

Downsampling (4B)

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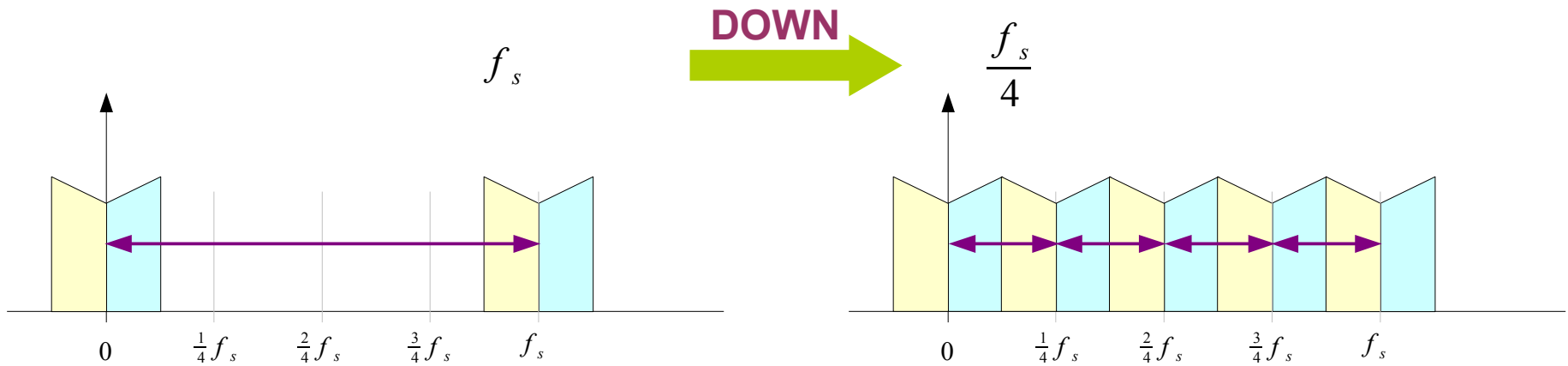
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Band-limited Signal



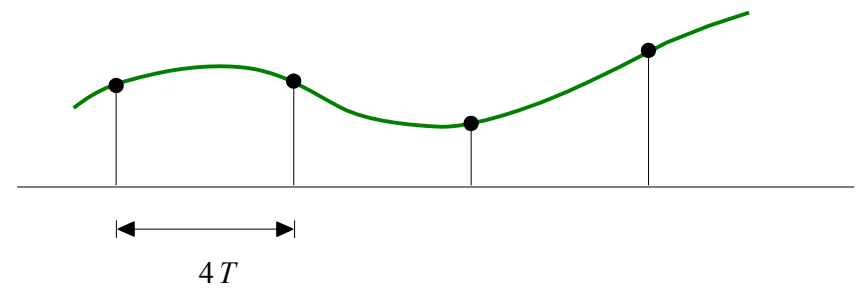
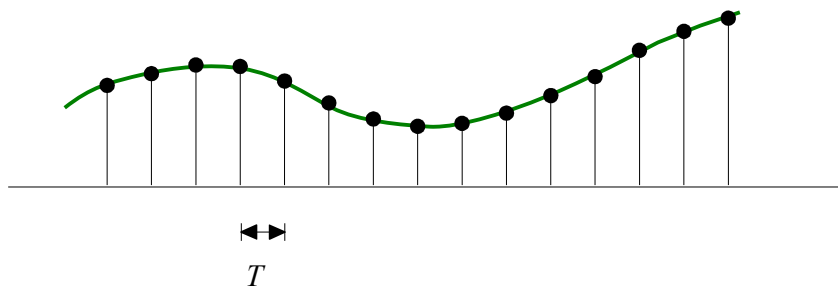
Sampling Frequency f_s

Sampling Time $T = \frac{1}{f_s}$

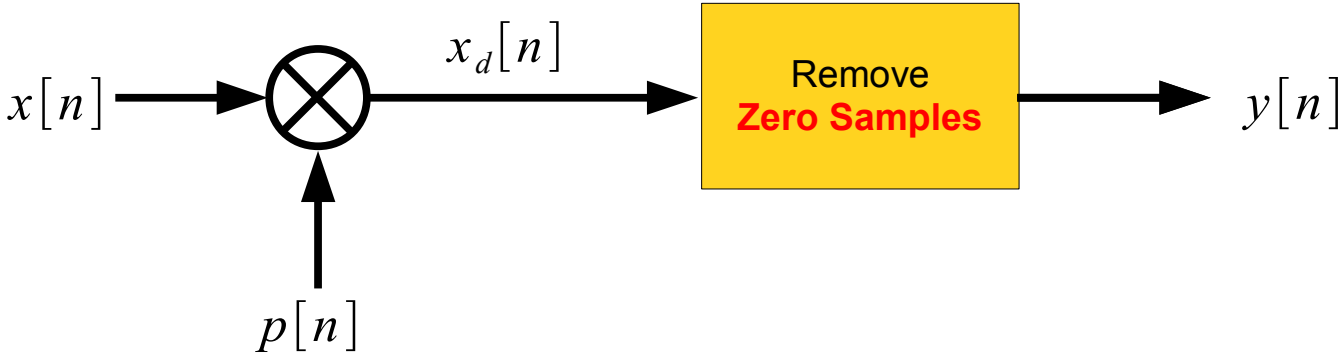
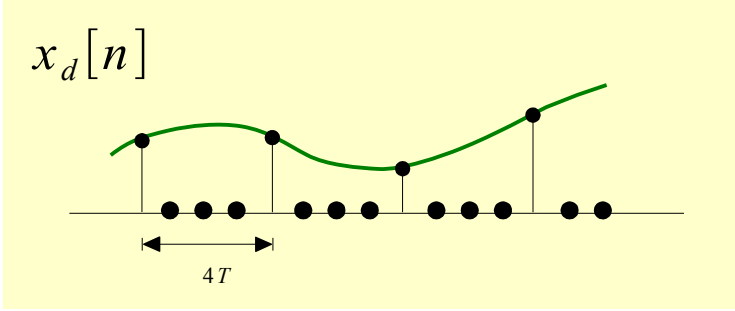
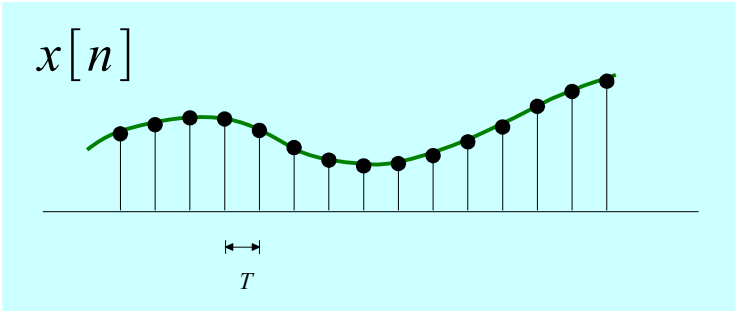


Sampling Frequency $f'_s = \frac{1}{4} f_s$

Sampling Time $T' = \frac{4}{f_s}$

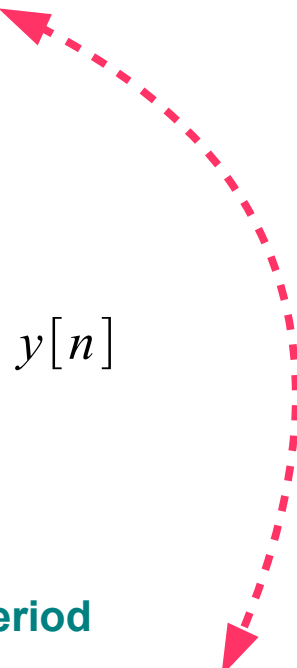
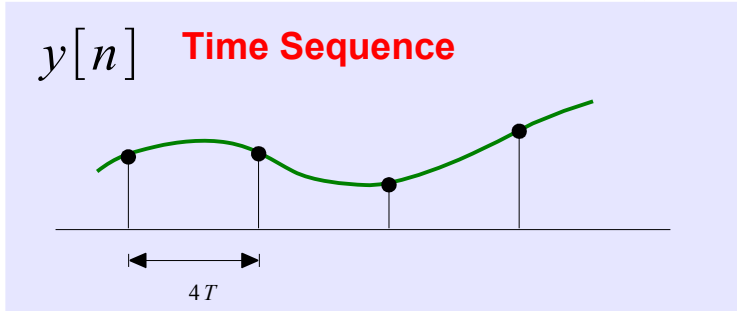
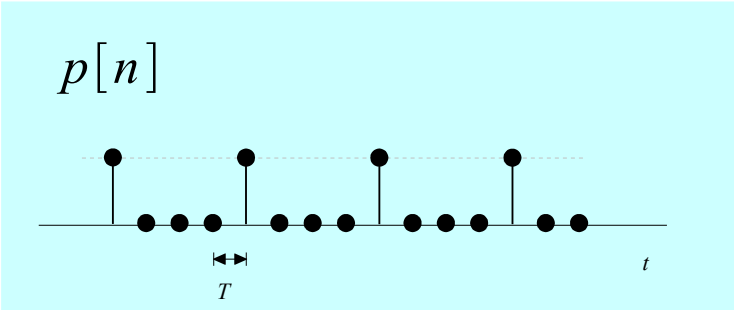


Time Sequence

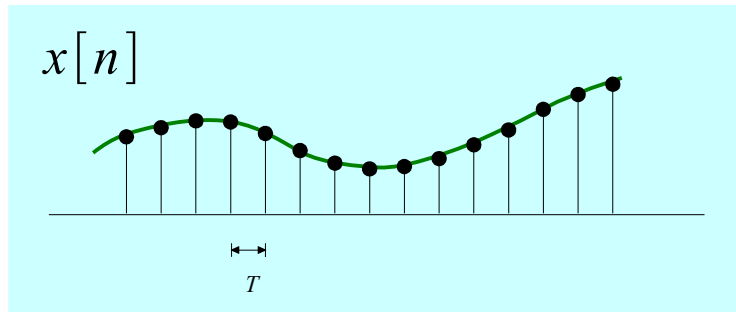


Ideal Sampling

T Sampling Period



Normalized Radian Frequency

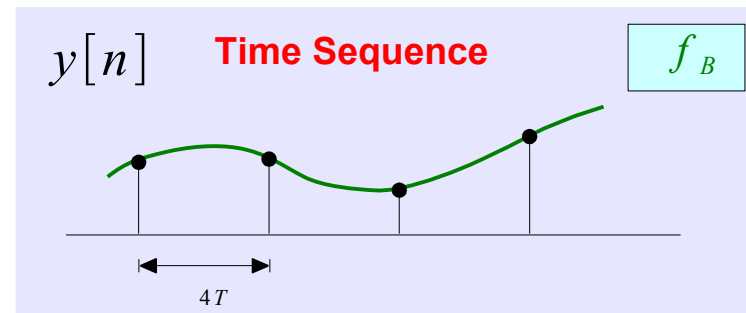


$$\hat{\omega} = \omega \cdot T_s = \frac{\omega}{1/T_s}$$

$$\hat{\omega} = \frac{\omega}{f_s} = 2\pi \frac{f}{f_s}$$

↑ ↑
Normalized to f_s

Normalized Radian Frequency

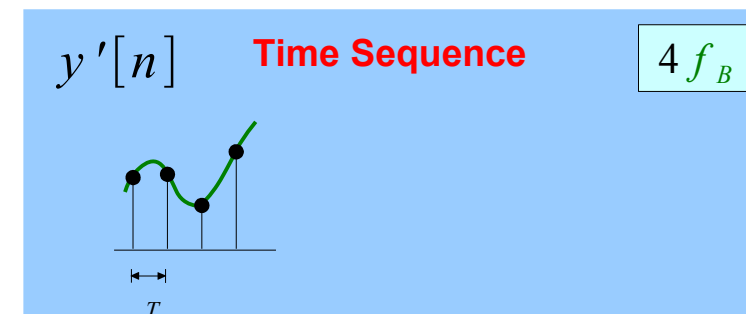


$$\frac{f}{f_s} = \frac{f_B}{1/4T} = f_B \cdot 4T$$

The Same

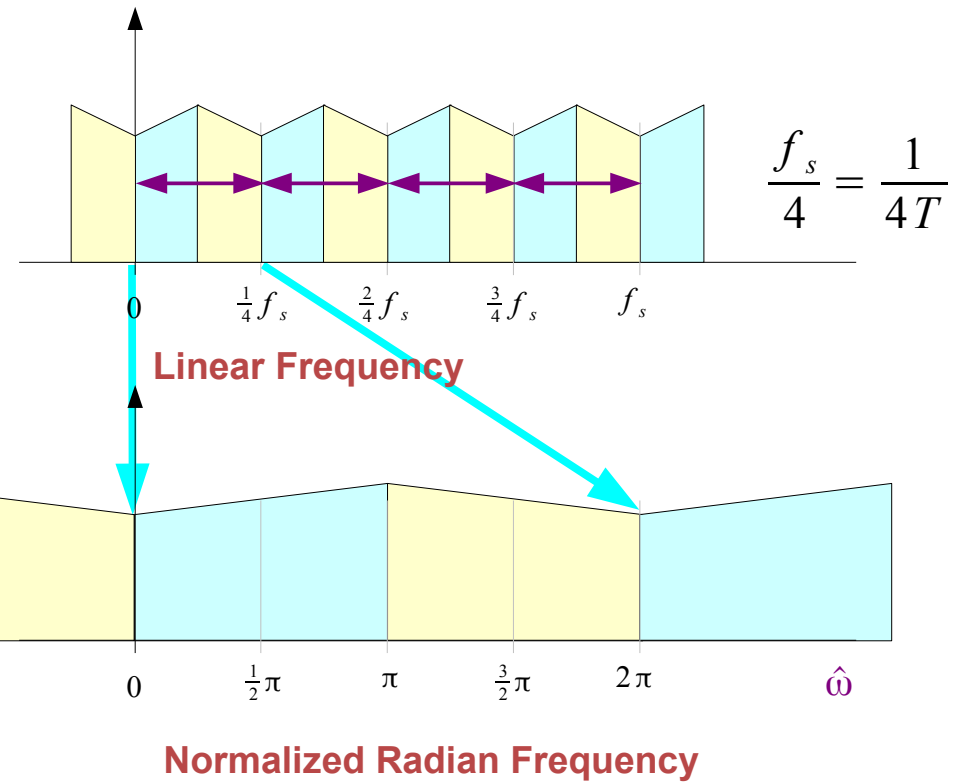
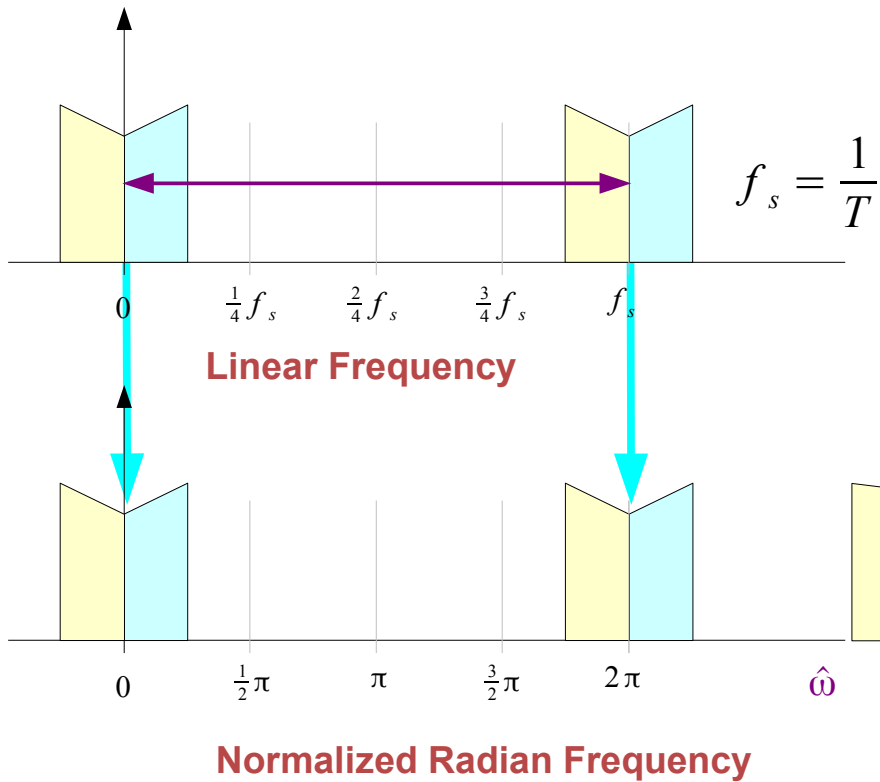
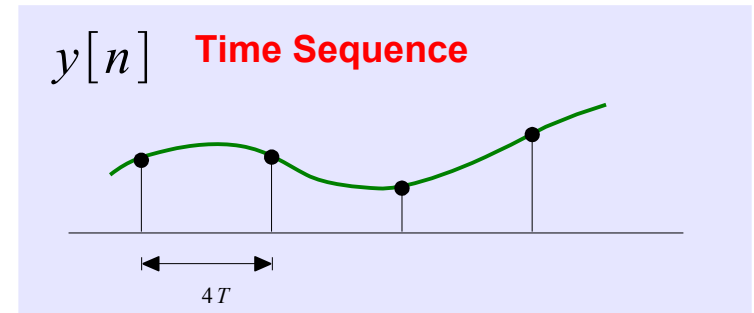
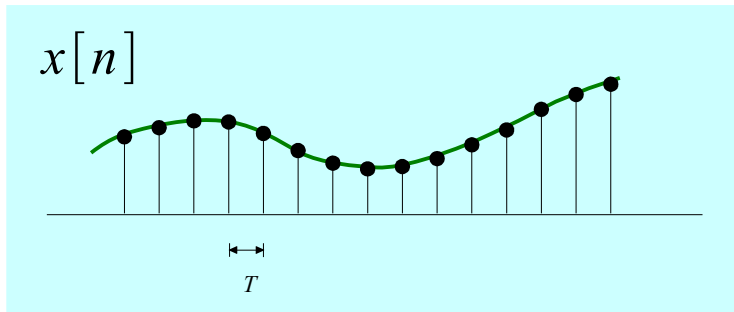
- Time Sequence
- Normalized Radian Frequency

$$\frac{f}{f_s} = \frac{4f_B}{1/T} = f_B \cdot 4T$$

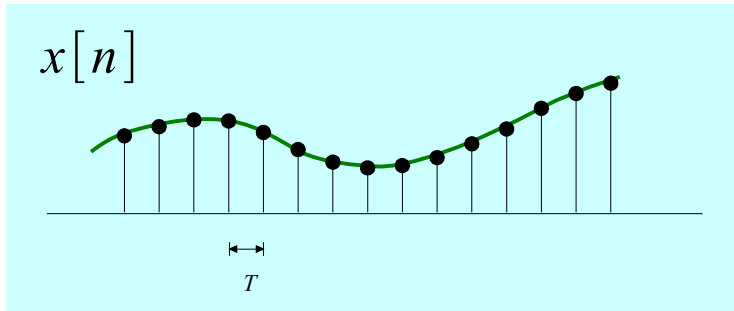


The Highest Frequency:

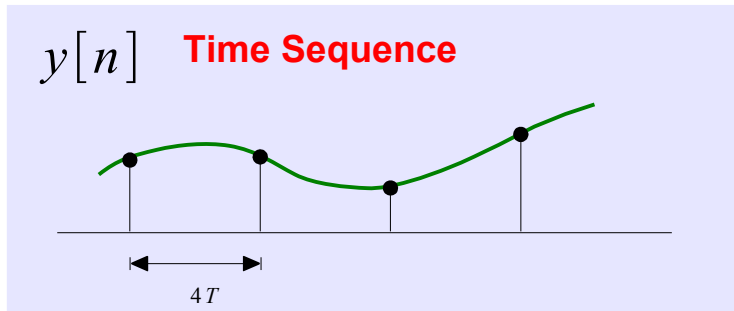
Time Sequence & Spectrum



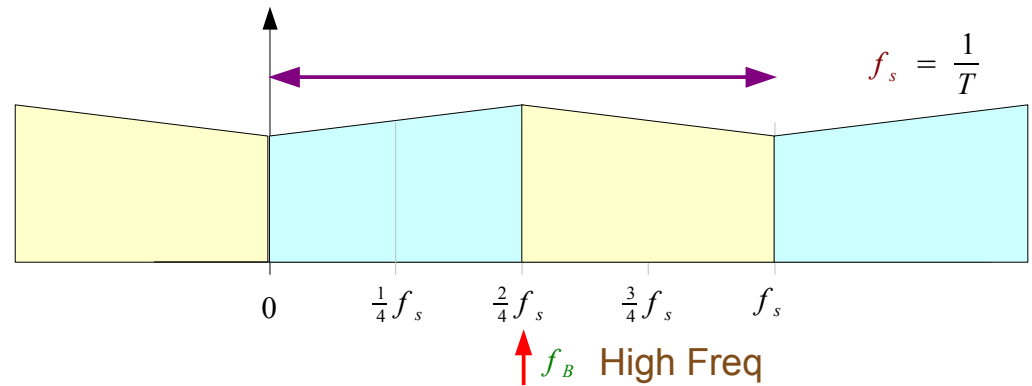
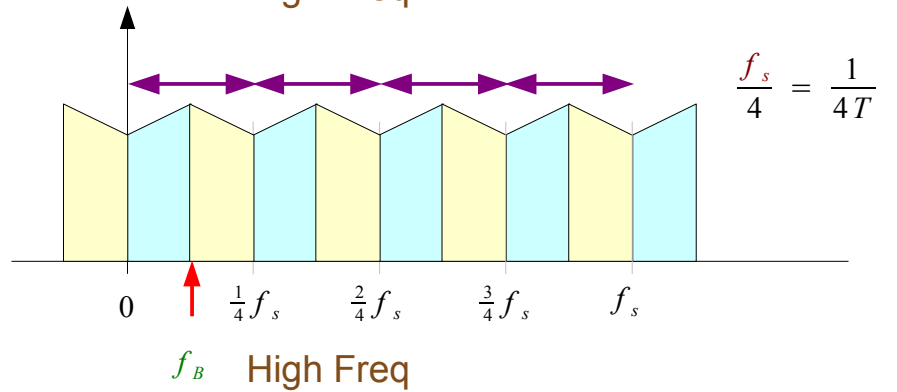
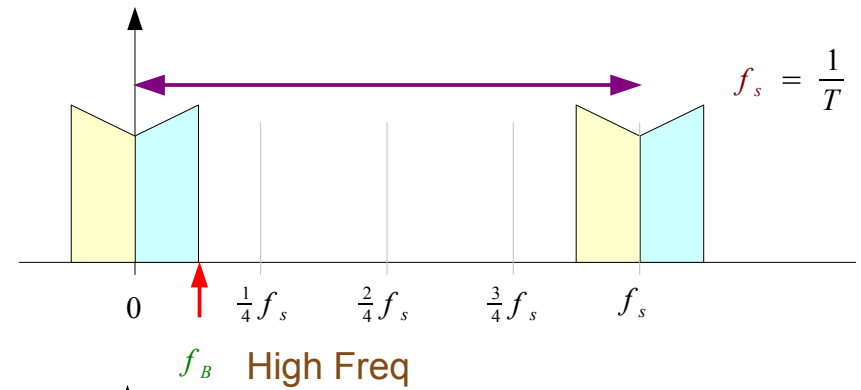
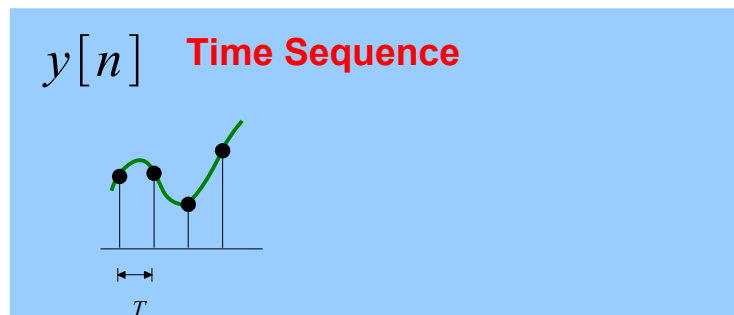
Spectrum in Linear Frequency



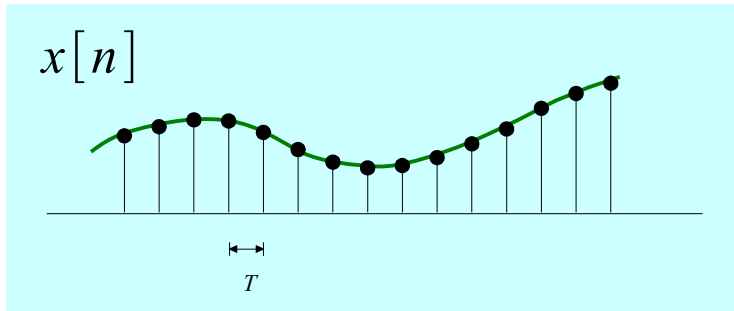
DOWN



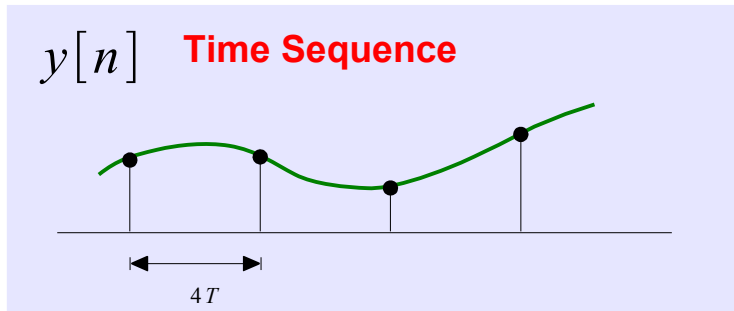
|| The Same Time Sequence



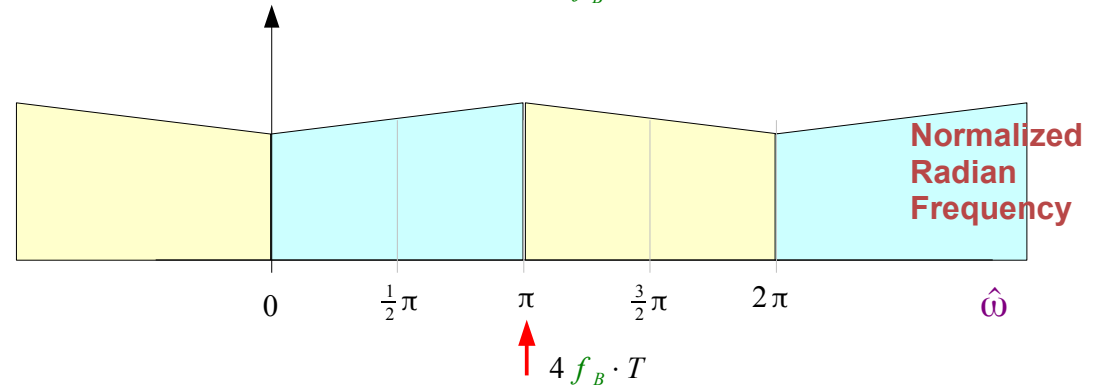
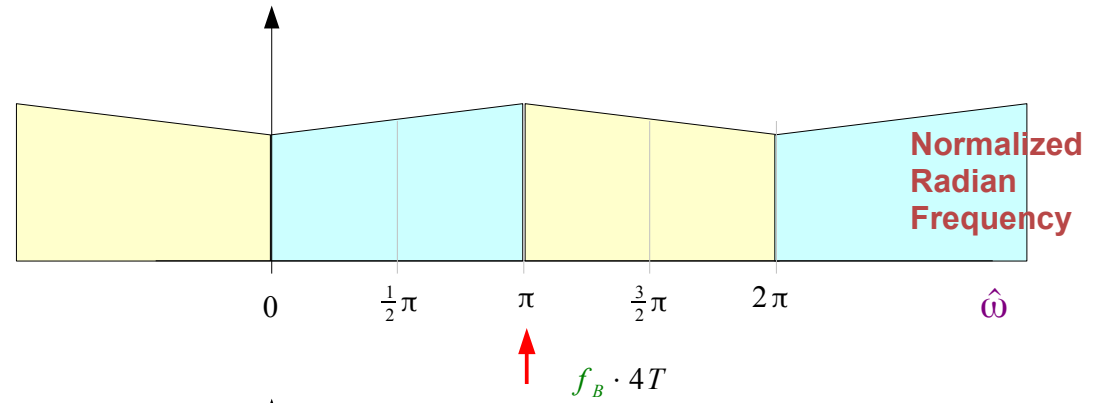
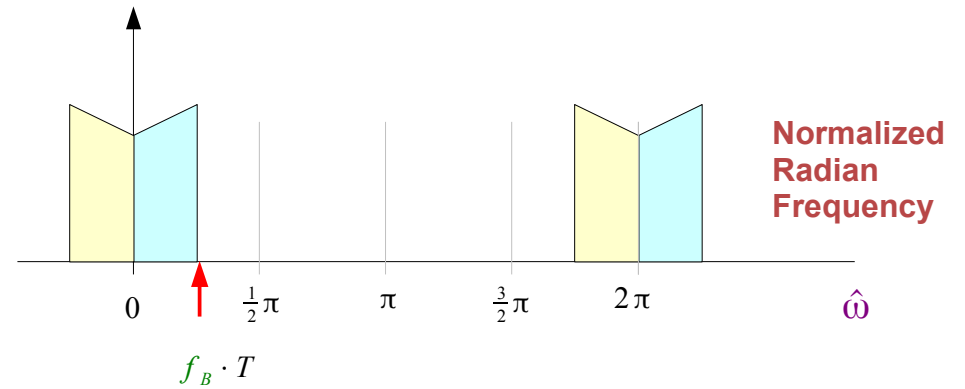
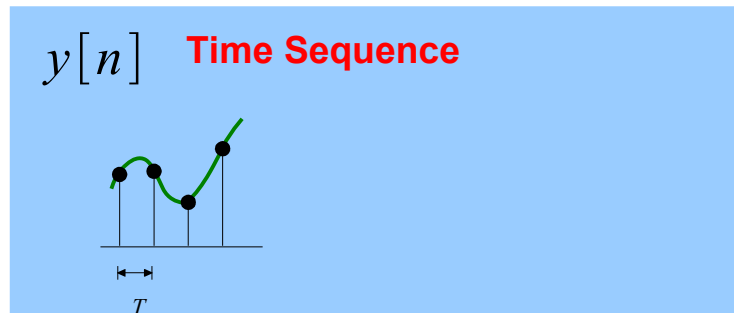
Spectrum in Normalized Frequency



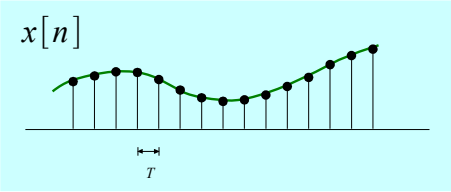
DOWN



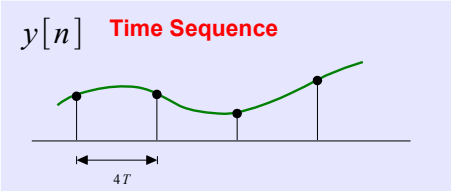
|| The Same Time Sequence



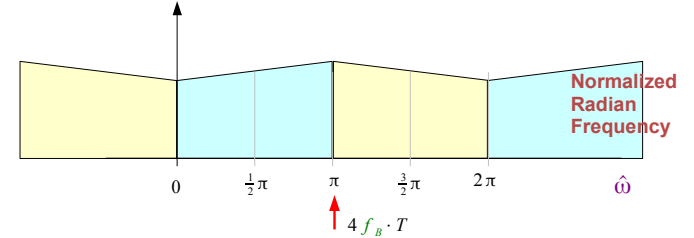
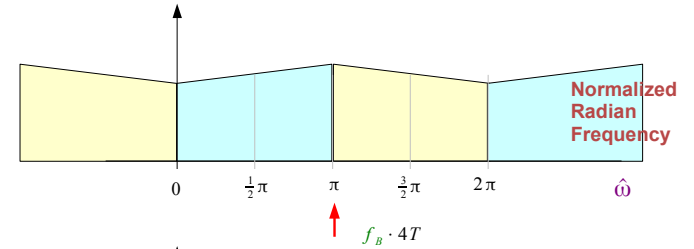
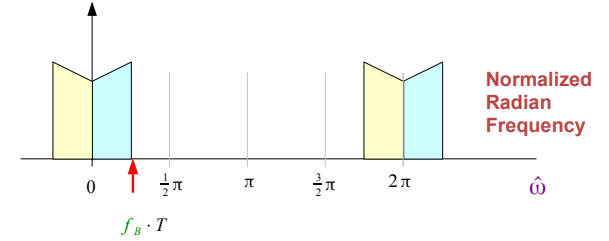
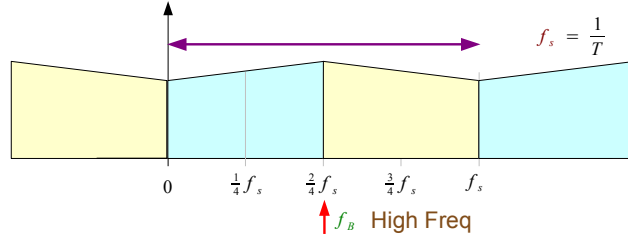
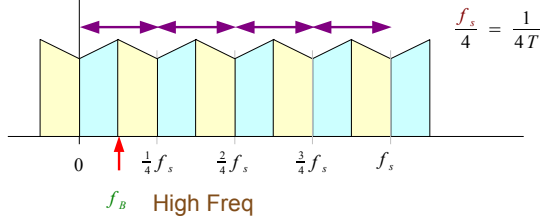
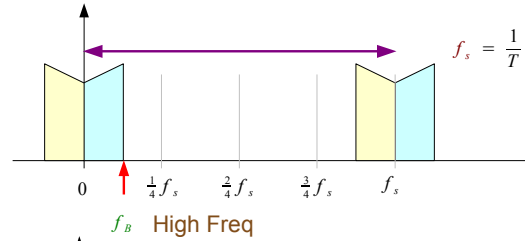
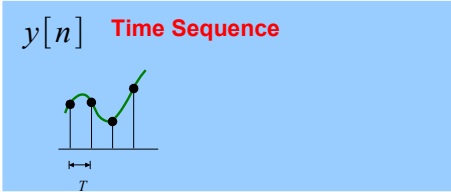
Time Sequence Spectrum



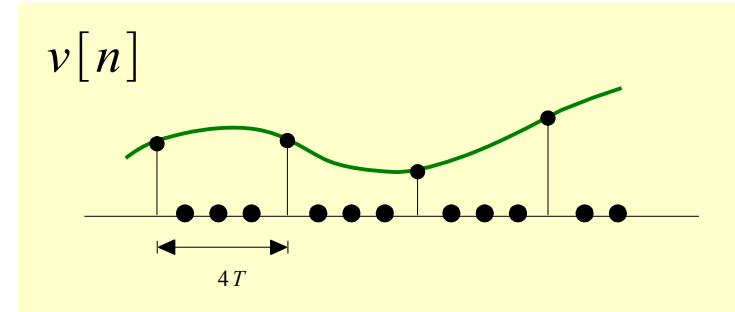
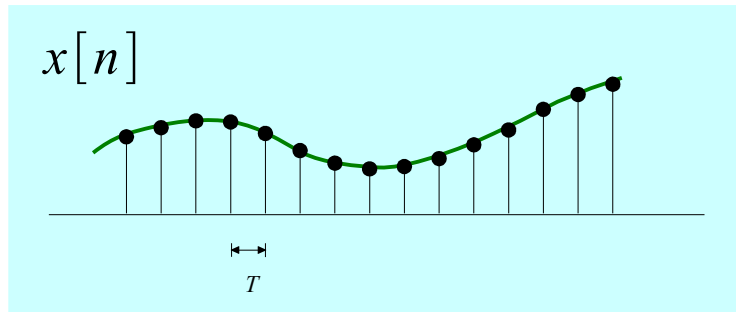
↓ DOWN



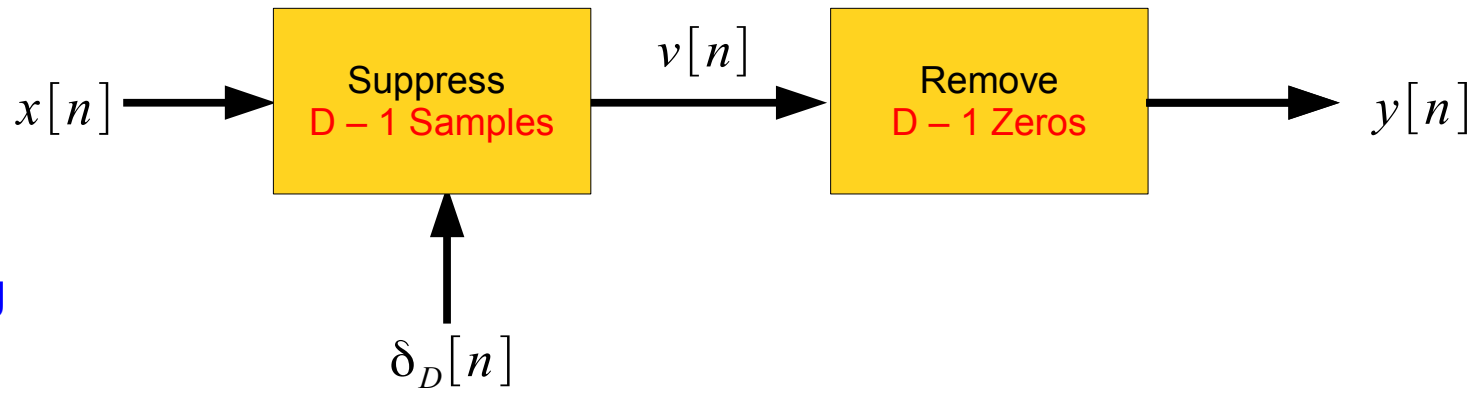
|| The Same Time Sequence



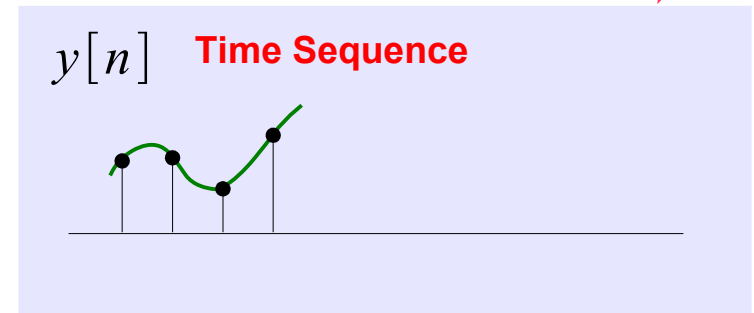
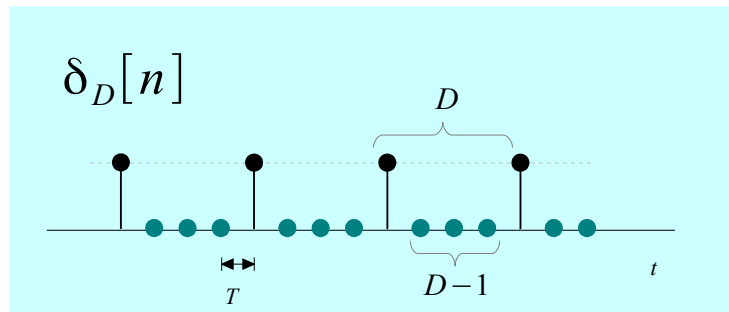
Z-Transform and Down Sampling



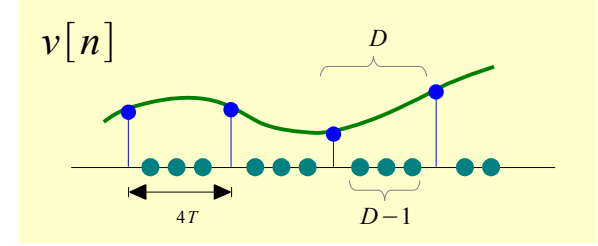
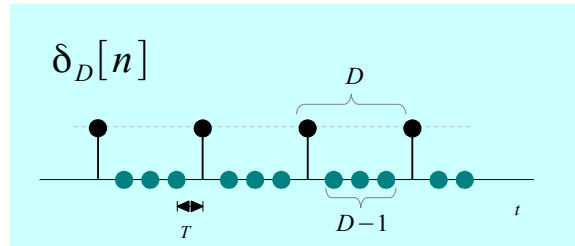
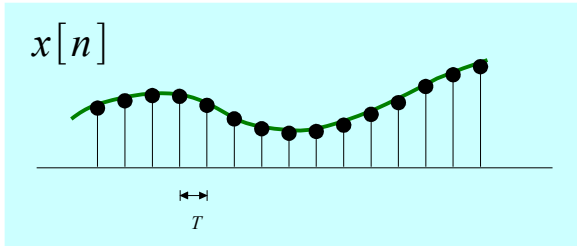
T Sampling Period



Ideal Sampling



Suppressing D-1 Samples

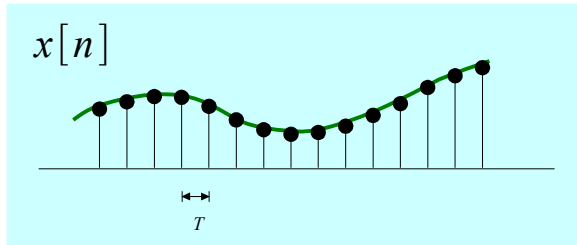


$$\delta_D[n] = \begin{cases} 1 & \text{if } \text{mod}(n, D) = 0 \\ 0 & \text{otherwise} \end{cases} \quad v[n] = \delta_D[n]x[n]$$

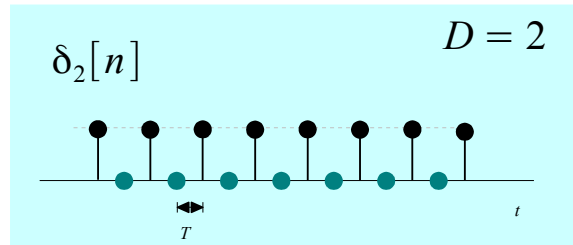
$$V[z] = \dots + v[0]z^0 + v[D]z^{-D} + v[2D]z^{-2D} + \dots$$

$$V[z] = \sum_{n=-\infty}^{+\infty} v[n]z^{-n} = \sum_{m=-\infty}^{+\infty} v[mD]z^{-Dm} = F(z^D)$$

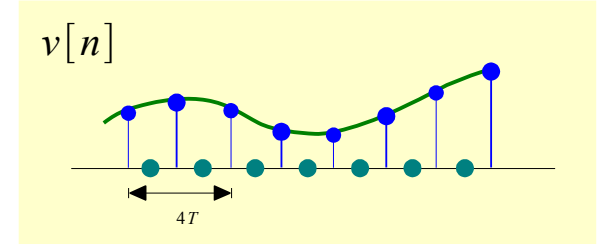
Example When D=2 (1)



$$x[n] = e^{j\omega n}$$



$$\begin{aligned} \delta_2[n] &= \frac{1}{2}(1 + (-1)^n) \\ &= \frac{1}{2}(1 + e^{-j\pi n}) \\ &\quad (e^{-j\pi} = -1) \end{aligned}$$



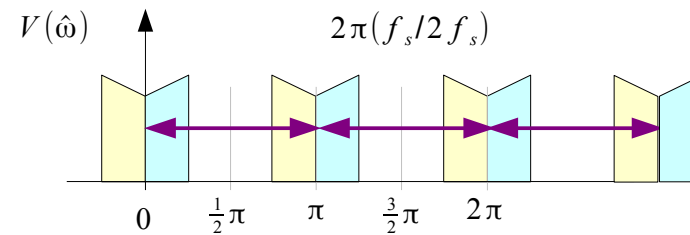
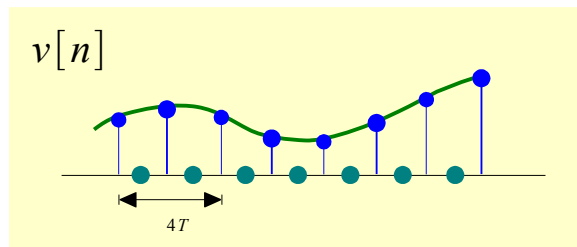
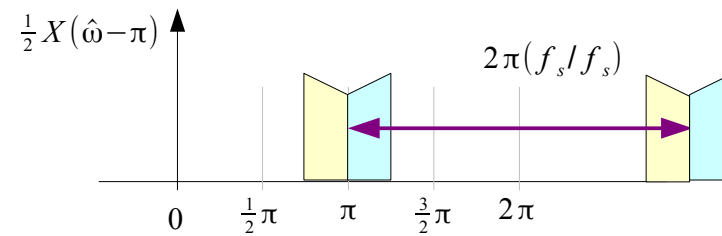
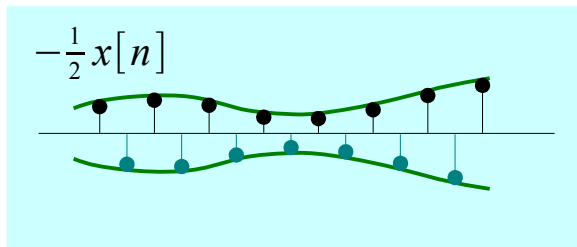
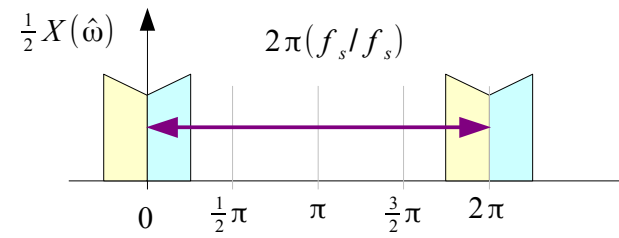
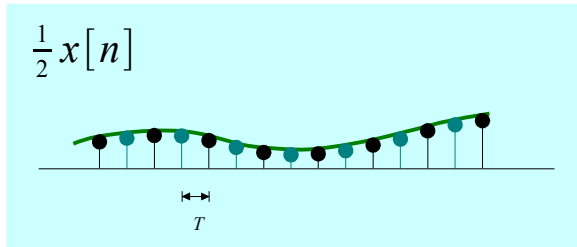
$$\begin{aligned} v[n] &= \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n] \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{-j\pi n}e^{j\omega n} \\ &= \frac{1}{2}e^{j\omega n} + \frac{1}{2}e^{+j(\omega-\pi)n} \end{aligned}$$

$$V(z) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} (x[n]z^{-n} + x[n](-z)^{-n}) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

Example When D=2 (2)



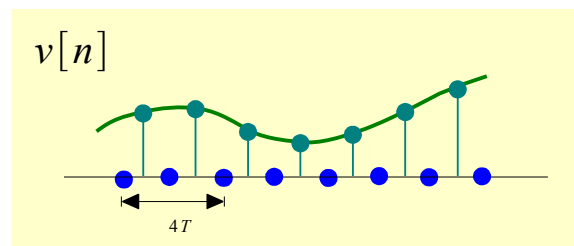
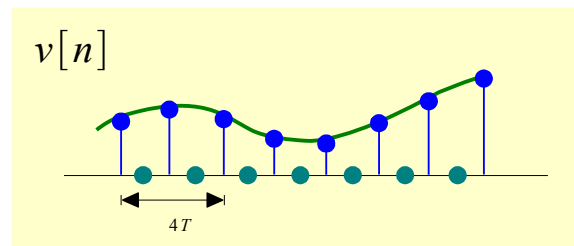
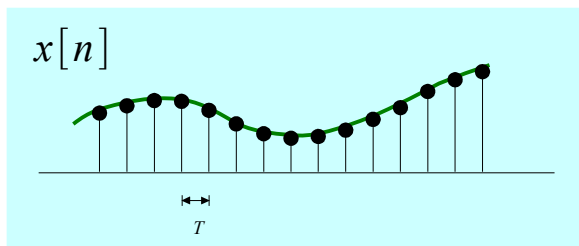
$$v[n] = \frac{1}{2}x[n] + \frac{1}{2}e^{-j\pi n}x[n]$$

$$V(z) = \frac{1}{2}X(z) + \frac{1}{2}X(-z)$$

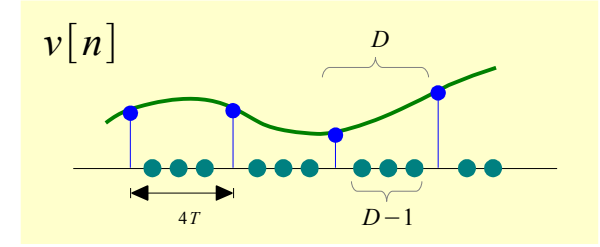
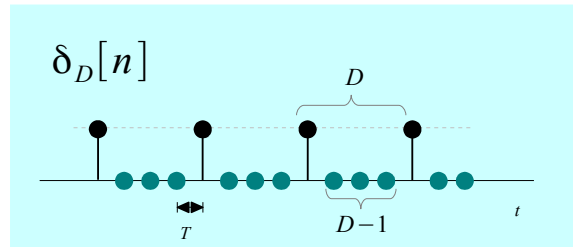
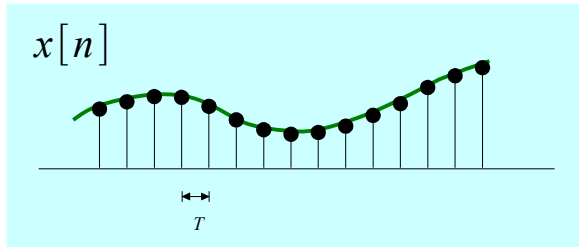
$$V(e^{j\hat{\omega}}) = \frac{1}{2}X(e^{j\hat{\omega}}) + \frac{1}{2}X(e^{-j\pi}e^{j\hat{\omega}})$$

$$V(\hat{\omega}) = \frac{1}{2}X(\hat{\omega}) + \frac{1}{2}X(\hat{\omega} - \pi)$$

Example When $D=2$ (3)



Z-Transform & Suppressing D-1 Samples (1)



$$\delta_D[n] = \frac{1}{D} \sum_{k=0}^{D-1} e^{-j2\pi kn/D} = \begin{cases} \frac{1}{D} D = 1 & \text{if } \text{mod}(n, D) = 0 \\ \frac{1}{D} \frac{1 - e^{-j2\pi n}}{1 - e^{-j2\pi n/D}} = 0 & \text{otherwise} \end{cases} \quad v[n] = \delta_D[n]x[n]$$

$$v[n] = \delta_D[n]x[n] = \frac{1}{D} \sum_{k=0}^{D-1} x[n] e^{-j2\pi kn/D} \quad Z\{x[n] e^{-j\hat{\omega}_k n}\} = \sum_{n=-\infty}^{+\infty} x[n] (e^{j\hat{\omega}_k \cdot z})^{-n} = X(e^{j\hat{\omega}_k \cdot z})$$

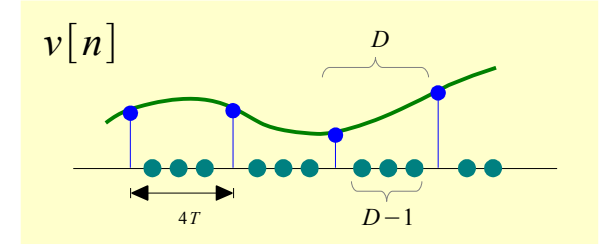
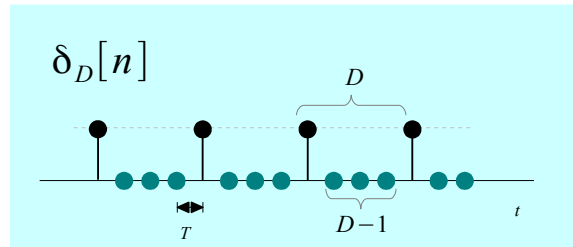
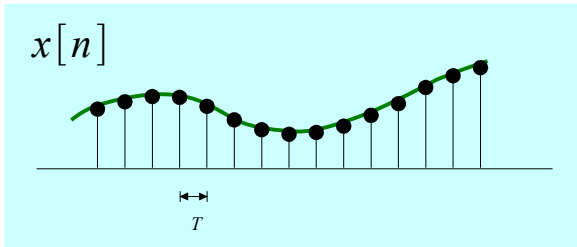
$$\hat{\omega}_k = 2\pi k/D$$

$$e^{-j2\pi kn/D} \cdot z = e^{-j2\pi k/D} e^{j\hat{\omega}_k n} = e^{j(\hat{\omega}_k - 2\pi k/D)n}$$

$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

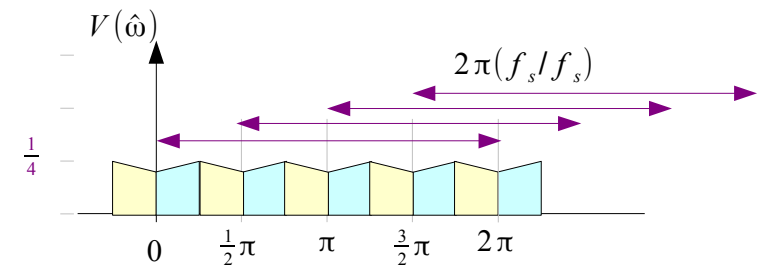
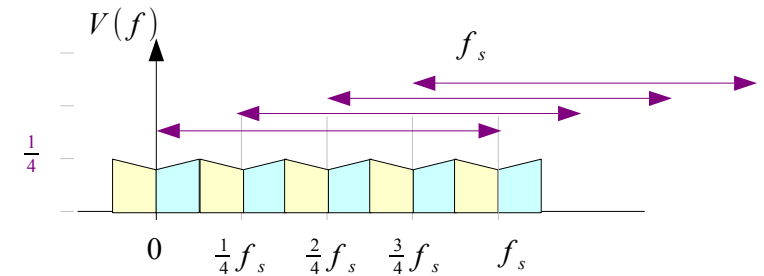
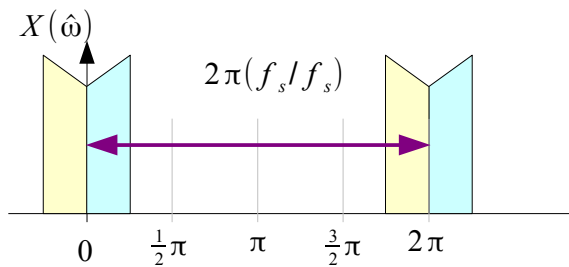
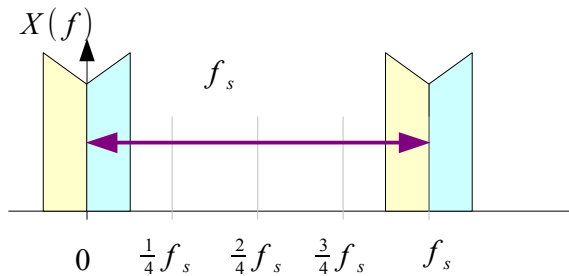
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

Z-Transform & Suppressing D-1 Samples (2)

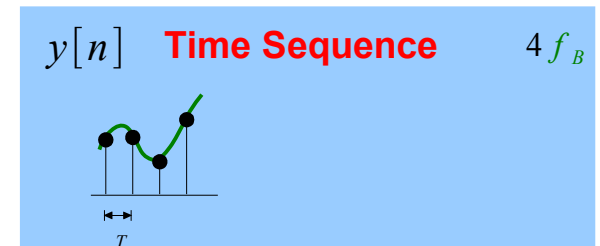
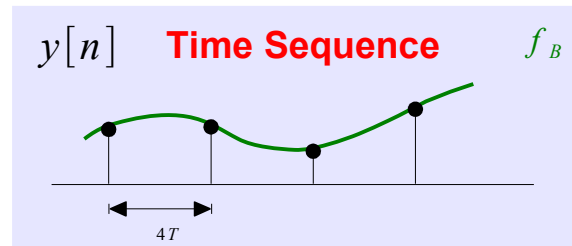
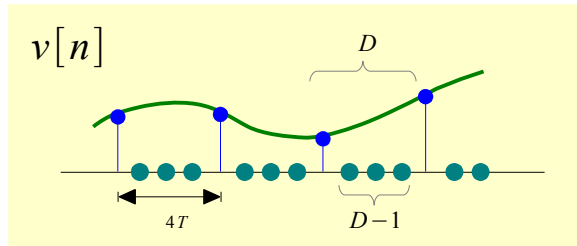


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$



Z-Transform & Removing D-1 Zeros (1)



$$\begin{aligned}
 Y(z) &= \cdots + \underline{y[0]} + \underline{y[1]}z^{-1} + \underline{y[2]}z^{-2} + \cdots + \underline{y[n]}z^{-n} + \cdots \\
 &= \cdots + \underline{v[0]} + \underline{v[D]}z^{-1} + \underline{v[2D]}z^{-2} + \cdots + \underline{v[nD]}z^{-n} + \cdots
 \end{aligned}$$

$$Y(z) = V(z^{1/D})$$

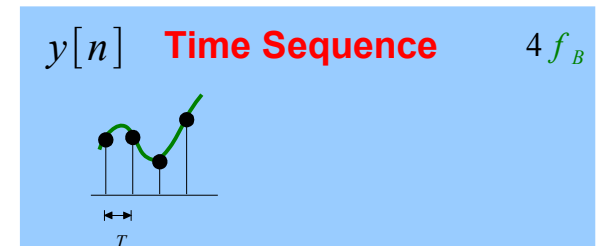
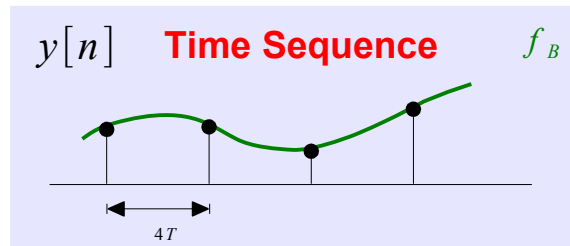
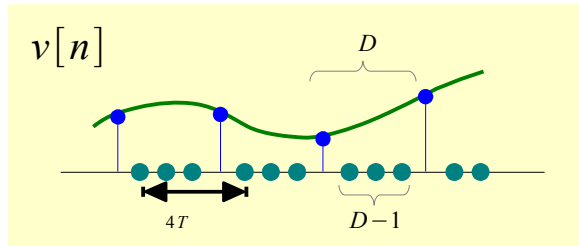
$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$

Z-Transform & Removing D-1 Zeros (2)

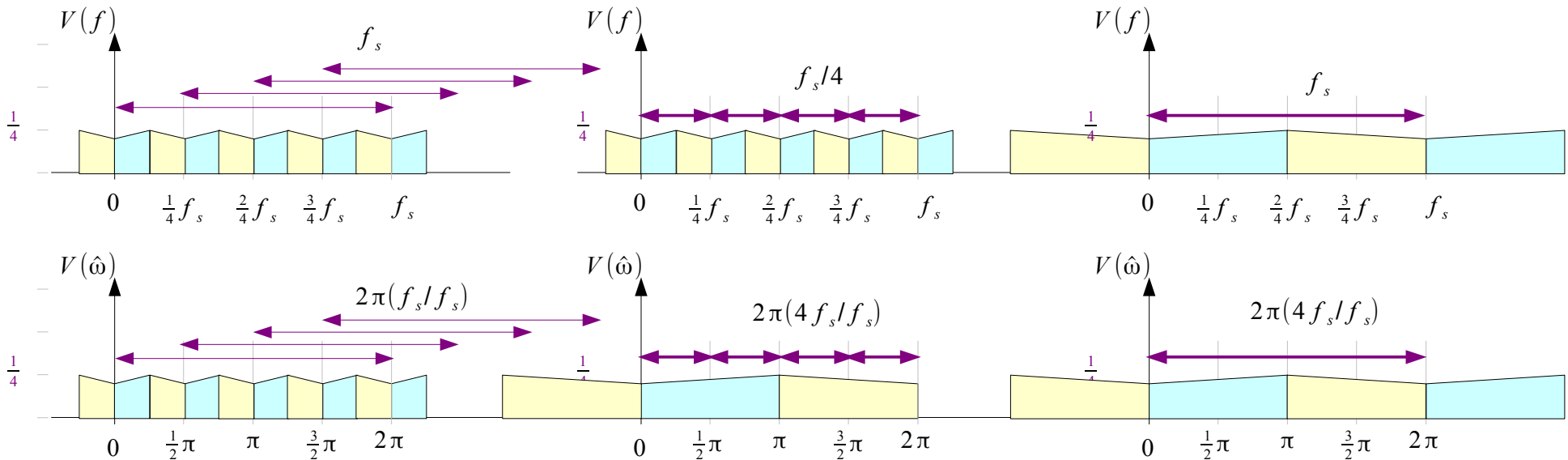


$$V(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z)$$

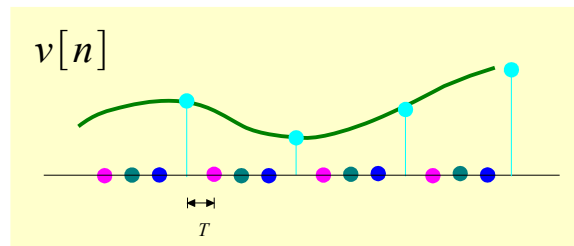
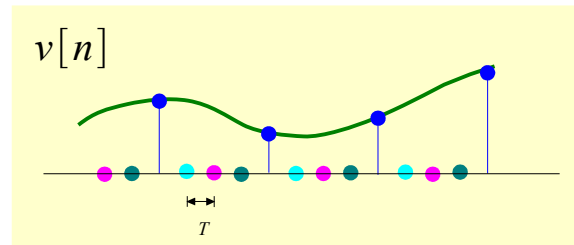
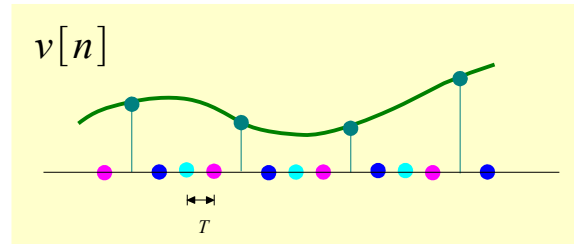
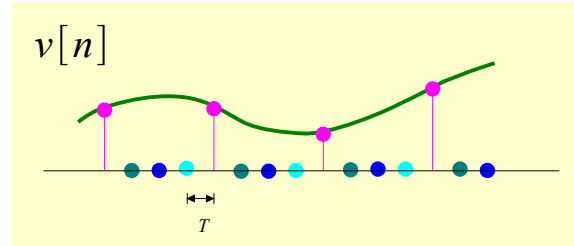
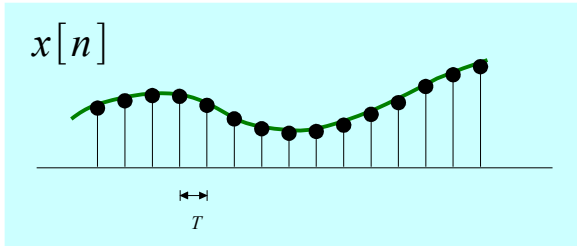
$$Y(z) = \frac{1}{D} \sum_{k=0}^{D-1} X(e^{-j2\pi k/D} \cdot z^{1/D})$$

$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega} - 2\pi k/D)$$

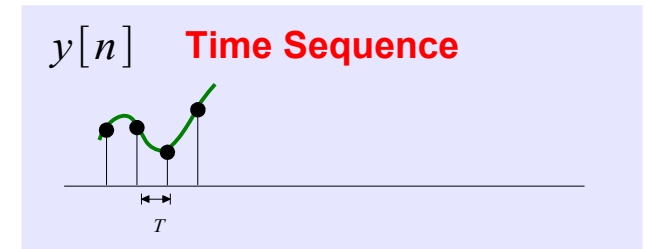
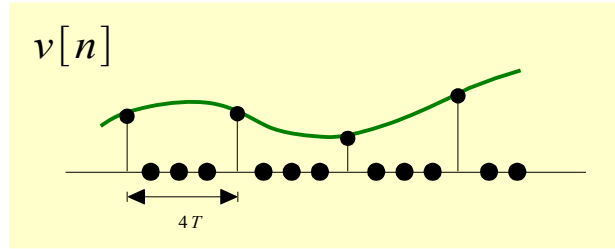
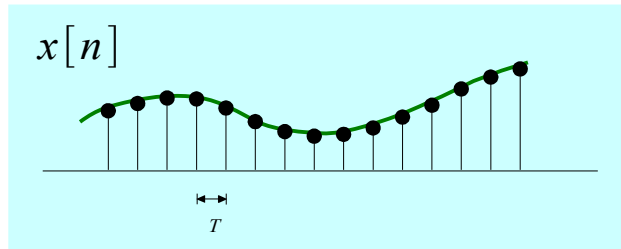
$$V(\hat{\omega}) = \frac{1}{D} \sum_{k=0}^{D-1} X(\hat{\omega}/D - 2\pi k/D)$$



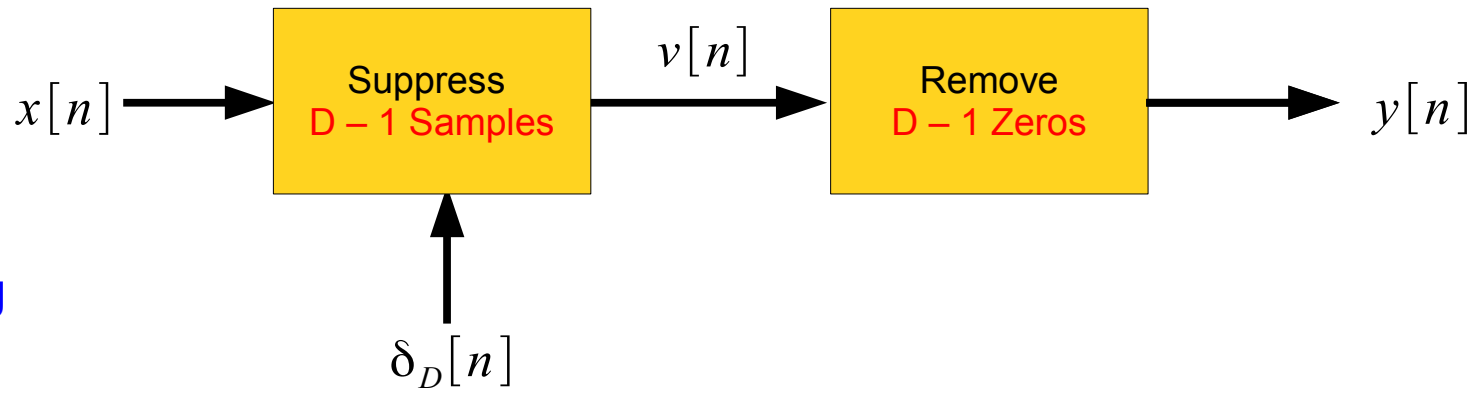
Example When $D=2$ (3)



Down Sampling



T Sampling Period



Ideal Sampling

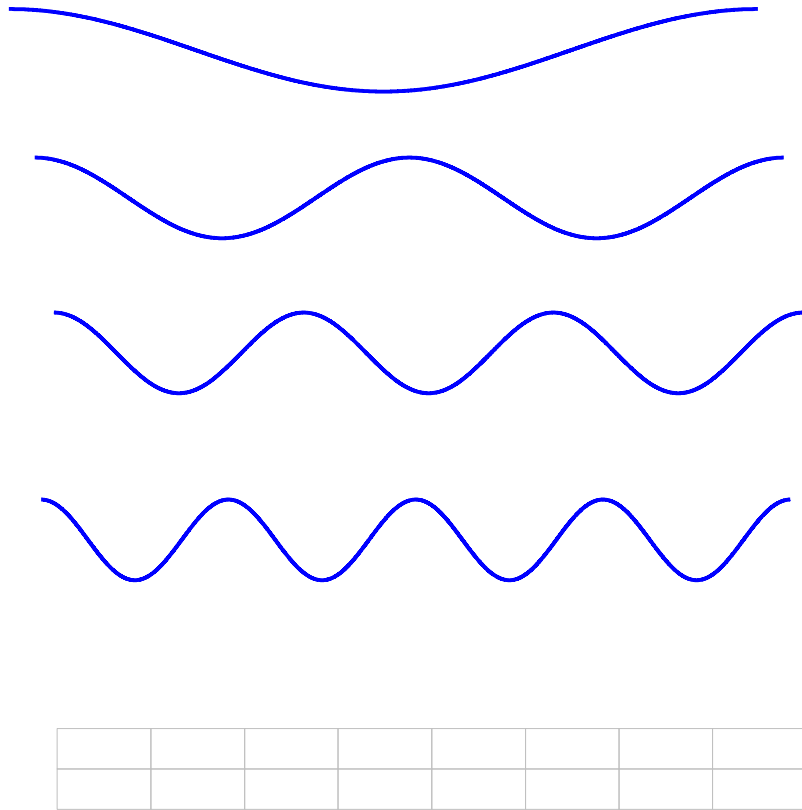
Amplitude Axis Scaling
 $1 / D$

Time Axis Scaling
 $1 / D$

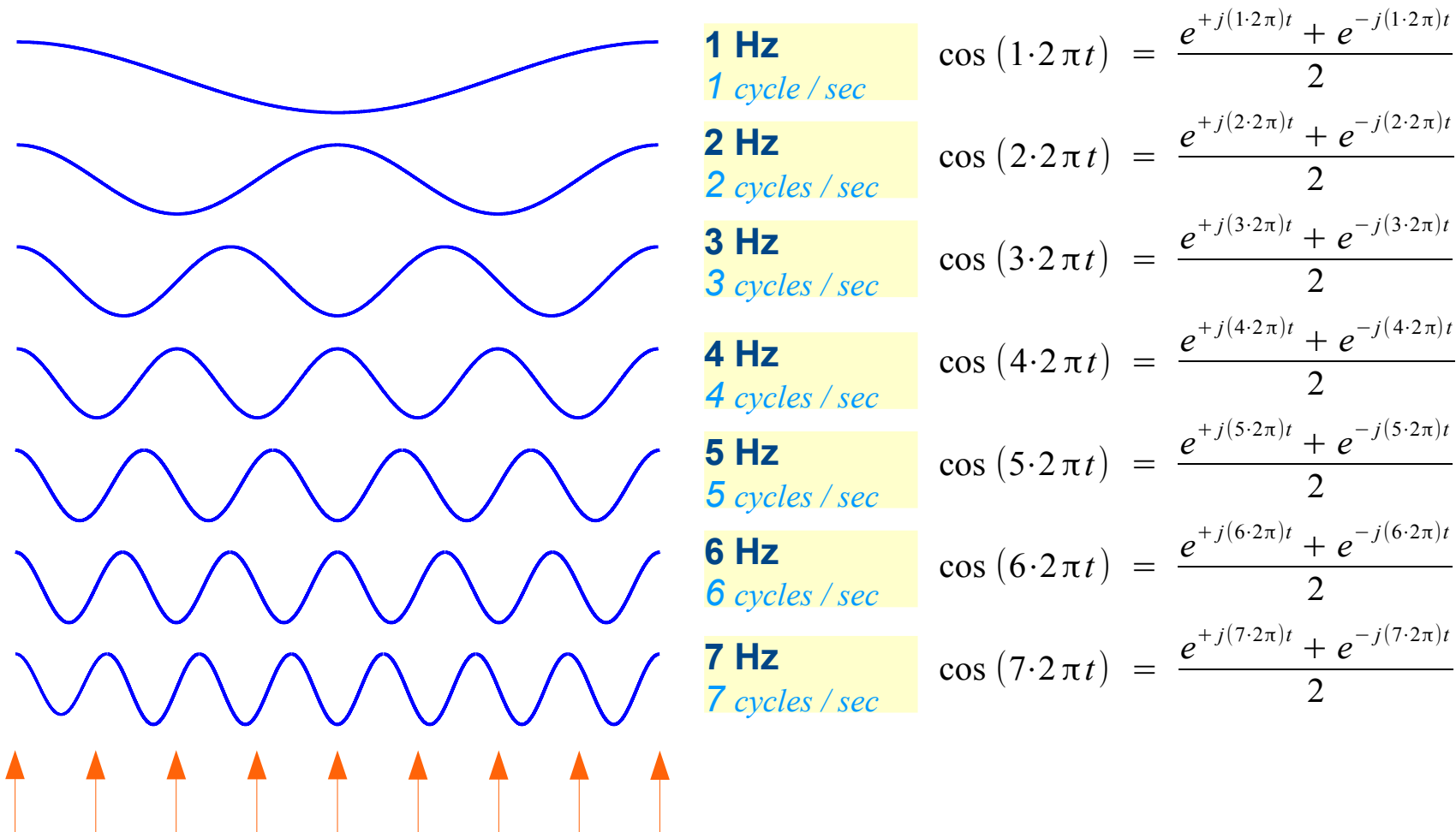
Spectrum Replication
 f_s / D

Stretching Spectrum
 ω / D

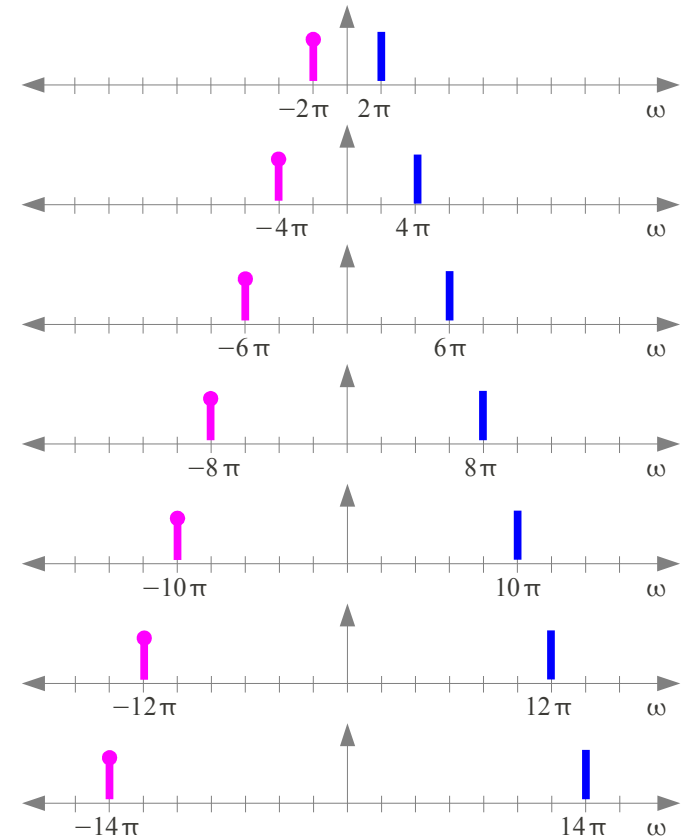
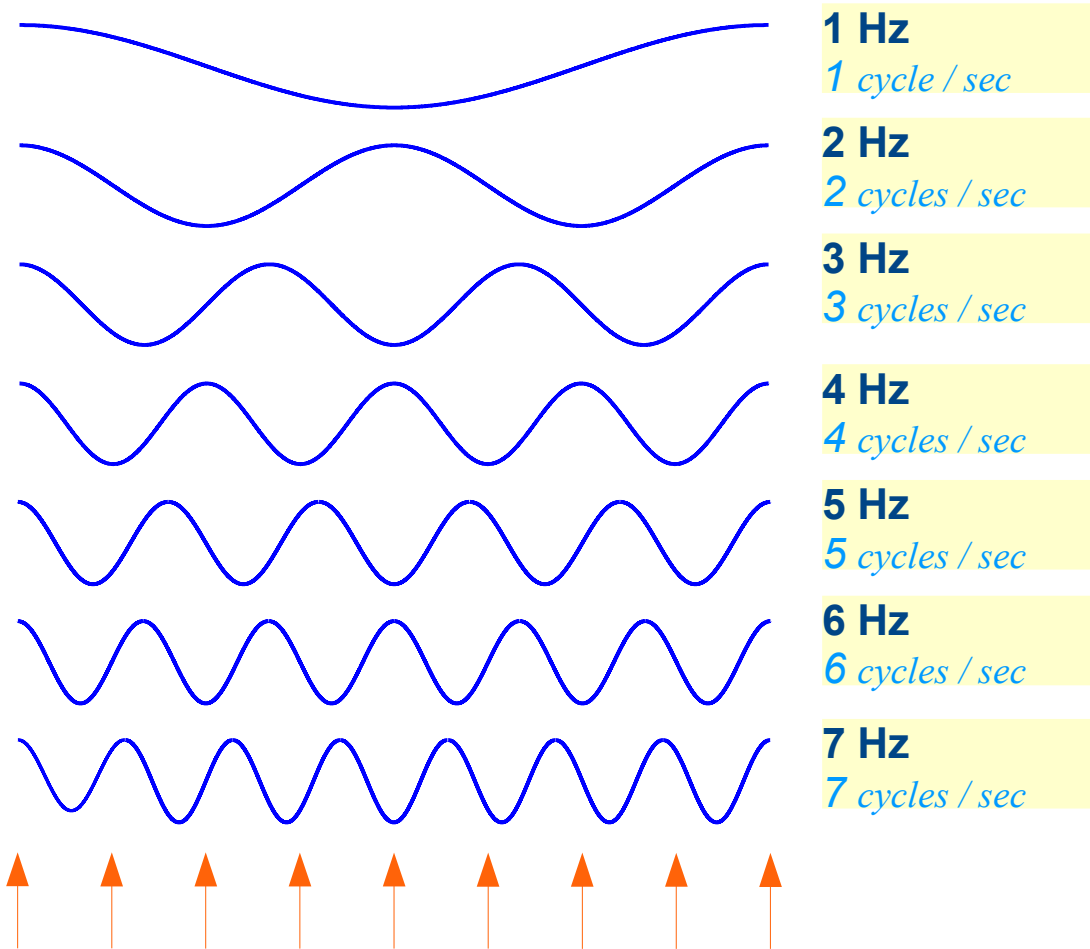
Measuring Rotation Rate



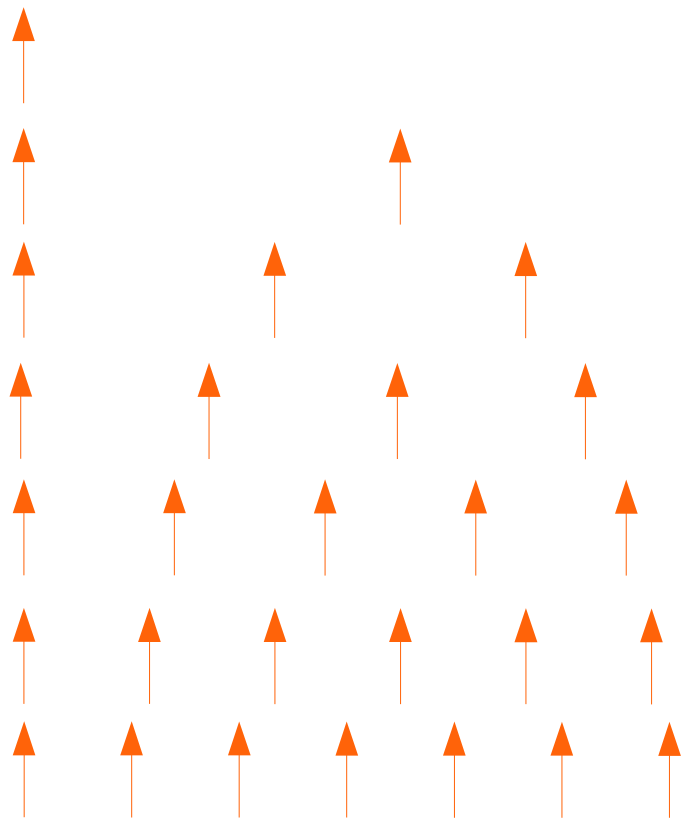
Signals with Harmonic Frequencies (1)



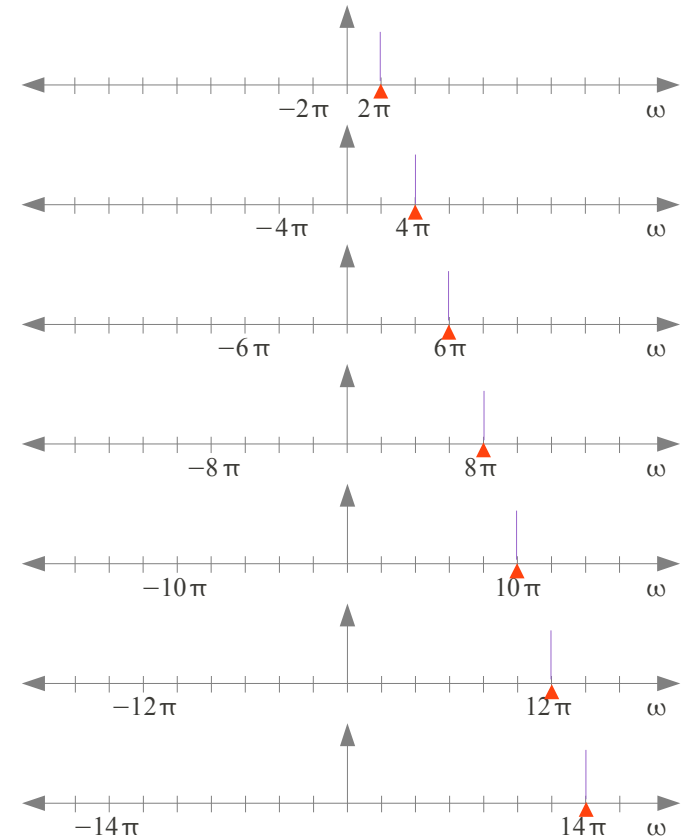
Signals with Harmonic Frequencies (2)



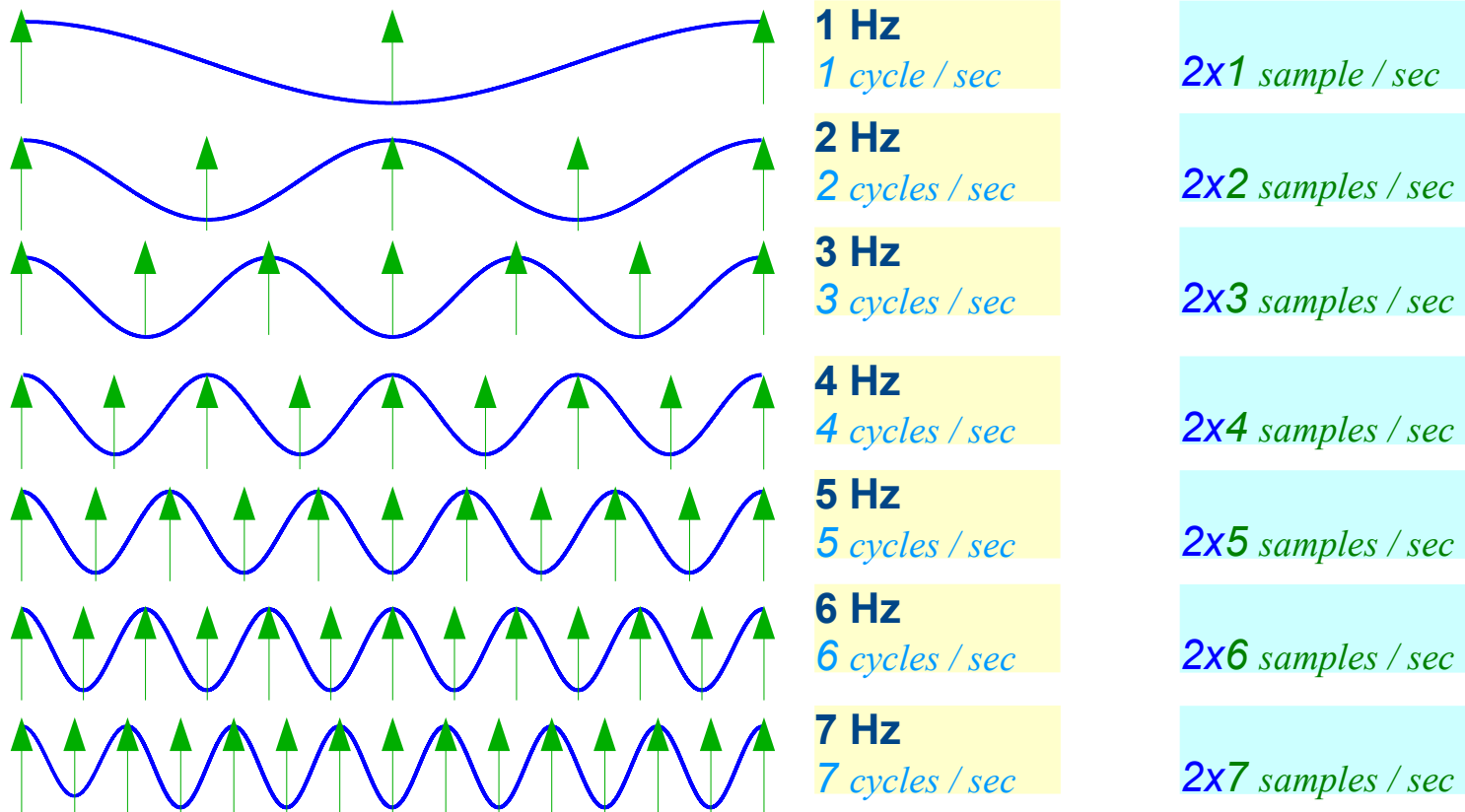
Sampling Frequency



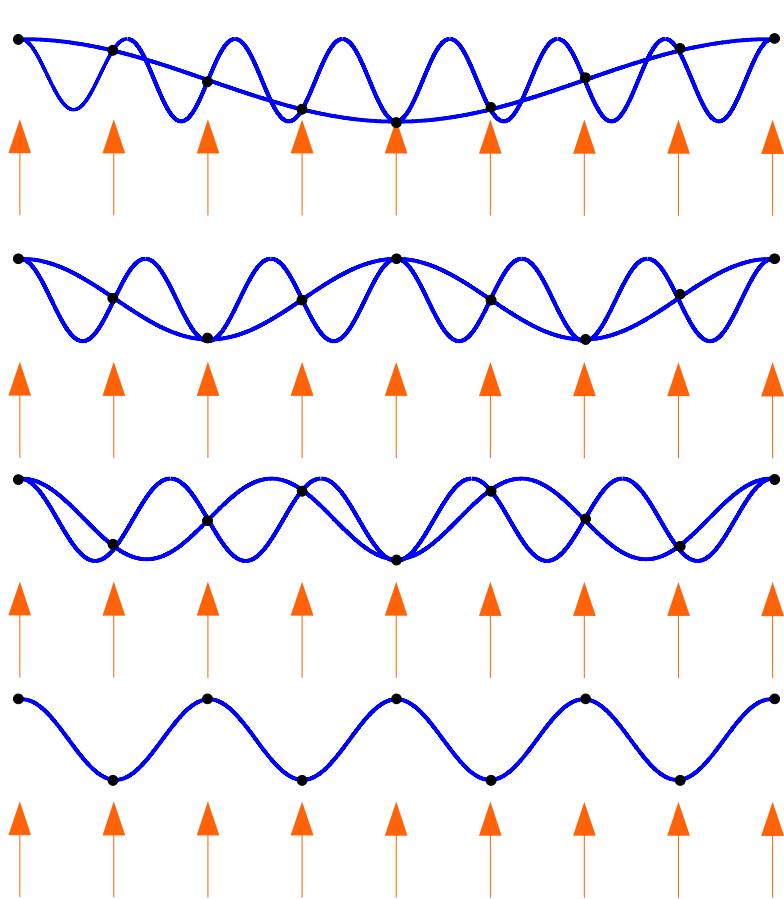
- 1 Hz
1 sample / sec
- 2 Hz
2 samples / sec
- 3 Hz
3 samples / sec
- 4 Hz
4 samples / sec
- 5 Hz
5 samples / sec
- 6 Hz
6 samples / sec
- 7 Hz
7 samples / sec



Nyquist Frequency



Aliasing



1 Hz
7 Hz

2x4 samples / sec

2 Hz
6 Hz

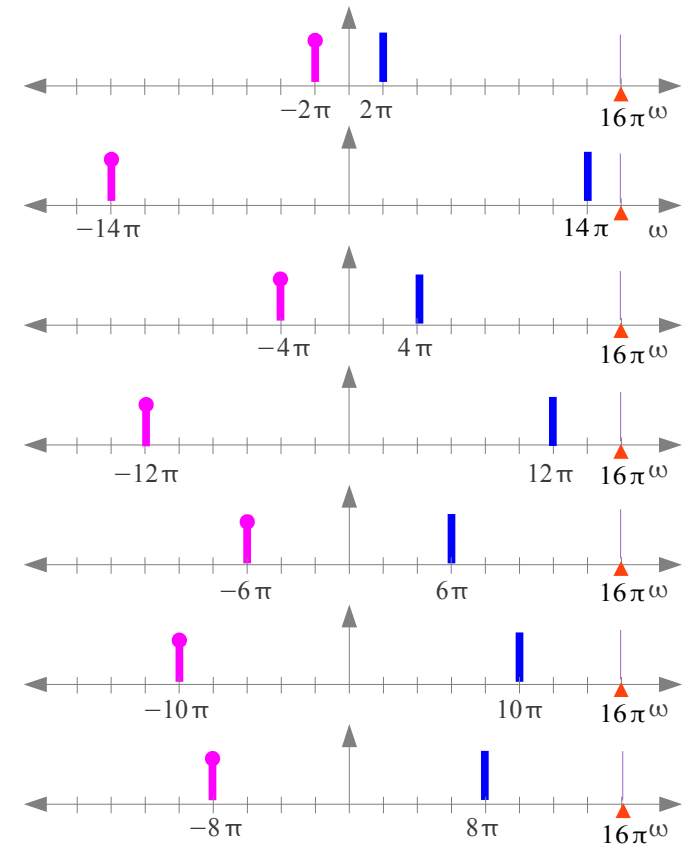
2x4 samples / sec

3 Hz
5 Hz

2x4 samples / sec

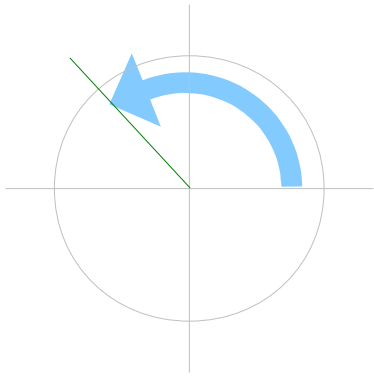
4 Hz

2x4 samples / sec

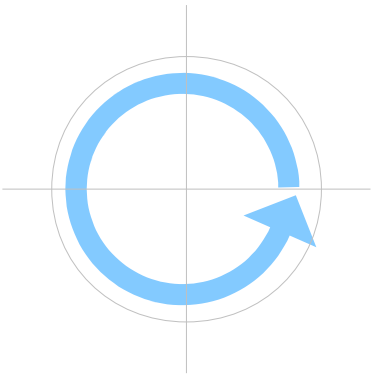


Sampling

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$



$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$

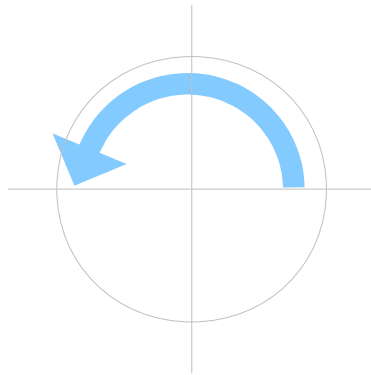


$$\omega_1 = 2\pi f_1$$

$$\omega_1 = \frac{\omega_s}{2} \text{ (rad/sec)}$$

$$f_1 = \frac{f_s}{2} \text{ (rad/sec)}$$

$$\pi \text{ (rad)} / T_s \text{ (sec)}$$

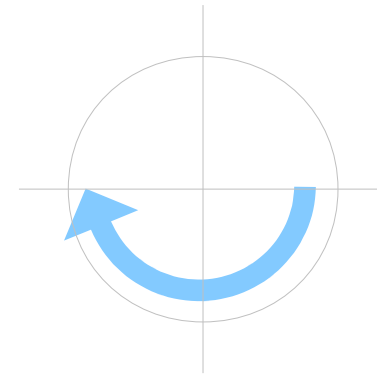


$$\omega_2 = 2\pi f_2$$

$$\omega_2 = -\frac{\omega_s}{2} \text{ (rad/sec)}$$

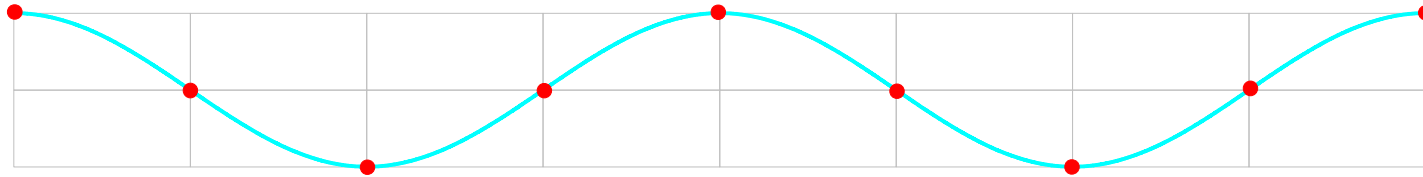
$$f_2 = -\frac{f_s}{2} \text{ (rad/sec)}$$

$$-\pi \text{ (rad)} / T_s \text{ (sec)}$$

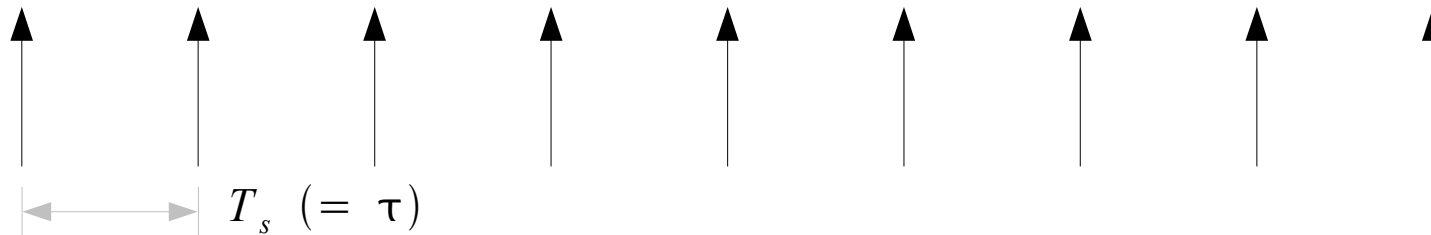


Sampling

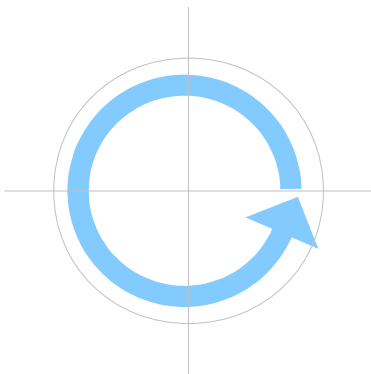
$$\omega_1 = 2\pi f_1 \text{ (rad/sec)}$$



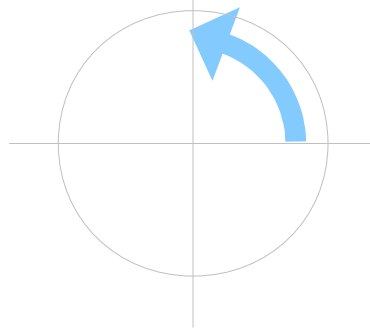
$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$



$$2\pi \text{ (rad)} / T_s \text{ (sec)}$$



$$\frac{\pi}{2} \text{ (rad)} / T_s \text{ (sec)}$$



For the period of T_s
Angular displacement $\frac{\pi}{2}$ (rad)

$$\begin{aligned} \hat{\omega} &= \omega \cdot T_s \text{ (rad)} \\ &= 2\pi f_1 \cdot T_s \text{ (rad)} \\ &= 2\pi \frac{f_s}{4} \cdot T_s \text{ (rad)} \\ &= \frac{\pi}{2} \text{ (rad)} \end{aligned}$$

Angular Frequencies in Sampling

continuous-time signals

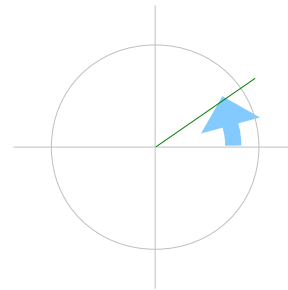
Signal Frequency

$$f_0 = \frac{1}{T_0}$$

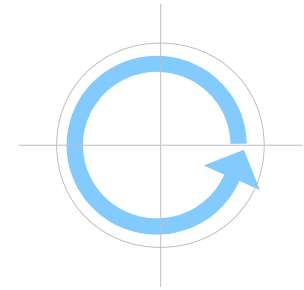
Signal Angular Frequency

$$\omega_0 = 2\pi f_0 \text{ (rad/sec)}$$

For 1 second
 $2\pi f_0 \text{ (rad/sec)}$



For 1 revolution
 $2\pi \text{ (rad)}$
 $T_0 \text{ (sec)}$



sampling sequence

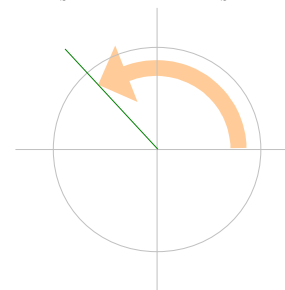
Sampling Frequency

$$f_s = \frac{1}{T_s}$$

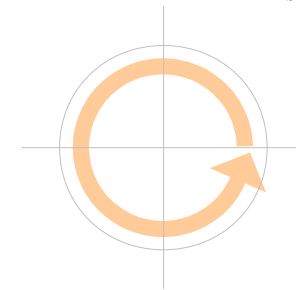
Sampling Angular Frequency

$$\omega_s = 2\pi f_s \text{ (rad/sec)}$$

For 1 second
 $2\pi f_s \text{ (rad/sec)}$



For 1 revolution
 $2\pi \text{ (rad)}$
 $T_s \text{ (sec)}$



References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] A "graphical interpretation" of the DFT and FFT, by Steve Mann
- [4] R. Cristi, "Modern Digital Signal Processing"