Divergence and Curl (3A)

- Divergence
- Curl
- Green's Theorem

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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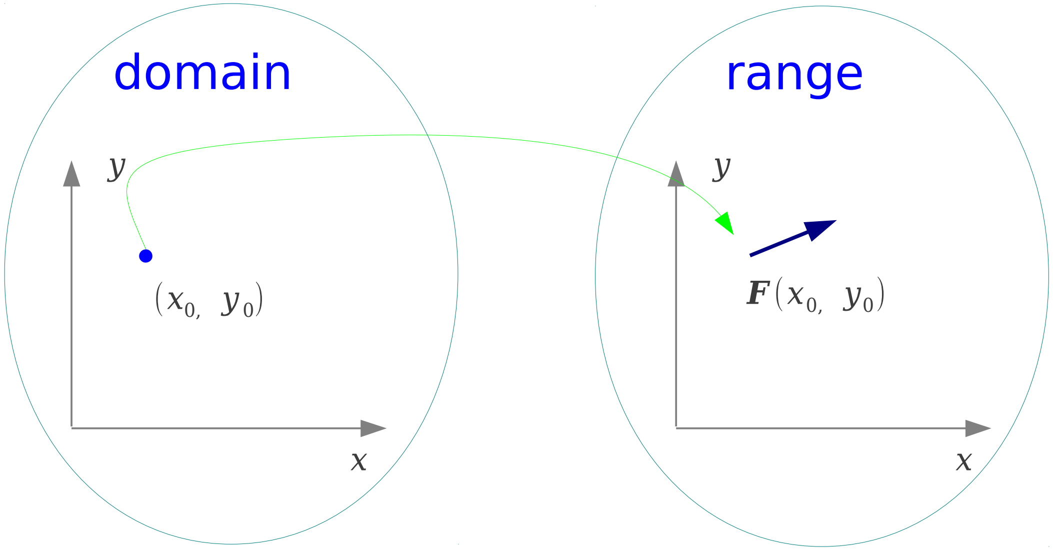
2-D Vector Field

a given point in a 2-d space ➡ A vector

$(x_{0,}\ \ y_0)$ $\qquad\qquad$ $\langle\ M(x_{0,}y_0),\ \ N(x_{0,}y_0)\ \rangle$



2 functions

$(x_{0,}\ \ y_0) \longrightarrow M(x_{0,}y_0)$

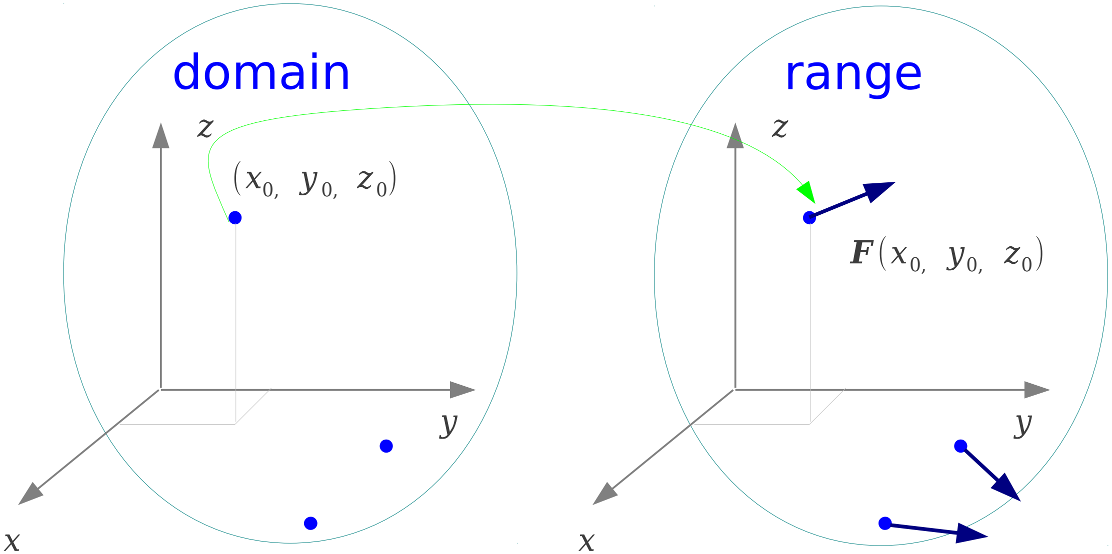
$(x_{0,}\ \ y_0) \longrightarrow N(x_{0,}y_0)$

$$(x_{0,}\ \ y_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_0)\ =\ M(x_{0,}y_0)\boldsymbol{i}\ +\ N(x_{0,}y_0)\boldsymbol{j}$$

3-D Vector Field

A given point in a 3-d space ➡ A vector

$(x_{0,}\ \ y_{0,}\ \ z_0)$ $\langle\, M(x_0,y_0,z_0),\ N(x_0,y_0,z_0),\ P(x_0,y_0,z_0)\,\rangle$



3 functions

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow M(x_{0,} y_{0,} Z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow N(x_{0,} y_{0,} z_0)$

$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow P(x_{0,} y_{0,} z_0)$

$$(x_{0,}\ \ y_{0,}\ \ z_0) \longrightarrow \boldsymbol{F}(x_{0,}\ \ y_{0,}\ \ z_0) = M(x_{0,}y_{0,}z_0)\boldsymbol{i} + N(x_{0,}y_{0,}z_0)\boldsymbol{j} + P(x_{0,}y_{0,}z_0)\boldsymbol{k}$$

2-Divergence (1 – 5)

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F**

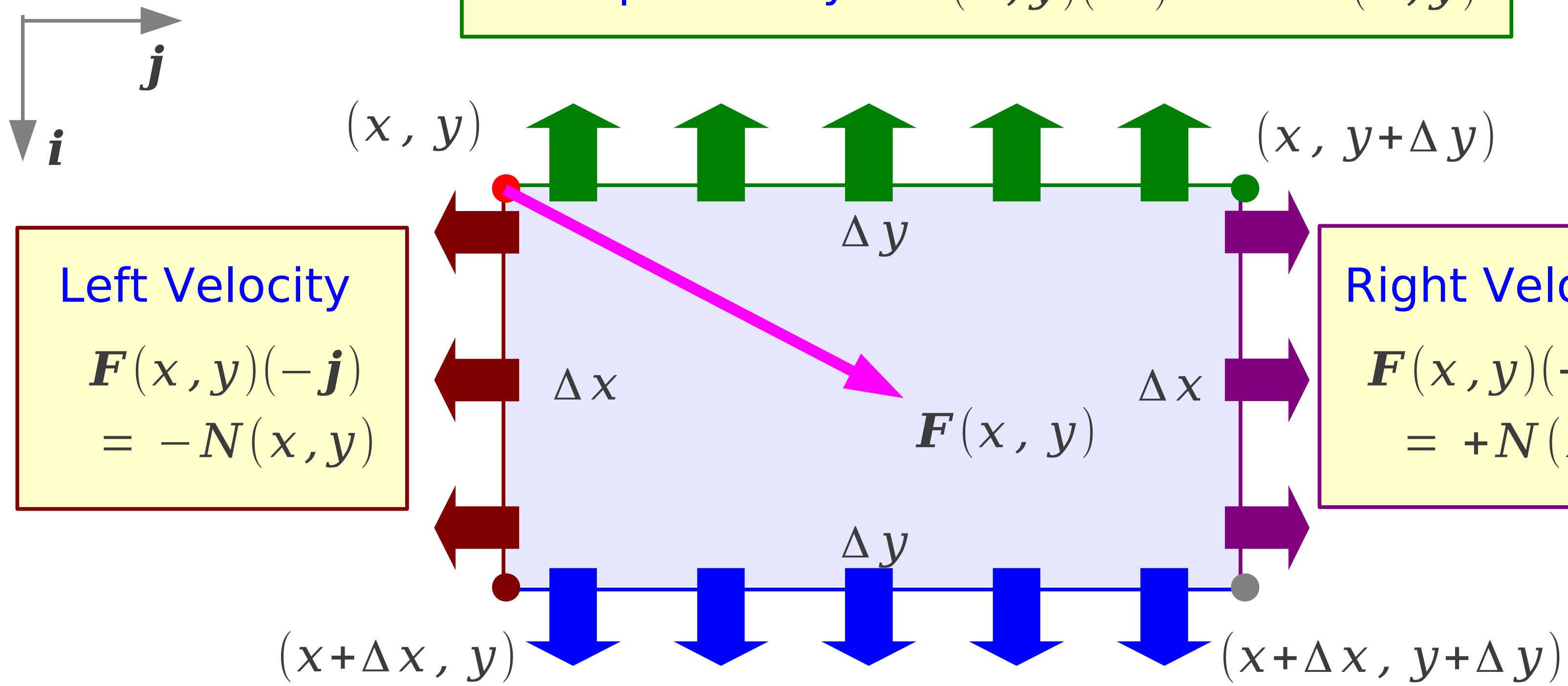
Flux Density

2-D Divergence (1)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Top Velocity $\boldsymbol{F}(x,y)(-\boldsymbol{i}) = -M(x,y)$



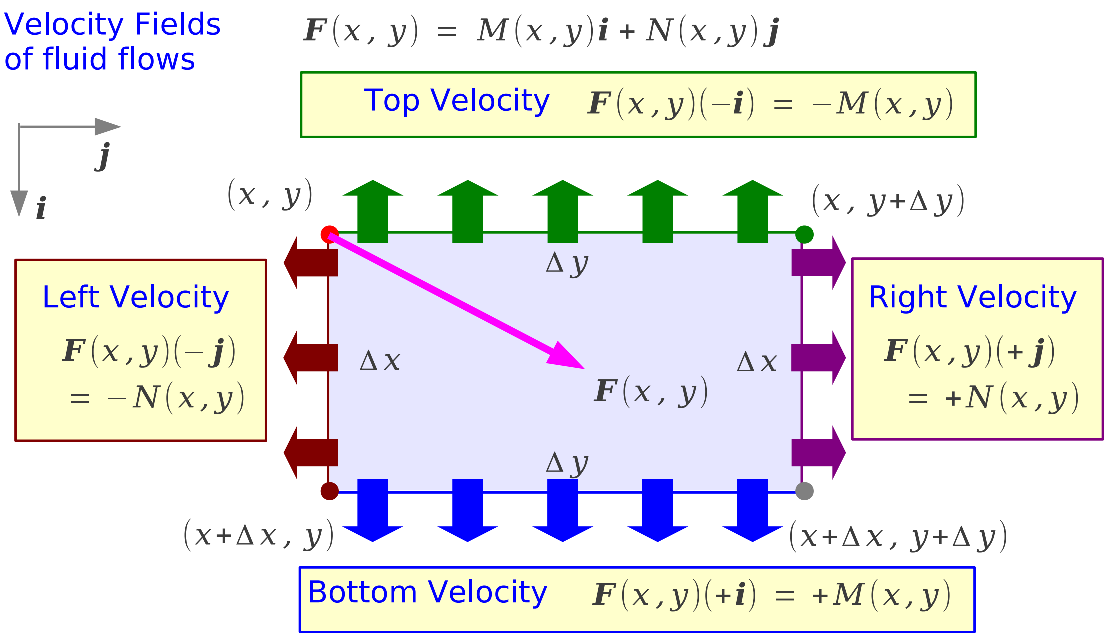
Left Velocity
$\boldsymbol{F}(x,y)(-\boldsymbol{j})$
$= -N(x,y)$

Right Velocity
$\boldsymbol{F}(x,y)(+\boldsymbol{j})$
$= +N(x,y)$

Bottom Velocity $\boldsymbol{F}(x,y)(+\boldsymbol{i}) = +M(x,y)$

Flow rate of outward bound fluid

2-D Divergence (2)



Flow rate of outward bound fluid

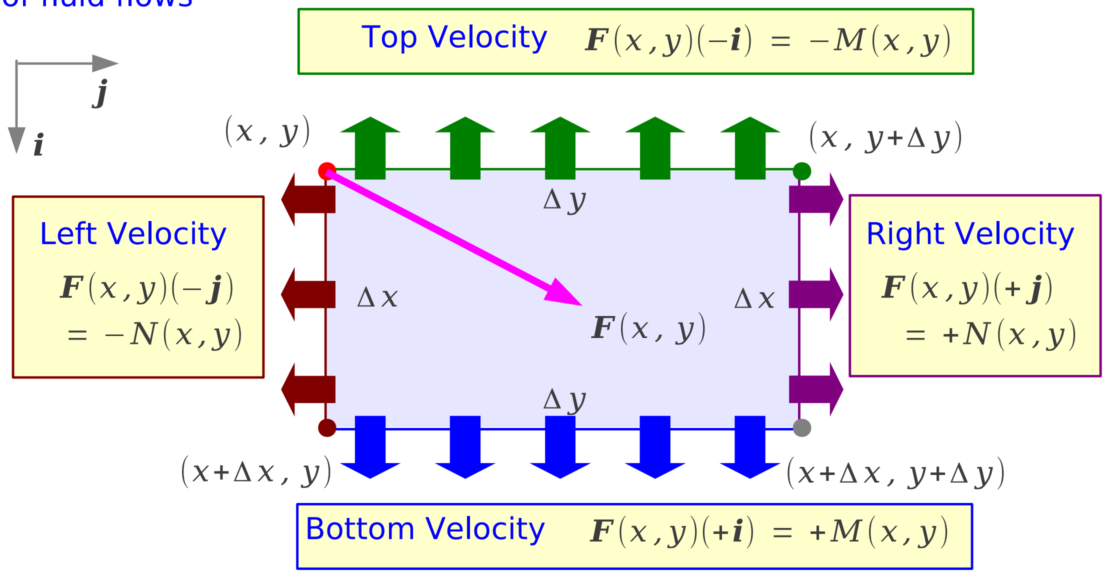
The rate at which fluid leave the rectangle

Across top	$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{i})\Delta y = -M(x, y)\Delta y$
Across bottom	$\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{i})\Delta y = M(x+\Delta x, y)\Delta y$
Across left	$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta x = -N(x, y)\Delta x$
Across right	$\boldsymbol{F}(x, y+\Delta y)\cdot(+\boldsymbol{j})\Delta x = N(x, y+\Delta y)\Delta x$

2-D Divergence (3)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{i})\Delta y = -M(x, y)\Delta y$$
$$\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{i})\Delta y = M(x+\Delta x, y)\Delta y$$
$$\boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta x = -N(x, y)\Delta x$$
$$\boldsymbol{F}(x, y+\Delta y)\cdot(+\boldsymbol{j})\Delta x = N(x, y+\Delta y)\Delta x$$

Flow rate of outward bound fluid

The rate at which fluid leave the rectangle

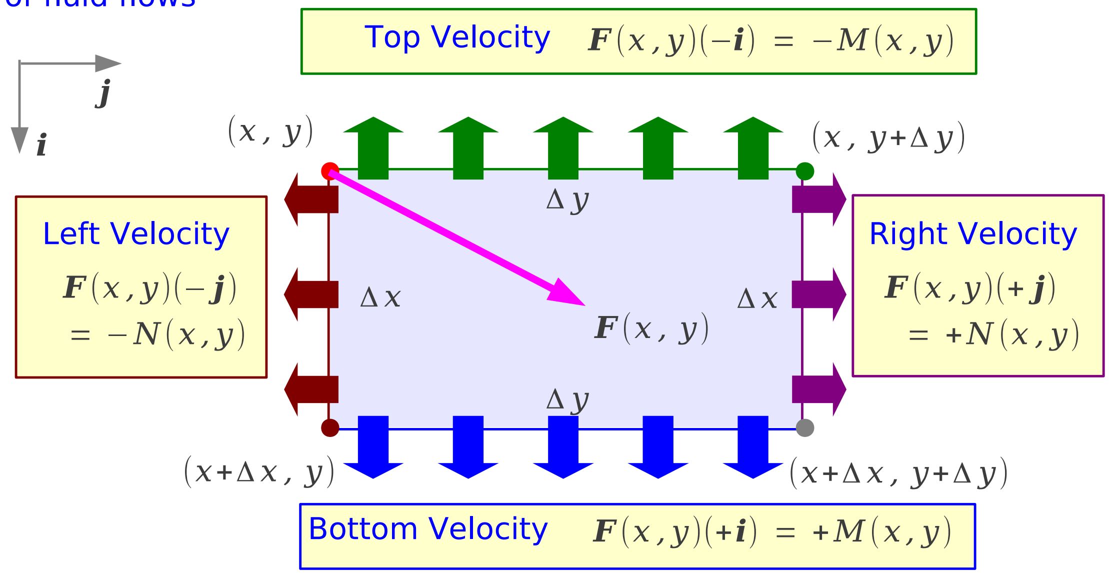
Across top + bottom $\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$

Across left + right $\{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$

2-D Divergence (4)

Velocity Fields of fluid flows

$\boldsymbol{F}(x,y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

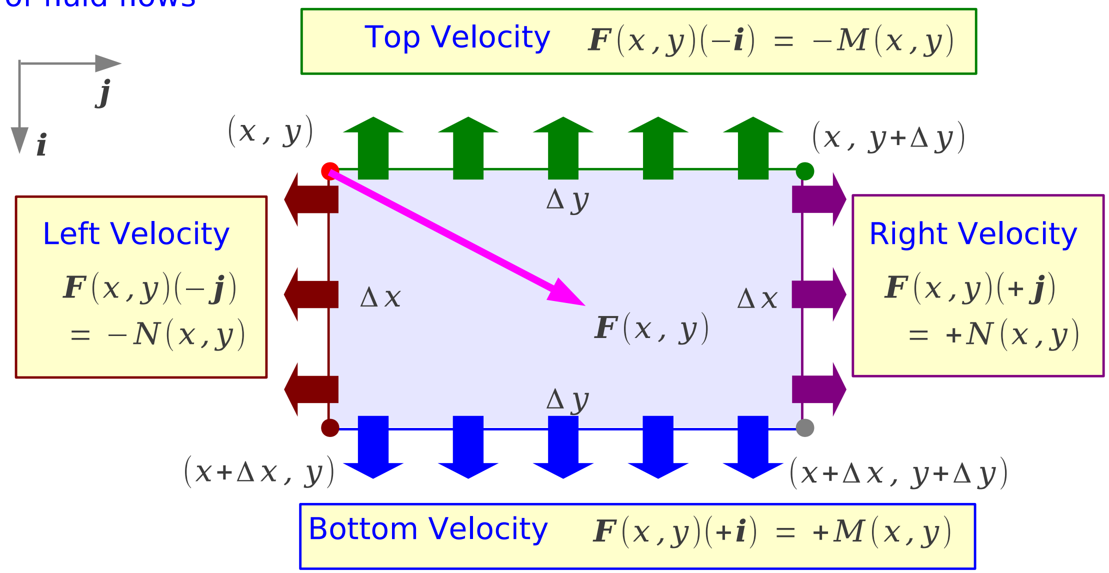
Divergence of **F**

Flux Density

2-D Divergence (5)

Velocity Fields
of fluid flows

$$\boldsymbol{F}(x,y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$

$$\{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

Flow rate of outward bound fluid

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \quad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$

Flux density

$$= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$$

Divergence of **F** Flux Density

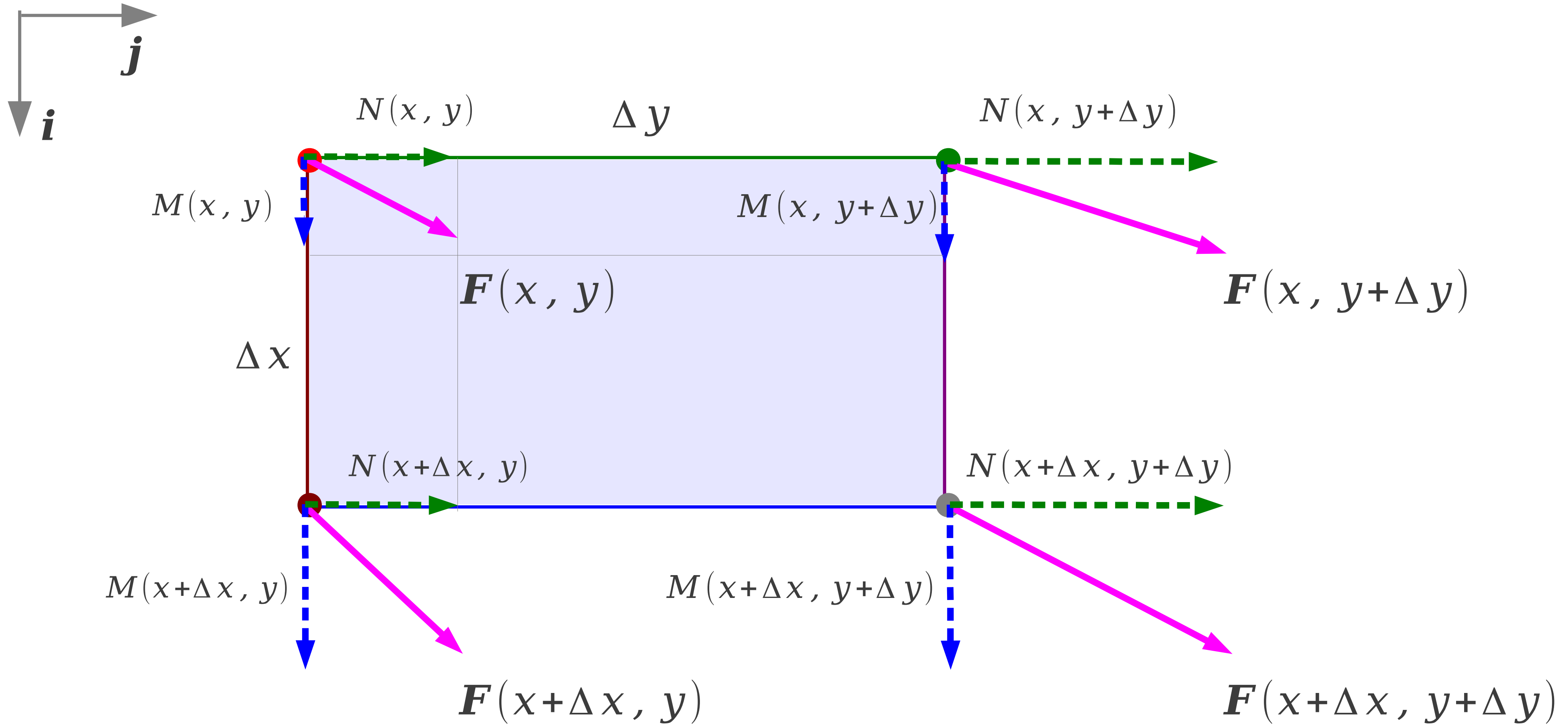
2-D Divergence and Del Operator

3-D Divergence and Del Operator

$\nabla \cdot \boldsymbol{F}$

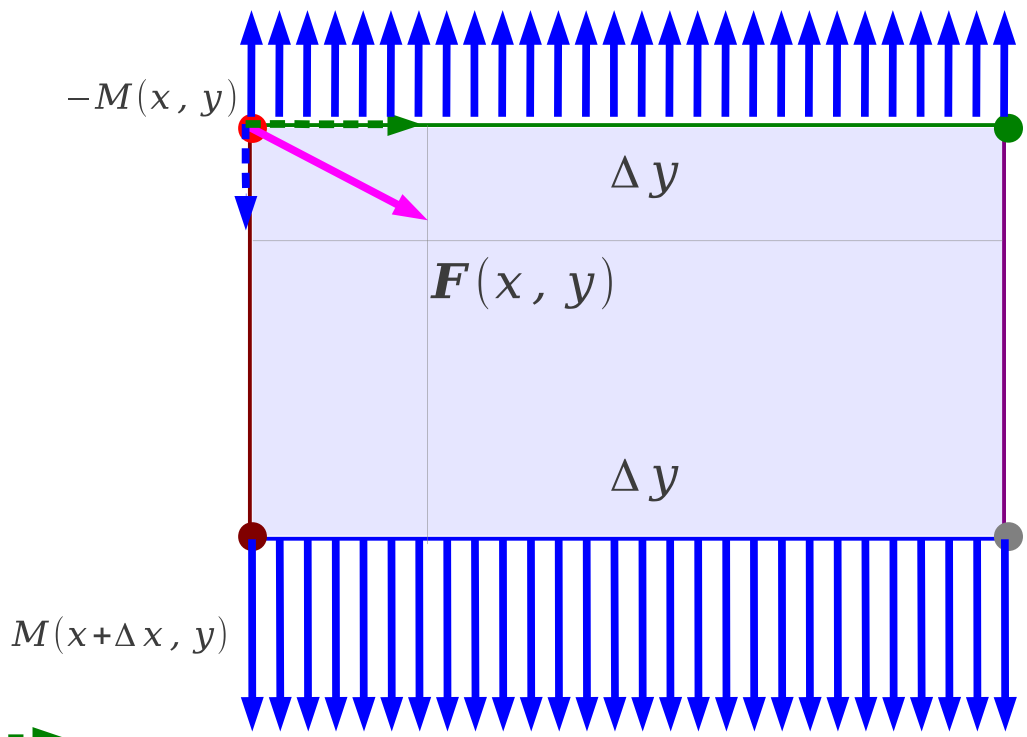
2-D Divergence (a)

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



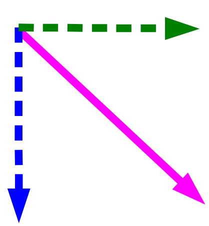
$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) \cdot (-\boldsymbol{i})\Delta y = -M(x, y)\Delta y$$



$$\frac{\{M(x+\Delta x, y) - M(x, y)\}}{\Delta x} \approx \frac{\partial M}{\partial x}$$

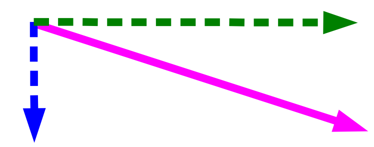
$$\{M(x+\Delta x, y) - M(x, y)\}\Delta y = \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y$$



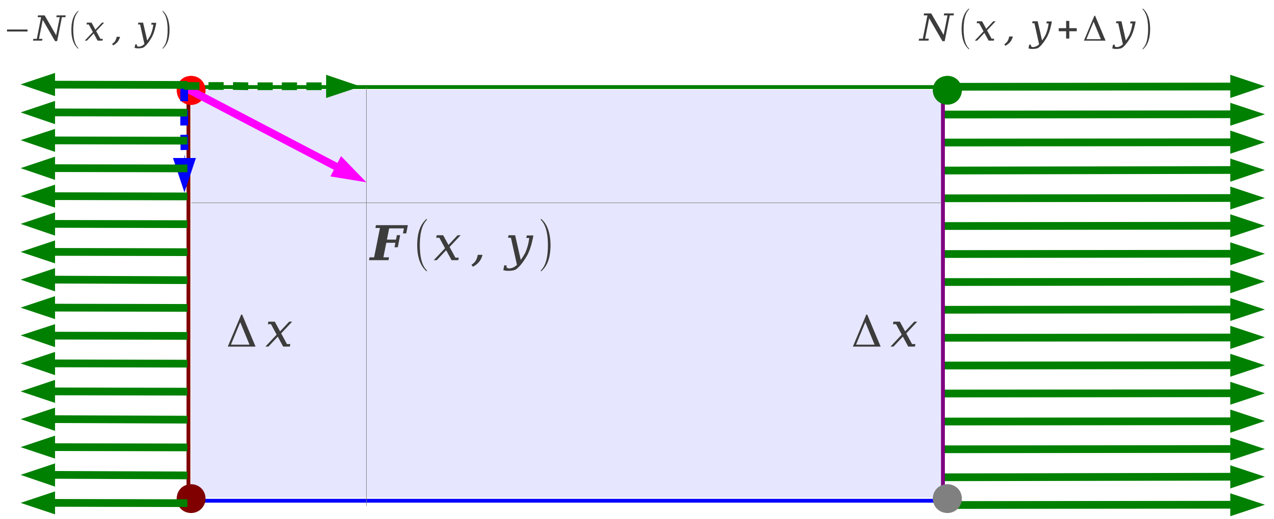
$$\boldsymbol{F}(x+\Delta x, y) \cdot (+\boldsymbol{i})\Delta y = M(x+\Delta x, y)\Delta y$$

2-D Divergence (left, right) (c)

$$\mathbf{F}(x, y) = M(x,y)\,\mathbf{i} + N(x,y)\,\mathbf{j}$$



$$\mathbf{F}(x, y)\cdot(-\mathbf{j})\Delta x = -N(x, y)\Delta x$$



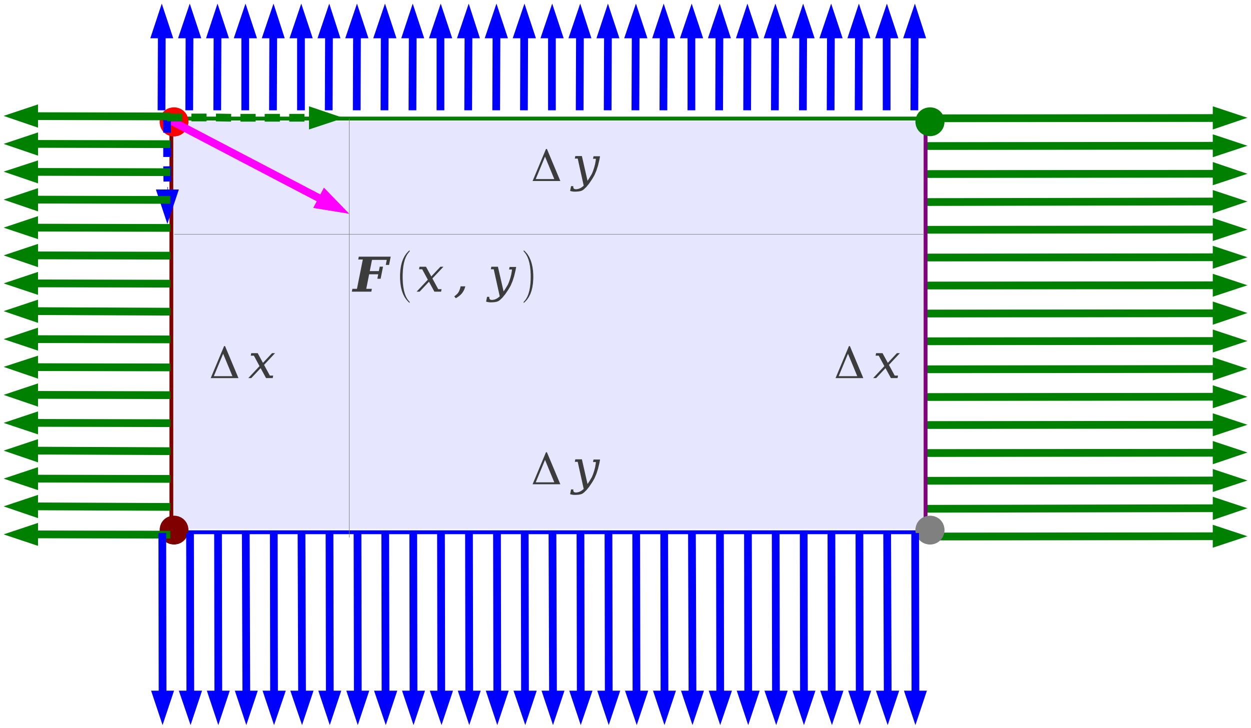
$$\mathbf{F}(x, y+\Delta y)\cdot(+\mathbf{j})\Delta x = N(x, y+\Delta y)\Delta x$$

$$\frac{\{N(x, y+\Delta y) - N(x, y)\}}{\Delta y} \approx \frac{\partial N}{\partial y} \qquad \{N(x, y+\Delta y) - N(x, y)\}\Delta x = \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x$$

2-D Divergence (d)

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$



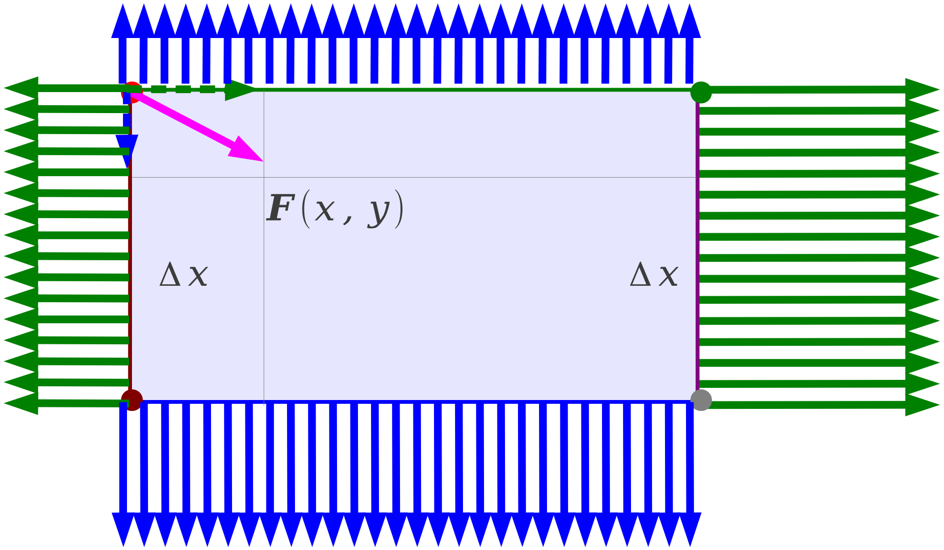
Flux density $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$ Divergence of **F** Flux Density

2-D Divergence and Del Operator

2-D Vector Field $\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Flux across rectangle boundary

$$\approx \left(\frac{\partial M}{\partial x}\Delta x\right)\Delta y + \left(\frac{\partial N}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\Delta x \Delta y$$



Flux density $= \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j})$$

$$= \nabla \cdot \boldsymbol{F}$$

Divergence of **F**

3-D Divergence and Del Operator

2-D Vector Field $\boldsymbol{F}(x,\ y) \ = \ M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Divergence of **F** $= \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\right)$

$= \left(\dfrac{\partial}{\partial x}\boldsymbol{i} + \dfrac{\partial}{\partial y}\boldsymbol{j}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j}) \ = \ \nabla \cdot \boldsymbol{F}$

3-D Vector Field $\boldsymbol{F}(x,\ y,\ z) \ = \ M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$

Divergence of **F** $= \left(\dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y} + \dfrac{\partial P}{\partial z}\right)$

$= \left(\dfrac{\partial}{\partial x}\boldsymbol{i} + \dfrac{\partial}{\partial y}\boldsymbol{j} + \dfrac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) \ = \ \nabla \cdot \boldsymbol{F}$

2-Curl (1 – 5)

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y \quad - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

Circulation density

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

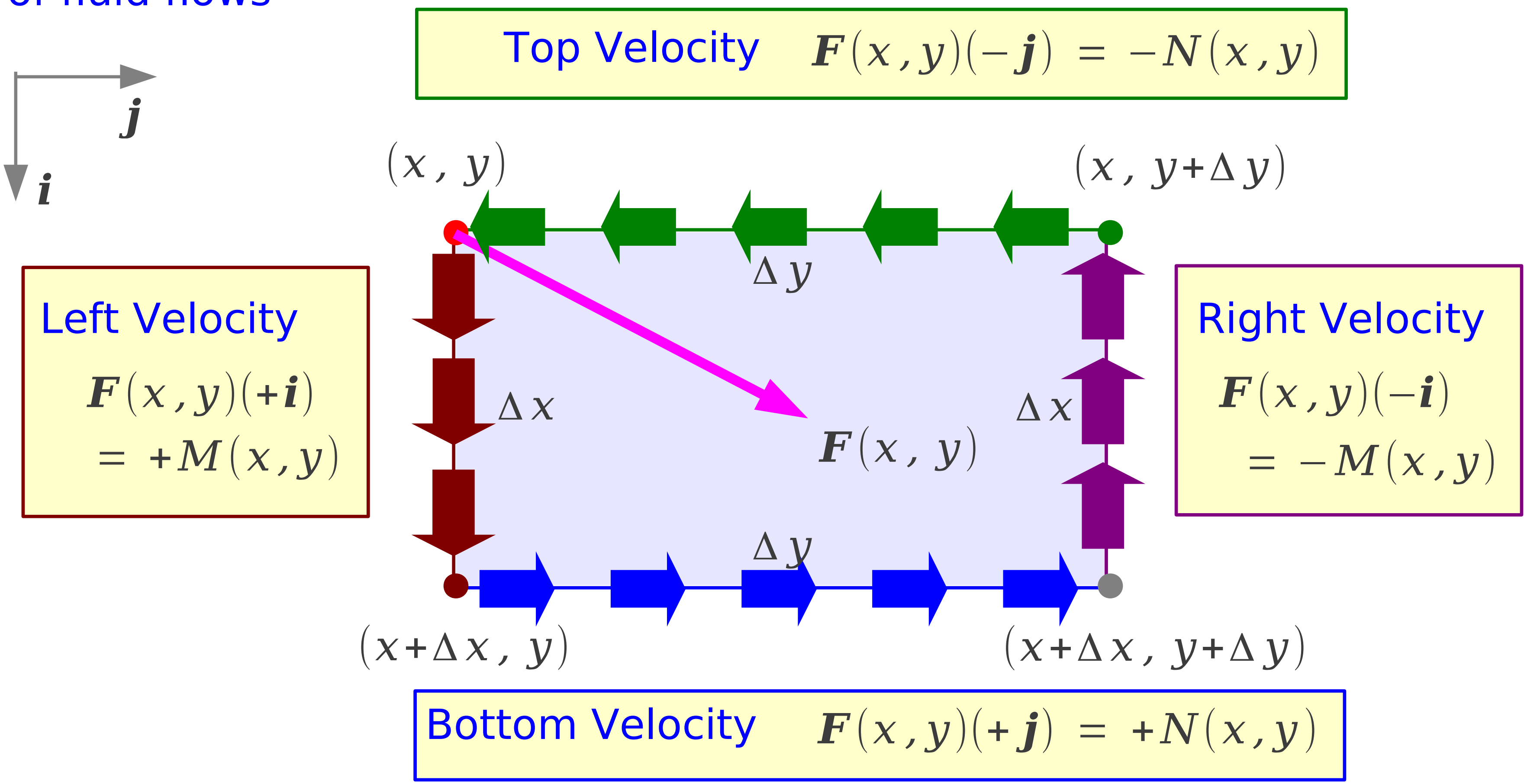
k-component
Curl of **F**

Circulation Density

2-D Curl (1)

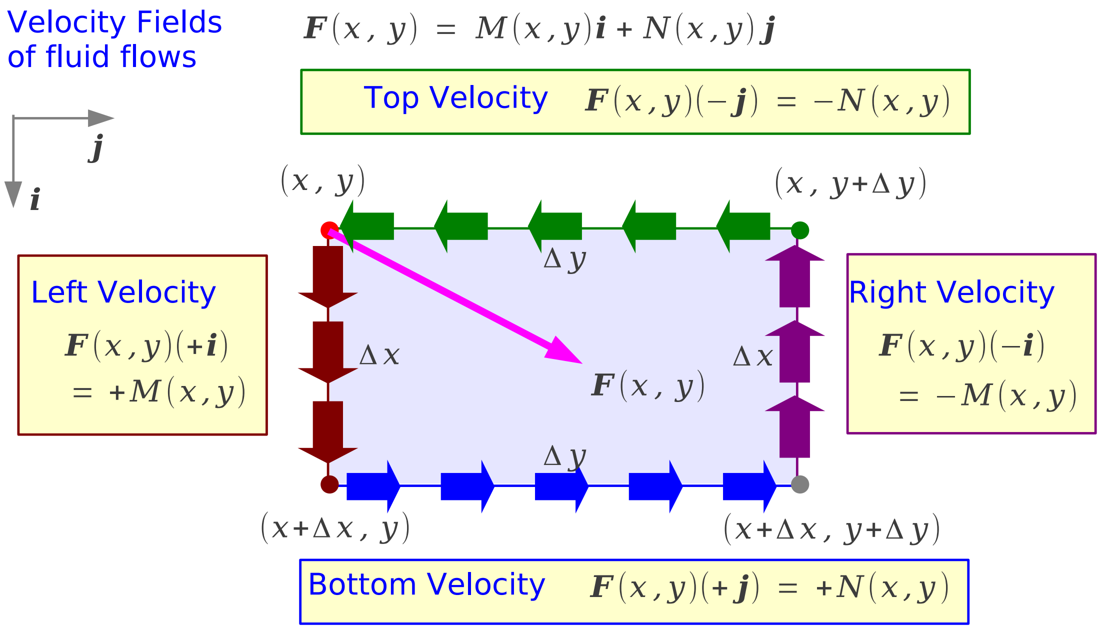
Velocity Fields
of fluid flows

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



Flow rate of counter clock wise circulating fluid

2-D Curl (2)



Flow rate of counter clock wise circulating fluid

The flow rate of counter clock wise circulation

Across top $\boldsymbol{F}(x, y)\cdot(-\boldsymbol{j})\Delta y = -N(x, y)\Delta y$

Across bottom $\boldsymbol{F}(x+\Delta x, y)\cdot(+\boldsymbol{j})\Delta y = N(x+\Delta x, y)\Delta y$

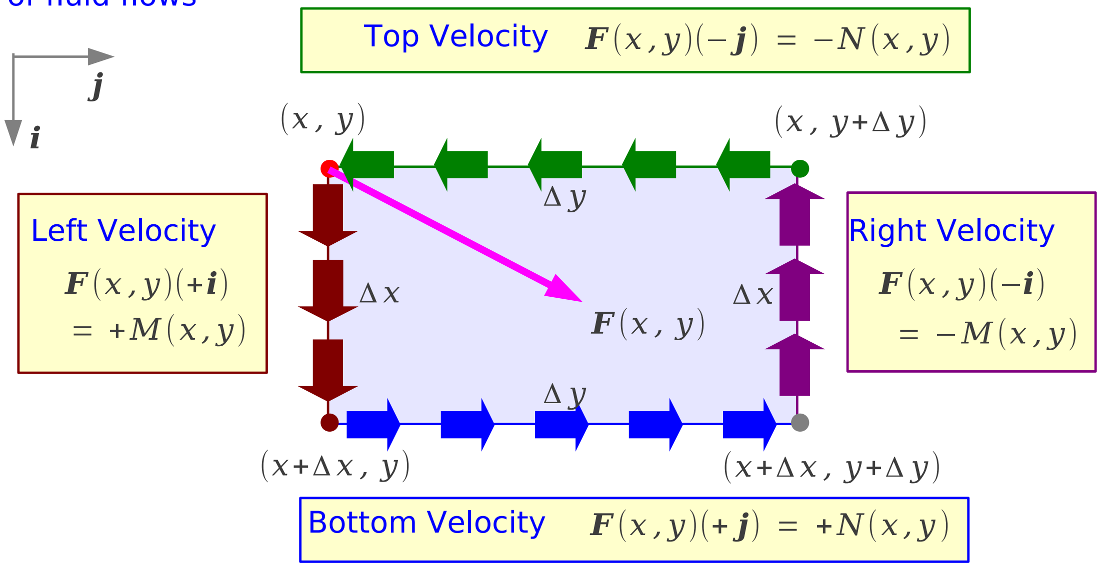
Across left $\boldsymbol{F}(x, y)\cdot(+\boldsymbol{i})\Delta x = M(x, y)\Delta x$

Across right $\boldsymbol{F}(x, y+\Delta y)\cdot(-\boldsymbol{i})\Delta x = -M(x, y+\Delta y)\Delta x$

2-D Curl (3)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x,y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\boldsymbol{F}(x,y)\cdot(-\boldsymbol{j})\Delta y = -N(x,y)\Delta y$$

$$\boldsymbol{F}(x+\Delta x,y)\cdot(+\boldsymbol{j})\Delta y = N(x+\Delta x,y)\Delta y$$

$$\boldsymbol{F}(x,y)\cdot(+\boldsymbol{i})\Delta x = M(x,y)\Delta x$$

$$\boldsymbol{F}(x,y+\Delta y)\cdot(-\boldsymbol{i})\Delta x = -M(x,y+\Delta y)\Delta x$$

Flow rate of counter clock wise circulating fluid

The flow rate of counter clock wise circulation

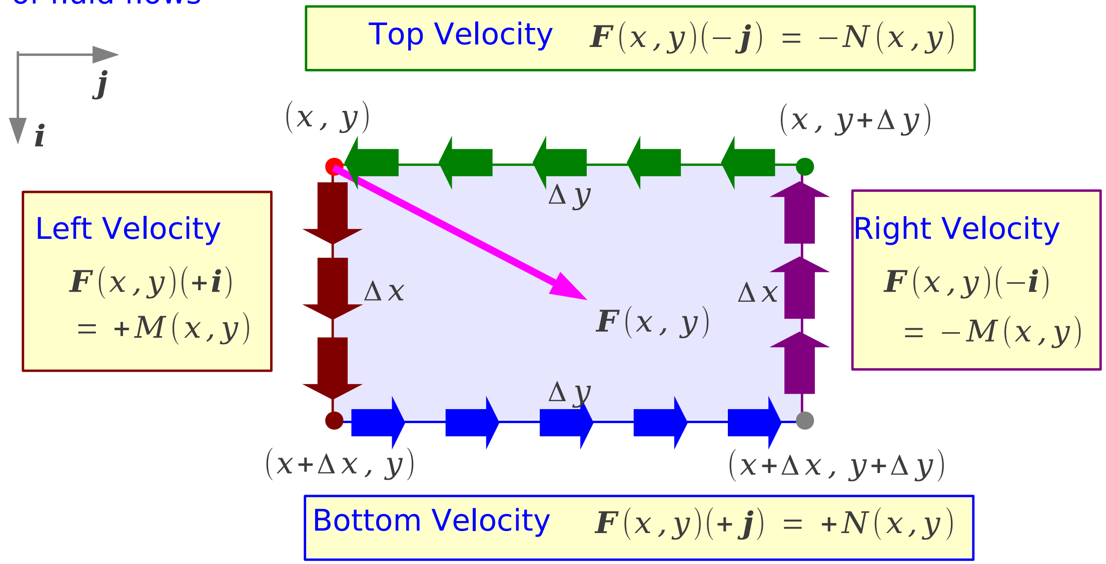
Across top + bottom $\quad \{N(x+\Delta x,y) - N(x,y)\}\Delta y \quad = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$

Across left + right $\quad -\{M(x,y+\Delta y) - M(x,y)\}\Delta x \quad = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$

2-D Curl (4)

Velocity Fields
of fluid flows

$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$



$$\{N(x+\Delta x, y) - N(x, y)\}\Delta y = \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y$$

$$-\{M(x, y+\Delta y) - M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

Flow rate of counter clock wise circulating fluid

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$

Circulation density

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

k-component
Curl of **F**

Circulation Density

2-D Curl (a – d)

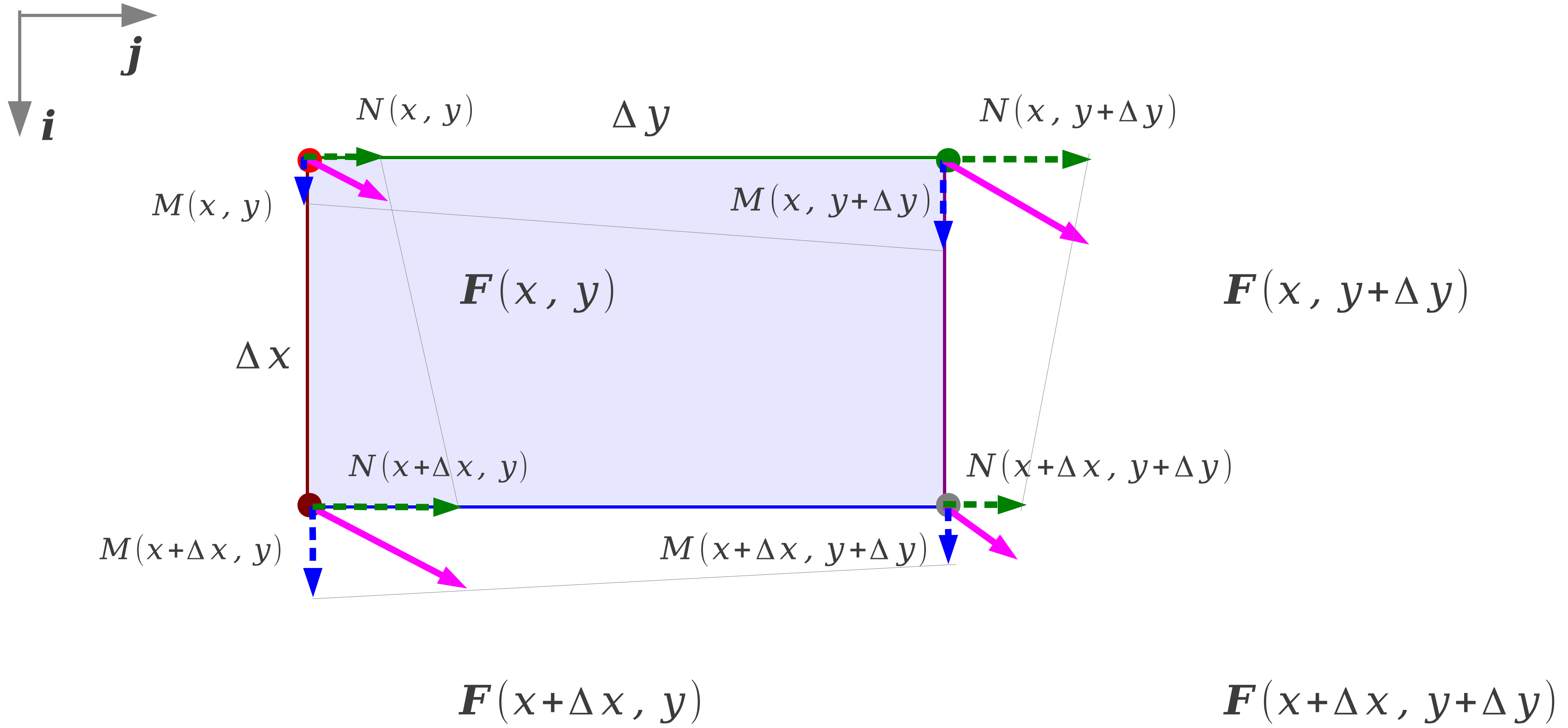
2-D Curl and Del Operator

3-D Curl and Del Operator

$$\nabla \times \boldsymbol{F}$$

2-D Curl (a)

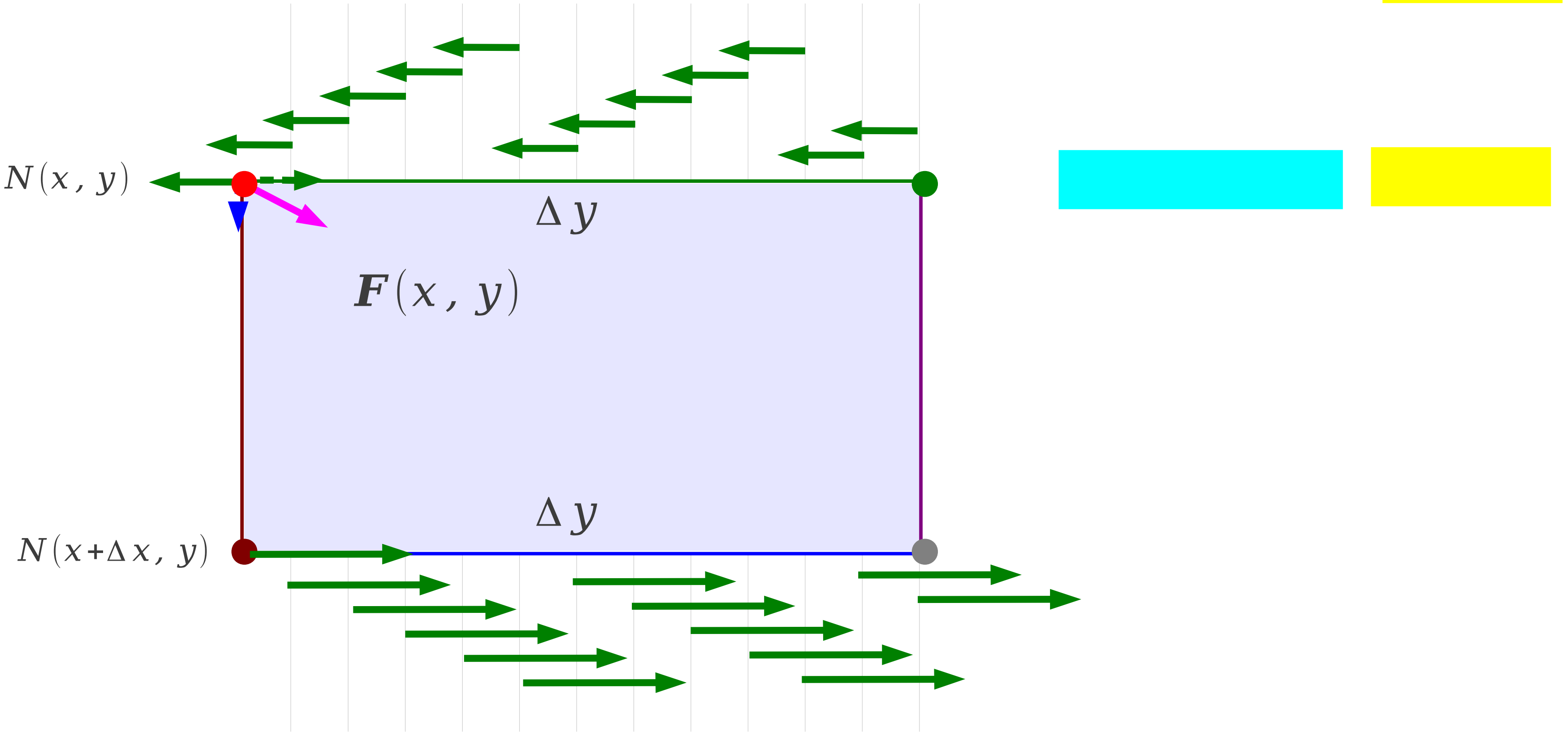
$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$



2-D Curl (top, bottom) (b)

$$\boldsymbol{F}(x, y) = M(x, y)\boldsymbol{i} + N(x, y)\boldsymbol{j}$$

$$\boldsymbol{F}(x, y) \cdot (-\boldsymbol{j})\Delta y = -N(x, y)\Delta y$$

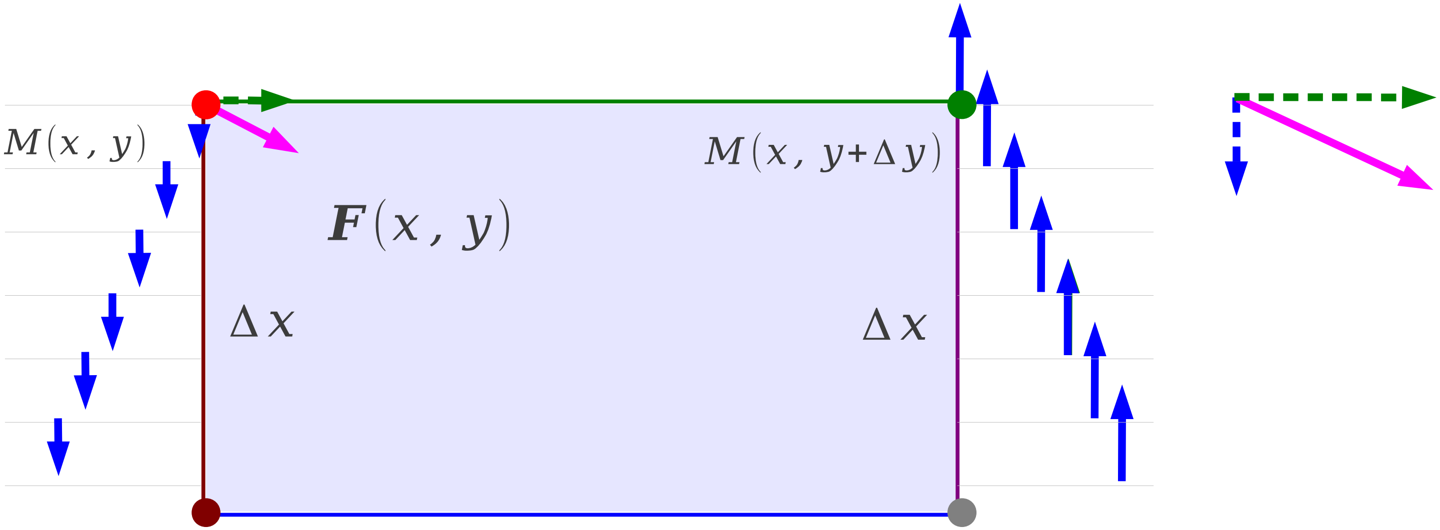


$$\boldsymbol{F}(x+\Delta x, y) \cdot (+\boldsymbol{j})\Delta y = N(x+\Delta x, y)\Delta y$$

2-D Curl (left, right) (c)

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$

$$\boldsymbol{F}(x, y)\cdot(+\boldsymbol{i})\Delta x = M(x, y)\Delta x \qquad \boldsymbol{F}(x, y+\Delta y)\cdot(-\boldsymbol{i})\Delta x = -M(x, y+\Delta y)\Delta x$$



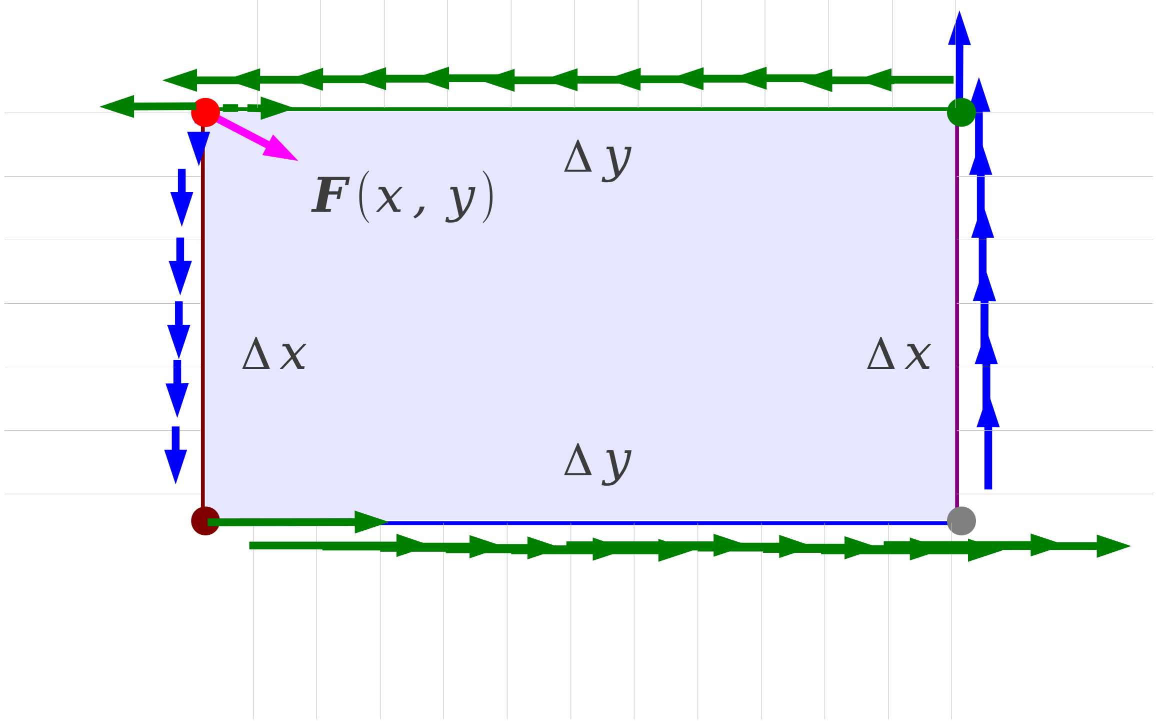
$$\frac{-\{M(x, y+\Delta y)-M(x, y)\}}{\Delta y} = -\left(\frac{\partial M}{\partial y}\right)$$

$$-\{M(x, y+\Delta y)-M(x, y)\}\Delta x = -\left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x$$

2-D Curl (d)

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y \; - \; \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x \qquad = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$



Circulation density $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

k-component
Curl of **F**

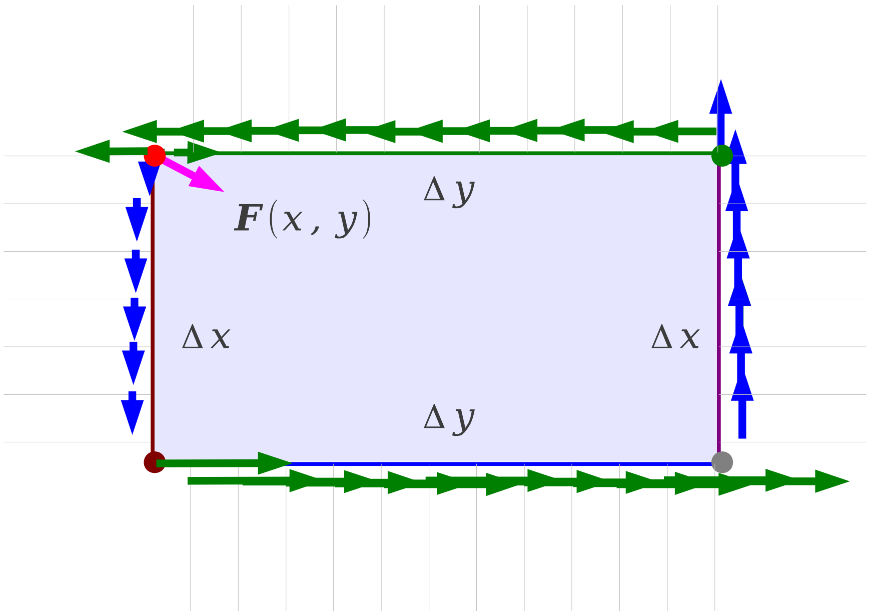
Circulation Density

2-D Curl and Del Operator

2-D Vector Field $\qquad \boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$

Circulation around rectangle boundary

$$\approx \left(\frac{\partial N}{\partial x}\Delta x\right)\Delta y - \left(\frac{\partial M}{\partial y}\Delta y\right)\Delta x = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\Delta x \Delta y$$



Circulation density $= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k})$$

Curl of **F** $= \nabla \times \boldsymbol{F}$

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

3-D Curl and Del Operator

2-D Vector Field

$$\boldsymbol{F}(x, y) = M(x,y)\boldsymbol{i} + N(x,y)\boldsymbol{j}$$

Curl of **F**

$$= \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k} \qquad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + 0\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + 0\boldsymbol{k}) = \nabla \times \boldsymbol{F}$$

3-D Vector Field

$$\boldsymbol{F}(x, y, z) = M(x,y,z)\boldsymbol{i} + N(x,y,z)\boldsymbol{j} + P(x,y,z)\boldsymbol{k}$$

Curl of **F**

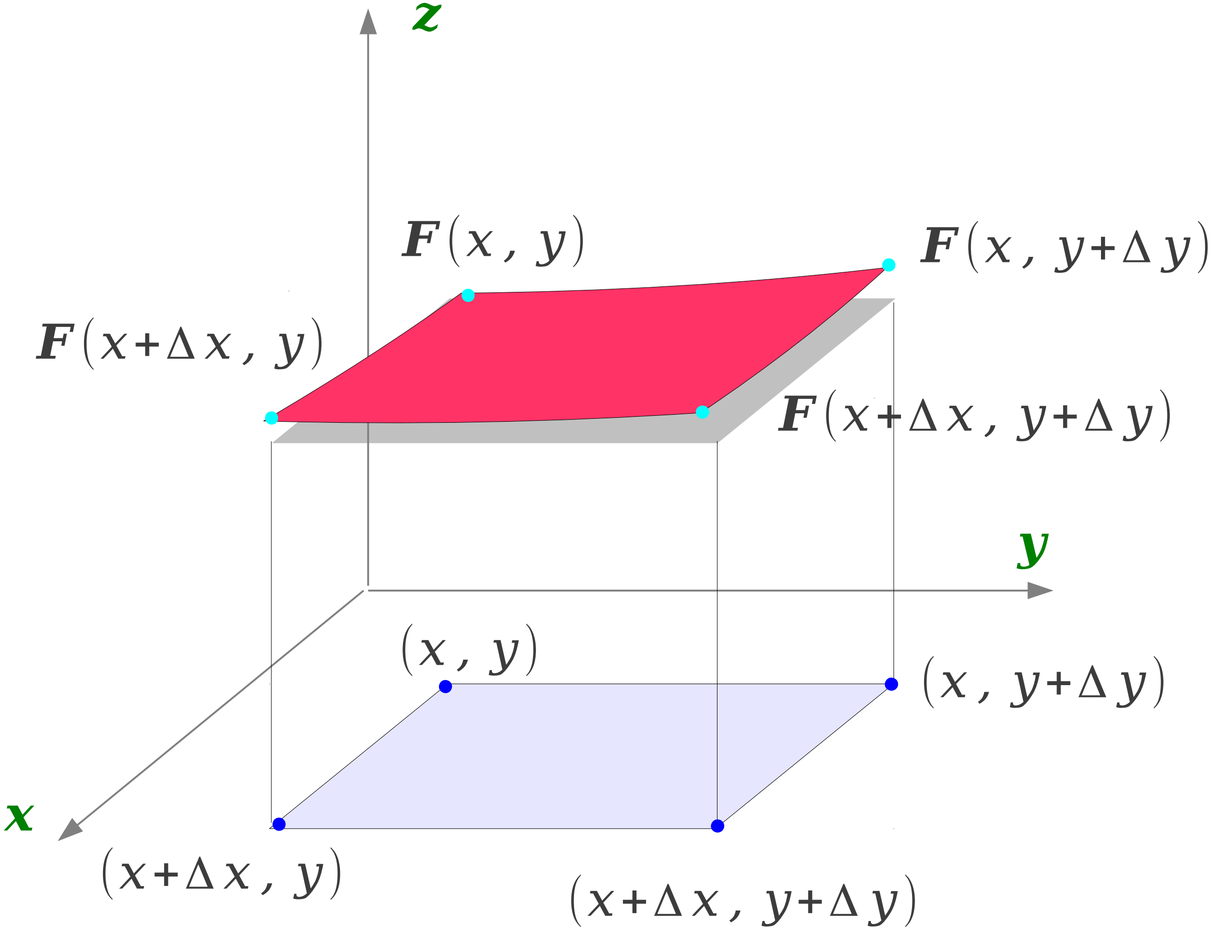
$$= \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\boldsymbol{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\boldsymbol{k} \qquad \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (M\boldsymbol{i} + N\boldsymbol{j} + P\boldsymbol{k}) = \nabla \times \boldsymbol{F}$$

2

i

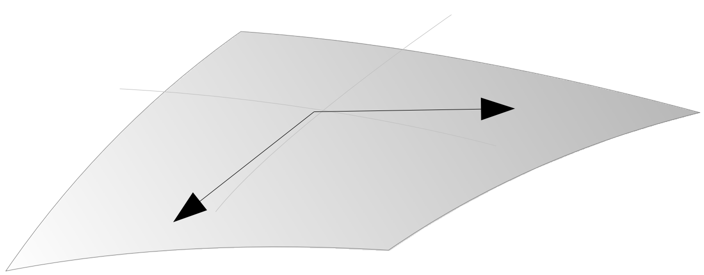
2-D Divergence

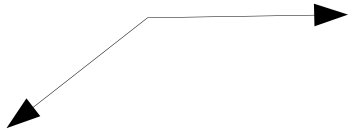


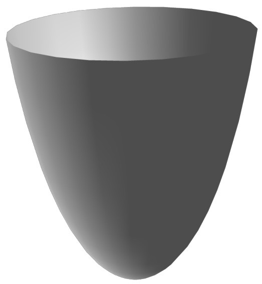
Chain Rule

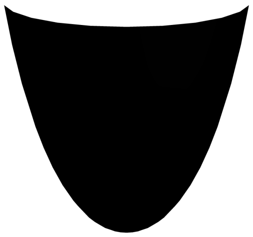
Function of two variable $y = f(u, v)$

$u = g(x, y)$ $v = h(x, y)$









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