

DFT Analysis (5B)

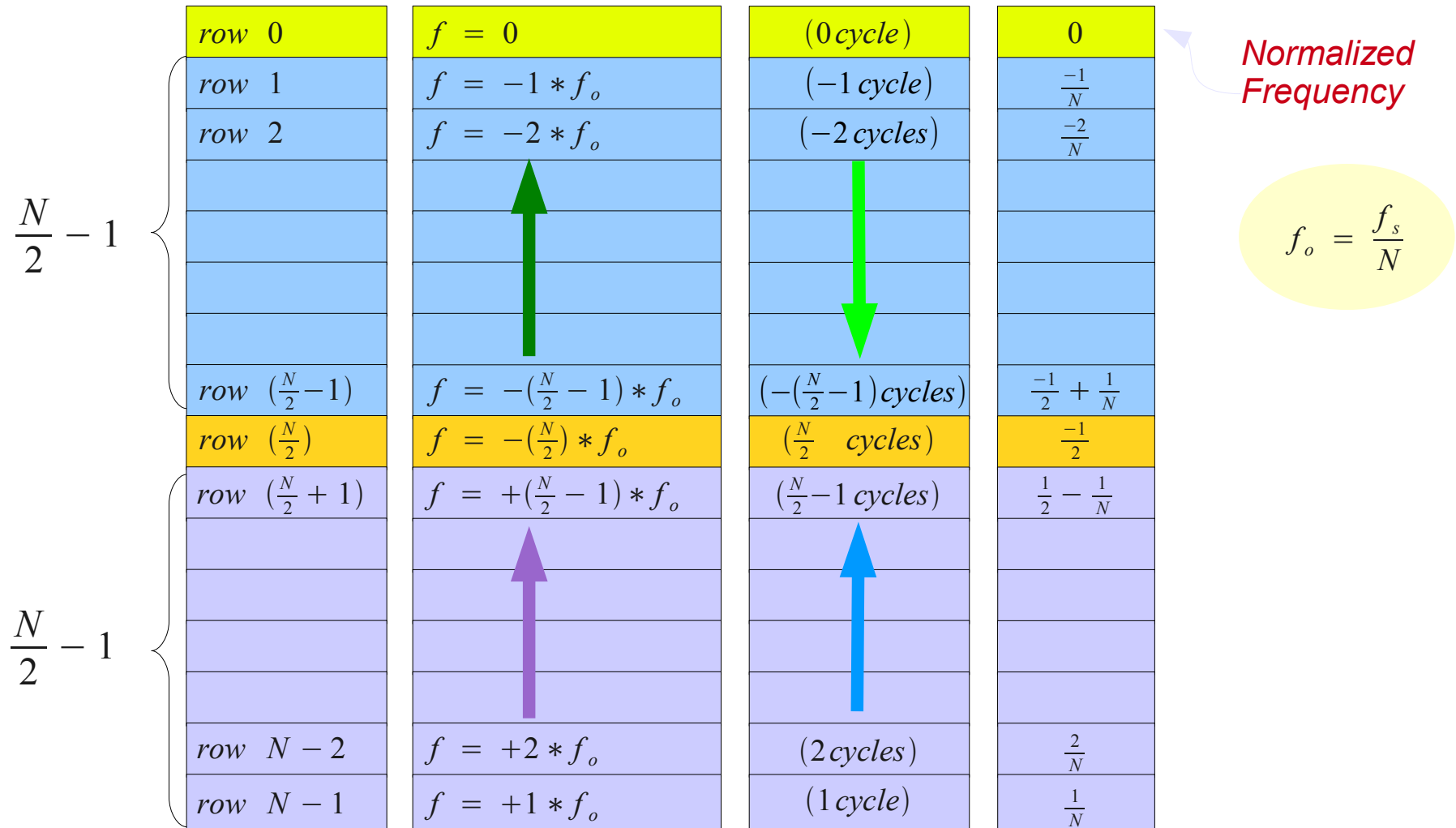
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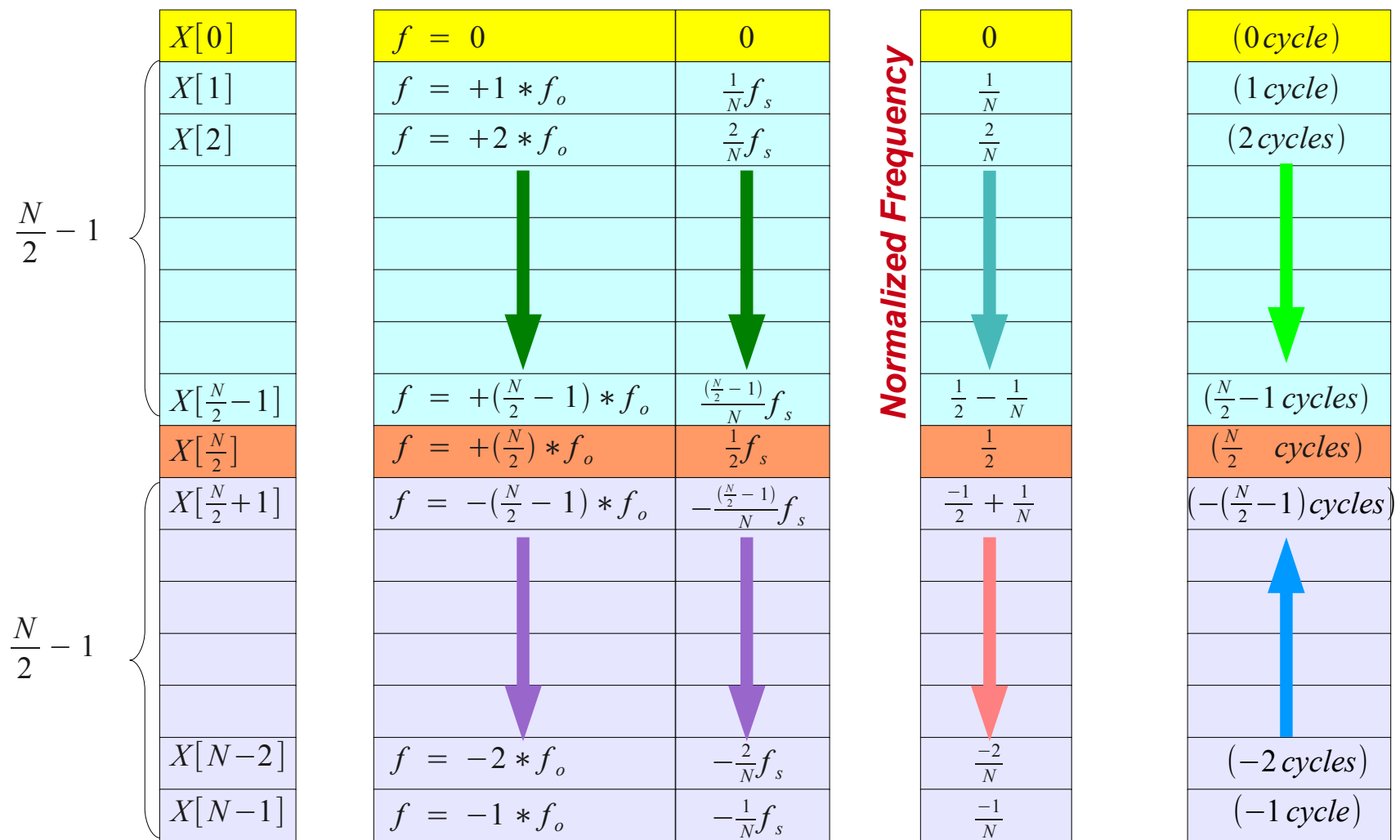
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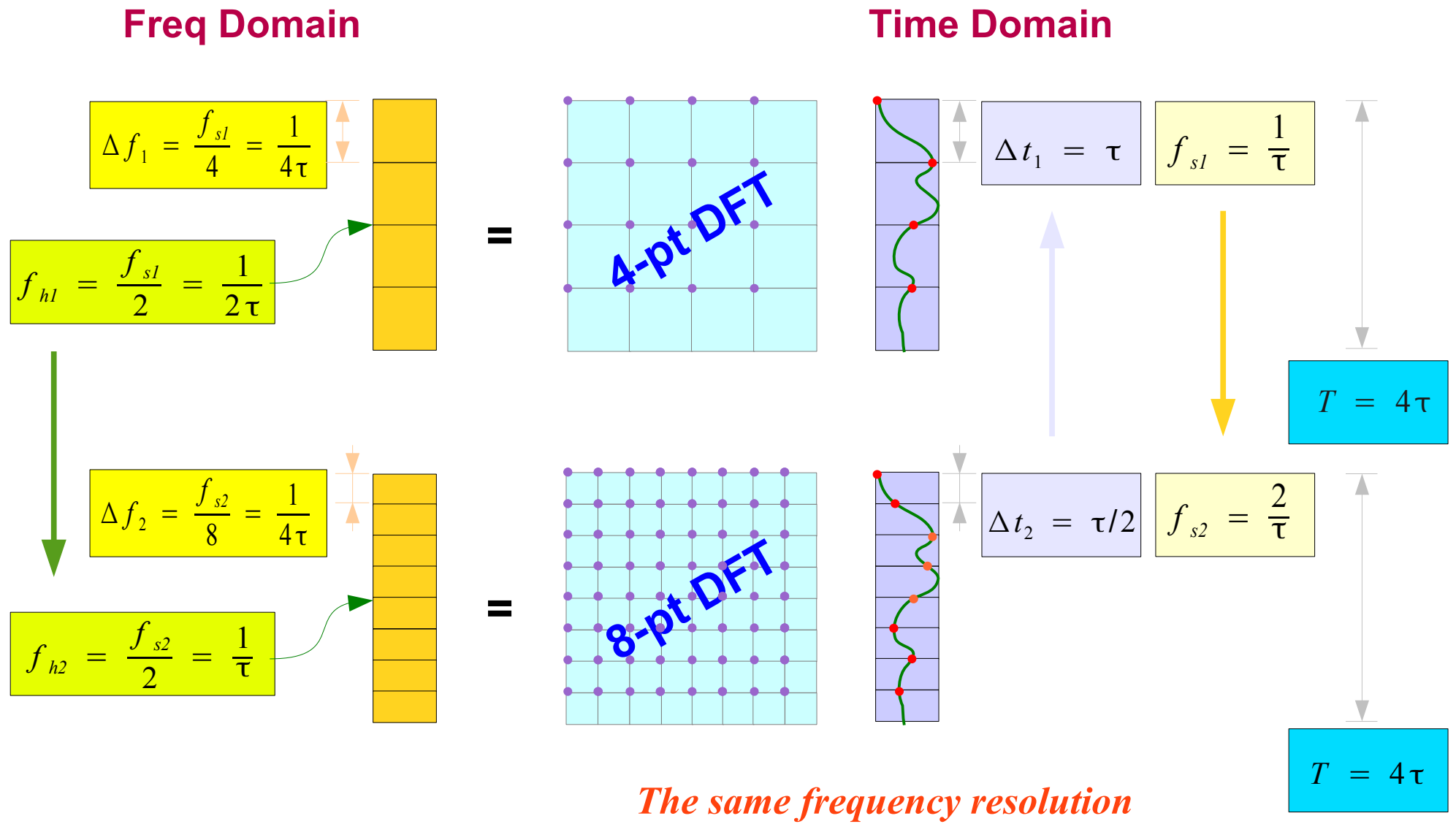
Frequency View of a DFT Matrix



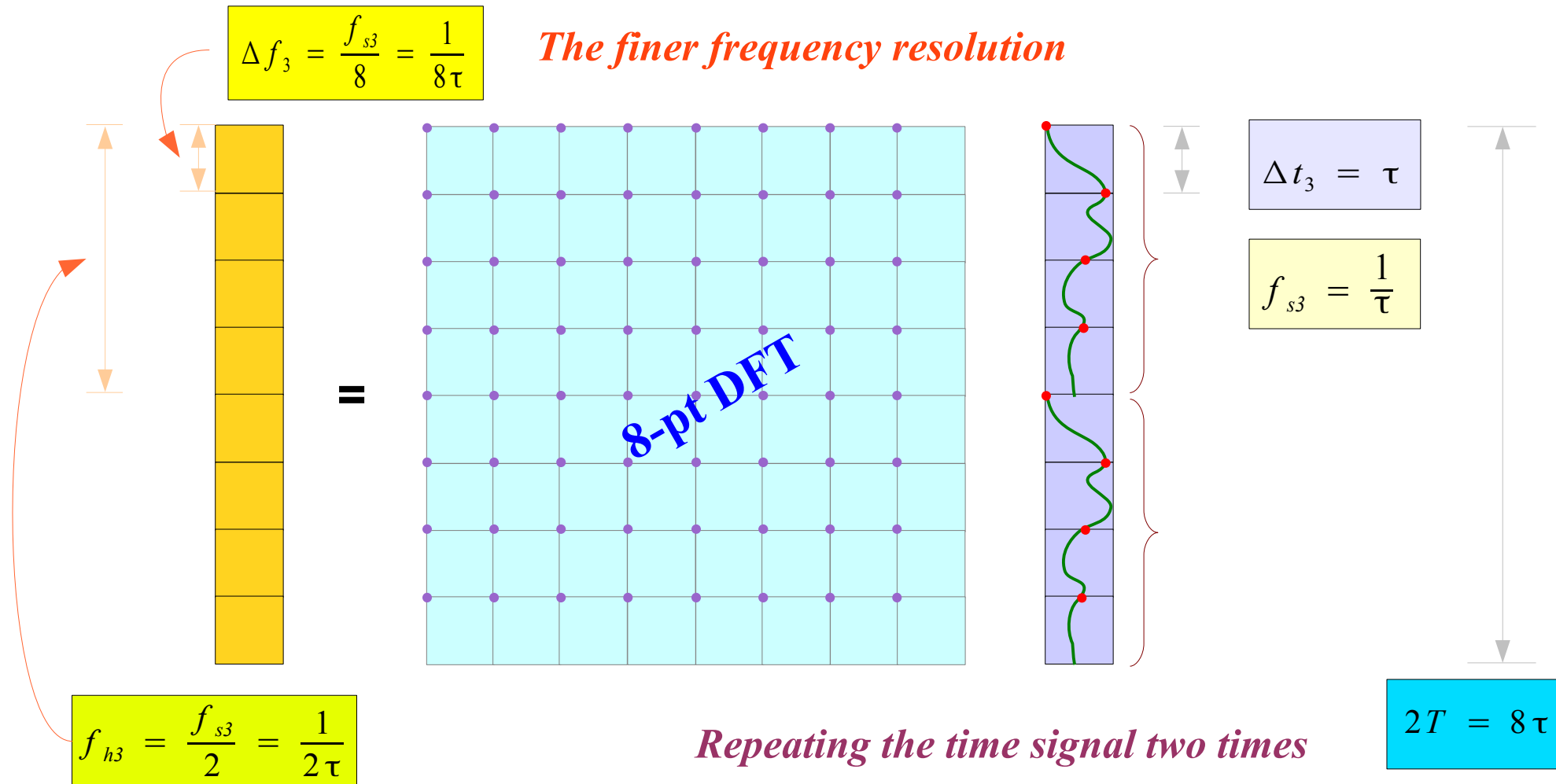
Frequency View of a X[i] Vector



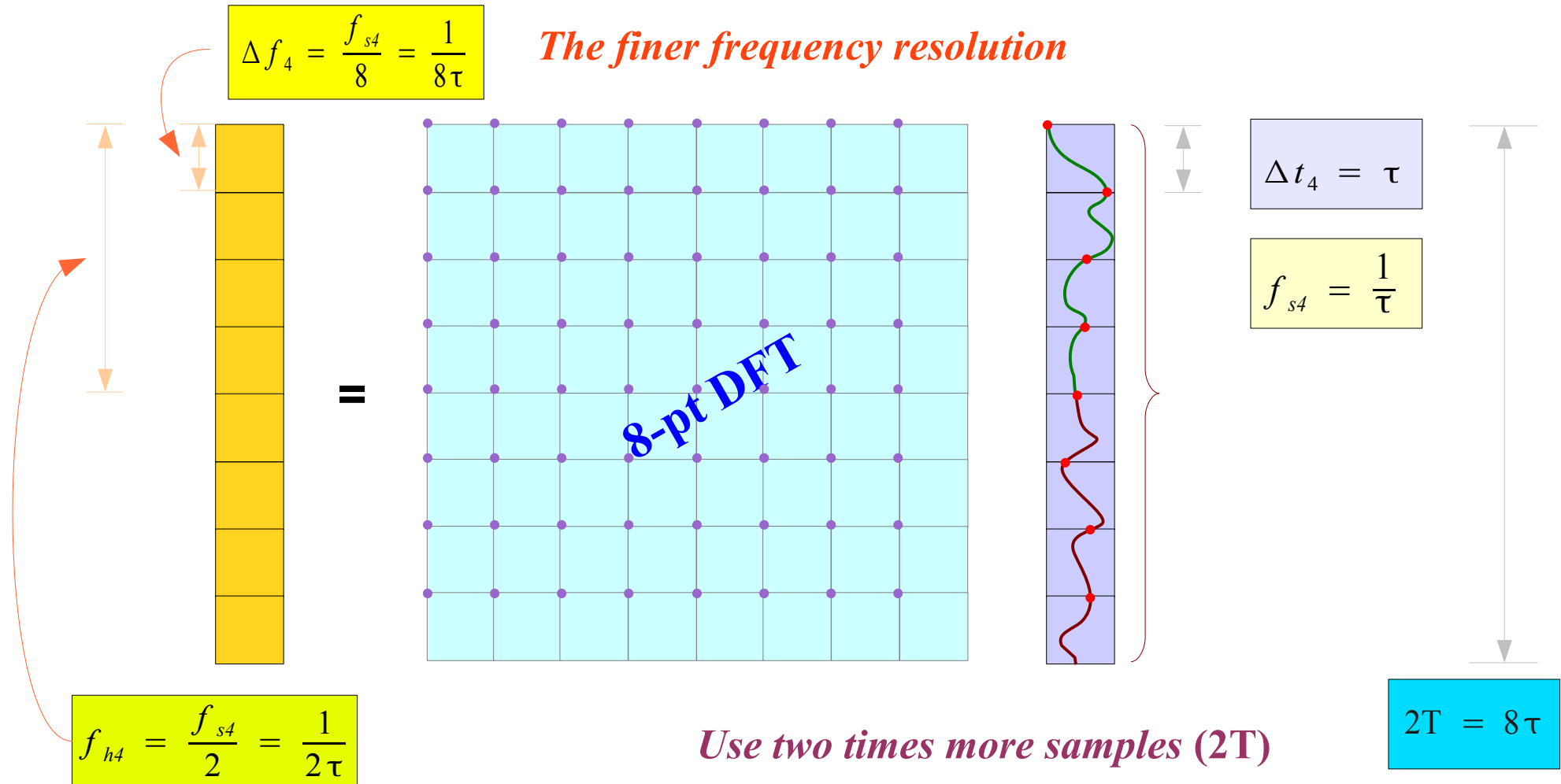
Frequency and Time Interval (1)



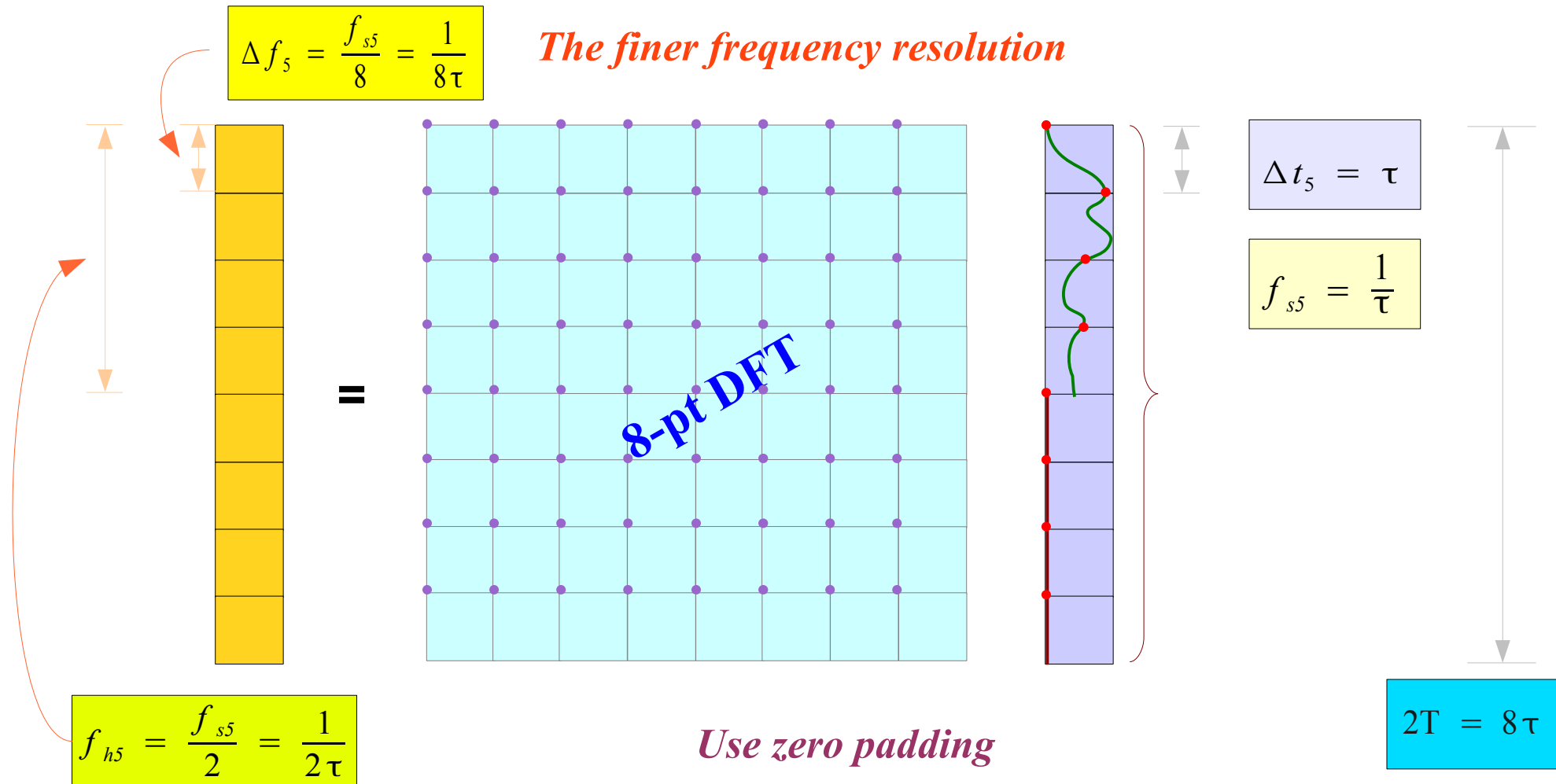
Frequency and Time Interval (2)



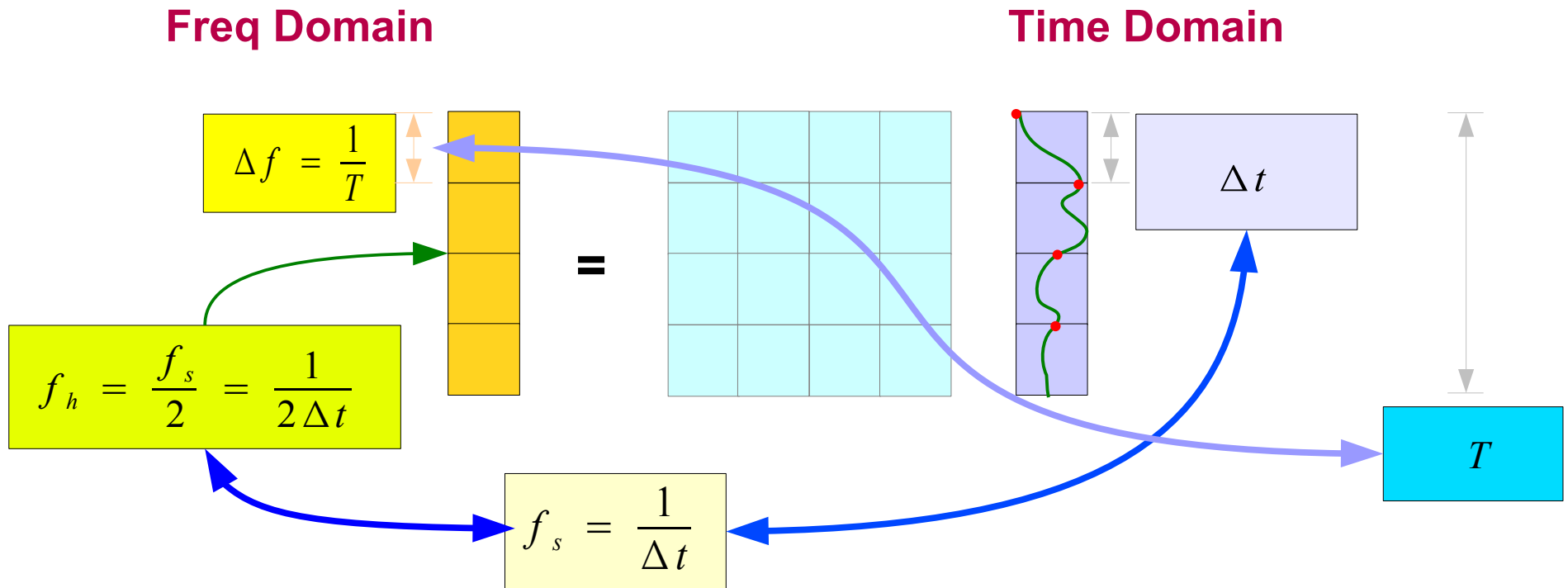
Frequency and Time Interval (3)



Frequency and Time Interval (4)



Frequency and Time Interval (5)



Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = +1, +2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left(-\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha) \cos(\beta)} - \underline{\sin(\alpha) \sin(\beta)}$$

$$g_k \cos(k \omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k) \cos(k \omega_0 t)} - \underline{g_k \sin(\phi_k) \sin(k \omega_0 t)}$$

$$\underline{a_k \cos(k \omega_0 t)} + \underline{b_k \sin(k \omega_0 t)}$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$\underline{|C_k|^2 + |C_{-k}|^2} = \frac{1}{2}g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}g_k e^{+j\phi_k} & (k > 0) \\ \frac{1}{2}g_k e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = \underline{g_k} = \underline{\sqrt{a_k^2 + b_k^2}}$$

CTFS and DTFS (1)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t} \quad \text{CTFS}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

N Time Samples: N equations

$$t \rightarrow nT_s = n\left(\frac{T}{N}\right)$$

$$jk\omega_0 t \rightarrow k\left(\frac{2\pi}{T}\right)n\left(\frac{T}{N}\right) = \left(\frac{2\pi}{T}\right)nk$$



$$x(t) \approx \sum_{k=-M}^{+M} C_k e^{+jk\omega_0 t} \quad N = 2M + 1$$

Truncate coefficients

$$k = -M, \dots, -1, 0, +1, \dots, +M$$

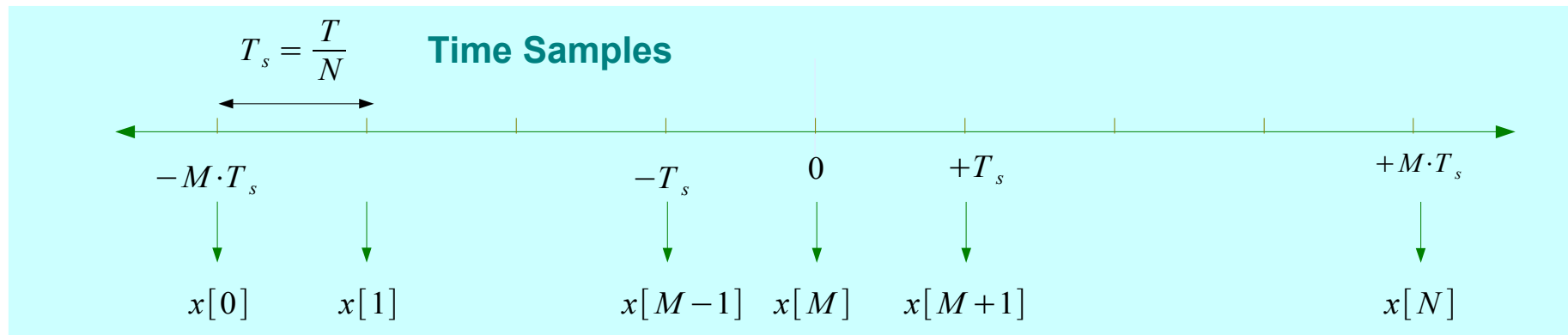


$$x[n] = \sum_{k=-M}^{+M} \gamma_k e^{+jk\omega_0 n\left(\frac{T}{N}\right)}$$



$$x[n] = \sum_{k=-M}^{+M} \gamma_k e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \dots, N-1,$$



CTFS and DTFS (2)

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t} \quad \text{CTFS}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \dots, N-1,$$

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$k = -M, \dots, 0, \dots, +M$$

Approximate Continuous Signal
With the truncated coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k e^{+jk\omega_0 t} \quad \text{DTFS}$$

$$C_k \approx \gamma_k = \frac{X[k]}{N} \quad \text{Approximated Fourier Coefficients}$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn} \quad \text{DFT}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn} \quad \text{IDFT}$$

CTFS and DTFS (3)

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

$$x[n] = \sum_{k=-M}^{+M} \gamma_k e^{+j\left(\frac{2\pi}{N}\right)nk}$$

$$n = 0, 1, 2, \dots, N-1,$$

$$\gamma_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2\pi}{N}\right)nk}$$

$$k = -M, \dots, 0, \dots, +M$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t} \quad \text{CTFS}$$

$$C_k \approx \gamma_k = \frac{X[k]}{N} \quad \text{Approximated Fourier Coefficients}$$



Truncate Fourier Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k e^{+jk\omega_0 t} \quad \text{DTFS}$$



Power Spectrum using FFT

Discrete Fourier Transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j(2\pi/N)kn}$$

$$X = \text{fft}(x)$$

$$x = \text{ifft}(X)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t} \quad \text{CTFS}$$

Approximated
Fourier Series Coefficients

$$x_{FS}(t) = \sum_{k=-M}^{+M} \gamma_k e^{+jk\omega_0 t} \quad \text{DTFS}$$

$$fc = \text{fft}(x)/N = X/N$$

$$C_k \approx \gamma_k = \frac{X[k]}{N} \quad \text{Approximated Fourier Coefficients}$$

$$x = \text{ifft}(fc)*N$$

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2} \quad \text{Approximated Power Spectrum}$$

Periodogram using FFT

$$C_k \approx y_k = \frac{X[k]}{N} \quad \begin{array}{l} \text{Approximated} \\ \text{Fourier Coefficients} \end{array}$$

$$|C_k|^2 \approx \frac{|X[k]|^2}{N^2} \quad \begin{array}{l} \text{Approximated} \\ \text{Power Spectrum} \end{array}$$

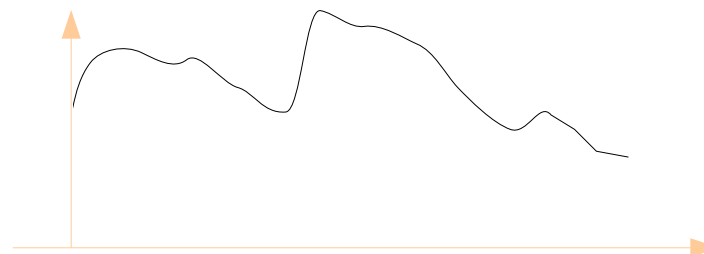
$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2 \quad \begin{array}{l} \text{Average} \\ \text{Power} \end{array}$$

$$\rightarrow \left(\sqrt{\frac{\sum_{k=0}^{N-1} \frac{|X[k]|^2}{N}}{N}} \right)^2 \quad \begin{array}{l} \text{RMS of sq root} \\ \text{Periodogram} \end{array}$$

$$\frac{|X[k]|^2}{N} \quad k=0,1,\dots,N-1 \quad \begin{array}{l} \text{Approximated} \\ \text{Periodogram} \end{array}$$

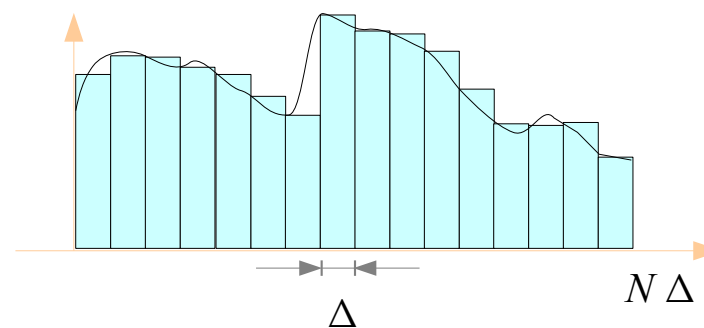
$$\frac{|X[k]|}{\sqrt{N}} \quad k=0,1,\dots,N-1 \quad \begin{array}{l} \text{Square root} \\ \text{Periodogram} \end{array}$$

RMS in continuous time



$$\frac{1}{T} \int_0^T g^2(t) dt$$

RMS in discrete time



$$\frac{1}{N\Delta} \sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N} \sum_{k=0}^{N-1} |g[k]|^2$$

From CTFS to CTFT

Continuous Time Fourier Series

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=-\infty}^{\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$x_{T_0}(t) = \sum_{n=-\infty}^{\infty} C_k e^{+jk\omega_0 t} \cdot \frac{2\pi}{2\pi} \cdot \frac{T_0}{T_0}$$

$$C_k T_0 = \int_{-T_0/2}^{+T_0/2} x_{T_0}(t) e^{-jk\omega_0 t} dt$$

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} C_k T_0 e^{+jk\omega_0 t} \cdot \frac{2\pi}{T_0}$$

$$aT_0 \rightarrow \infty \quad C_k T_0 \rightarrow X(j\omega) \quad x_{T_0} \rightarrow x(t) \quad \omega = \frac{2\pi}{T_0} \rightarrow d\omega$$

Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

Periodic Signals

Aperiodic Signals

Random Signals

Frequency Spacing

$$\Delta f = \frac{1}{N\Delta t}$$

$$\Delta f = \frac{1}{N\Delta t}$$

$$\sum S \Delta f = \frac{1}{N\Delta t} \sum S \quad \frac{1}{N\Delta t} \sum x^2 \Delta t$$

Two Sided FC Coeff

$$\frac{1}{N} X(k)$$

$$\frac{\Delta t}{N} X(k)$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2 \quad P = \sum_{k=0}^{N-1} S(k) \Delta f$$

One Sided

$$k=0, \frac{N}{2}$$

$$\frac{1}{N} X(k)$$

$$\frac{\Delta t}{N} X(k)$$

$$S_1(k) = 2S(k) \quad P = \sum_{k=0}^{N/2} S_1(k) \Delta f$$

$$k=1, \dots, \frac{N}{2}-1$$

$$\frac{2}{N} X(k)$$

$$\frac{2\Delta t}{N} X(k)$$

$$S_1(k) = S(k)$$

Frequency Scale

$$k \Delta f$$

$$k \Delta f$$

$$k \Delta f$$

Periodic Signals

Frequency Spacing

$$\Delta f = \frac{1}{N\Delta t}$$

Two Sided Fourier Series Coefficient

$$\frac{1}{N} X(k)$$

One Sided Fourier Series Coefficient

$$\frac{1}{N} X(k) \quad k=0, \frac{N}{2}$$

$$\frac{2}{N} X(k) \quad k=1, \dots, \frac{N}{2}-1$$

Frequency Scale

$$k \Delta f$$

Aperiodic Signals

$$\Delta f = \frac{1}{N\Delta t}$$

Two Sided Fourier Series Coefficient

$$\frac{\Delta t}{N} X(k)$$

One Sided Fourier Series Coefficient

$$\frac{\Delta t}{N} X(k) \quad k=0, \frac{N}{2}$$

$$\frac{2\Delta t}{N} X(k) \quad k=1, \dots, \frac{N}{2}-1$$

$$k \Delta f$$

Random Signals

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N-1} S(k) \Delta f$$

One-sided Power Spectral Density

$$P = \sum_{k=0}^{N/2} S_1(k) \Delta f$$

$$S_1(k) = 2S(k) \quad k = 1, \dots, \frac{N}{2} - 1$$

$$S_1(k) = S(k) \quad k = 0, \frac{N}{2}$$

Two Sided Fourier Series Coefficient

$$\frac{1}{N \Delta t} \sum x^2 \Delta t$$

$$\sum S \Delta f = \frac{1}{N \Delta t} \sum S$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2$$

$$k \Delta f$$

Amplitude Spectrum

$$A_k = \frac{1}{N}|X(k)| = \frac{1}{N}\sqrt{\Re^2(X(k)) + \Im^2(X(k))}$$

$$k = 0, 1, 2, \dots, N-1$$

Power Spectrum

$$P_k = \frac{1}{N^2}|X(k)|^2 = \frac{1}{N^2}\{\Re^2(X(k)) + \Im^2(X(k))\}$$

$$k = 0, 1, 2, \dots, N-1$$

One Sided Amplitude Spectrum

$$\bar{A}_k = \frac{1}{N}|X(0)| \quad k=0$$

$$\bar{A}_k = \frac{2}{N}|X(0)| \quad k=1, 2, \dots, N/2$$

One Sided Power Spectrum

$$\bar{P}_k = \frac{1}{N^2}|X(0)|^2 \quad k=0$$

$$\bar{P}_k = \frac{2}{N^2}|X(0)|^2 \quad k=1, 2, \dots, N/2$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Frequency Bin

$$f = \frac{k f_s}{N}$$

Phase Spectrum

$$\phi_k = \tan^{-1}\left(\frac{\Im(X(k))}{\Re(X(k))}\right) \quad k=0, 1, 2, \dots, N-1$$

Data Truncation
Frequency Resolution
Zero Padding
Periodogram
Spectral Plot

Amplitude spectrum in quantity peak
Phase spectrum in radians
Amplitude spectrum in volts rms
Phase spectrum in degrees
Power spectrum

Signals without discontinuity
Signals with discontinuity

Sampling frequency is not an integer multiple
of the FFT length

Leakage

$$\left[0, \frac{f_s}{2}\right]$$

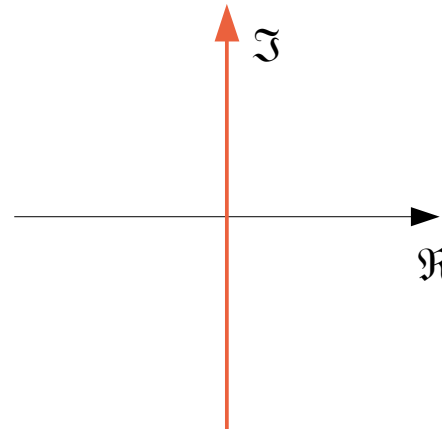
Fourier Transform

$f(t)$ A continuous sum of weighted exponential functions :

$$f(t) e^{-j\omega t}$$

$$-\infty < \omega < +\infty$$

Not so useful in transient analysis

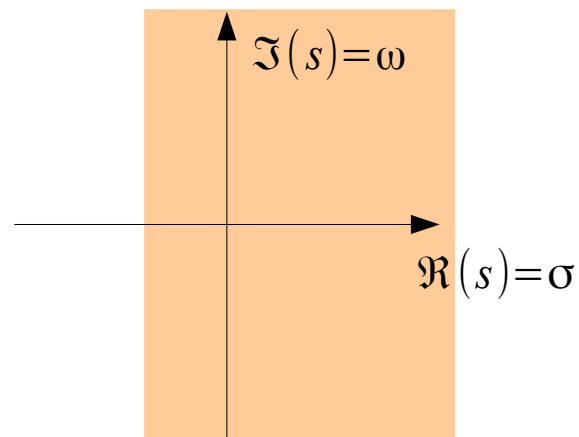


Laplace Transform

$$f(t) e^{-st} = f(t) e^{-(\sigma + j\omega)t}$$

Linear Time Domain Analysis

Initial Condition



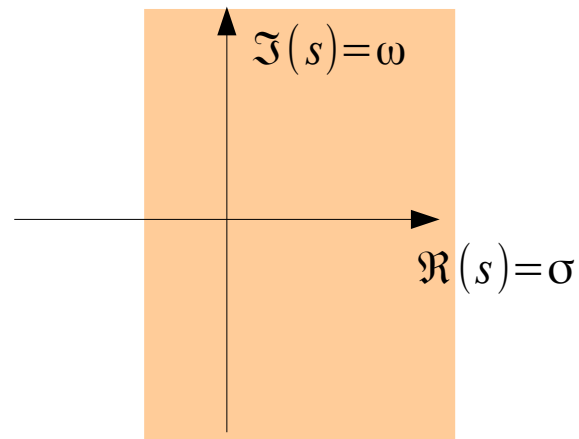
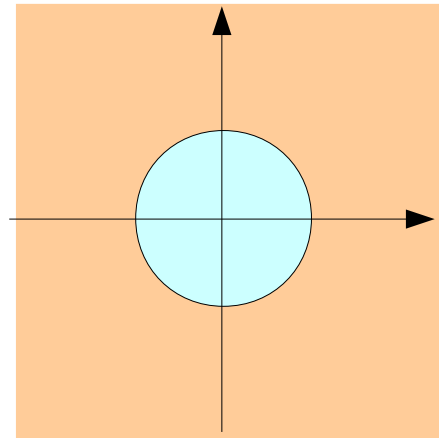
z Transform

$$f[n] z^{-n}$$

Discrete Time System

Difference Equation

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}$$



References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A “graphical interpretation” of the DFT and FFT, by Steve Mann