

DFT Matrix Properties (3B)

- $X[1]$
- $X[2]$
- $X[3]$
- $X[4]$
- $X[5]$
- $X[6]$
- $X[7]$

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N=8 DFT

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^1 & W_8^2 & W_8^3 & W_8^4 & W_8^5 & W_8^6 & W_8^7 \\ W_8^0 & W_8^2 & W_8^4 & W_8^6 & W_8^8 & W_8^{10} & W_8^{12} & W_8^{14} \\ W_8^0 & W_8^3 & W_8^6 & W_8^9 & W_8^{12} & W_8^{15} & W_8^{18} & W_8^{21} \\ W_8^0 & W_8^4 & W_8^8 & W_8^{12} & W_8^{16} & W_8^{20} & W_8^{24} & W_8^{28} \\ W_8^0 & W_8^5 & W_8^{10} & W_8^{15} & W_8^{20} & W_8^{25} & W_8^{30} & W_8^{35} \\ W_8^0 & W_8^6 & W_8^{12} & W_8^{18} & W_8^{24} & W_8^{30} & W_8^{36} & W_8^{42} \\ W_8^0 & W_8^7 & W_8^{14} & W_8^{21} & W_8^{28} & W_8^{35} & W_8^{42} & W_8^{49} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 IDFT

$$x[n] = \frac{1}{N} \sum_{k=0}^7 W_8^{-kn} X[k] \quad W_8^{-kn} = e^{+j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \frac{1}{N} \begin{bmatrix} W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 & W_8^0 \\ W_8^0 & W_8^{-1} & W_8^{-2} & W_8^{-3} & W_8^{-4} & W_8^{-5} & W_8^{-6} & W_8^{-7} \\ W_8^0 & W_8^{-2} & W_8^{-4} & W_8^{-6} & W_8^{-8} & W_8^{-10} & W_8^{-12} & W_8^{-14} \\ W_8^0 & W_8^{-3} & W_8^{-6} & W_8^{-9} & W_8^{-12} & W_8^{-15} & W_8^{-18} & W_8^{-21} \\ W_8^0 & W_8^{-4} & W_8^{-8} & W_8^{-12} & W_8^{-16} & W_8^{-20} & W_8^{-24} & W_8^{-28} \\ W_8^0 & W_8^{-5} & W_8^{-10} & W_8^{-15} & W_8^{-20} & W_8^{-25} & W_8^{-30} & W_8^{-35} \\ W_8^0 & W_8^{-6} & W_8^{-12} & W_8^{-18} & W_8^{-24} & W_8^{-30} & W_8^{-36} & W_8^{-42} \\ W_8^0 & W_8^{-7} & W_8^{-14} & W_8^{-21} & W_8^{-28} & W_8^{-35} & W_8^{-42} & W_8^{-49} \end{bmatrix} \begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix}$$

N=8 DFT Matrix (1)

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \\ X[4] \\ X[5] \\ X[6] \\ X[7] \end{bmatrix} = \begin{bmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 0} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 7} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 6} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 5} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 3} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} \\ e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 1} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix}$$

N=8 IDFT Matrix (1)

$$x[n] = \frac{1}{N} \sum_{k=0}^7 W_8^{-kn} X[k] \quad W_8^{-kn} = e^{+j\left(\frac{2\pi}{8}\right)kn}$$

$$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ x[7] \end{bmatrix} = \begin{bmatrix} e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 0} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 7} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 6} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 5} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 4} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 3} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 2} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 7} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 1} \end{bmatrix} \begin{bmatrix} \frac{X[0]}{N} \\ \frac{X[1]}{N} \\ \frac{X[2]}{N} \\ \frac{X[3]}{N} \\ \frac{X[4]}{N} \\ \frac{X[5]}{N} \\ \frac{X[6]}{N} \\ \frac{X[7]}{N} \end{bmatrix}$$

Symmetric DFT Matrix – Index (1)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

	n=0	n=1	n=2	...	n=N-1
k=0	0·0	0·1	0·2	...	0·(N-1)
k=1	1·0	1·1	1·2	...	1·(N-1)
k=2	2·0	2·1	2·2	...	2·(N-1)
...
k=N-1	(N-1)·0	(N-1)·1	(N-1)·2	...	(N-1)·(N-1)

Exponents in
DFT matrix **A** and
IDFT matrix **B**

$$A = A^T$$

$$B = B^T$$

Symmetric DFT Matrix – Index (2)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

	n=0	n=1	n=2	...	n=N-1	
k=0	0·0 → +0	0·1 → +0	0·2 → +0	...	0·(N-1) → +0	+ 0 (mod N)
k=1	1·0 → +1	1·1 → +1	1·2 → +1	...	1·(N-1) → +1	+ 1 (mod N)
k=2	2·0 → +2	2·1 → +2	2·2 → +2	...	2·(N-1) → +2	+ 2 (mod N)
•	•	•	•	•	•	
•	•	•	•	•	•	
•	•	•	•	•	•	
k=N-1	(N-1)·0 → +(N-1)	(N-1)·1 → +(N-1)	(N-1)·2 → +(N-1)	...	(N-1)·(N-1) → +(N-1)	+ N-1 (mod N)

$$A = A^T$$

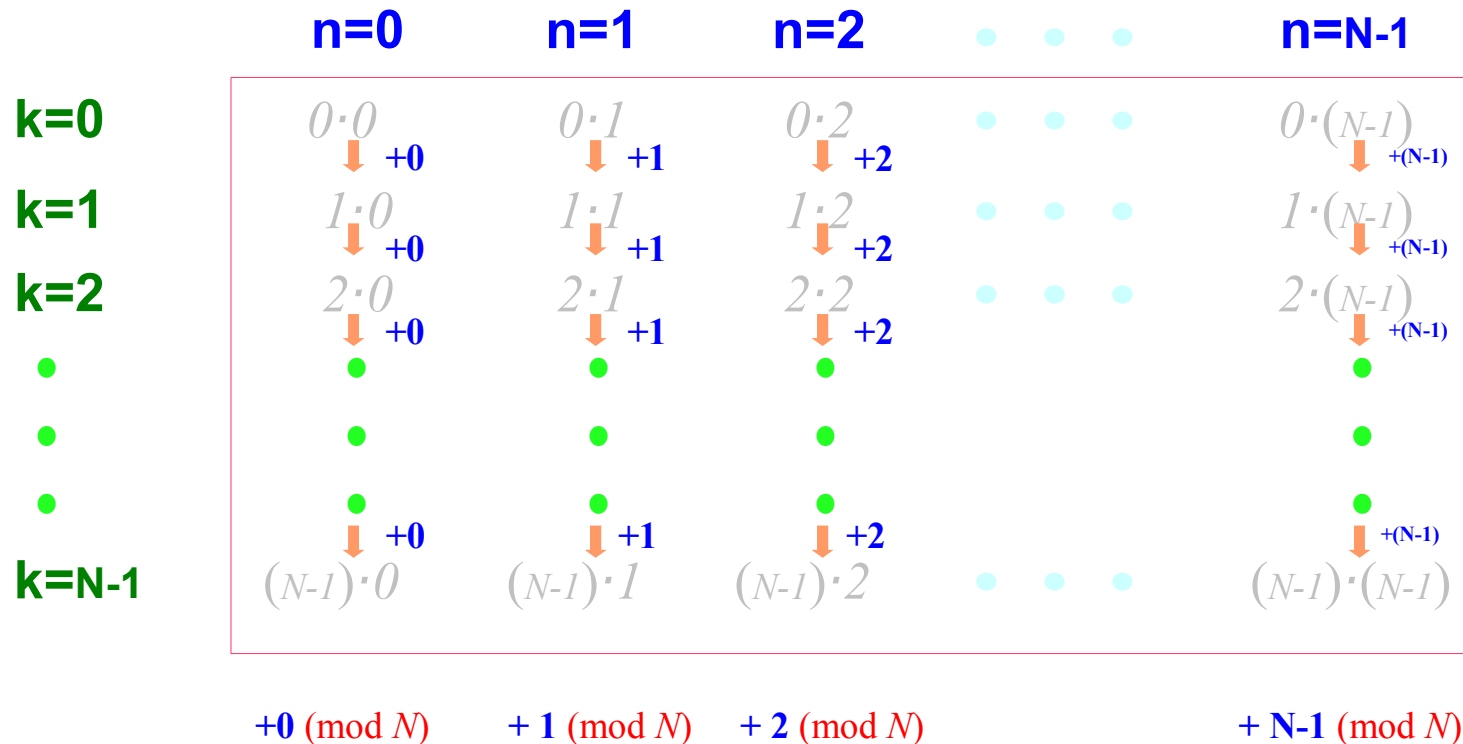
$$B = B^T$$

Exponents in
DFT matrix **A** and
IDFT matrix **B**

Symmetric DFT Matrix – Index (3)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



$$A = A^T$$

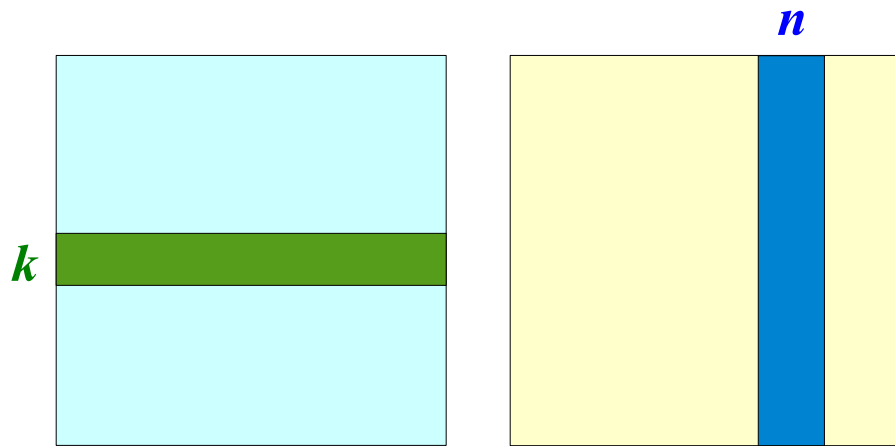
$$B = B^T$$

Exponents in DFT matrix **A** and IDFT matrix **B**

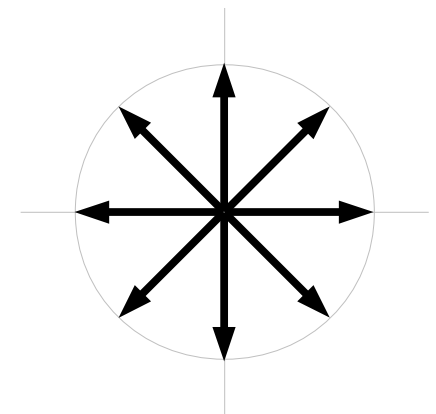
Product of DFT & IDFT Matrix

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$



$$\left\{ \begin{array}{l} e^{+j(\frac{2\pi}{N})0 \cdot n} \\ e^{+j(\frac{2\pi}{N})1 \cdot n} \\ \vdots \\ e^{+j(\frac{2\pi}{N})(N-1) \cdot n} \end{array} \right\}$$



$$\{ e^{-j(\frac{2\pi}{N})k \cdot 0}, e^{-j(\frac{2\pi}{N})k \cdot 1}, \dots, e^{-j(\frac{2\pi}{N})k \cdot (N-1)} \}$$

Inner product

$$e^{-j(\frac{2\pi}{N})(k-n) \cdot 0} + e^{-j(\frac{2\pi}{N})(k-n) \cdot 1} + \dots + e^{-j(\frac{2\pi}{N})(k-n) \cdot (N-1)} = \begin{cases} 0 & (n \neq k) \\ N & (n = k) \end{cases}$$

N=8 DFT & IDFT Matrix (1)

	0	1	2	3	4	5	6	7	
Row 0	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 0}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 0}$	IDFT
Row 1	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 1}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 3}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 5}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 7}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 1}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 3}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 5}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 7}$	IDFT
Row 2	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 6}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 6}$	IDFT
Row 3	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 3}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 1}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 7}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 5}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 3}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 1}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 7}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 5}$	IDFT

N=8 DFT & IDFT Matrix (2)

	0	1	2	3	4	5	6	7	
Row 4	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 4}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 4}$	IDFT
Row 5	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 5}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 7}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 1}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 3}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 5}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 7}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 1}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 3}$	IDFT
Row 6	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 2}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 2}$	IDFT
Row 7	$e^{-j\frac{\pi}{4}\cdot 0}$	$e^{-j\frac{\pi}{4}\cdot 7}$	$e^{-j\frac{\pi}{4}\cdot 6}$	$e^{-j\frac{\pi}{4}\cdot 5}$	$e^{-j\frac{\pi}{4}\cdot 4}$	$e^{-j\frac{\pi}{4}\cdot 3}$	$e^{-j\frac{\pi}{4}\cdot 2}$	$e^{-j\frac{\pi}{4}\cdot 1}$	DFT
	$e^{+j\frac{\pi}{4}\cdot 0}$	$e^{+j\frac{\pi}{4}\cdot 7}$	$e^{+j\frac{\pi}{4}\cdot 6}$	$e^{+j\frac{\pi}{4}\cdot 5}$	$e^{+j\frac{\pi}{4}\cdot 4}$	$e^{+j\frac{\pi}{4}\cdot 3}$	$e^{+j\frac{\pi}{4}\cdot 2}$	$e^{+j\frac{\pi}{4}\cdot 1}$	IDFT

Product AB – Diagonal Elements

$$C = A \cdot B$$

$$[C]_{(i,j)} = [A]_{(row\ i)} \cdot [B]_{(col\ j)}$$

$$C_{(i,i)} = N$$

$$C_{(1,1)} \begin{pmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 3} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 7} \\ e^{+j\frac{\pi}{4}\cdot 0} & e^{+j\frac{\pi}{4}\cdot 1} & e^{+j\frac{\pi}{4}\cdot 2} & e^{+j\frac{\pi}{4}\cdot 3} & e^{+j\frac{\pi}{4}\cdot 4} & e^{+j\frac{\pi}{4}\cdot 5} & e^{+j\frac{\pi}{4}\cdot 6} & e^{+j\frac{\pi}{4}\cdot 7} \end{pmatrix} \bullet \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}^T = N$$

+1 +1 +1 +1 +1 +1 +1 +1

Product AB – Off-Diagonal Elements

$$C = A \cdot B$$

$$[C]_{(i,j)} = [A]_{(row\ i)} \cdot [B]_{(col\ j)}$$

$$C_{(i,j)} = 0$$

$$C_{(1,2)}$$

$$\begin{pmatrix} e^{-j \cdot \frac{\pi}{4} \cdot 0} & e^{-j \cdot \frac{\pi}{4} \cdot 1} & e^{-j \cdot \frac{\pi}{4} \cdot 2} & e^{-j \cdot \frac{\pi}{4} \cdot 3} & e^{-j \cdot \frac{\pi}{4} \cdot 4} & e^{-j \cdot \frac{\pi}{4} \cdot 5} & e^{-j \cdot \frac{\pi}{4} \cdot 6} & e^{-j \cdot \frac{\pi}{4} \cdot 7} \end{pmatrix} \cdot \begin{pmatrix} e^{+j \cdot \frac{\pi}{4} \cdot 0} & e^{+j \cdot \frac{\pi}{4} \cdot 2} & e^{+j \cdot \frac{\pi}{4} \cdot 4} & e^{+j \cdot \frac{\pi}{4} \cdot 6} & e^{+j \cdot \frac{\pi}{4} \cdot 0} & e^{+j \cdot \frac{\pi}{4} \cdot 2} & e^{+j \cdot \frac{\pi}{4} \cdot 4} & e^{+j \cdot \frac{\pi}{4} \cdot 6} \end{pmatrix}^T$$
$$e^{+j \cdot \frac{\pi}{4} \cdot 0} + e^{+j \cdot \frac{\pi}{4} \cdot 1} + e^{+j \cdot \frac{\pi}{4} \cdot 2} + e^{+j \cdot \frac{\pi}{4} \cdot 3} + e^{+j \cdot \frac{\pi}{4} \cdot 4} + e^{+j \cdot \frac{\pi}{4} \cdot 5} + e^{+j \cdot \frac{\pi}{4} \cdot 6} + e^{+j \cdot \frac{\pi}{4} \cdot 7} = 0$$

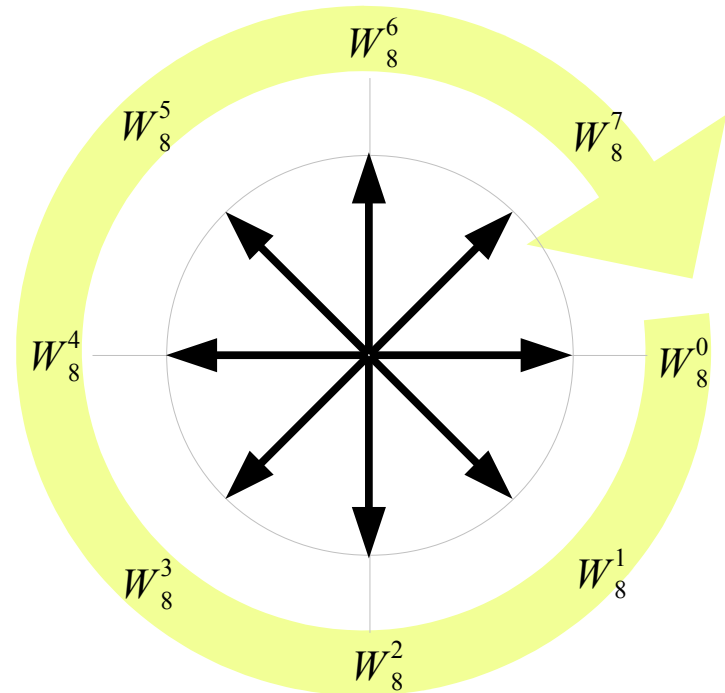
Root of Unity

$$\sum_{k=0}^{N-1} W_N^k = \sum_{k=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)k} = 0$$

$$z \equiv e^{-j\left(\frac{2\pi}{N}\right)}$$

$$z^N = e^{-j\left(\frac{2\pi}{N}\right)N} = 1$$

$$\sum_{k=0}^{N-1} e^{-j\left(\frac{2\pi}{N}\right)k} = \frac{z^N - 1}{z - 1} = 0$$



$$W_8^0 + W_8^1 + W_8^2 + W_8^3 + W_8^4 + W_8^5 + W_8^6 + W_8^7 = 0$$

Complex Vector Inner Product

Hermitian inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} = \sum x_i^* y_i$$

\mathbf{x}^H : conjugate transpose

Norm of Hermitian inner products

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{\mathbf{x}^H \cdot \mathbf{x}} = \sqrt{\sum x_i^* x_i}$$

the length of a vector

$$\mathbf{x} = \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix}$$

$$\langle \mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \cdot \mathbf{x} = \sum x_i^* x_i$$

$$\begin{pmatrix} a_1 - jb_1 & a_2 - jb_2 & \cdots & a_n - jb_n \end{pmatrix} \begin{pmatrix} a_1 + jb_1 \\ a_2 + jb_2 \\ \vdots \\ a_n + jb_n \end{pmatrix} = \sum_{i=1}^n a_i^2 + b_i^2$$

Cauchy-Schwartz Inequality

For all vectors \mathbf{x} and \mathbf{y} of an inner product space

$$|\langle \mathbf{x}, \mathbf{y} \rangle|^2 \leq \langle \mathbf{x}, \mathbf{x} \rangle \cdot \langle \mathbf{y}, \mathbf{y} \rangle$$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

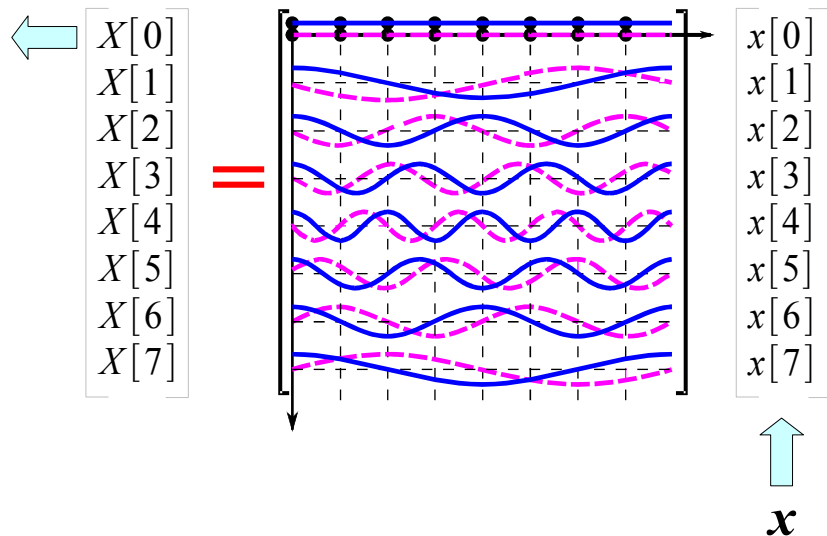
The equality holds if and only if \mathbf{x} and \mathbf{y} are linearly dependent \Rightarrow maximum

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^H \cdot \mathbf{y} \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\| \quad \mathbf{x} = \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix} \quad \mathbf{y} = k \begin{pmatrix} a_1 + j b_1 \\ a_2 + j b_2 \\ \vdots \\ a_n + j b_n \end{pmatrix}$$

Inner product is maximum
when $\mathbf{y} = k \mathbf{x}$

$$\langle \mathbf{x}, \mathbf{y} \rangle \leq k \left(\sum_{i=1}^n a_i^2 + b_i^2 \right)$$

N=8 DFT : Inner Product X[0]

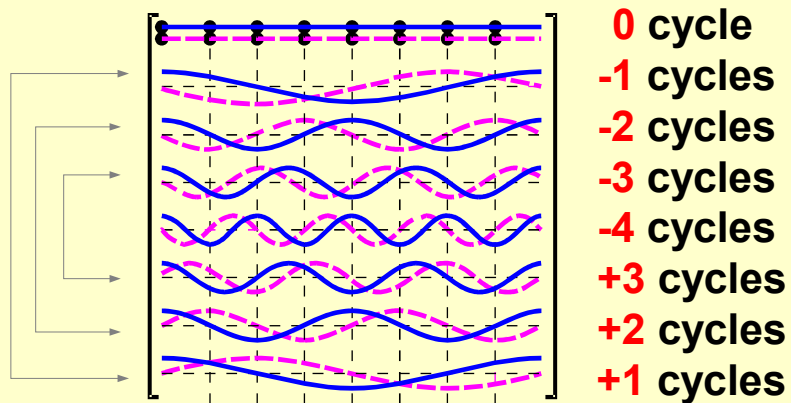


$X[0]$ measures “0 cycle” component in \mathbf{x}

$$\langle \mathbf{r}_0^H, \mathbf{x} \rangle = \mathbf{r}_0 \cdot \mathbf{x} \leq \|\mathbf{r}_0^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_0^H$

When \mathbf{x} looks like this, $X[0]$ is max.

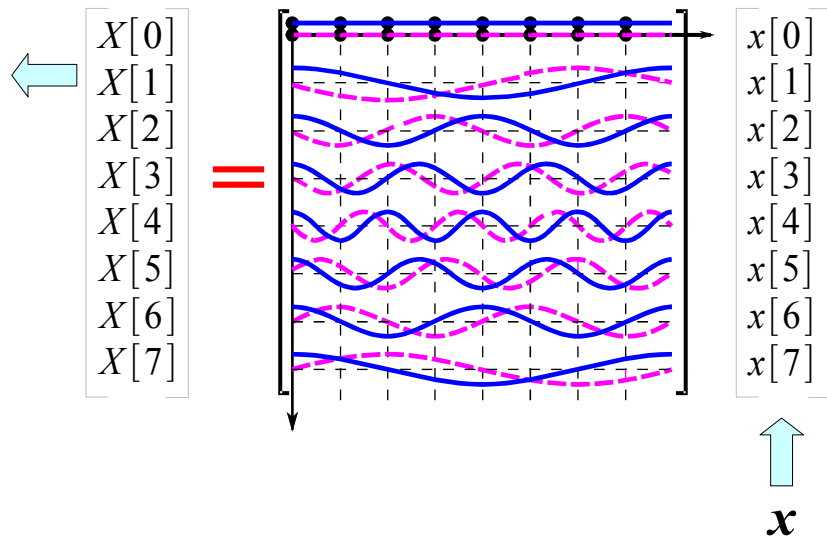


- 0 cycle
- 1 cycles
- 2 cycles
- 3 cycles
- 4 cycles
- +3 cycles
- +2 cycles
- +1 cycles

————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[1]

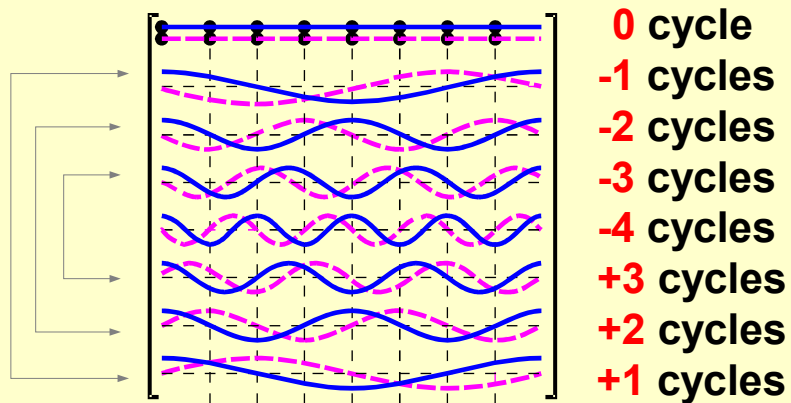


X[1] measures "+1 cycle" component in x

$$\langle \mathbf{r}_1^H, \mathbf{x} \rangle = \mathbf{r}_1 \cdot \mathbf{x} \leq \|\mathbf{r}_1^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_1^H$

When x looks like this, $X[1]$ is max.

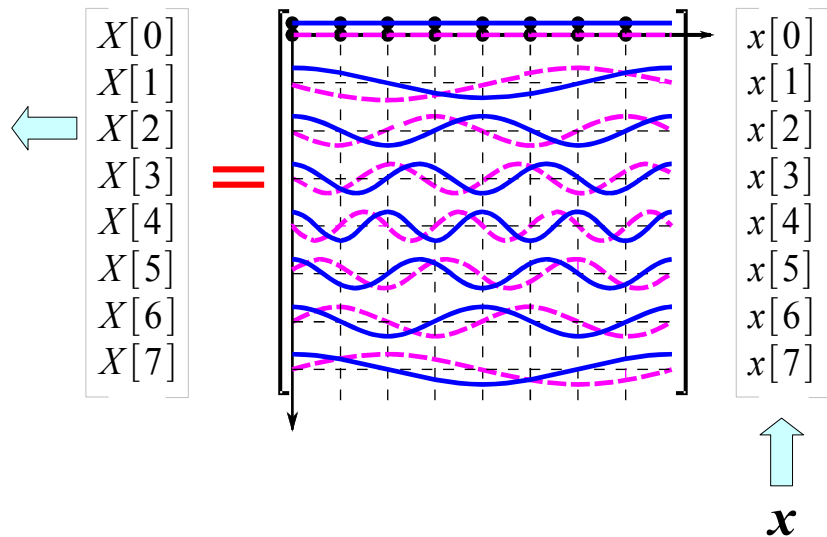


- 0 cycle
- 1 cycles
- 2 cycles
- 3 cycles
- 4 cycles
- +3 cycles
- +2 cycles
- +1 cycles

————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

..... $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[2]

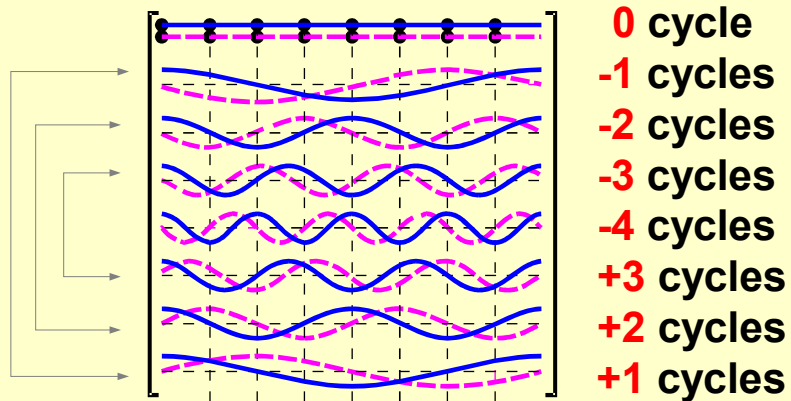


X[2] measures “+2 cycle” component in **x**

$$\langle \mathbf{r}_2^H, \mathbf{x} \rangle = \mathbf{r}_2 \cdot \mathbf{x} \leq \|\mathbf{r}_2^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_2^H$

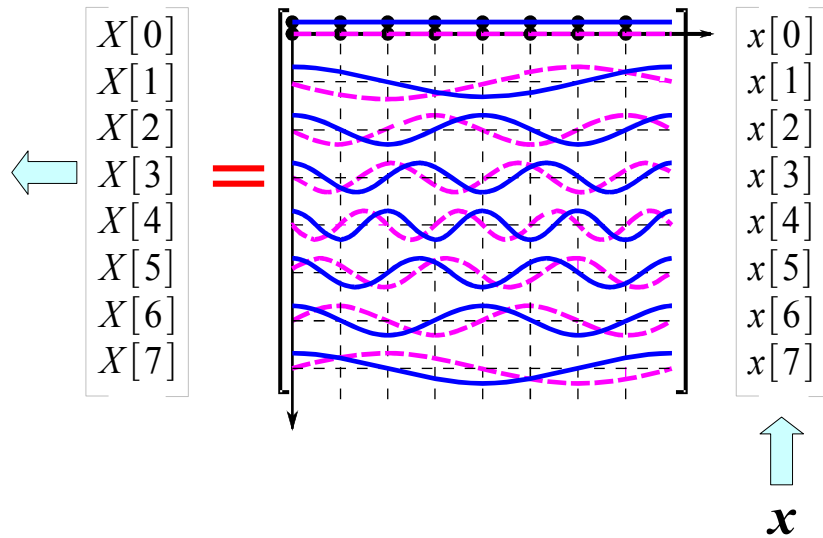
When x looks like this, $X[2]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[3]

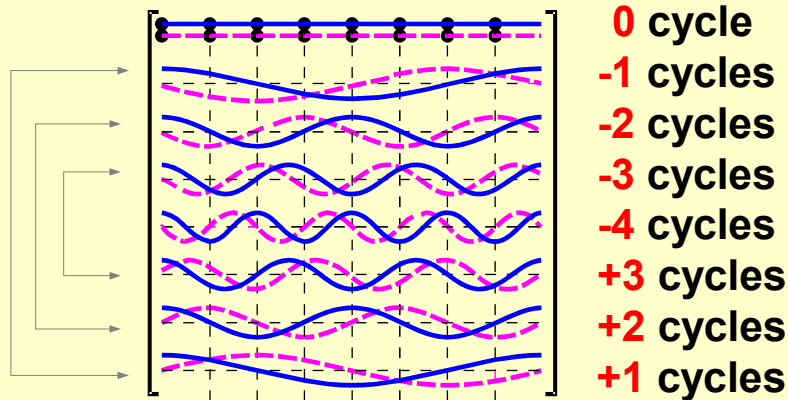


X[3] measures “+3 cycle” component in **x**

$$\langle \mathbf{r}_3^H, \mathbf{x} \rangle = \mathbf{r}_3 \cdot \mathbf{x} \leq \|\mathbf{r}_3^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_3^H$

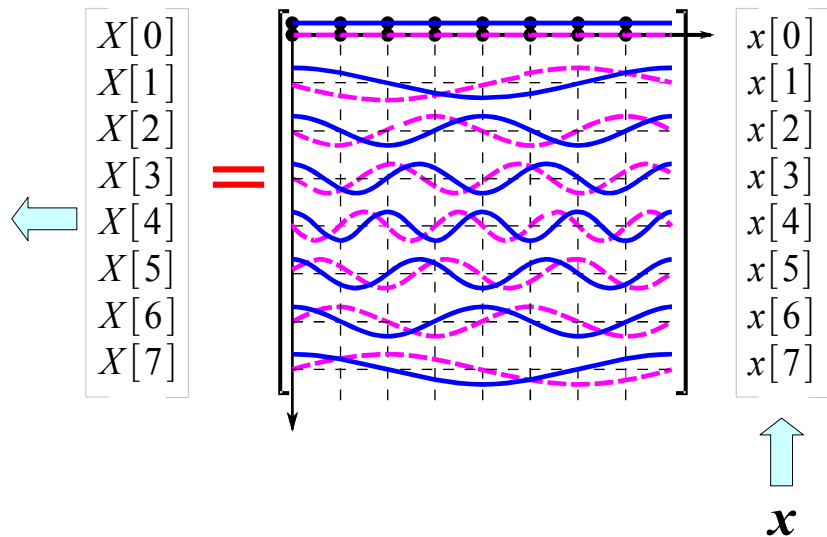
When x looks like this, $X[3]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[4]

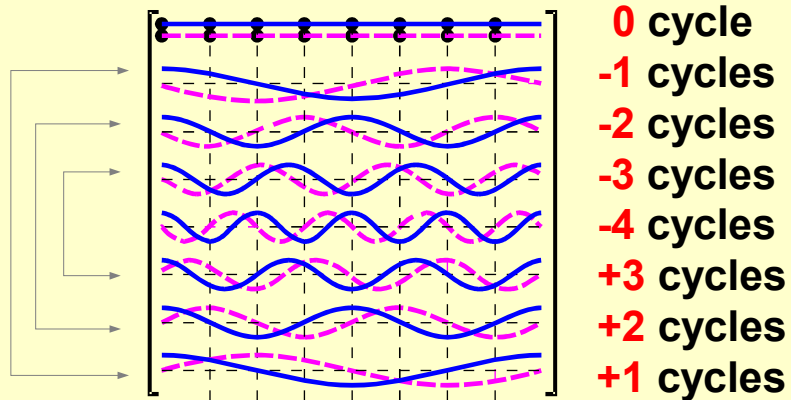


X[4] measures “+4 cycle” component in x

$$\langle \mathbf{r}_4^H, \mathbf{x} \rangle = \mathbf{r}_4 \cdot \mathbf{x} \leq \|\mathbf{r}_4^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_4^H$

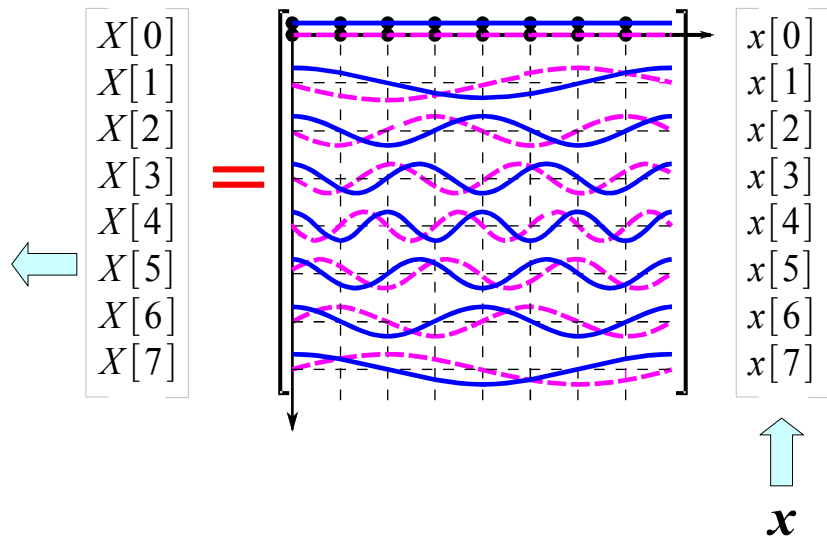
When x looks like this, $X[4]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[5]

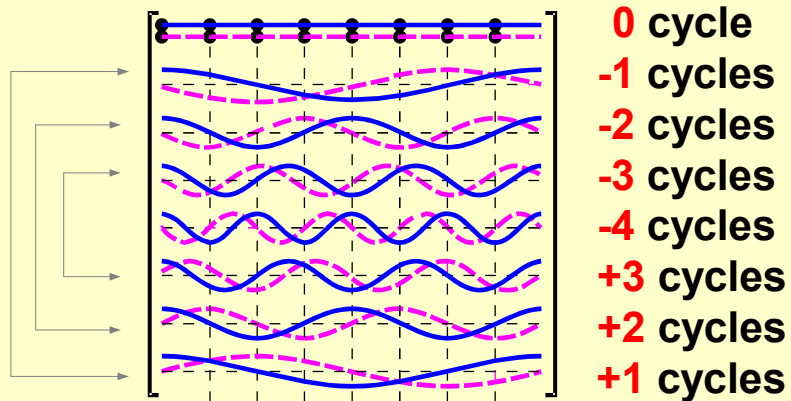


X[5] measures “-3 cycle” component in **x**

$$\langle \mathbf{r}_5^H, \mathbf{x} \rangle = \mathbf{r}_5 \cdot \mathbf{x} \leq \|\mathbf{r}_5^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_5^H$

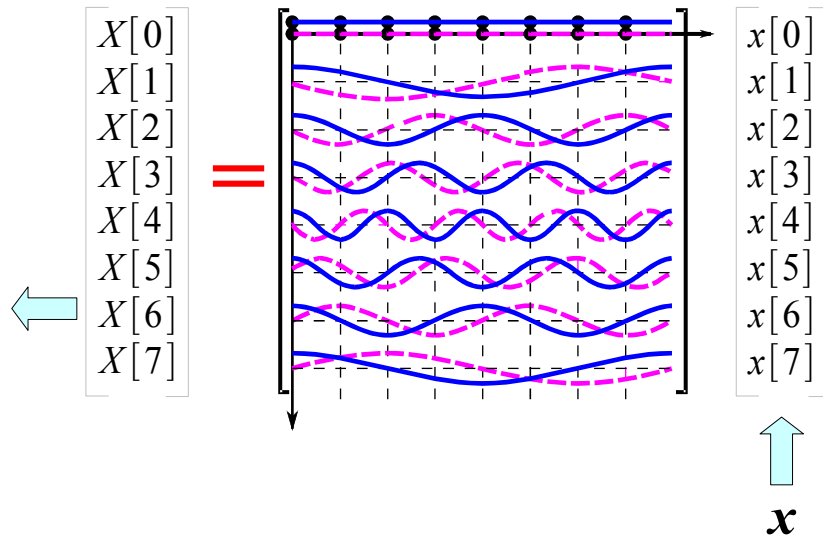
When \mathbf{x} looks like this, $X[5]$ is max.



————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : Inner Product X[6]

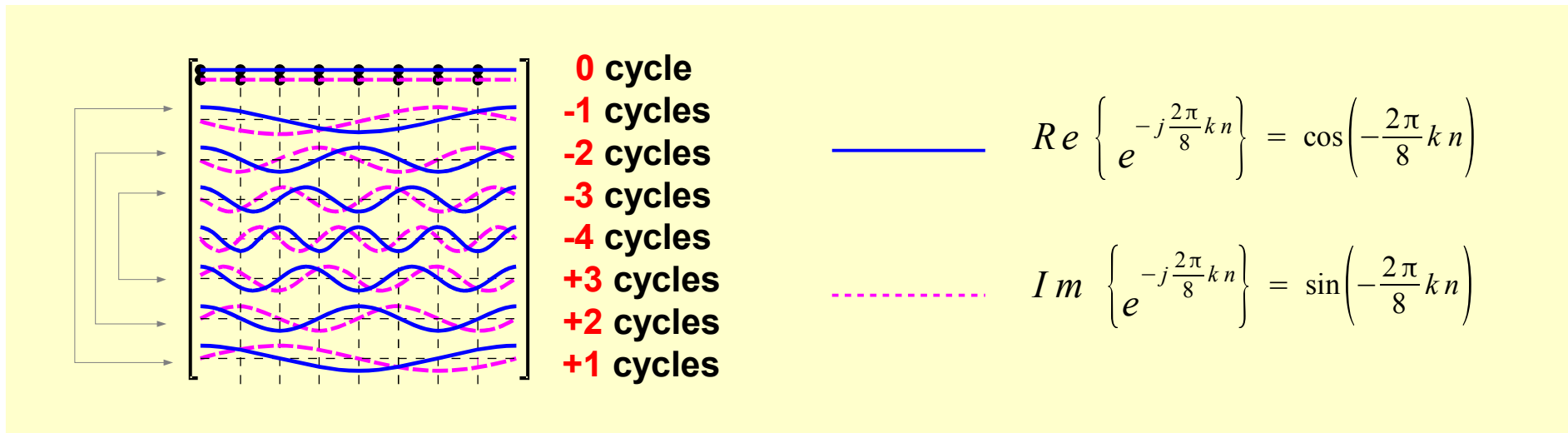


X[6] measures “-2 cycle” component in **x**

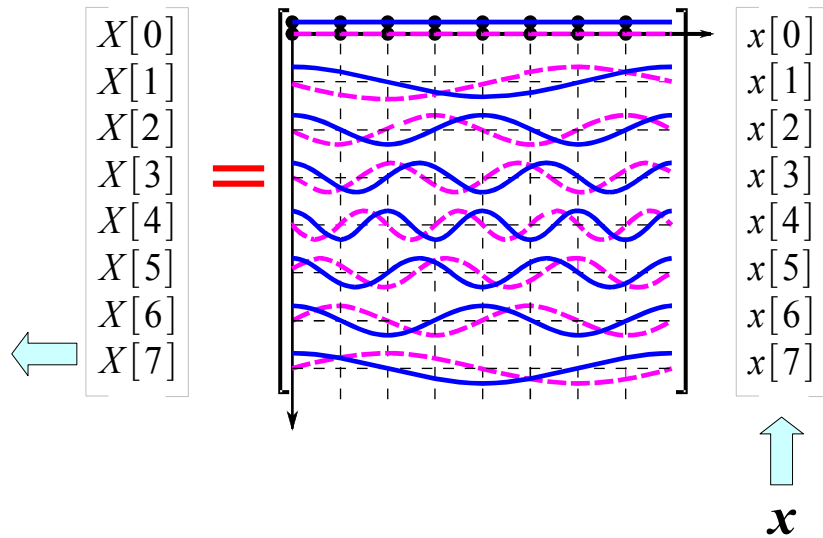
$$\langle \mathbf{r}_6^H, \mathbf{x} \rangle = \mathbf{r}_6 \cdot \mathbf{x} \leq \|\mathbf{r}_6^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_6^H$

When \mathbf{x} looks like this, $X[6]$ is max.



N=8 DFT : Inner Product X[7]

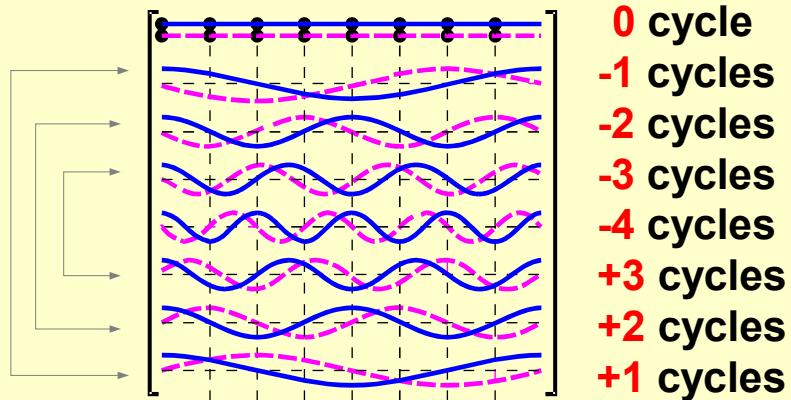


X[7] measures “-1 cycle” component in **x**

$$\langle \mathbf{r}_7^H, \mathbf{x} \rangle = \mathbf{r}_7 \cdot \mathbf{x} \leq \|\mathbf{r}_7^H\| \cdot \|\mathbf{x}\|$$

maximum when $\mathbf{x} = k \mathbf{r}_7^H$

When \mathbf{x} looks like this, $X[7]$ is max.

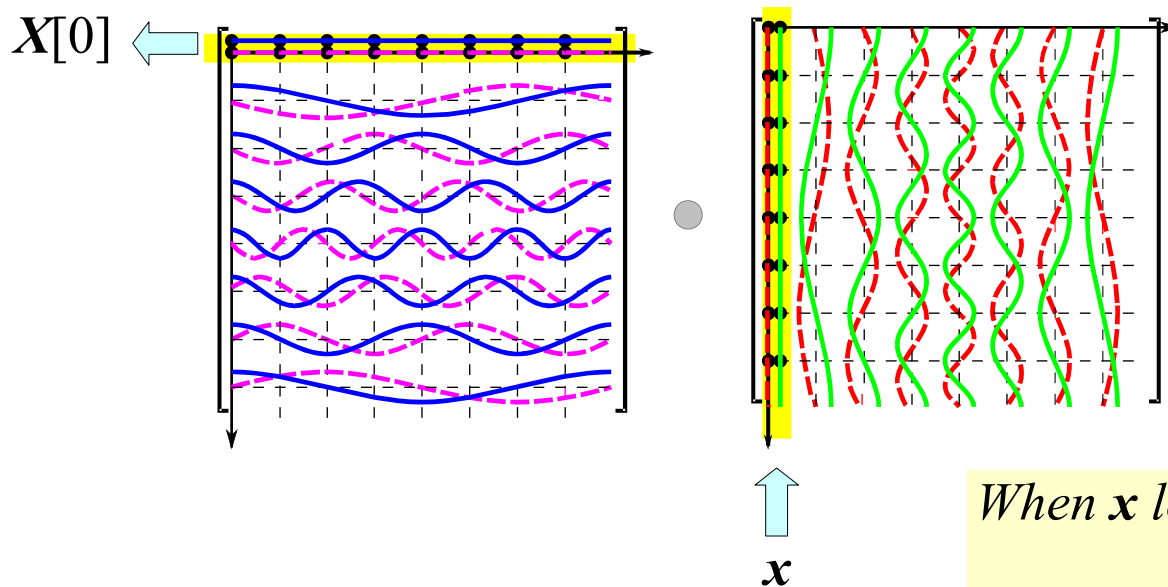


————— $Re \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \cos\left(-\frac{2\pi}{8}kn\right)$

----- $Im \left\{ e^{-j\frac{2\pi}{8}kn} \right\} = \sin\left(-\frac{2\pi}{8}kn\right)$

N=8 DFT : X[0] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

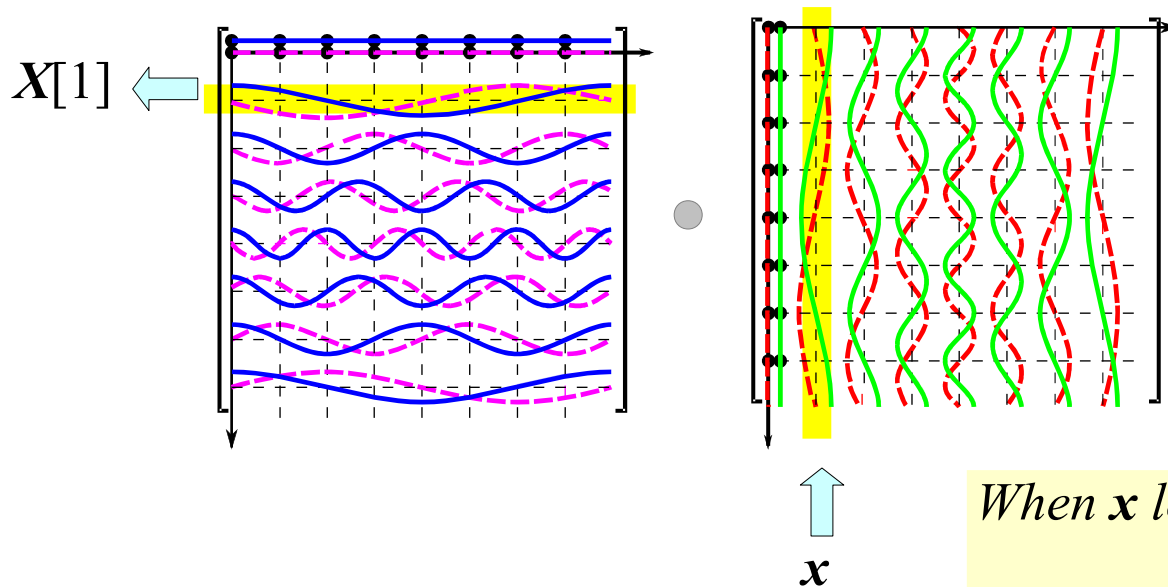
Unitary Matrix

When x looks like this, $X[0]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 0$

$$X[0] = \begin{bmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 0} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{bmatrix} \bullet \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}^T$$

N=8 DFT : X[1] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

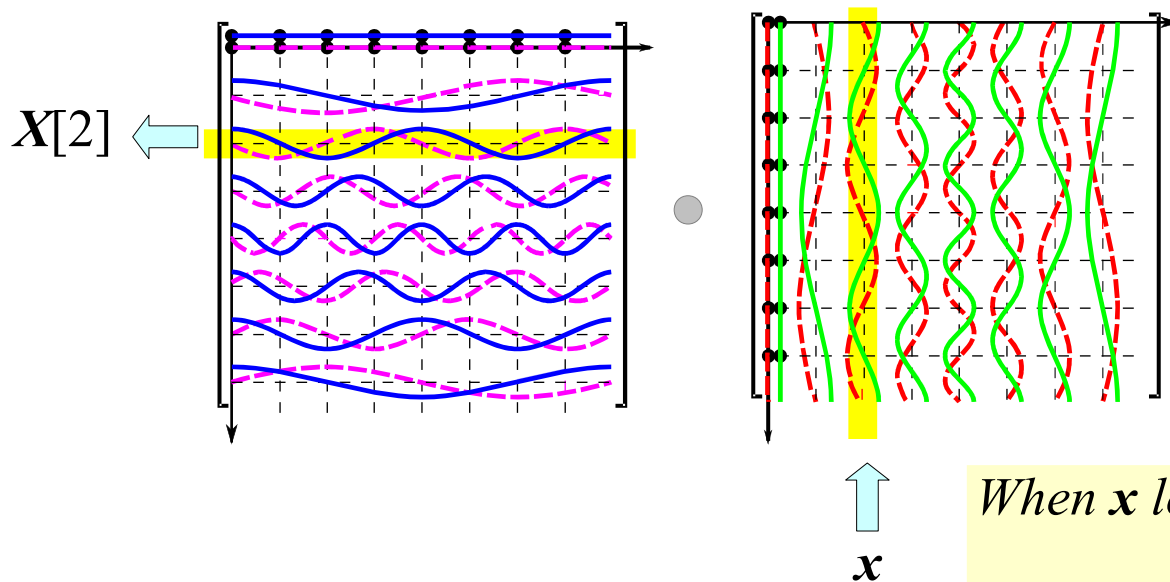
Unitary Matrix

When x looks like this, $X[1]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 1$

$$X[1] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 1} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 5} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 7} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[2] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

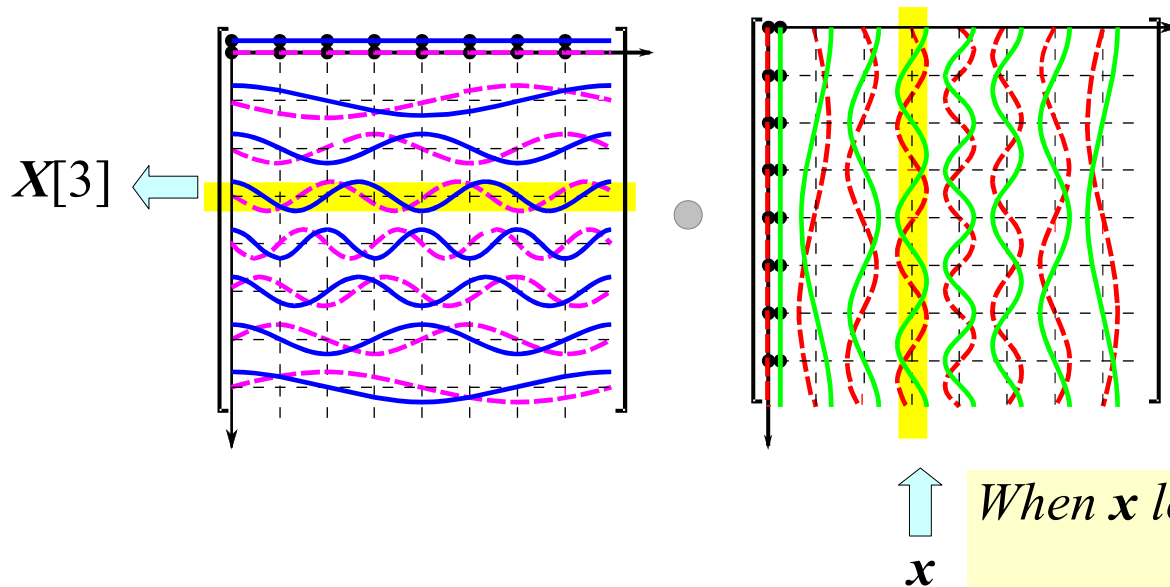
Unitary Matrix

When x looks like this, $X[2]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 2$

$$X[2] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 6} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[3] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

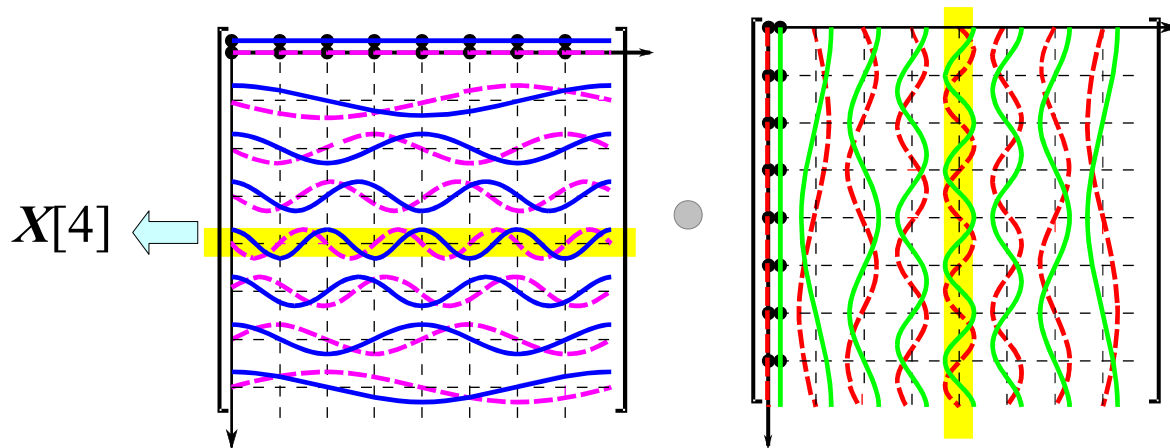
Unitary Matrix

When x looks like this, $X[3]$ is max. ($=N$)
 $X[k] = 0$ for $k \neq 3$

$$X[3] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 1} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 7} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 5} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix} \bullet$$

N=8 DFT : X[4] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

When x looks like this, $X[4]$ is max. ($=N$)

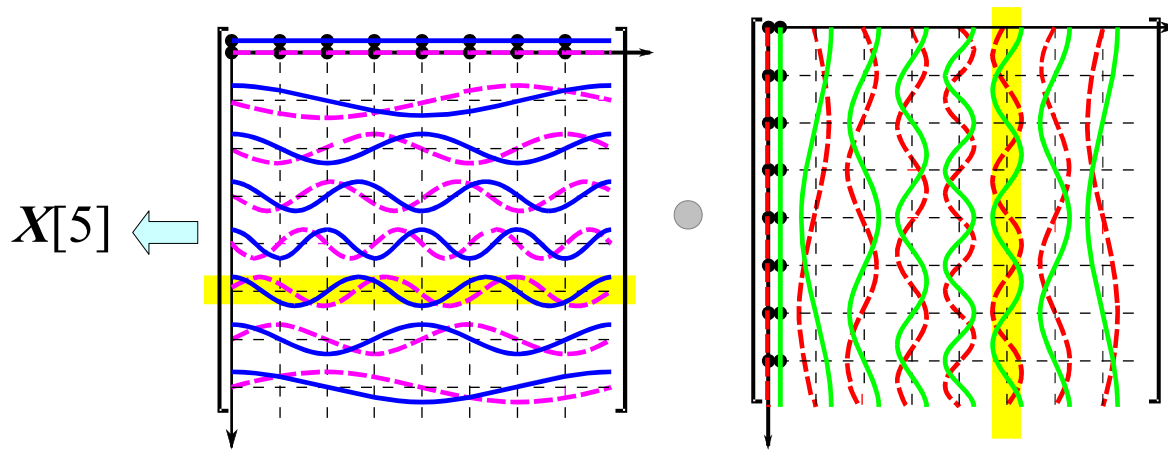
$$X[k] = 0 \text{ for } k \neq 4$$

\uparrow
 x

$$X[4] = \begin{pmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 4} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[5] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

When x looks like this, $X[5]$ is max. ($=N$)

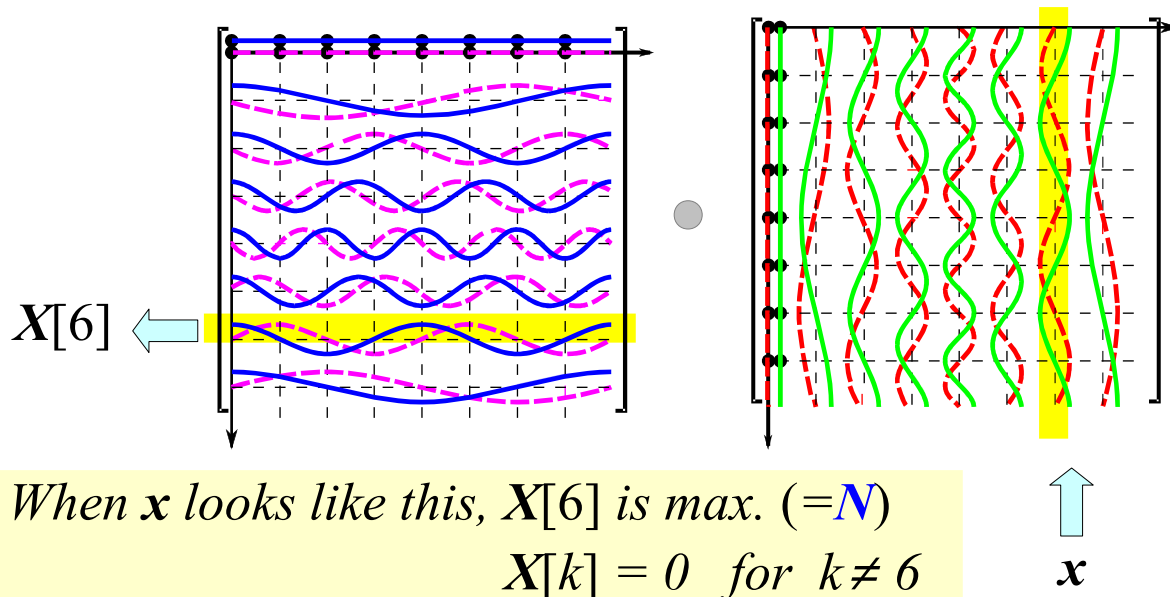
$$X[k] = 0 \text{ for } k \neq 5$$

x

$$X[5] = \begin{pmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 5} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 7} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 1} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 3} \end{pmatrix} \bullet \begin{pmatrix} x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix}^T$$

N=8 DFT : X[6] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

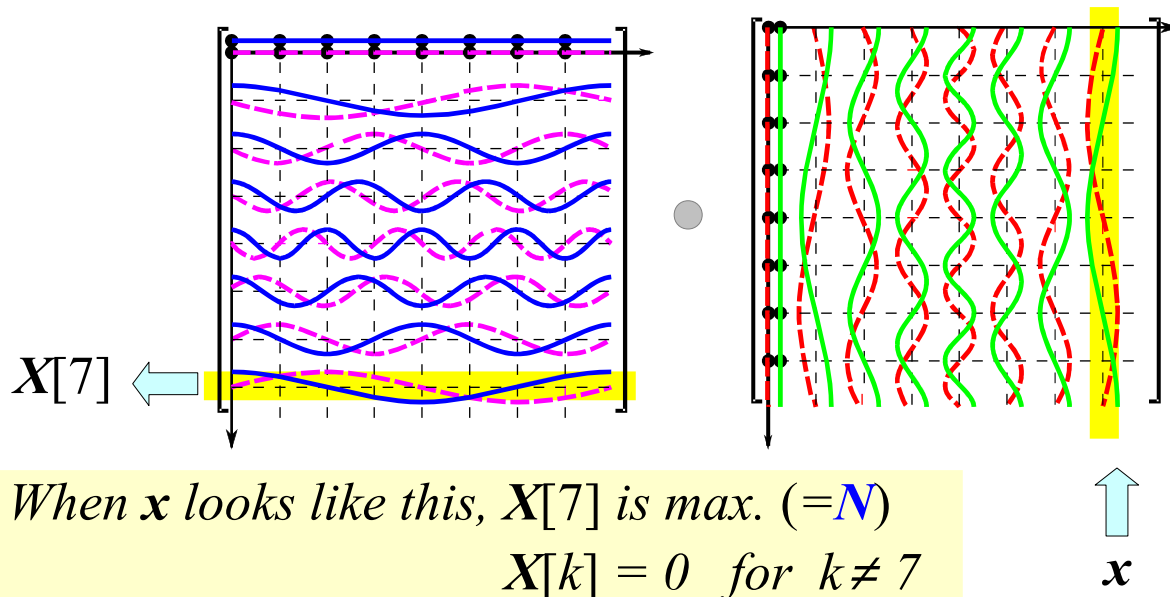
When x looks like this, $X[6]$ is max. ($=N$)

$$X[k] = 0 \text{ for } k \neq 6$$

$$X[6] = \begin{bmatrix} e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} & e^{-j\frac{\pi}{4}\cdot 0} & e^{-j\frac{\pi}{4}\cdot 6} & e^{-j\frac{\pi}{4}\cdot 4} & e^{-j\frac{\pi}{4}\cdot 2} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{bmatrix} \bullet \begin{matrix} \\ \\ \\ \\ \\ \\ \\ \end{matrix}^T$$

N=8 DFT : X[7] in A·B

$$X[k] = \sum_{n=0}^7 W_8^{kn} x[n] \quad W_8^{kn} = e^{-j\left(\frac{2\pi}{8}\right)kn}$$



$$A \cdot B = A \cdot A^H = N I$$

$$U \cdot U^H = I$$

Unitary Matrix

$$X[7] = \begin{pmatrix} e^{-j\frac{\pi}{4} \cdot 0} & e^{-j\frac{\pi}{4} \cdot 7} & e^{-j\frac{\pi}{4} \cdot 6} & e^{-j\frac{\pi}{4} \cdot 5} & e^{-j\frac{\pi}{4} \cdot 4} & e^{-j\frac{\pi}{4} \cdot 3} & e^{-j\frac{\pi}{4} \cdot 2} & e^{-j\frac{\pi}{4} \cdot 1} \\ x[0] & x[1] & x[2] & x[3] & x[4] & x[5] & x[6] & x[7] \end{pmatrix} \bullet T$$

References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann