Sampler Spectra (8B)

Young Won Lim

Please send corrections (or suggestions) to youngwlim@hotmail.com.

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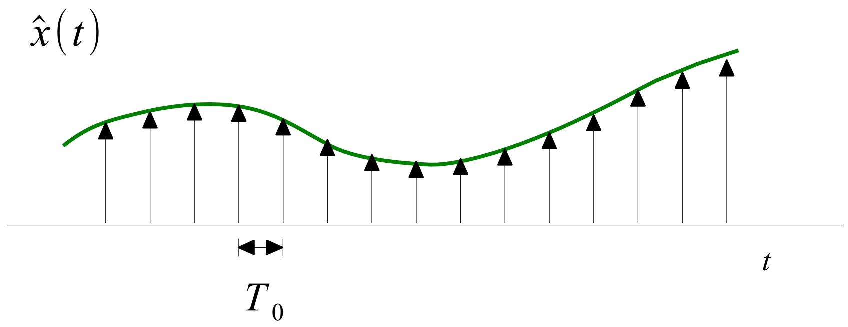
Sampler

Ideal Sampling

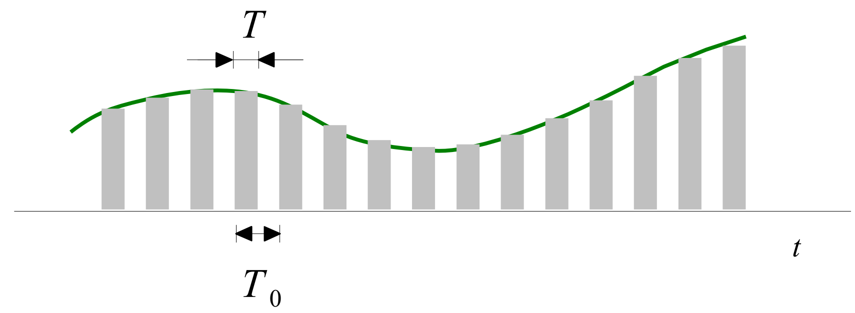


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t - nT_0)$$

CTFT

?

Practical Sampling



$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0)\,p(t - nT_0)$$

CTFT

?

Square Wave CTFS

$$C_k = \frac{1}{T_0}\cdot\frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$\omega_0 = \frac{2\pi}{T_0}$$

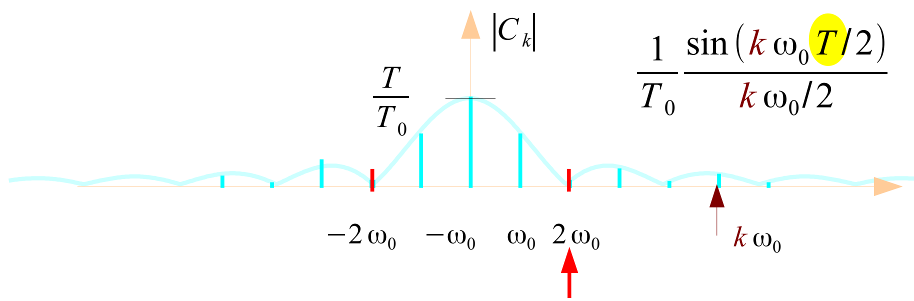
$$\omega_0\frac{T_0}{T} = \frac{2\pi}{T}$$

$$\omega_0 T = 2\pi\frac{T}{T_0}$$

$$C_k = 0 \quad \Rightarrow \quad \sin(k\omega_0 T/2) = 0$$

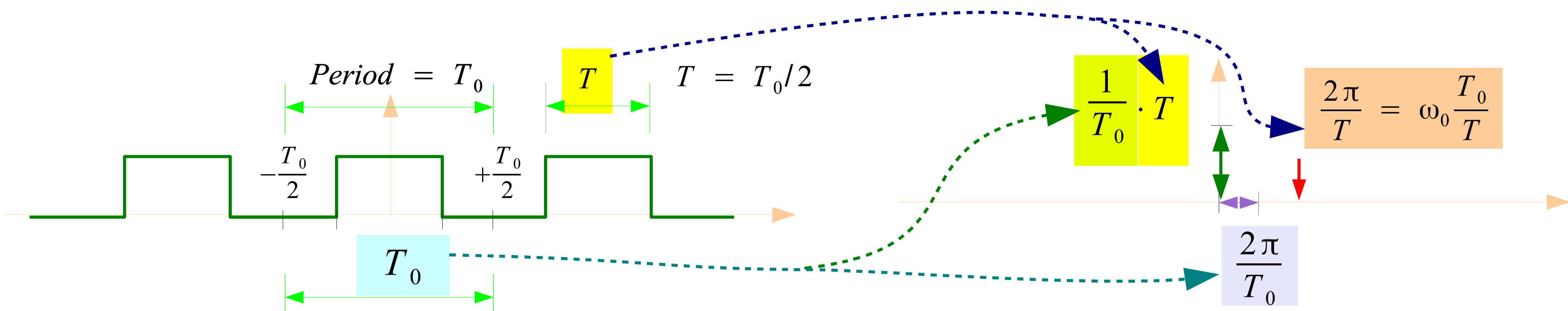
$$\sin\left(k\frac{2\pi}{T_0}\frac{T}{2}\right) = 0 \quad \Rightarrow \quad \sin(\pm n\pi) = 0$$

$$k = \pm n\frac{T_0}{T} \quad \Rightarrow \quad \omega = \pm n\frac{T_0}{T}\omega_0$$



$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{\sin(T k\omega_0/2)}{k\omega_0/2}$$

$$C_0 = \lim_{k\to 0}\frac{1}{T_0}\cdot\frac{(T\omega_0/2)\cos(T k\omega_0/2)}{\omega_0/2} = \frac{T}{T_0}$$

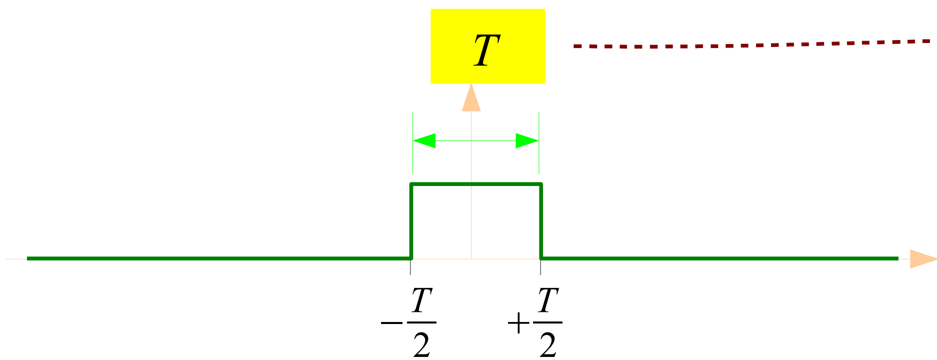


CTFT of a Rect(t/T) function

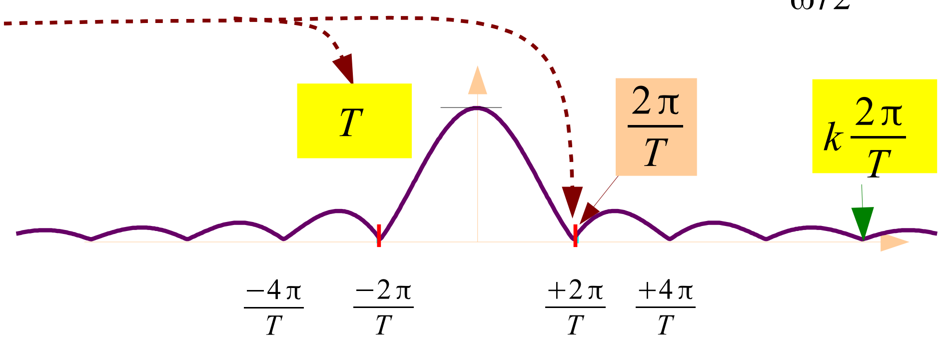
Continuous Time Fourier Transform   Aperiodic Continuous Time Signal

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\omega t}\, dt \quad \Longleftrightarrow \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)\, e^{+j\omega t}\, d\omega$$

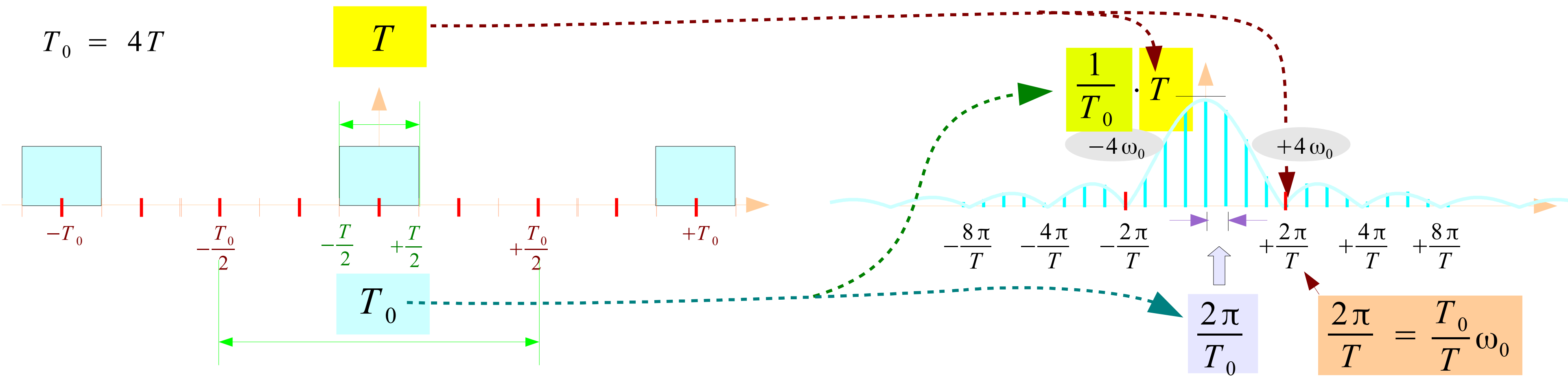
$$X(j\omega) = \int_{-T/2}^{+T/2} e^{-j\omega t}\, dt = \frac{\sin(\omega T/2)}{\omega/2}$$



$$|X(j\omega)| = \frac{\sin(\omega T/2)}{\omega/2}$$



CTFT of a Rect(t/T) function



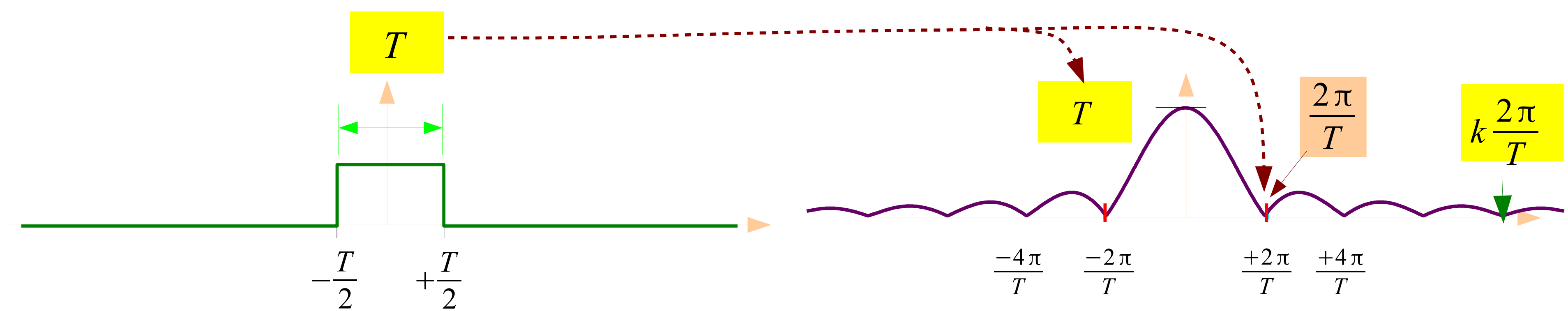
$$C_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2} = \frac{\sin(\omega T/2)}{\omega/2}$$

$$C_k = \frac{1}{T_0} \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

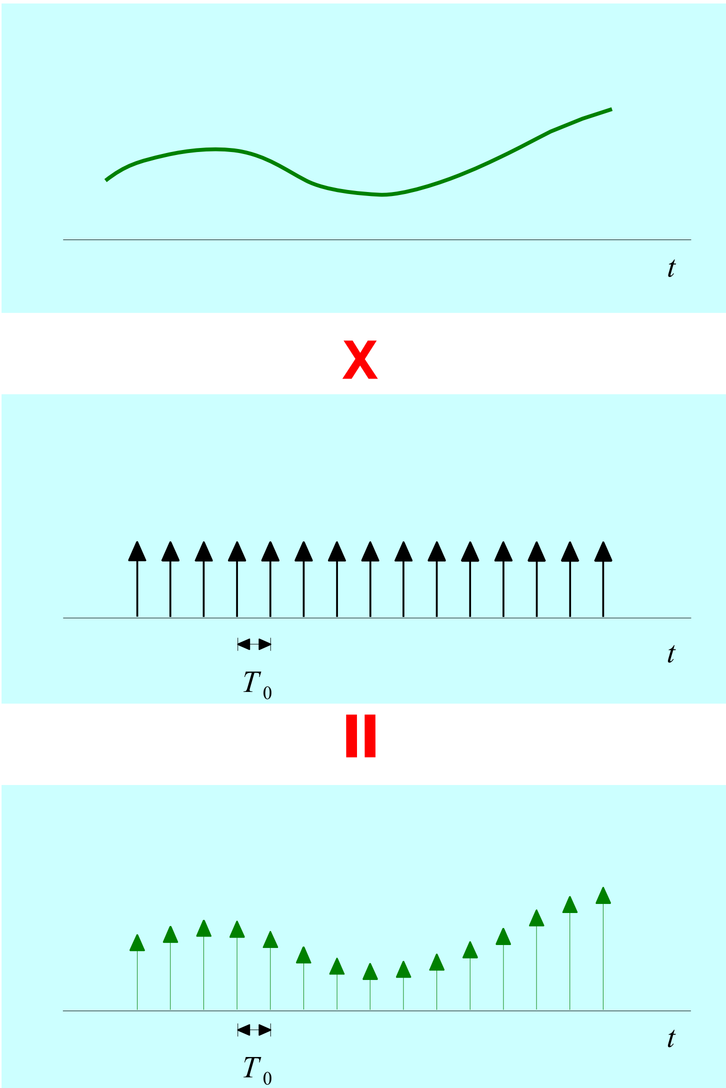
$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

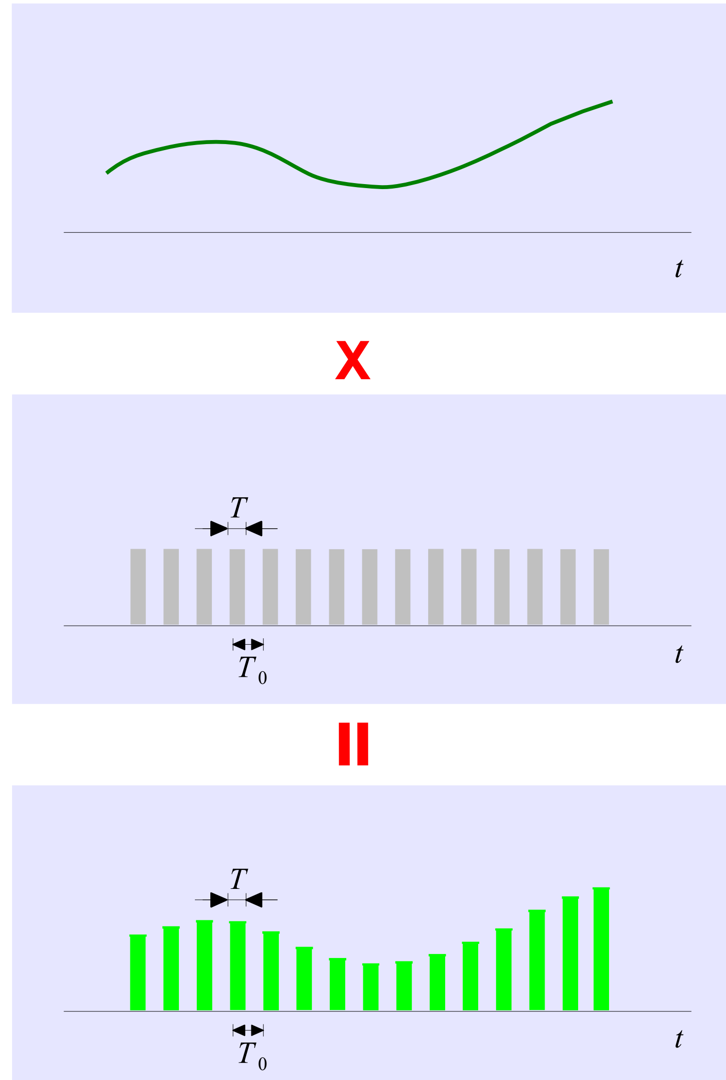


Sampling (1)

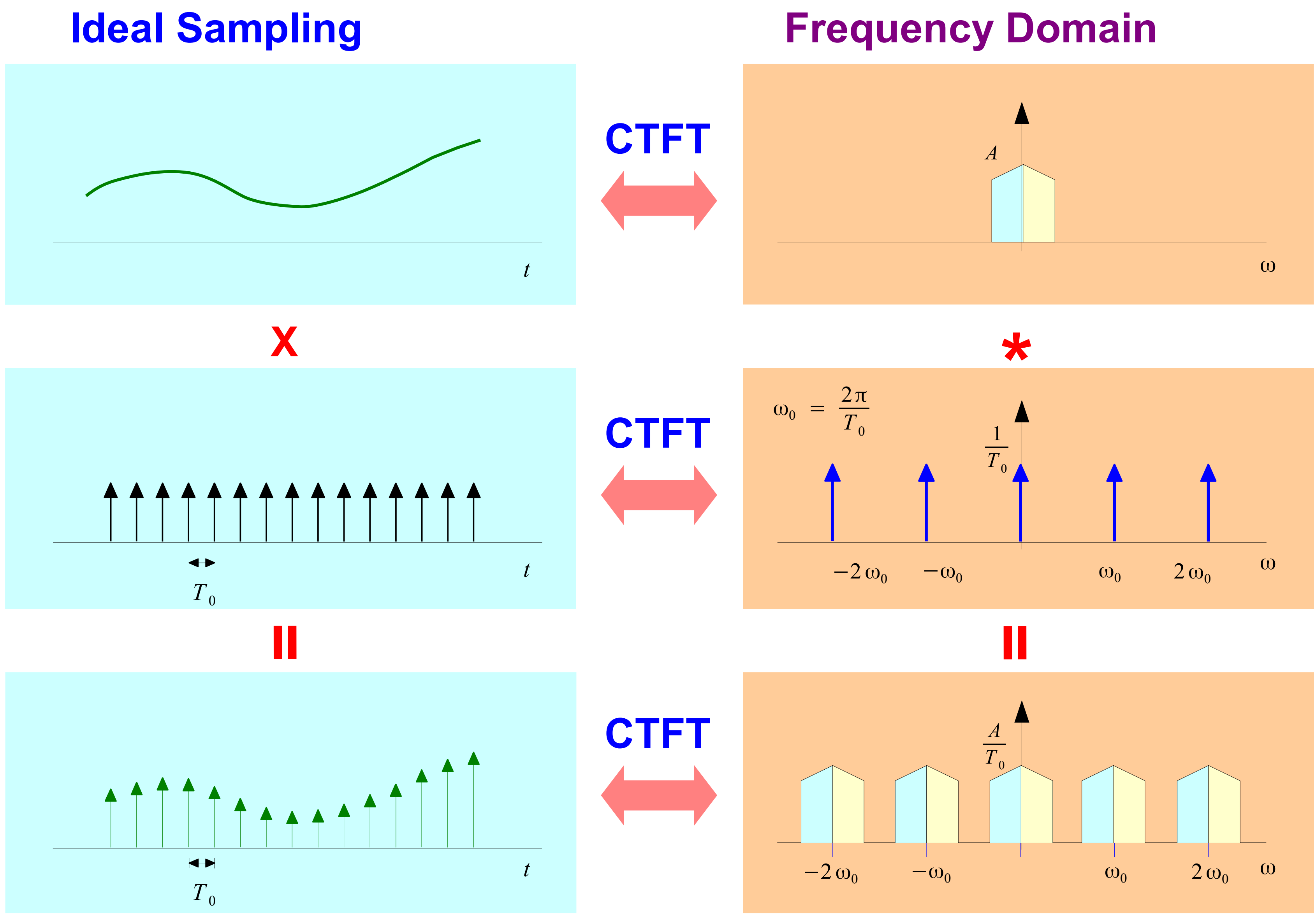
Ideal Sampling



Practical Sampling



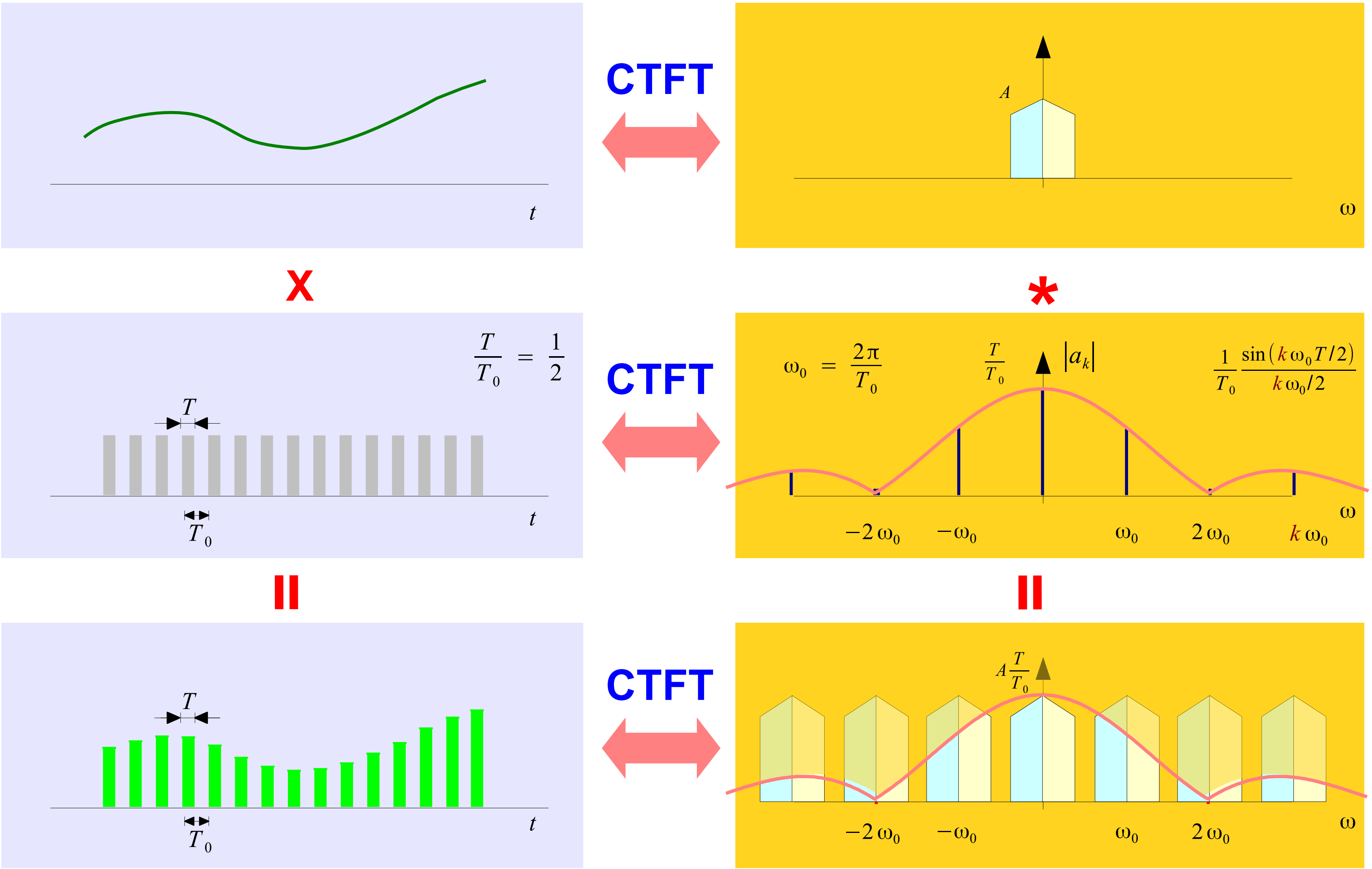
Sampling (2)



Ideal Sampling
Frequency Domain
CTFT
CTFT
CTFT
t
ω
A
X
*
$\omega_0 = \frac{2\pi}{T_0}$
$\frac{1}{T_0}$
$-2\omega_0$
$-\omega_0$
ω_0
$2\omega_0$
T_0
=
=
$\frac{A}{T_0}$

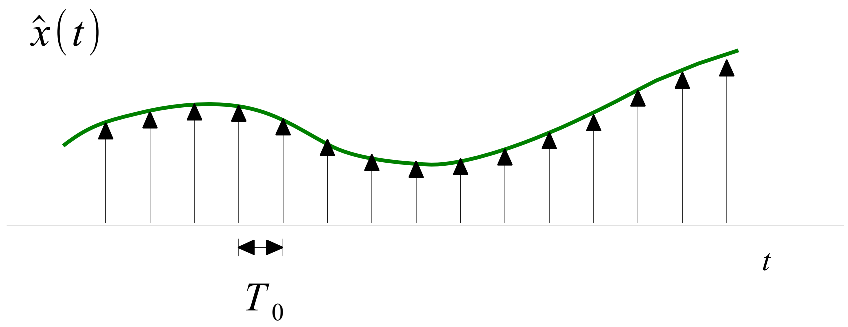
Sampling (3)

Practical Sampling | **Frequency Domain**



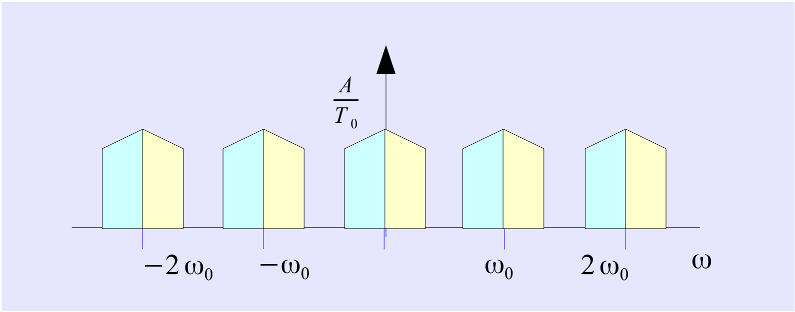
Sampling CTFT

Ideal Sampling

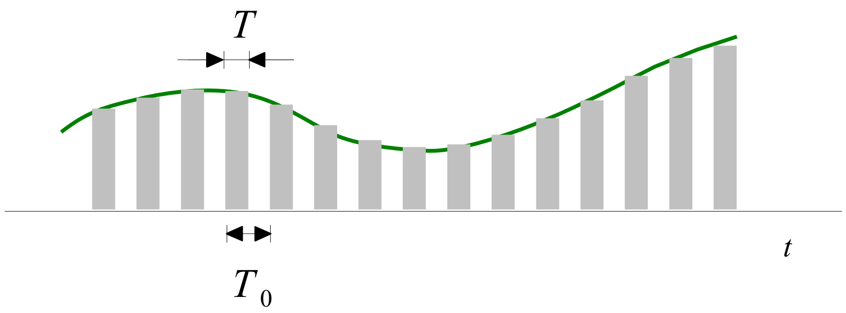


$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT

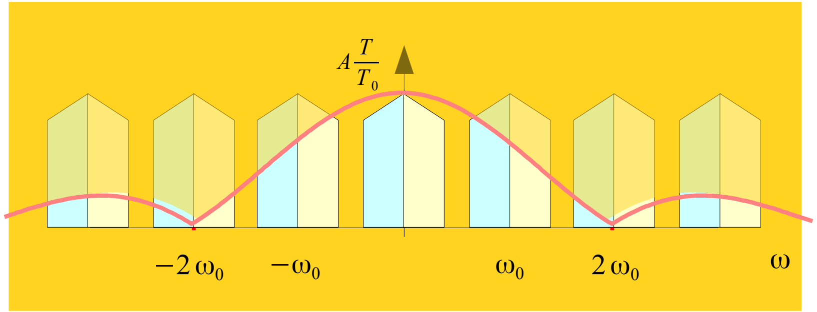


Practical Sampling

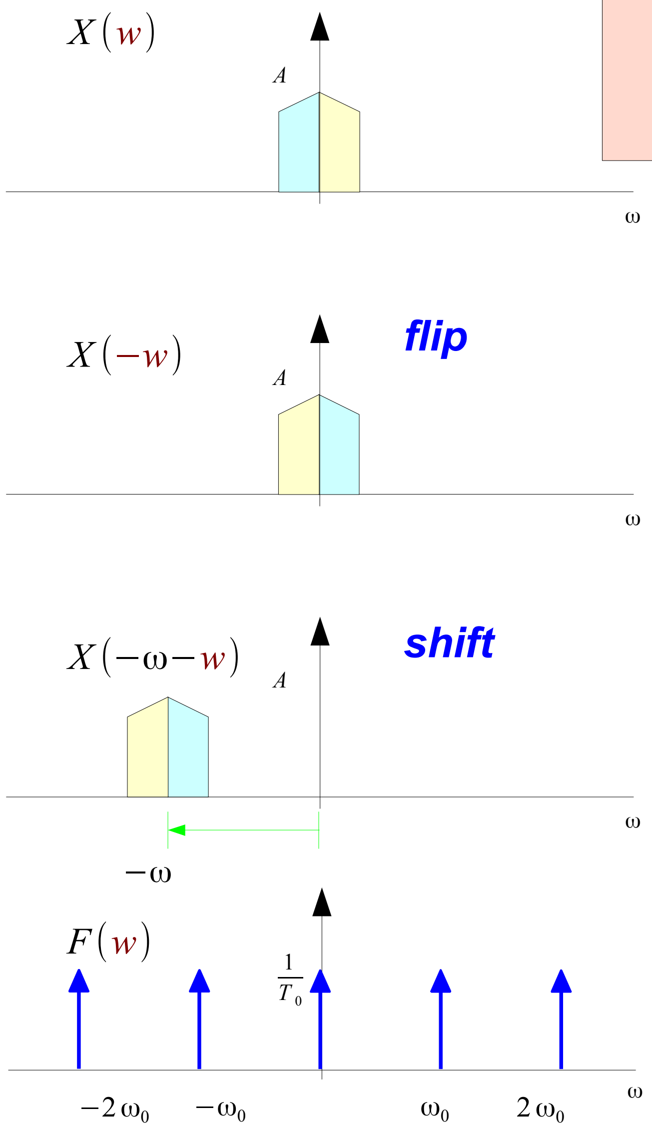


$$\hat{x}(t) \approx \sum_{n=-\infty}^{+\infty} x(nT_0)\,p(t-nT_0)$$

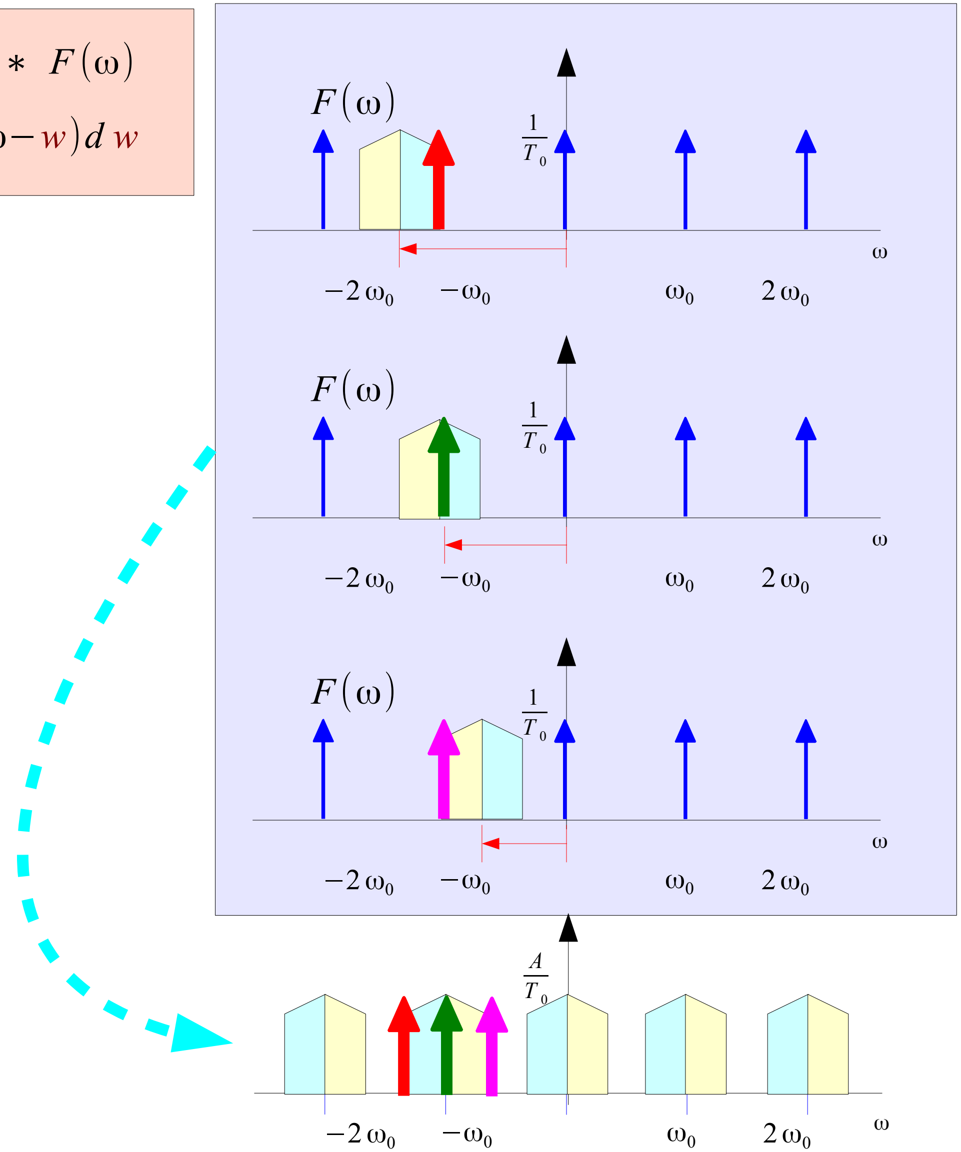
CTFT



Convolution with Impulse Train

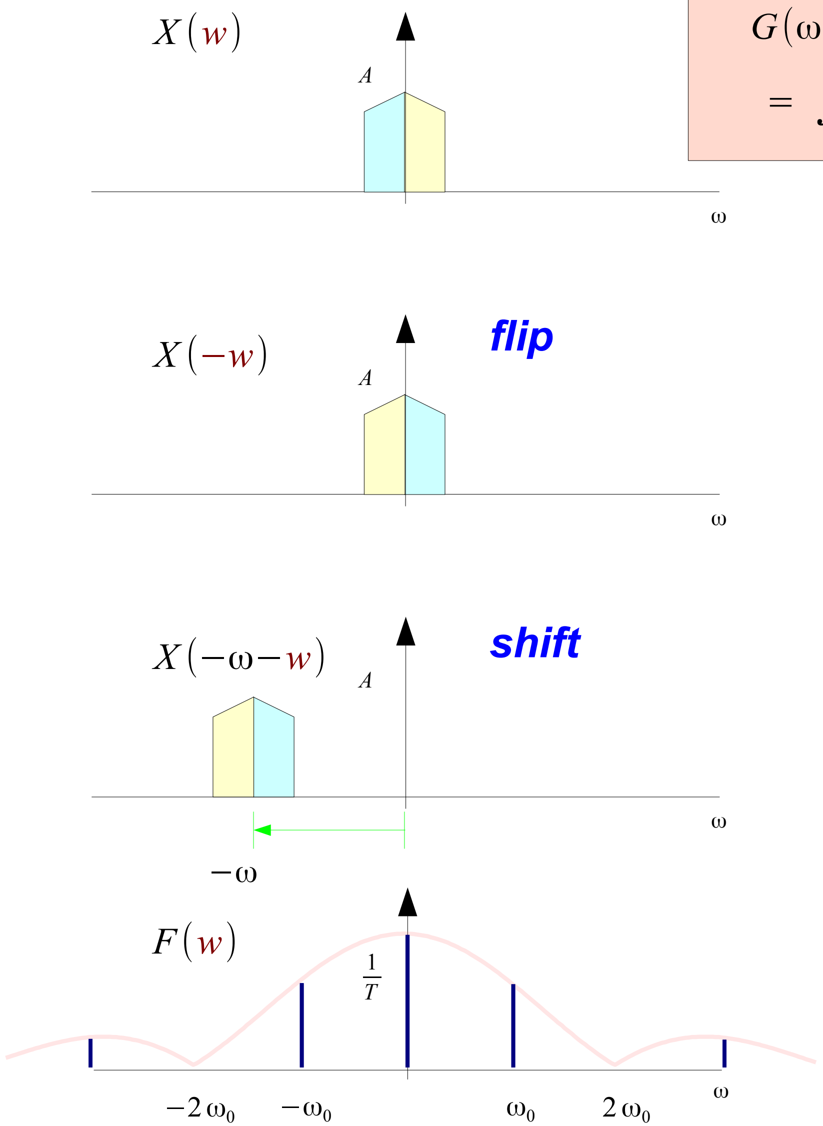


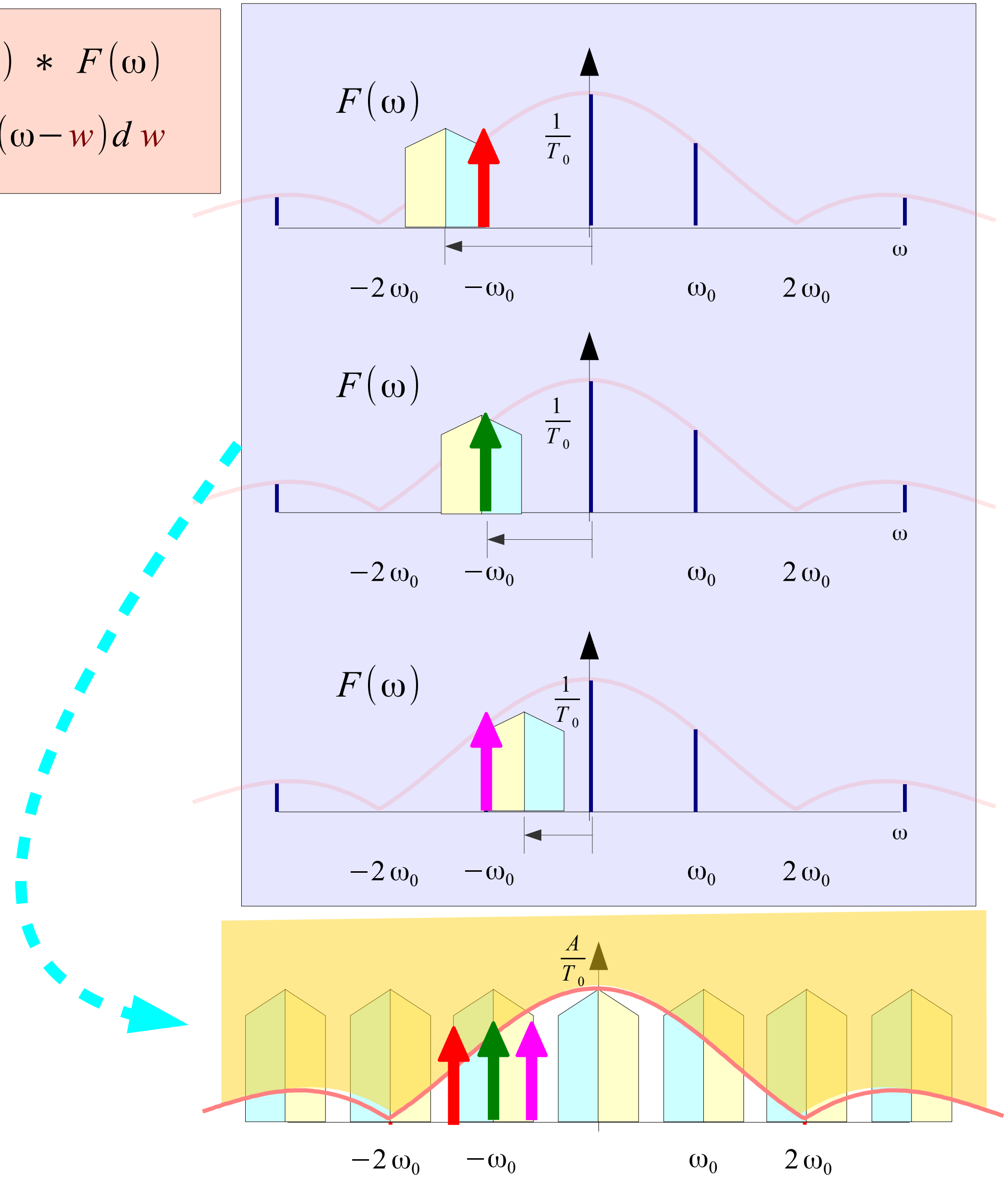
$$G(\omega) = X(\omega) * F(\omega)$$
$$= \int F(w) X(\omega - w)\, d w$$



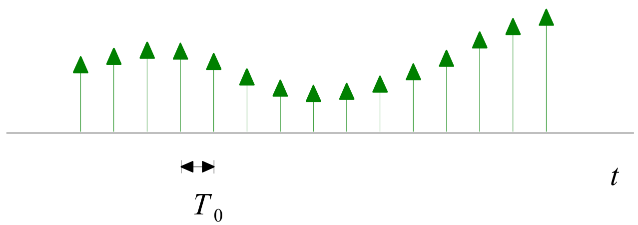
Convolution with Sinc Impulse Train

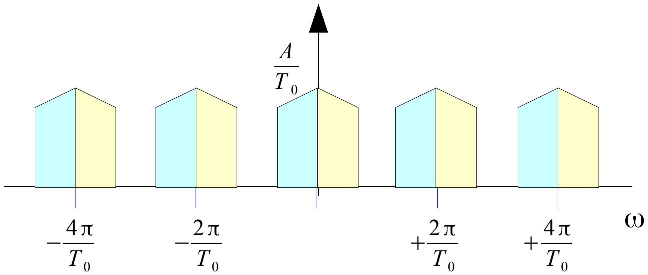
$$G(\omega) = X(\omega) * F(\omega)$$
$$= \int F(w)X(\omega - w)\,dw$$





CTFT of Sampled Signal





$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT

$$\hat{X}(f) = \int_{-\infty}^{+\infty} \hat{x}(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)\, e^{-j2\pi f t}\, dt$$

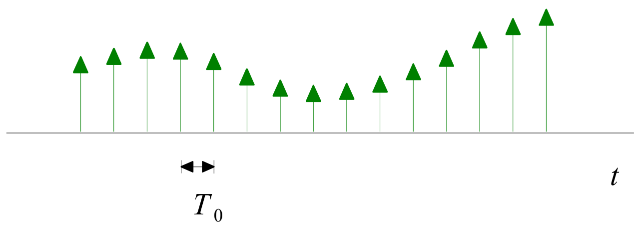
$$= \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}\, dt$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

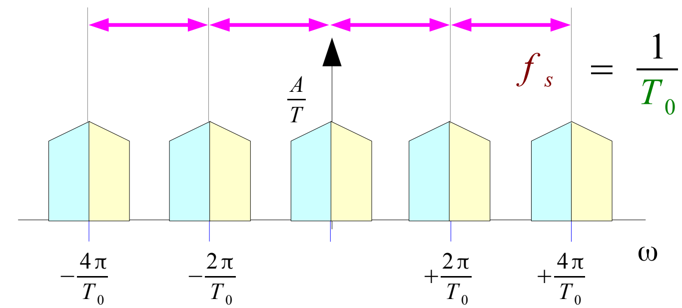
CTFT

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$

Periodicity in Frequency



$f_s = \frac{1}{T_0}$ $\qquad 2\pi f_s = \frac{2\pi}{T_0} = \omega_0$



$$2\pi f = \omega$$

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT

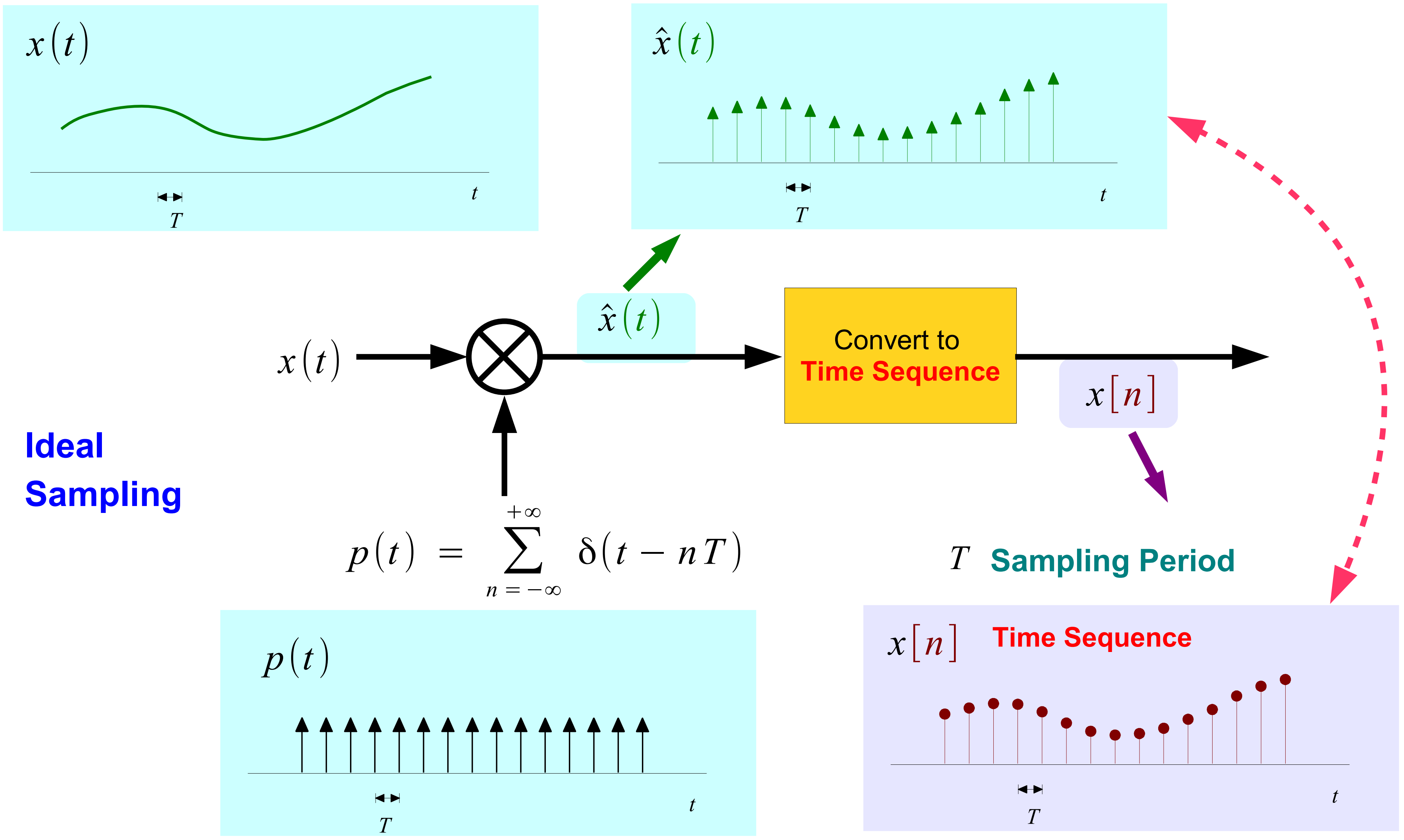
$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$

$$e^{-j2\pi(f+f_s)T_0 n} = e^{-j2\pi(f)T_0 n} \quad \Leftarrow \quad f_s T_0 = 1$$

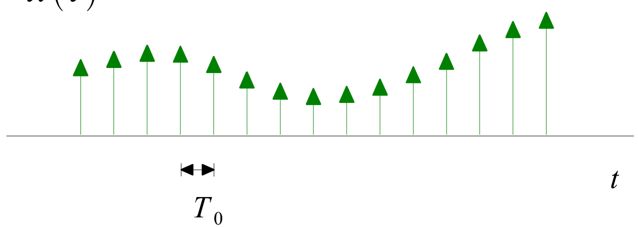
1 / Period = Sampling Frequency f_s

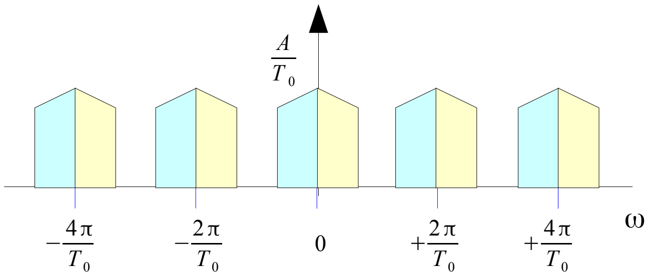
$$\hat{X}(f) = \hat{X}(f+f_s)$$

Time Sequence



DTFT of a Time Sequence



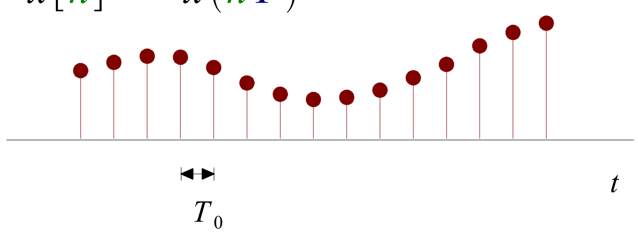


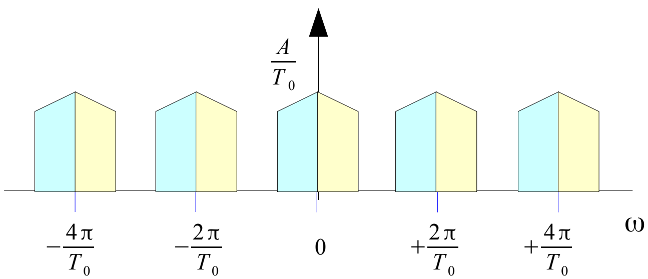
$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$

$$x[n] = x(nT)$$





$x[n]$, Sampling Period T_0

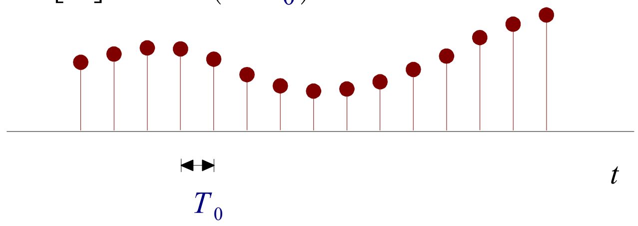
DTFT

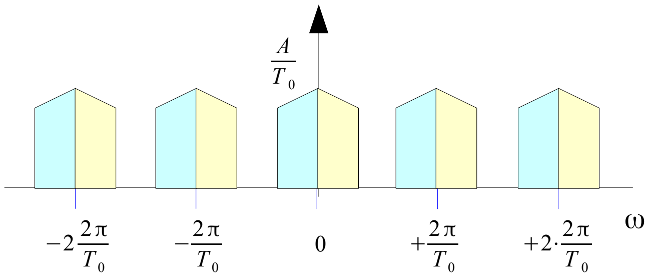
$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f T_0 n}$$

Here, $X(f)$ does not denote the CTFT of $x(t)$

Discrete Time Fourier Transform (1)

$x[n] = x(nT_0)$





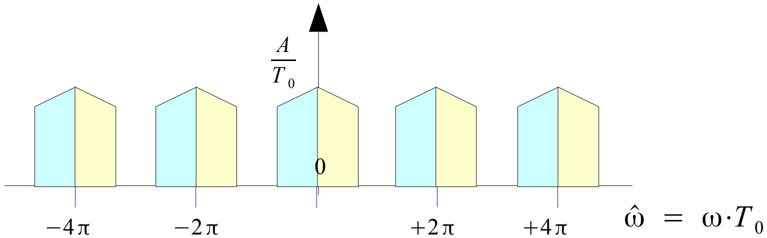
$x[n]$, Sampling Period T_0

DTFT

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j 2\pi f T_0 n}$$

Normalized Angular Frequency

$$2\pi f T_0 = \frac{2\pi f}{1/T_0} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

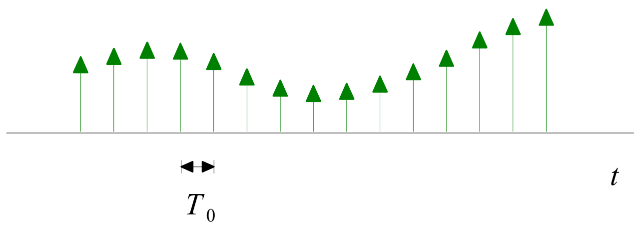


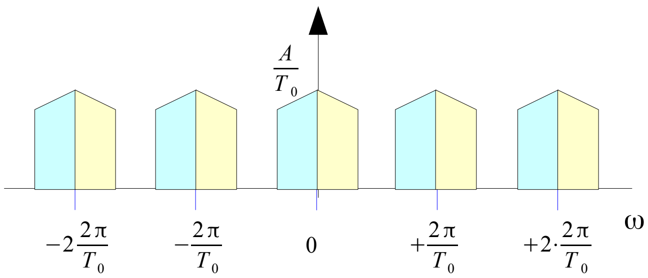
$x[n]$, Sampling Period T_0

DTFT

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

Discrete Time Fourier Transform (2)





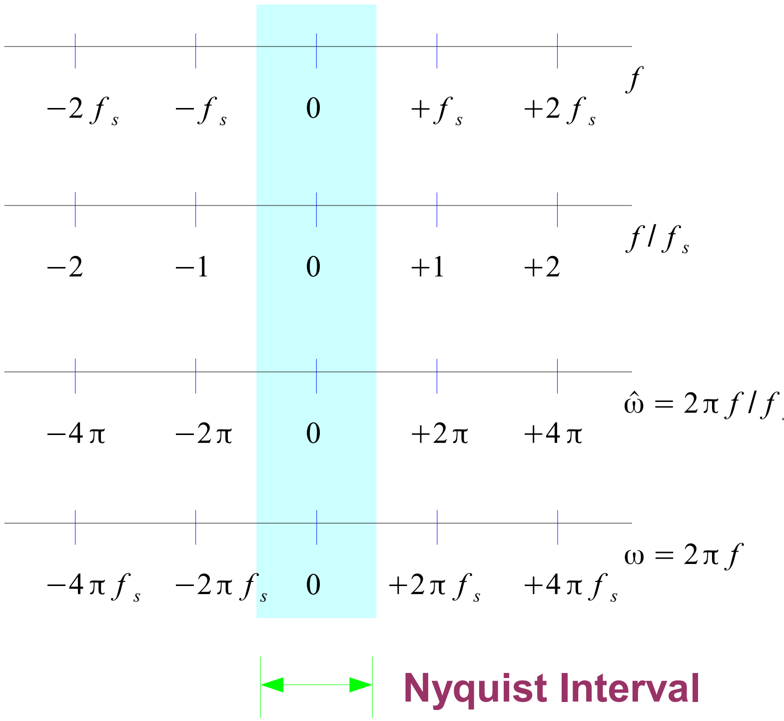
$$f_s = \frac{1}{T} \qquad 2\pi f_s = \frac{2\pi}{T} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j2\pi f T n}$$

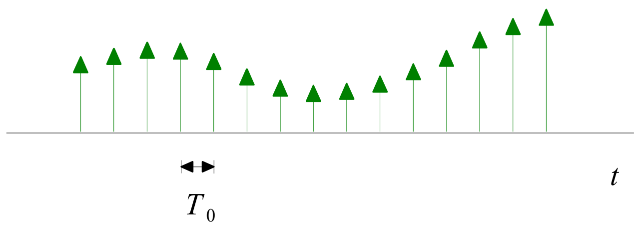
Normalized Angular Frequency

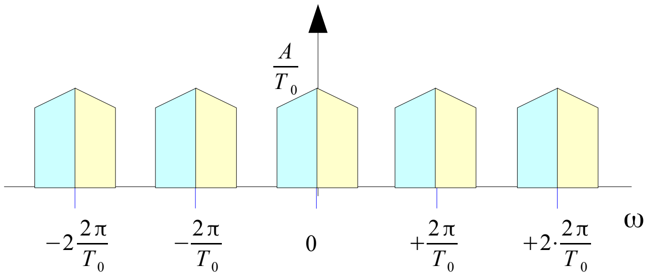
$$2\pi f T = \frac{2\pi f}{1/T} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$



Discrete Time Fourier Transform (3)



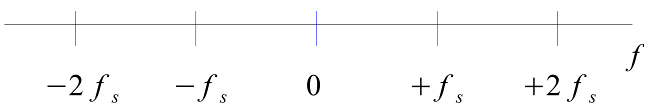


$$f_s = \frac{1}{T_0}$$

$$2\pi f_s = \frac{2\pi}{T_0} = \omega_0$$

$$X(f) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j 2\pi f T_0 n}$$

$\hat{X}(f)$ **Absolute Frequency**



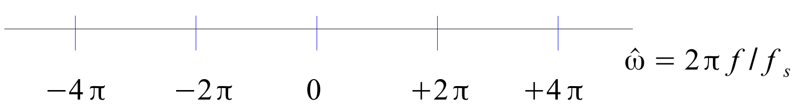
Normalized Angular Frequency

$$2\pi f T_0 = \frac{2\pi f}{1/T_0} = 2\pi \frac{f}{f_s} = \hat{\omega}$$

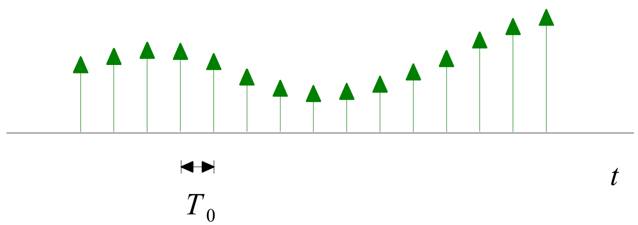
$$X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{+\infty} x[n]\, e^{-j\hat{\omega} n}$$

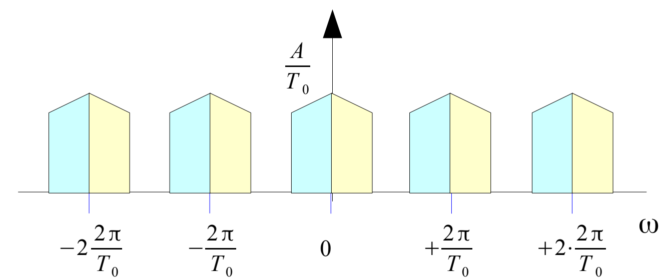
$\hat{X}(e^{j\hat{\omega}})$ **Normalized Angular Frequency**

unit circle →
emphasize the periodic nature



Fourier Series Interpretation





$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

CTFT →

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T n}$$

CTFS →

$$x(nT_0) = \frac{1}{f_s}\int_{+f_s/2}^{-f_s/2} \hat{X}(f)\, e^{+j2\pi f T_0 n}\, df$$

$$= \int_{-\pi}^{+\pi} \hat{X}(\omega)\, e^{+j\omega n}\, \frac{d\omega}{2\pi}$$

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$

Fourier Series Coefficients $x(nT_0)$

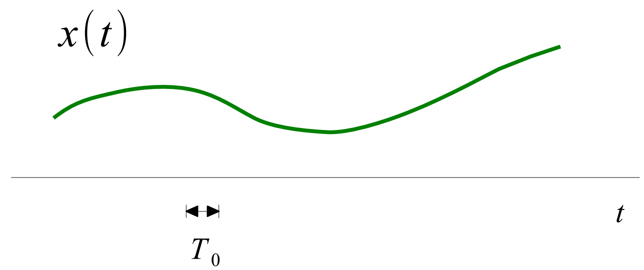
$\hat{X}(f)$ **Continuous Periodic Function**

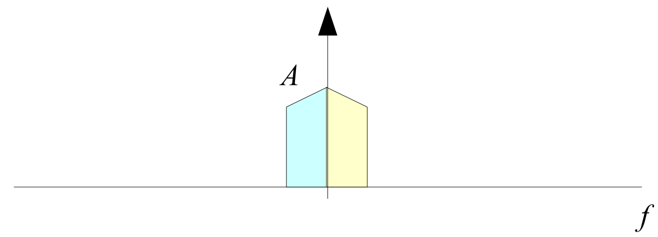
$$\omega = 2\pi f / f_s \qquad \frac{df}{f_s} = \frac{d\omega}{2\pi}$$

View as a Fourier Series Expansion

Numerical Approximation

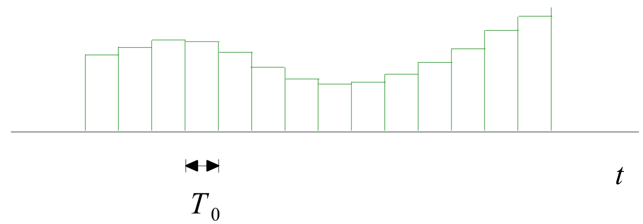
$$X(f) = \lim_{T_0 \to 0} T_0 \cdot \hat{X}(f)$$





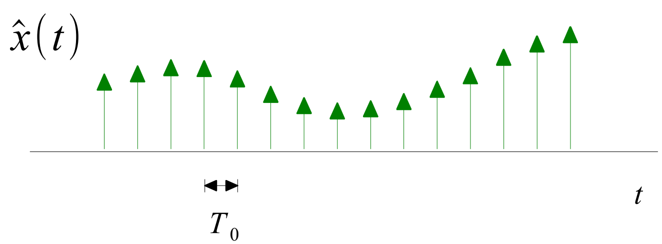
CTFT

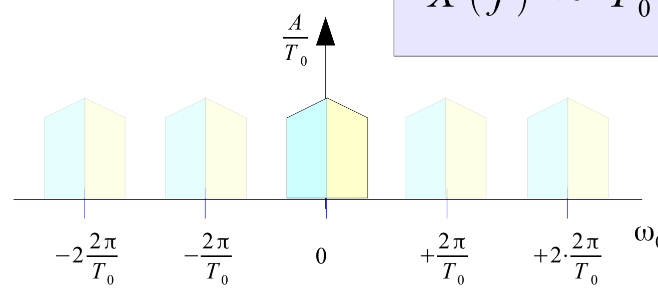
$$X(f) = \int_{-\infty}^{+\infty} x(t)\, e^{+j2\pi f t}\, dt$$



$$\approx \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n} \cdot T_0$$

$$X(f) \approx T_0 \cdot \hat{X}(f)$$





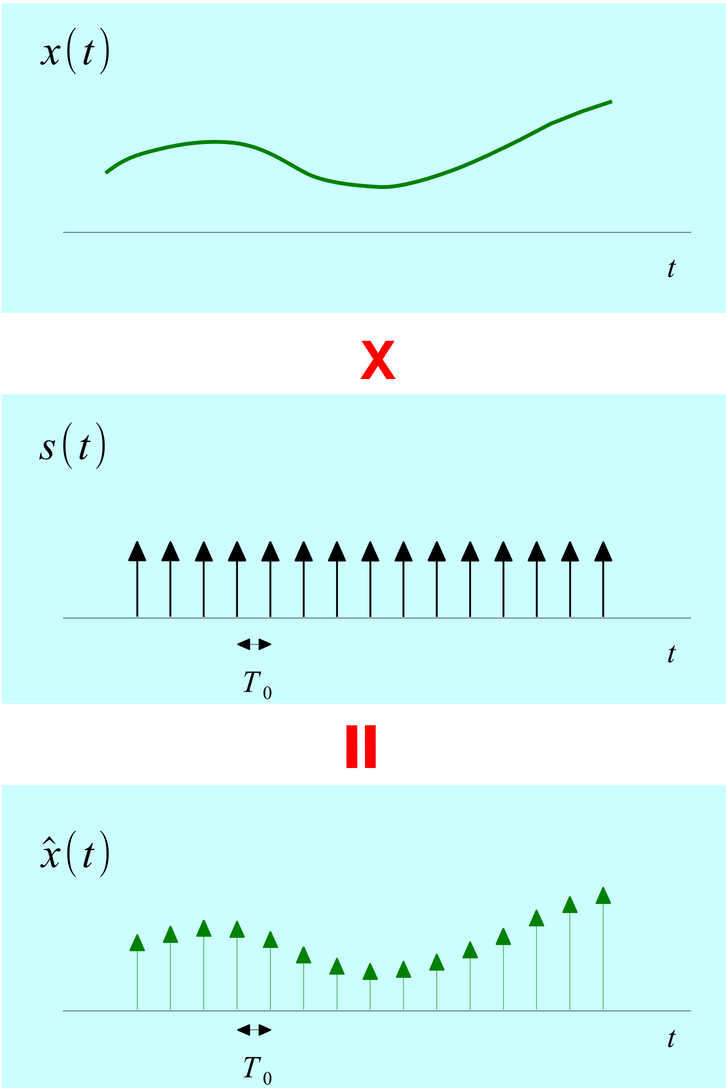
CTFT

$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, \delta(t - nT_0)$$

$$\hat{X}(f) = \sum_{n=-\infty}^{+\infty} x(nT_0)\, e^{-j2\pi f T_0 n}$$

Spectrum Replication (1)

Ideal Sampling



$$\hat{x}(t) = \sum_{n=-\infty}^{+\infty} x(nT_0)\,\delta(t-nT_0)$$

$$s(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT_0)$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} e^{+j2\pi m f_s t}$$

$$\hat{x}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} x(t)\, e^{+j2\pi m f_s t}$$

Shift Property

$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f-m f_s)$$

Spectrum Replication (2)

$$S(f) = \frac{1}{T} \sum_{m=-\infty}^{+\infty} \delta(f - m f_s)$$

Convolution in Frequency

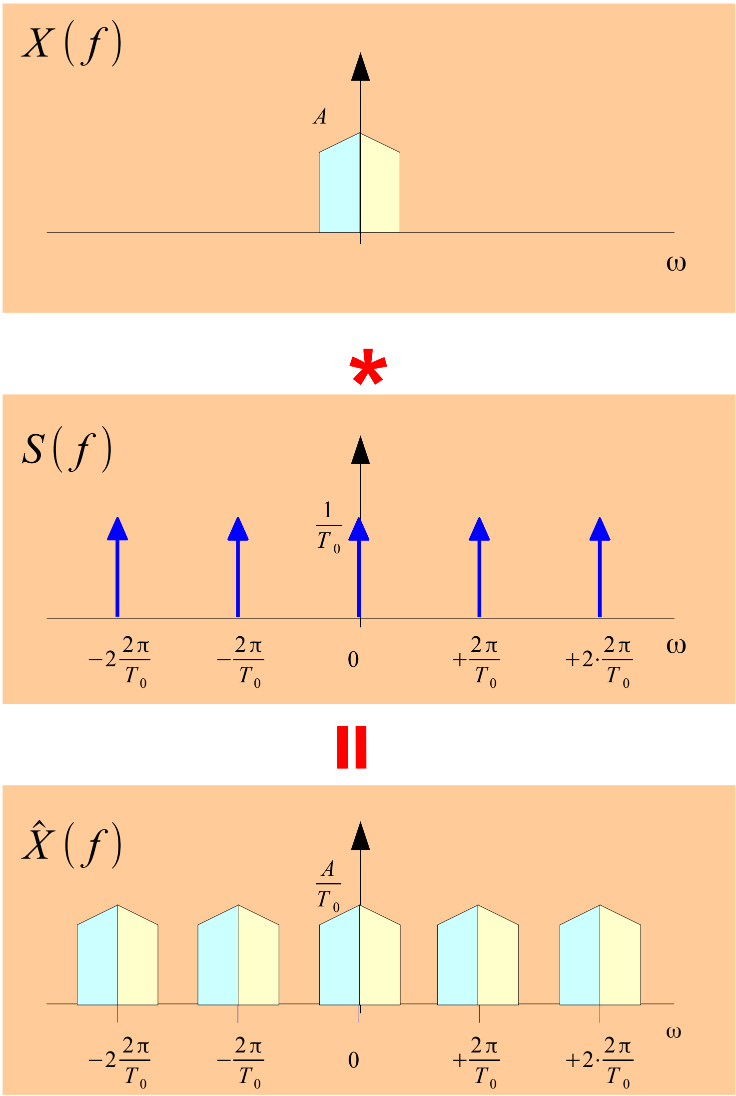
$$\hat{X}(f) = X(f) * S(f)$$

$$= \int_{-\infty}^{+\infty} X(f - f')S(f')\,df'$$

$$= \frac{1}{T_0} \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} X(f - f')\delta(f' - m f_s)\,df'$$

$$\hat{X}(f) = \frac{1}{T_0} \sum_{n=-\infty}^{+\infty} X(f - m f_s)$$

Frequency Domain



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