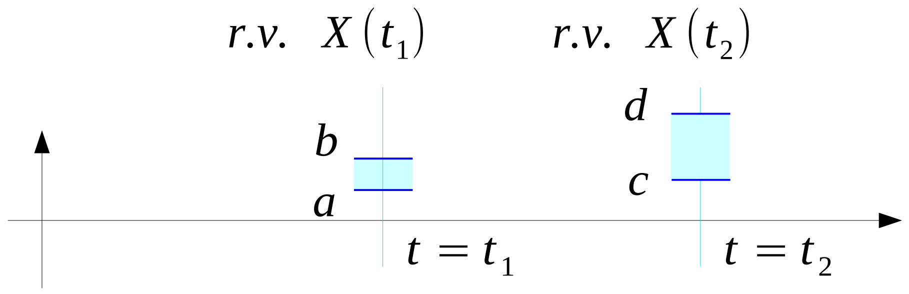
$$P[\{X(t_1) \le b\} \cap \{X(t_2) \le d\}] = \int_{-\infty}^{b} \int_{-\infty}^{d} f(x_1, x_2;\ t_1, t_2)\, dx_1\, dx_2 = F(b, d;\ t_1, t_2)$$

$X(t_1),\ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, x_2;\ t_1, t_2)$ *joint probability density function*

$P[\{X(t_1) = x_1\} \cap \{X(t_2) = x_2\}]$

$\Rightarrow\ f(x_1, x_2;\ t_1, t_2)$



Probability Density Function

$X(t_1),\ X(t_2)$ *random variables at time t_1 and t_2*

$f(x_1, x_2;\ t_1, t_2)$ *joint probability density function*

$P[\ \{X(t_1) = x_1\}\ \cap\ \{X(t_2) = x_2\}\ ]$

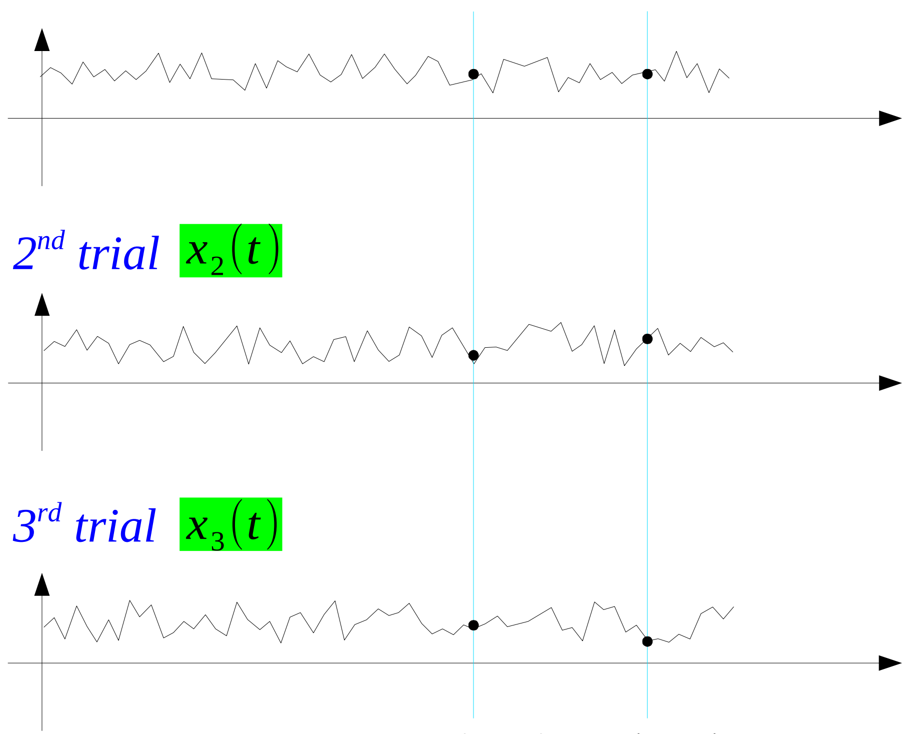
$\Rightarrow\ f(x_1, x_2;\ t_1, t_2)$

random variable at a given time t_0

1st trial $x_1(t)$ $\quad X(t_1) \quad X(t_2)$

2nd trial $x_2(t)$

3rd trial $x_3(t)$



$t = t_1 \qquad t = t_2$

$X(t_1) = x_1(t_1), \quad X(t_2) = x_1(t_2)$

r.v. $X(t_1)$ *has a value* $x_1(t_1)$

r.v. $X(t_2)$ *has a value* $x_1(t_2)$

at times $t_1,\ t_2$ *in the* ***1st trial***

$X(t_1) = x_2(t_1), \quad X(t_2) = x_2(t_2)$

r.v. $X(t_1)$ *has a value* $x_2(t_1)$

r.v. $X(t_2)$ *has a value* $x_2(t_2)$

at times $t_1,\ t_2$ *in the* ***2nd trial***

$X(t_1) = x_3(t_1), \quad X(t_2) = x_3(t_2)$

r.v. $X(t_1)$ *has a value* $x_3(t_1)$

r.v. $X(t_2)$ *has a value* $x_3(t_2)$

at times $t_{1,}\ t_2$ *in the* ***3rd trial***

Moments of a Random Process

The n-th Moment

$$E\left[\, X^n(t) \,\right] = \int_{-\infty}^{+\infty} x^n \, f(x\,;\,t)\, dx$$

The n-th Central Moment

$$E\left[\, (X(t) - \mu(t))^n \,\right] = \int_{-\infty}^{+\infty} (x - \mu(t))^n \, f(x\,;\,t)\, dx$$

1st Moment

$$E\left[\, X(t) \,\right] = \int_{-\infty}^{+\infty} x \; f(x\,;\,t)\, dx$$

$$= \mu(t)$$

1st Central Moment

$$E\left[\, (X(t) - \mu(t)) \,\right] = \int_{-\infty}^{+\infty} (x - \mu(t)) \; f(x\,;\,t)\, dx$$

2nd Moment

$$E\left[\, X^2(t) \,\right] = \int_{-\infty}^{+\infty} x^2 \, f(x\,;\,t)\, dx$$

2nd Central Moment

$$E\left[\, (X(t) - \mu(t))^2 \,\right] = \int_{-\infty}^{+\infty} (x - \mu(t))^2 \, f(x\,;\,t)\, dx$$

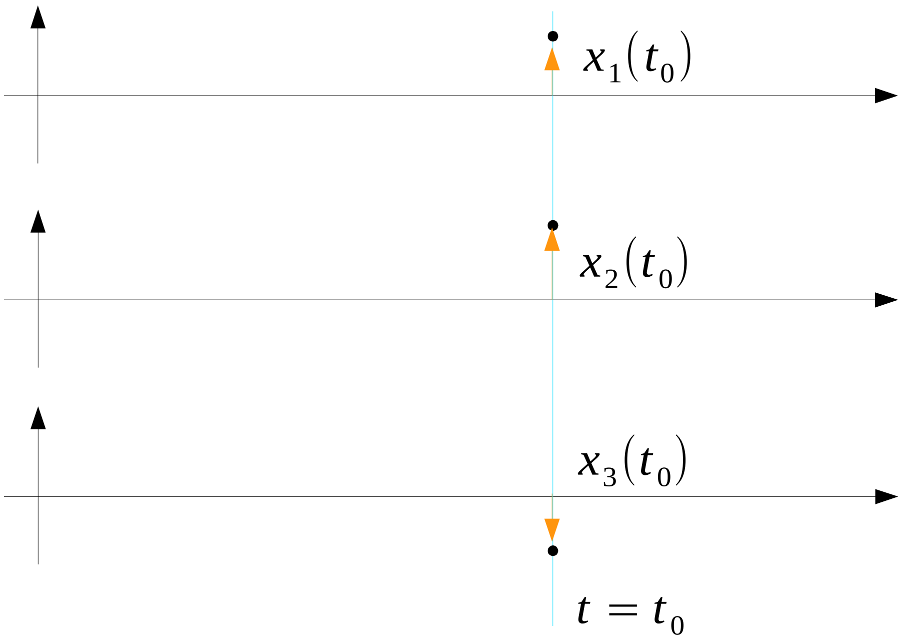
$$= \sigma^2(t)$$

Moments of a Random Process

$X(t)$ Random Variable at a given time t

$x_i(t)$ outcome of i[th] realization at a given time t

Ensemble



Ensemble Average

$$\bar{x}(t) = \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

$$N \rightarrow \infty$$

$$E[\, X(t) \,] = \int_{-\infty}^{+\infty} x \; f(x\,;\,t)\, dx$$

$$= \mu(t)$$

$$E[\, X(t_0) \,] = \int_{-\infty}^{+\infty} x \; f(x\,;\,t_0)\, dx = \mu(t_0)$$

Stationarity

First-Order Stationary Process

$$f(x_1;\ t_1) = f(x_1;\ t_1+k) \qquad \forall k$$

$$f(x;\ t) \Rightarrow f(x)$$

Second-Order Stationary Process

$$f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ t_1+k, t_2+k) \qquad \forall k$$

$$f(x_1, x_2;\ t_1, t_2) = f(x_1, x_2;\ 0, t_2-t_1) \qquad k = -t_1$$

$$f(x_1, x_2;\ t_1, t_2) \Rightarrow f(x_1, x_2;\ t_2-t_1)$$

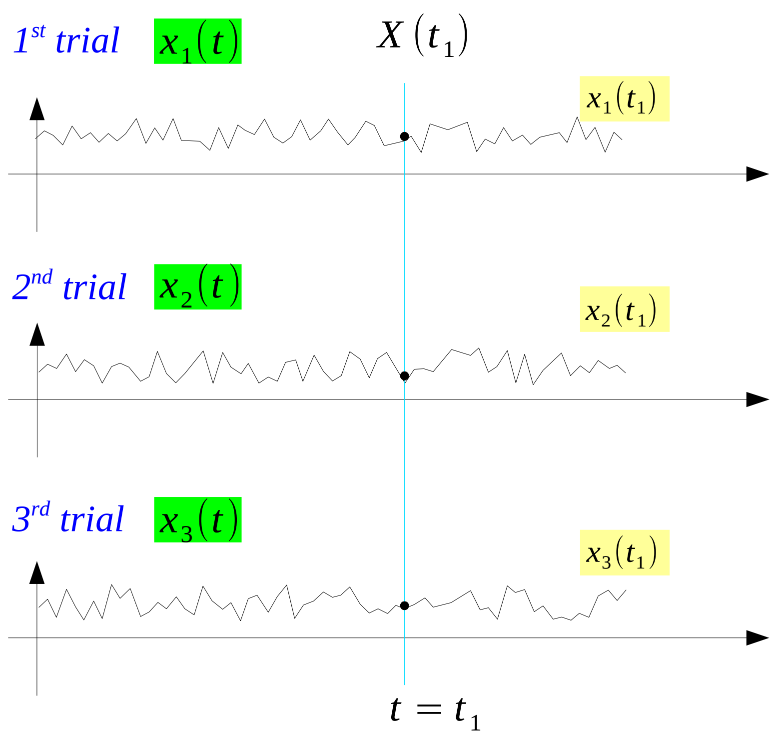
Nth-Order Stationary Process

$$f(x_1, x_2, \cdots, x_n;\ t_1, t_2, \cdots, t_n) = f(x_1, x_2, \cdots, x_n;\ t_1+k, t_2+k, \cdots, t_n+k) \qquad \forall k$$

Mean Function

Mean Function

$$\mu_x(t) = E[\,X(t)\,]$$



Ensemble Average

$$\overline{x(t)} = \frac{1}{N}\sum_{n=1}^{N} x_n(t)$$

$$N \rightarrow \infty$$

$$E[\,X(t)\,] = \int_{-\infty}^{+\infty} x\, f(x\,;t)\,dx$$

$$= \mu(t)$$

First Order Stationary Process

The n-th Moment

$$E\left[\, X^n(t) \,\right] = \int_{-\infty}^{+\infty} x^n \, f(x\,;\,t)\, dx$$

The n-th Central Moment

$$E\left[\, (X(t) - \mu(t))^n \,\right] = \int_{-\infty}^{+\infty} (x - \mu(t))^n \, f(x\,;\,t)\, dx$$

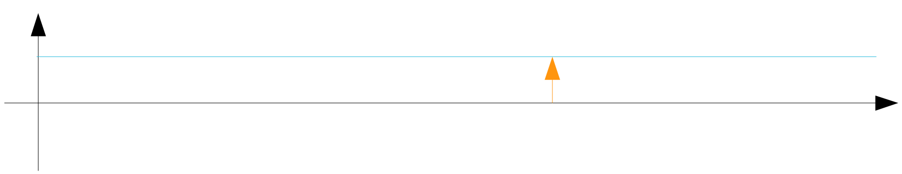
$$f(x, t) \Rightarrow f(x)$$

if ***f*** *does not change with time*

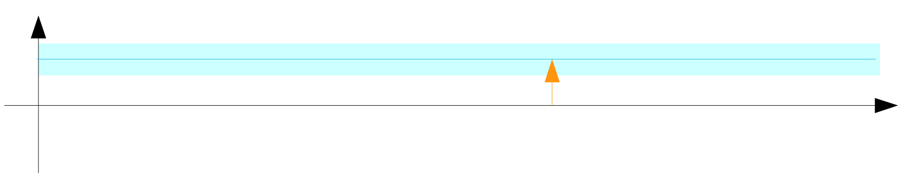
$$E\left[\, X^n(t) \,\right] \Rightarrow E\left[\, X^n \,\right]$$

$$E\left[\, (X(t) - \mu(t))^n \,\right] \Rightarrow E\left[\, (X - \mu)^n \,\right]$$

First Order Stationary Process



$$E\left[\, X(t) \,\right] \Rightarrow E\left[\, X \,\right] = \boxed{\mu}$$

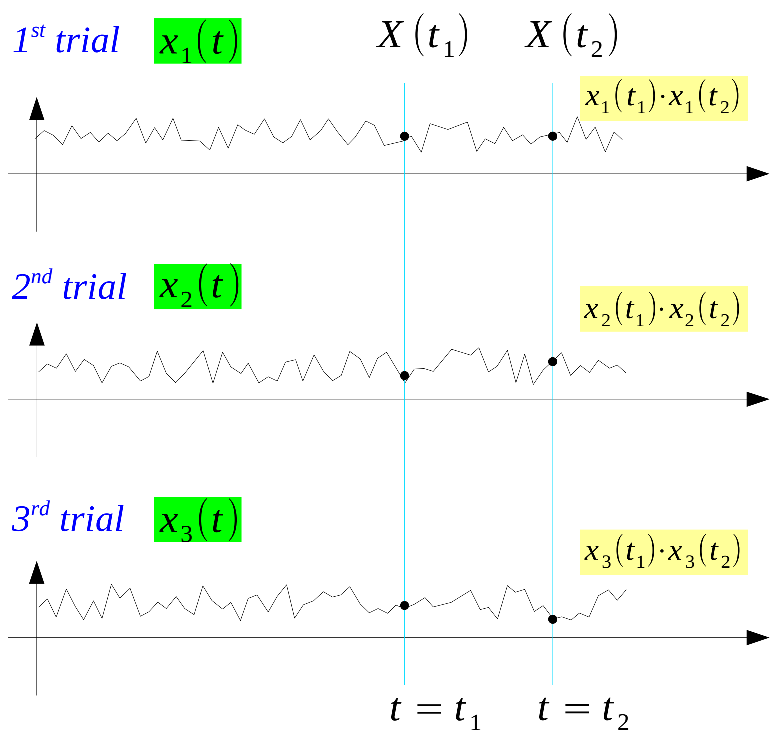


$$E\left[\, (X(t) - \mu(t))^2 \,\right] \Rightarrow E\left[\, (X - \mu)^2 \,\right] = \boxed{\sigma^2}$$

AutoCorrelation Functions

Auto-correlation Function

$$R_{xx}(t_1, t_2) = E\left[\, \underline{X(t_1)}\ \underline{X(t_2)} \,\right]$$



Ensemble Average

$$\overline{x(t_1)\,x(t_2)} = \frac{1}{N}\sum_{n=1}^{N} x_n(t_1)\cdot x_n(t_2)$$

$$N \rightarrow \infty$$

$$E\left[\, X(t_1)\,X(t_2) \,\right] = R_{xx}(t_1, t_2)$$

$$= \int_{-\infty}^{+\infty} x_1\ x_2\ f(x_1, x_2;\ t_1, t_2)\ dx_1\ dx_2$$

AutoCorrelation Function Examples

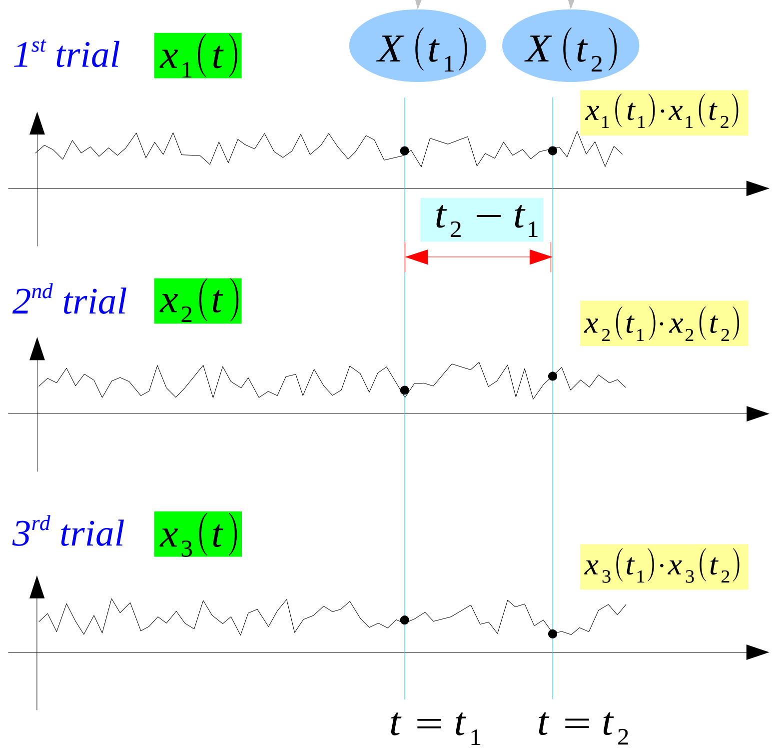
auto-correlation function

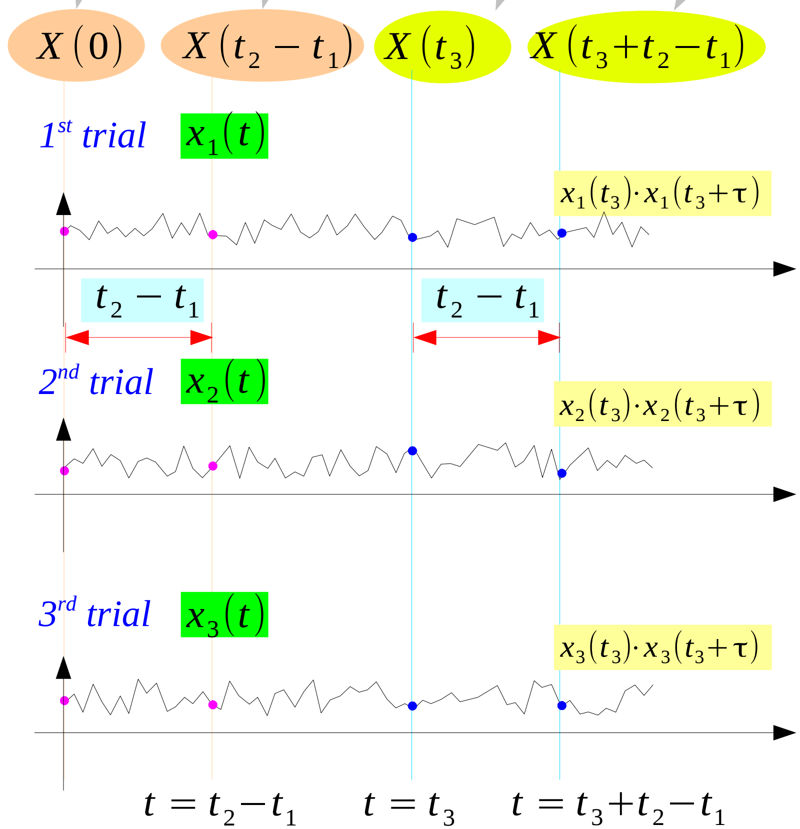
$\tau = t_2 - t_1$

$$R_{xx}(t_1 , t_2) = E[\, X(t_1)\ X(t_2) \,]$$

$$R_{xx}(0, t_2 - t_1) = E[\, X(0)\ X(t_2 - t_1) \,]$$

$$R_{xx}(t_3 , t_3 + t_2 - t_1) = E[\, X(t_3)\ X(t_3 + t_2 - t_1) \,]$$





Second Order Stationary Process

auto-correlation function

$$R_{xx}(t_1, t_2) = E[\,X(t_1)\,X(t_2)\,]$$

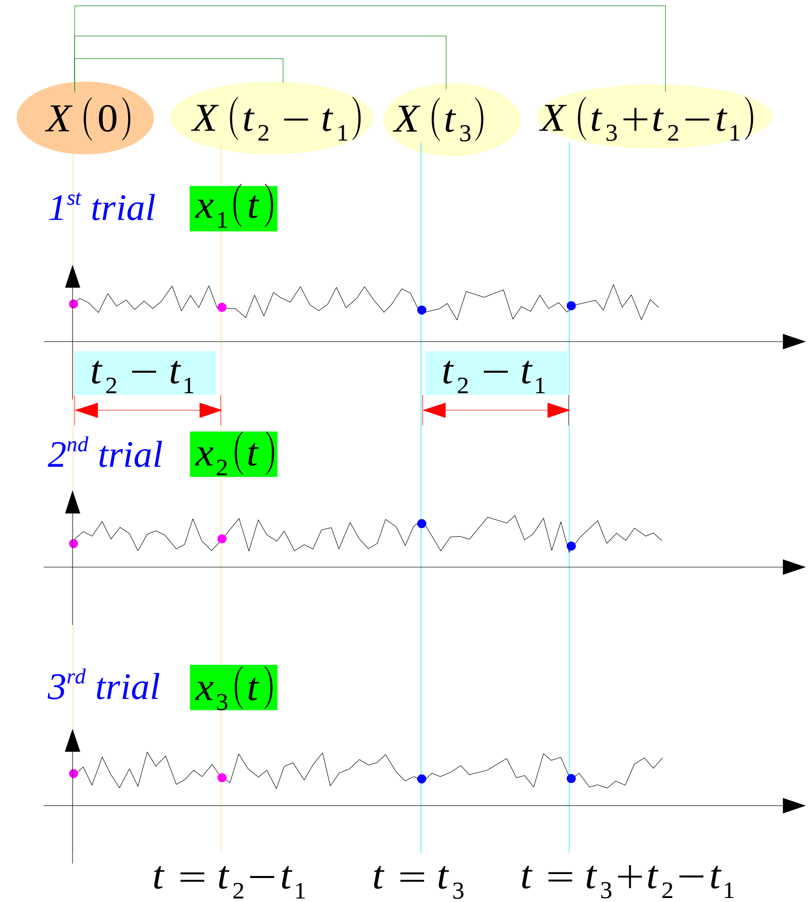
2nd Order Stationary Process

$$f(x_1, x_2;\ t_1, t_2) \Rightarrow f(x_1, x_2;\ t_2 - t_1)$$

$$R_{xx}(t_3+t_2-t_1) = E[\,X(0)\,X(t_3+t_2-t_1)\,]$$

$$R_{xx}(t_3) = E[\,X(0)\,X(t_3)\,]$$

$$R_{xx}(t_2-t_1) = E[\,X(0)\,X(t_2-t_1)\,]$$



$$R_{xx}(t_1, t_2) = R_{xx}(0, t_2-t_1) = R_{xx}(t_3, t_3+t_2-t_1) =$$

$$E[\,X(t_1)\,X(t_2)\,] = E[\,X(0)\,X(t_2-t_1)\,] =$$

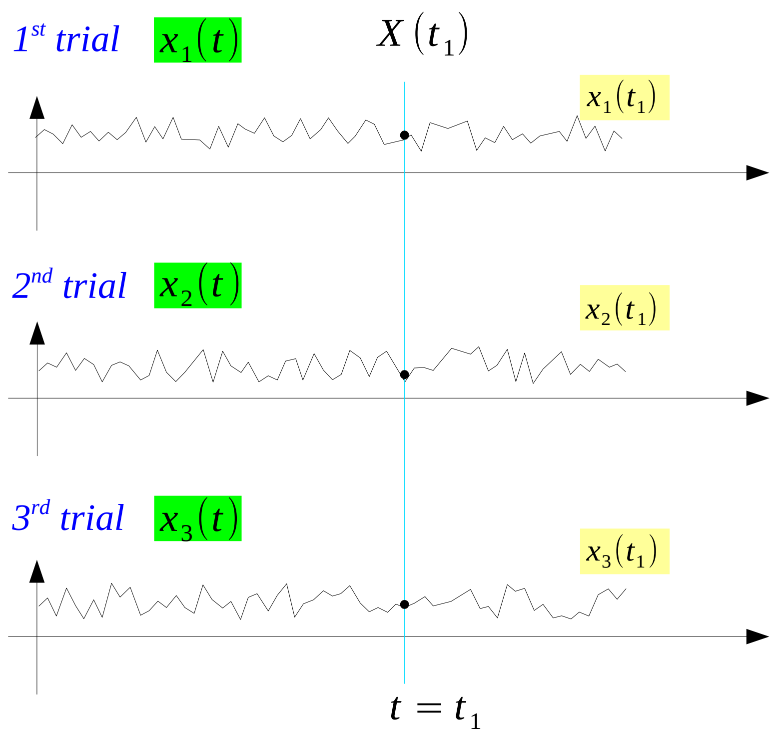
$$E[\,X(t_3)\,X(t_3+t_2-t_1)\,]$$

$$\Rightarrow R_{xx}(t_1 - t_2) = R_{xx}(\tau) \qquad \tau = t_2 - t_1$$

$$R_{xx}(\tau) = E[\,X(t)\,X(t+\tau)\,]$$

Time Average & Mean

mean function



Time Average

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt = \boxed{\mu_x}$$

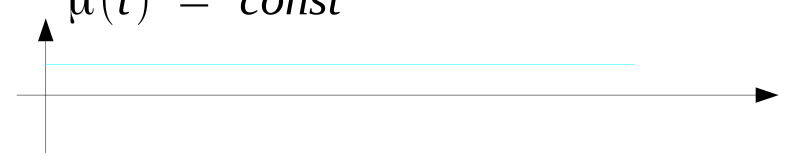
Ensemble Average

$$\overline{x(t)} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} x_n(t)$$

$$E[\, X(t) \,] = \int_{-\infty}^{+\infty} x\ f(x\,;t)\, dx = \boxed{\mu(t)}$$

Stationary Process

$$\mu(t) = const$$

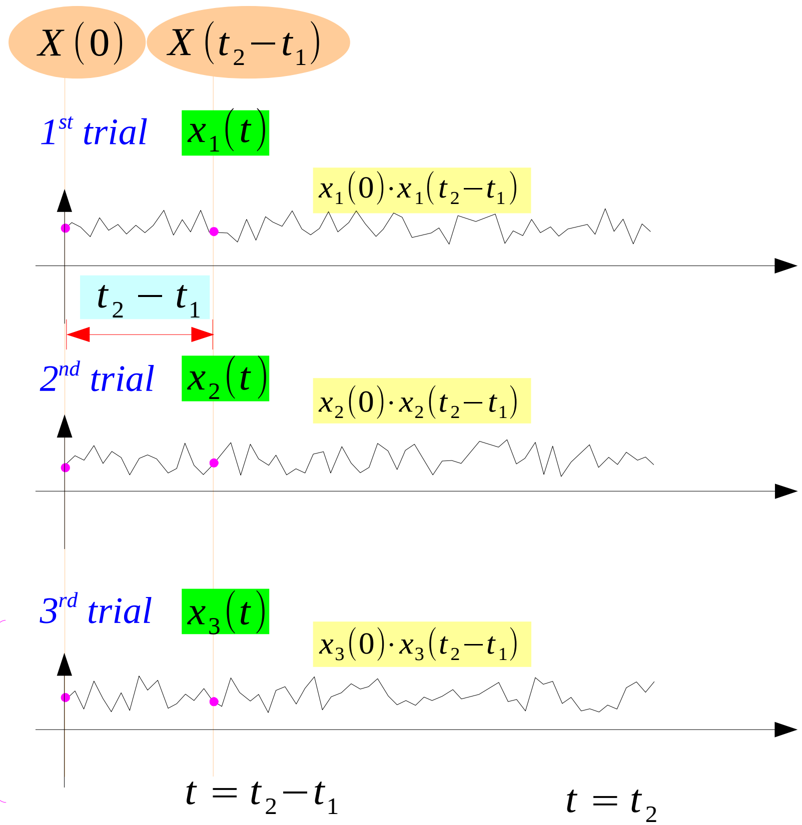


Ergodic Process

$$\boxed{\mu(t) = \mu_x}$$

Time Average & AutoCorrelation

autocorrelation function



Time Average

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)\,x(t+t_2-t_1)\,dt$$

Ensemble Average

$$\overline{x(t_1)\,x(t_2)} = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} x_n(t_1)\cdot x_n(t_2)$$

$$E[\,X(t_1)\,X(t_2)\,] = R_{xx}(t_1,\ t_2)$$

Stationary Process

$$E[\,X(t_1)\,X(t_2)\,] = R_{xx}(t_2-t_1)$$

Ergodic Process

$$R_{xx}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_0^T x(t)\,x(t+\tau)\,dt$$

Time Average & Ensemble Average

autocorrelation function

Ensemble Average

Time Average

Time Average & Ensemble Average

autocorrelation function

Ensemble Average

Moving Average Filter

Matched Filter

Correlation Functions (1)

auto-covariance function

$$C_{xx}(t_{1,}\ t_2) \;=\; E\Big[\ \underline{(X(t_1)-\mu_x(t_1))}\ \ \underline{(X(t_2)-\mu_x(t_2))}\ \Big]$$

$$C_{xx}(t_2\ -\ t_1) \;=\; E\Big[\ (X(t_1)-\mu_x(t_1))\ \ (X(t_2)-\mu_x(t_2))\ \Big]$$

$$C_{xx}(\tau) \;=\; E\Big[\ (X(t)-\mu_x)\ \ (X(t+\tau)-\mu_x)\ \Big]$$

auto-correlation function

$$R_{xx}(t_{1,}\ t_2) \;=\; E\Big[\ \underline{X(t_1)}\ \underline{X(t_2)}\ \Big]$$

$$R_{xx}(\tau) \;=\; E\Big[\ X(t)\ X(t+\tau)\ \Big]$$

Correlation Functions (2)

cross-covariance function

$$C_{xy}(t_1, t_2) = E\left[\, \underline{(X(t_1) - \mu_x(t_1))}\ \underline{(Y(t_2) - \mu_y(t_2))} \,\right]$$

$$C_{xy}(\tau) = E\left[\, (X(t) - \mu_x)\ (Y(t+\tau) - \mu_y) \,\right]$$

cross-correlation function

$$R_{xy}(t_1, t_2) = E\left[\, \underline{X(t_1)}\ \underline{Y(t_2)} \,\right]$$

$$R_{xy}(\tau) = E\left[\, X(t)\, Y(t+\tau) \,\right]$$

Ergodicity

$$\mu_x = \lim_{T \to \infty} \frac{1}{T} \int_0^T x(t)\, dt$$

$$\hat{\mu}_x = \frac{1}{T} \int_0^T x(t)\, dt = \bar{x}$$

$$\bar{\psi}_x^2 = \frac{1}{T} \int_0^T x^2(t)\, dt = \overline{x^2}$$

$$C_{xy}(\tau) = E\left[(X(t) - \mu_x)\ (Y(t+\tau) - \mu_y) \right]$$

$$C_{xy}(\tau) = \lim_{T \to 0} \frac{1}{T} \int_0^T (x(t) - \mu_x)\ (y(t+\tau) - \mu_y)\, dt$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-\tau} \int_0^{T-\tau} (x(t) - \bar{x})\ (y(t+\tau) - \bar{y})\, dt \qquad 0 \le \tau < T$$

$$\hat{C}_{xy}(\tau) = \frac{1}{T-|\tau|} \int_0^{T-|\tau|} (x(t) - \bar{x})\ (y(t+\tau) - \bar{y})\, dt \qquad -T < \tau \le 0$$

Linear System

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$\delta(t) \longrightarrow \boxed{h(t)} \longrightarrow h(t)$$

$$A\, e^{j\Phi}\, e^{jwt} \longrightarrow \boxed{h(t)} \longrightarrow \boxed{H(j\omega)}\, A\, e^{j\Phi}\, e^{j\omega t}$$

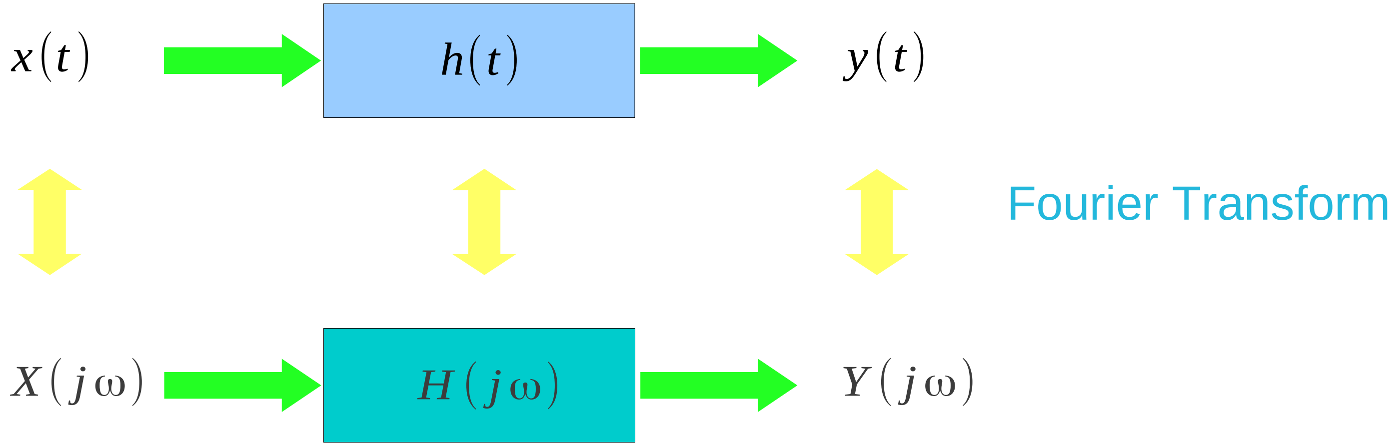
single frequency component : ω

single frequency component : ω

$$\boxed{H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}$$

Impulse Response & Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$



$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$Y(j\omega) = \int_{-\infty}^{+\infty} y(\tau)\, e^{-j\omega\tau}\, d\tau \qquad H(j\omega) = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

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Linear System & Random Variables

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$$

R.V		R.V
$X(t)$	$\rightarrow \boxed{h(t)} \rightarrow$	$Y(t)$

mean

$$E[X(t)] \rightarrow \boxed{h(t)} \rightarrow E[Y(t)] = \int_0^t E[X(\tau)] h(t-\tau)\, d\tau$$

correlation

$$E[X(t_1)X(t_2)] \rightarrow \boxed{h(t)} \rightarrow E[Y(t_1)Y(t_2)]$$

$$= \int_0^{t_1}\int_0^{t_2} E[X(\tau)] h(t_1-\tau) h(t_2-\alpha)\, d\tau\, d\alpha$$

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Output Correlation

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

correlation

$$E[X(t_1)X(t_2)] \longrightarrow \boxed{h(t)} \longrightarrow E[Y(t_1)Y(t_2)]$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[X(\alpha)X(\beta)]h(\underline{t_1-\alpha})h(\underline{t_2-\beta})\, d\alpha\, d\beta$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E[X(\underline{t_1-\alpha})X(\underline{t_2-\beta})]h(\alpha)h(\beta)\, d\alpha\, d\beta$$

WSS

$$R_{xx}(\tau) \longrightarrow \boxed{h(t)} \longrightarrow R_{YY}(\tau)$$

$$= \int_0^{t_1}\int_0^{t_2} E[X(\underline{t-\alpha})X(\underline{t+\tau-\beta})]h(\alpha)h(\beta)\, d\alpha\, d\beta$$

$$= \int_0^{t_1}\int_0^{t_2} R_{XX}(\underline{\tau+\alpha-\beta})h(\alpha)h(\beta)\, d\alpha\, d\beta$$

Output Spectral Density

WSS

$R_{xx}(\tau)$ → $h(t)$ → $R_{YY}(\tau) = \int_0^{t_1}\int_0^{t_2} R_{XX}(\tau+\alpha-\beta)h(\alpha)h(\beta)\,d\alpha\,d\beta$

Fourier Transform

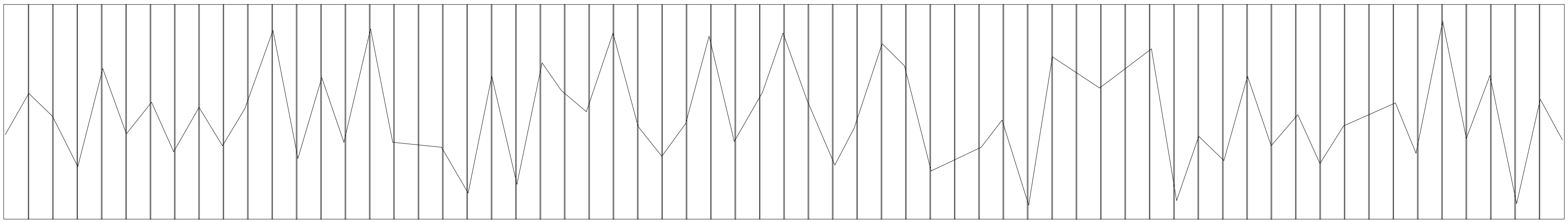
WSS

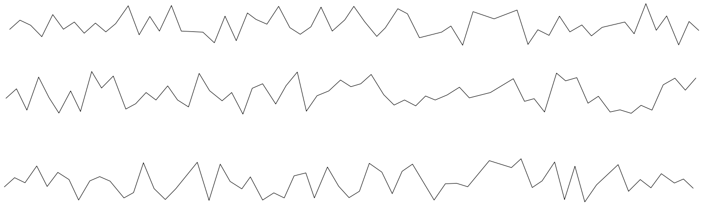
$S_{xx}(\omega)$ → $H(\omega)$ → $S_{YY}(\omega)$ → $= |H(\omega)|^2\, S_{XX}(\omega)$

$$= \int_{-\infty}^{+\infty} R_{xx}(\tau)e^{-j\omega\tau}\,d\tau \qquad\qquad = \int_{-\infty}^{+\infty} R_{YY}(\tau)e^{-j\omega\tau}\,d\tau$$

$$\begin{aligned}
S_{YY}(\omega) &= \int_{-\infty}^{+\infty} \underline{e^{-j\omega\tau}}\,d\tau\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\alpha-\beta)h(\alpha)h(\beta)\,d\alpha\,d\beta\right\} \\
&= \int_{-\infty}^{+\infty} \underline{e^{-j\omega(\tau-\beta+\alpha)}}\,\underline{e^{+j\omega\alpha}}\,\underline{e^{-j\omega\beta}}\,d\tau\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\alpha-\beta)h(\alpha)h(\beta)\,d\alpha\,d\beta\right\} \\
&= \left\{\int_{-\infty}^{+\infty} \underline{e^{-j\omega(\tau-\beta+\alpha)}} R_{XX}(\tau+\alpha-\beta)\,d\tau\right\} \left\{\int_{-\infty}^{+\infty} h(\alpha)\underline{e^{+j\omega\alpha}}\,d\alpha\right\} \left\{\int_{-\infty}^{+\infty} h(\beta)\underline{e^{-j\omega\beta}}\,d\beta\right\} \\
&= S_{XX}(\omega)H^*(\omega)H(\omega) \qquad = |H(\omega)|^2\, S_{XX}(\omega)
\end{aligned}$$

Time Average





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