CORDIC Background (4A)

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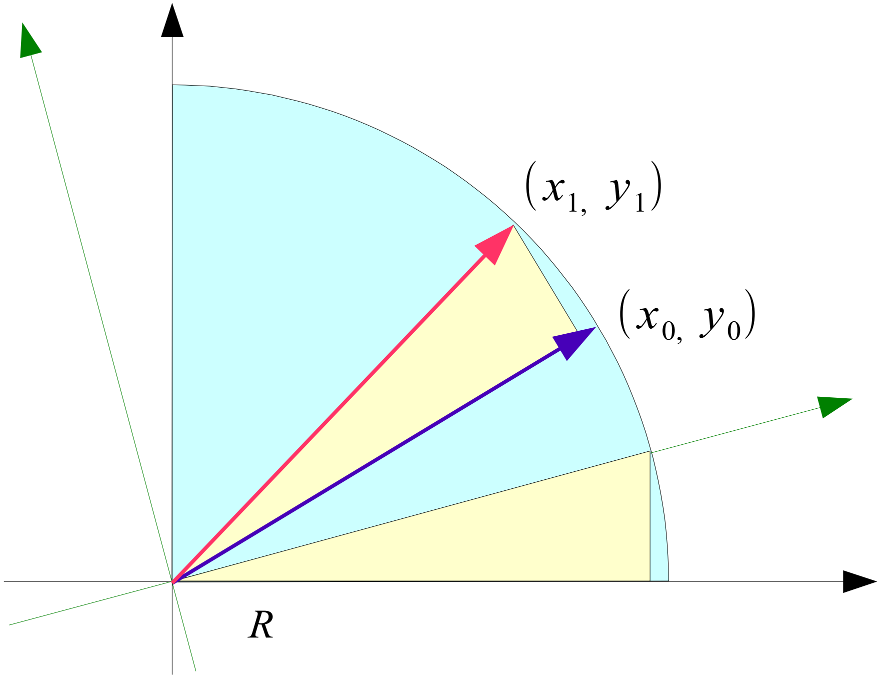
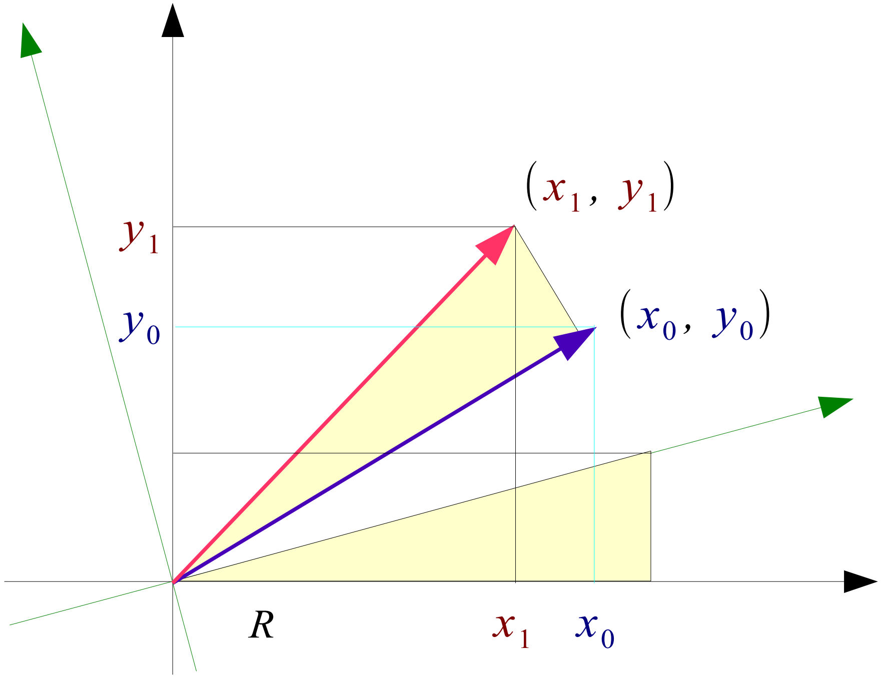
CORDIC Background

J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits

Vector Rotation (1)

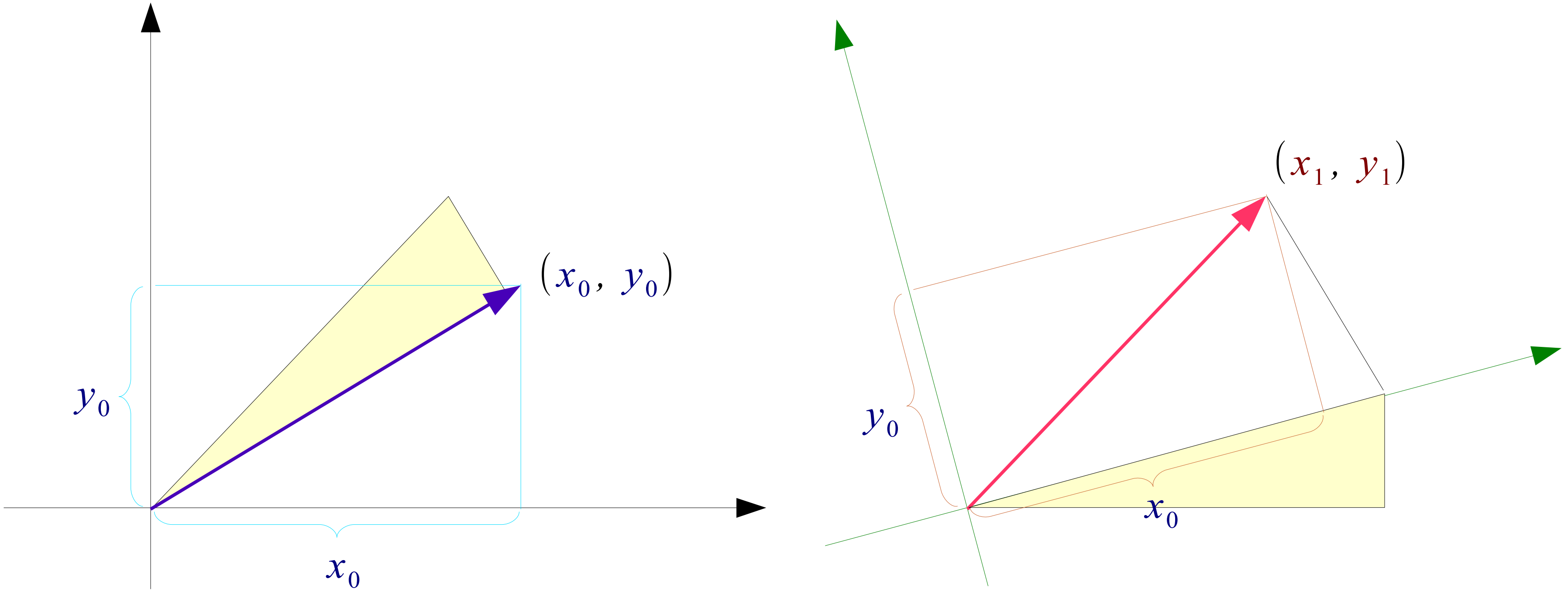
$$y_1 = x_0 \sin\alpha + y_0 \cos\alpha$$



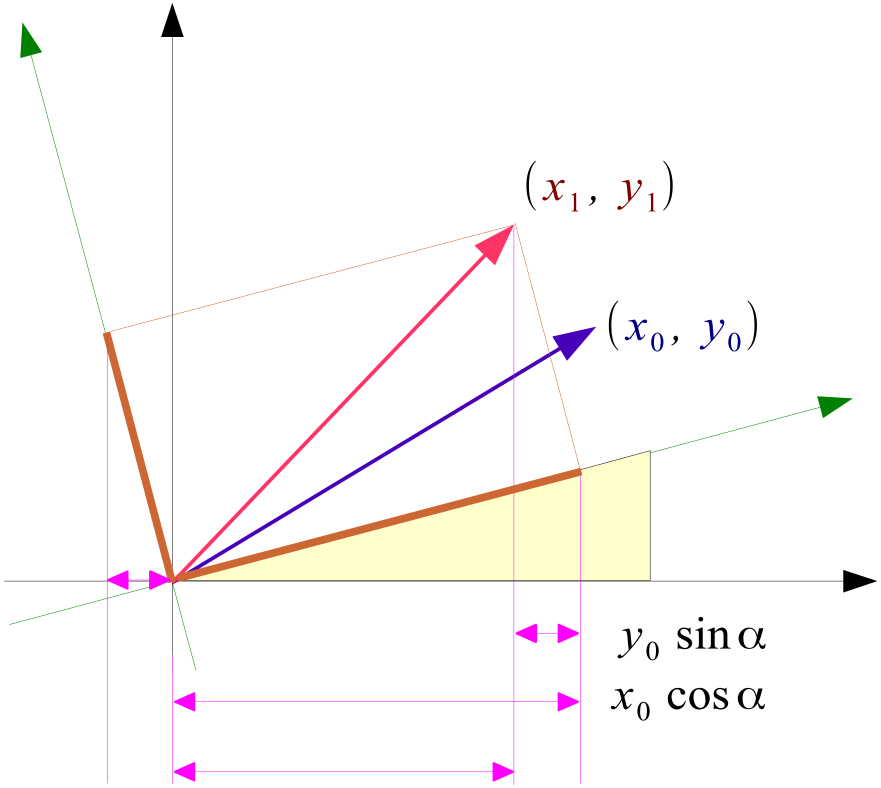


$$x_1 = x_0 \cos\alpha - y_0 \sin\alpha$$

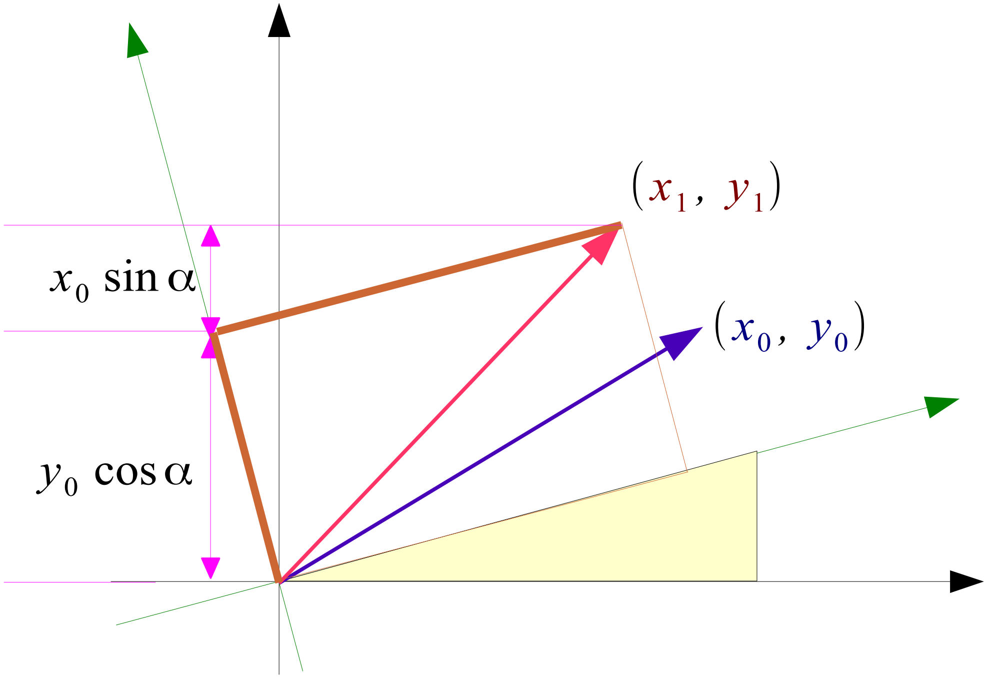
Vector Rotation (2)



Vector Rotation (3)



$$x_1 = x_0 \cos\alpha - y_0 \sin\alpha$$



$$y_1 = x_0 \sin\alpha + y_0 \cos\alpha$$

Iterative Rotation

Given Vector
Given Angle

Rotate →

Computing
new coordinates

Given Unit Vector
Given Angle α

Rotate →

$$x = \cos\alpha$$
$$y = \sin\alpha$$

Given Vector $(x_{0,} \;\; y_0)$
Given Angle α

Rotate →

$$x_n = x_0 \cos\alpha - y_0 \sin\alpha$$
$$y_n = x_0 \sin\alpha + y_0 \cos\alpha$$

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

CORDIC Rotation

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \to 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$\begin{cases} x_{i+1} = x_i \cos\alpha_i - y_i \sin\alpha_i \\ y_{i+1} = x_i \sin\alpha_i + y_i \cos\alpha_i \end{cases}$$

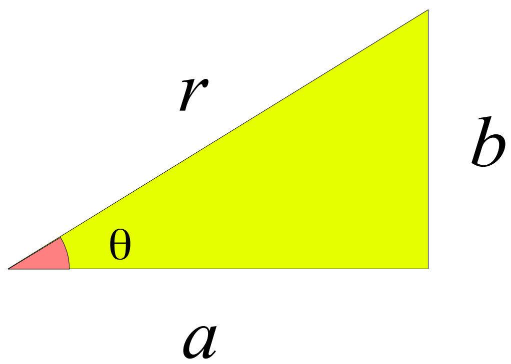
$$\begin{cases} x_{i+1} = \cos\alpha_i \left(x_i - y_i \tan\alpha_i\right) \\ y_{i+1} = \cos\alpha_i \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

$$\begin{cases} x_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\alpha_i}} \left(x_i - y_i \tan\alpha_i\right) \\[2ex] y_{i+1} = \dfrac{1}{\sqrt{1 + \tan^2\alpha_i}} \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

Pseudo-rotation

$$\begin{cases} x'_{i+1} = \left(x_i - y_i \tan\alpha_i\right) \\ y'_{i+1} = \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

$\cos\theta$



$$r = \sqrt{a^2 + b^2}$$

$$\cos\theta = \frac{a}{r} = \frac{a}{\sqrt{a^2 + b^2}}$$

$$= \frac{1}{\sqrt{1 + (b/a)^2}}$$

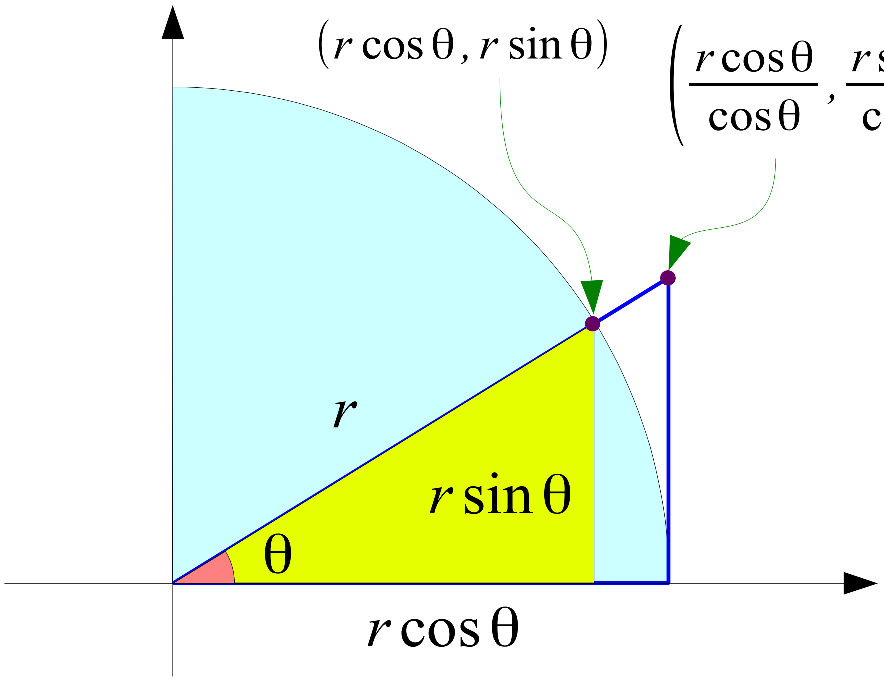
$$= \frac{1}{\sqrt{1 + \tan^2\theta}}$$

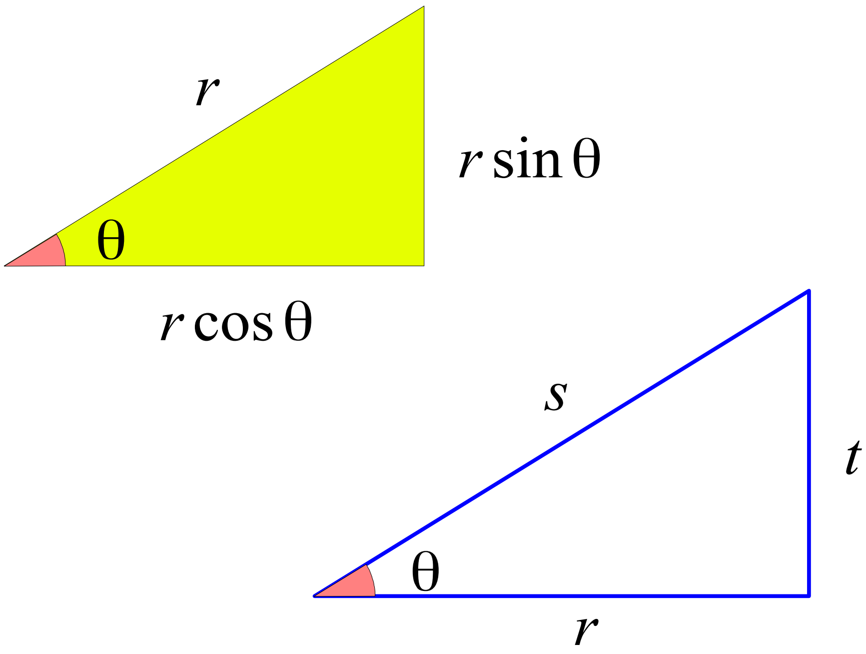
$$\cos\theta = \frac{a}{r}$$

$$\sin\theta = \frac{b}{r}$$

$$\tan\theta = \frac{b}{a}$$

Pseudo-rotation – factor of 1/cos θ





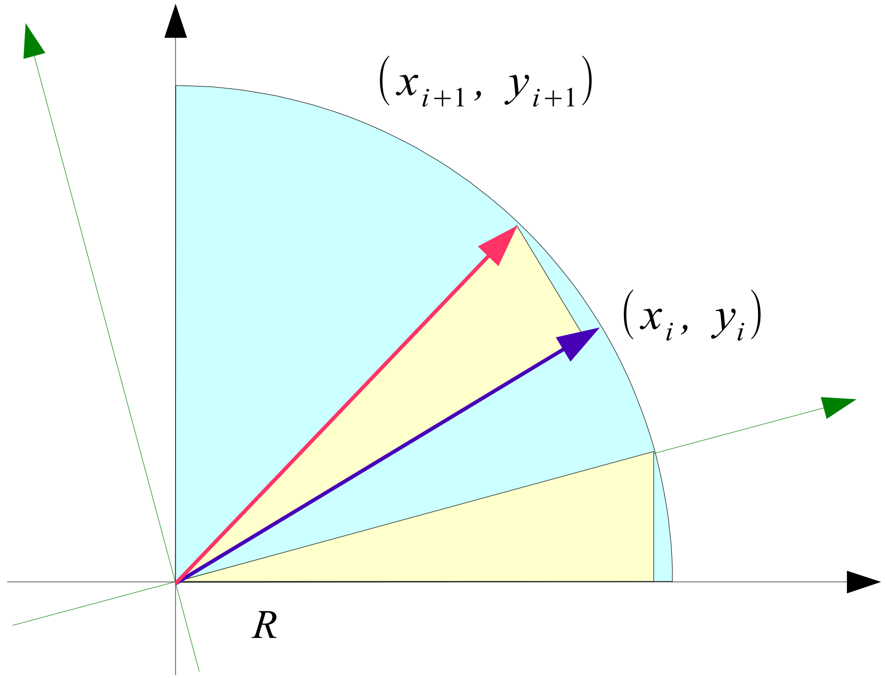
$$r : r\cos\theta = s : r$$

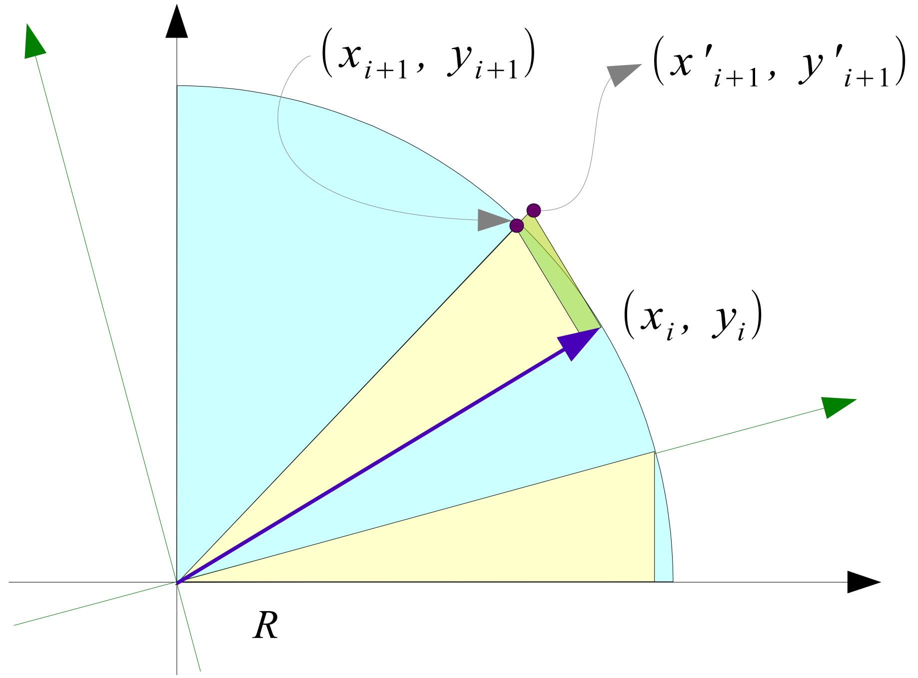
$$r\cos\theta : r\sin\theta = r : t$$

$$s = \frac{r}{\cos\theta}$$

$$s = \frac{r\sin\theta}{\cos\theta}$$

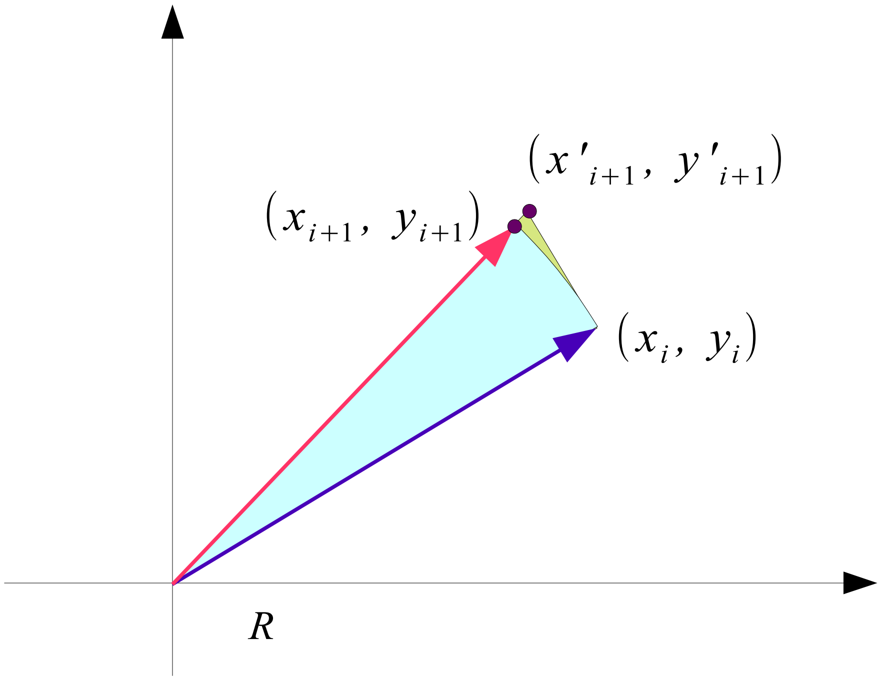
Pseudo-rotation (1)

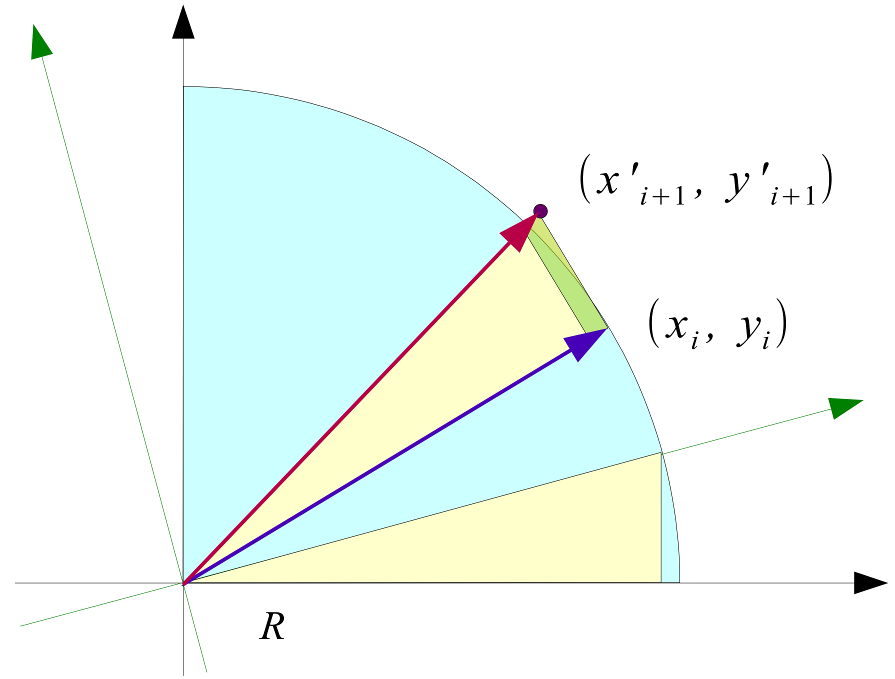




$$\begin{cases} x_{i+1} = \cos\alpha_i \left(x_i - y_i \tan\alpha_i\right) \\ y_{i+1} = \cos\alpha_i \left(x_i \tan\alpha_i + y_i\right) \end{cases} \Rightarrow \begin{cases} x'_{i+1} = \left(x_i - y_i \tan\alpha_i\right) \\ y'_{i+1} = \left(x_i \tan\alpha_i + y_i\right) \end{cases}$$

Pseudo-rotation (2)





$$x'_{i+1} = x_{i+1} / \cos\alpha_i$$
$$y'_{i+1} = y_{i+1} / \cos\alpha_i$$

$$x'_{i+1} > x_{i+1}$$
$$y'_{i+1} > y_{i+1}$$

CORDIC Iteration Equations (1)

$$\alpha_0 \rightarrow \cdots \rightarrow \alpha_i \rightarrow \alpha_{i+1} \rightarrow \cdots \rightarrow \alpha_n \rightarrow 0$$

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_i \\ y_i \end{pmatrix} \rightarrow \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

$$x_{i+1} = x_i \cos\alpha_i - y_i \sin\alpha_i = \cos\alpha_i \left(x_i - y_i \tan\alpha_i\right) = \left(1/\sqrt{1+\tan^2\alpha_i}\right)\left(x_i - y_i \tan\alpha_i\right)$$

$$y_{i+1} = x_i \sin\alpha_i + y_i \cos\alpha_i = \cos\alpha_i \left(x_i \tan\alpha_i + y_i\right) = \left(1/\sqrt{1+\tan^2\alpha_i}\right)\left(x_i \tan\alpha_i + y_i\right)$$

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = \left(x'_i - y'_i \tan\alpha_i\right) = \left(x'_i \cos\alpha_i - y'_i \sin\alpha_i\right)\sqrt{1+\tan^2\alpha_i}$$

$$y'_{i+1} = \left(x'_i \tan\alpha_i + y'_i\right) = \left(x'_i \sin\alpha_i + y'_i \cos\alpha_i\right)\sqrt{1+\tan^2\alpha_i}$$

$$x'_n = \left\{x_0 \cos\left(\Sigma\,\alpha_i\right) - y_0 \sin\left(\Sigma\,\alpha_i\right)\right\} \cdot \prod \left\{\sqrt{1+\tan^2\alpha_i}\right\}$$

$$y'_n = \left\{x_0 \sin\left(\Sigma\,\alpha_i\right) + y_0 \cos\left(\Sigma\,\alpha_i\right)\right\} \cdot \prod \left\{\sqrt{1+\tan^2\alpha_i}\right\}$$

$$\alpha_n = \alpha - \Sigma\,\alpha_i$$

CORDIC Iteration Equations (2)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = (x'_i - y'_i \tan\alpha_i) = (x'_i \cos\alpha_i - y'_i \sin\alpha_i)\sqrt{1+\tan^2\alpha_i}$$

$$y'_{i+1} = (x'_i \tan\alpha_i + y'_i) = (x'_i \sin\alpha_i + y'_i \cos\alpha_i)\sqrt{1+\tan^2\alpha_i}$$

$$x'_n = \{x_0 \cos(\Sigma\alpha_i) - y_0 \sin(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1+\tan^2\alpha_i}\}$$

$$y'_n = \{x_0 \sin(\Sigma\alpha_i) + y_0 \cos(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1+\tan^2\alpha_i}\}$$

$$\alpha_n = \alpha - \Sigma\alpha_i$$

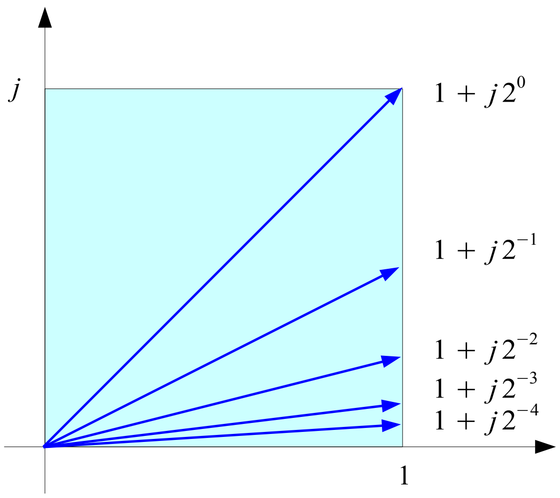
Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$

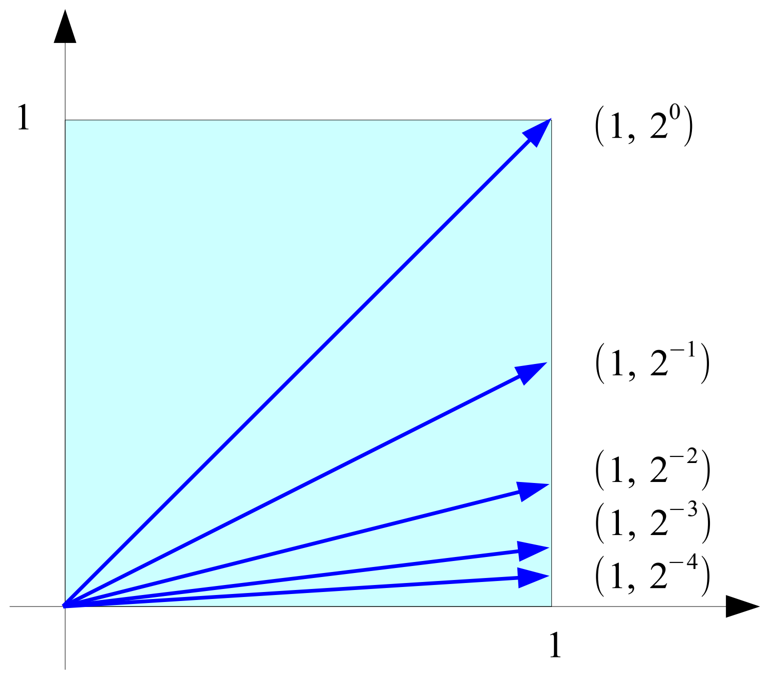


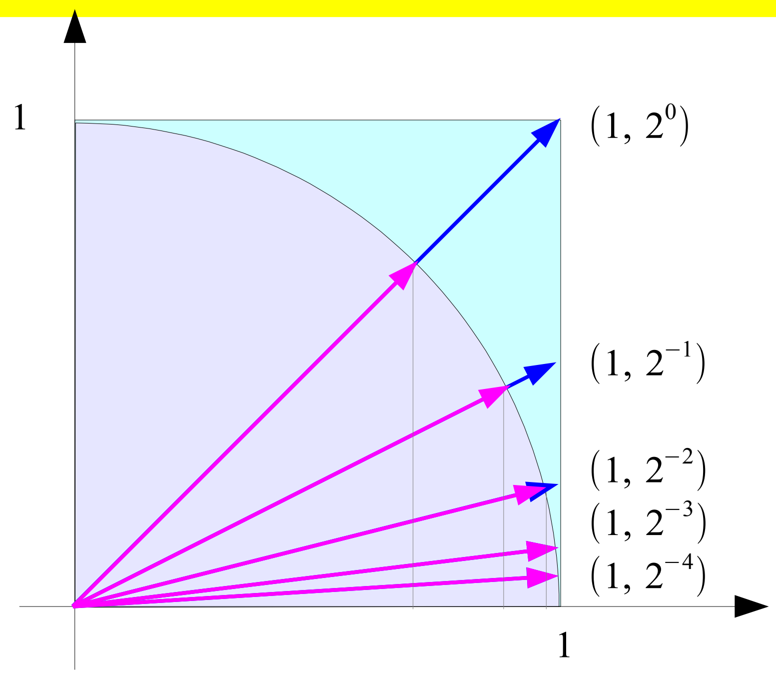
CORDIC Iteration Equations (2)

Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$x'_{i+1} = (x'_i - y'_i\,\sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i\,\sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan(\sigma_i 2^{-i})$$



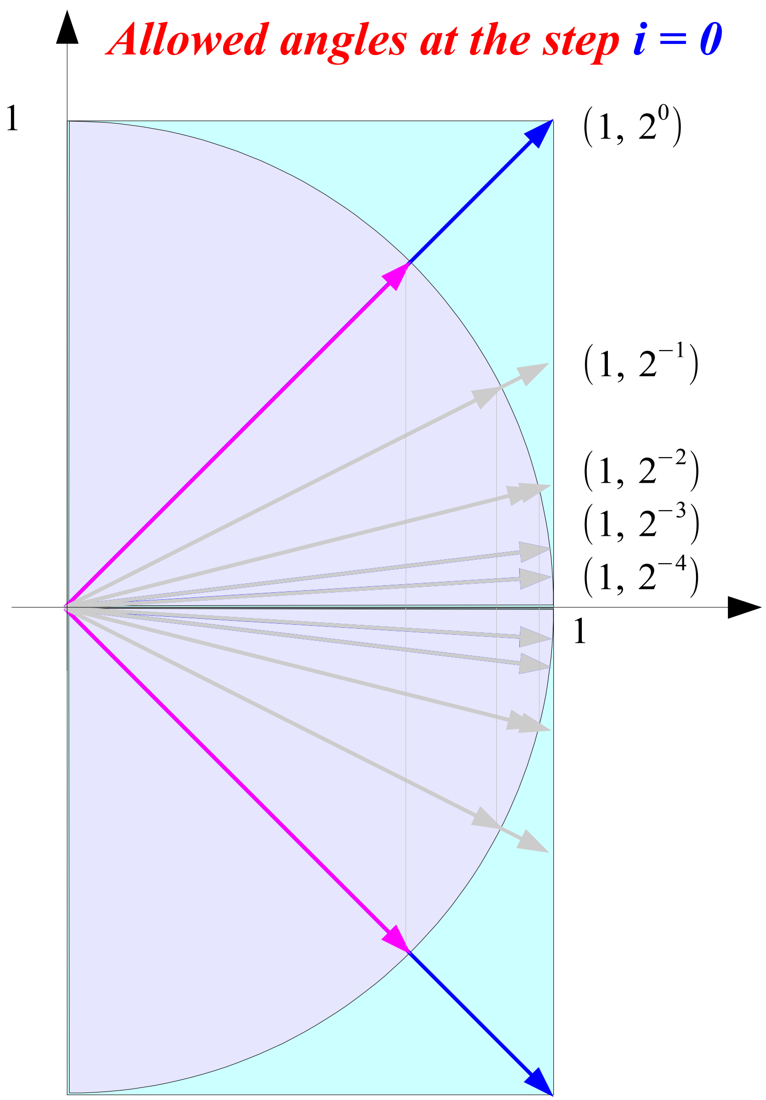


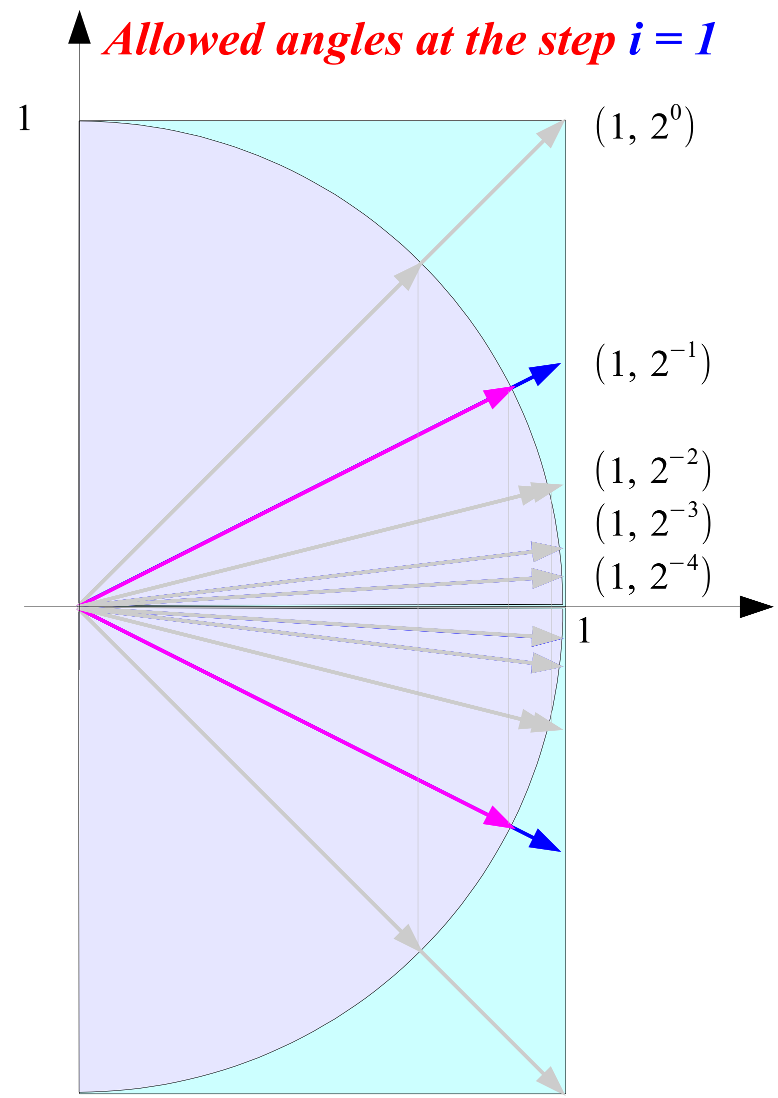
$$\tan\alpha_i = \pm 2^{-i} \qquad \cos\alpha_i = \frac{+1}{\sqrt{1 + 2^{-2i}}}$$

$$\sin\alpha_i = \frac{\pm 2^{-i}}{\sqrt{1 + 2^{-2i}}}$$

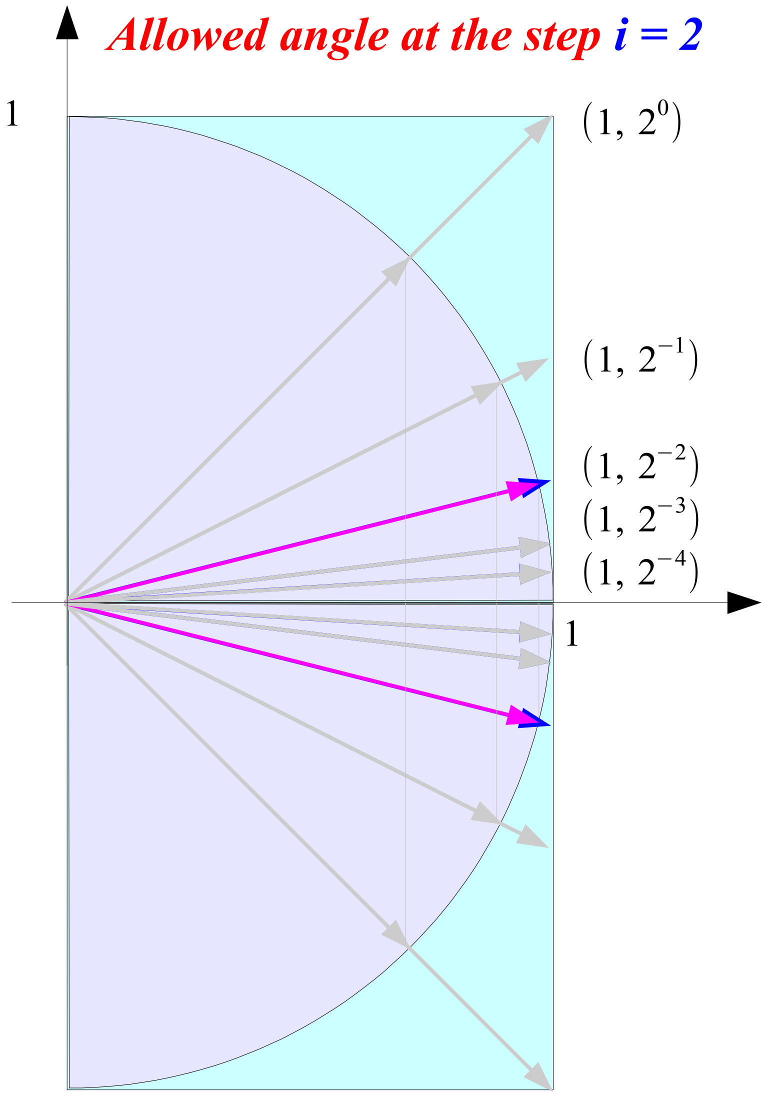
$$\begin{pmatrix} +\cos\alpha_i & -\sin\alpha_i \\ +\sin\alpha_i & +\cos\alpha_i \end{pmatrix} = \frac{1}{\sqrt{1 + 2^{-2i}}} \begin{pmatrix} +1 & \mp 2^{-i} \\ \pm 2^{-i} & +1 \end{pmatrix}$$

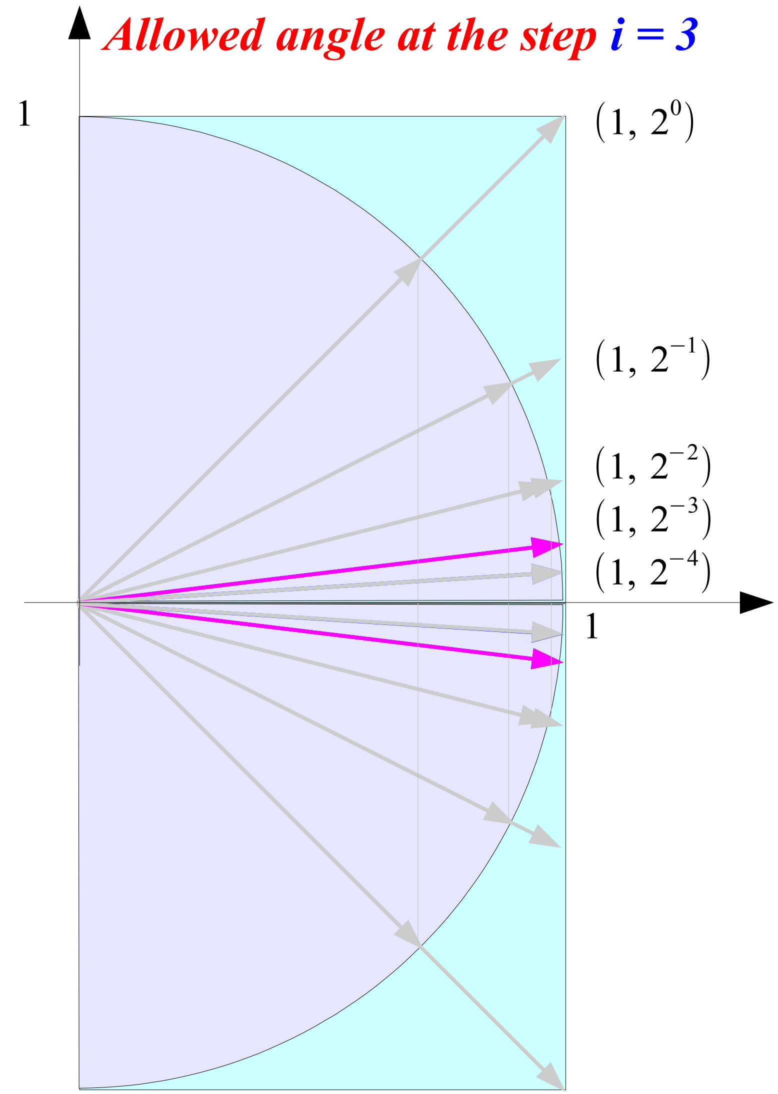
CORDIC Iteration Equations (3)





CORDIC Iteration Equations (4)





CORDIC Iteration Equations (2)

Choose α_i such that $\tan\alpha_i = \begin{cases} +2^{-i} \\ -2^{-i} \end{cases}$

$$\tan\alpha_i = \sigma_i 2^{-i} \qquad \sigma_i \in \{+1, -1\}$$

$$\begin{aligned} x'_{i+1} &= (x'_i - y'_i \sigma_i 2^{-i}) \\ y'_{i+1} &= (x'_i \sigma_i 2^{-i} + y'_i) \\ \alpha_{i+1} &= \alpha_i - \tan(\sigma_i 2^{-i}) \end{aligned}$$

$$\begin{pmatrix} +\cos\alpha & -\sin\alpha \\ +\sin\alpha & +\cos\alpha \end{pmatrix}$$

$$= \frac{1}{\sqrt{1+2^{-2\cdot 0}}}\begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix} \cdot \frac{1}{\sqrt{1+2^{-2\cdot 1}}}\begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \frac{1}{\sqrt{1+2^{-2\cdot n}}}\begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{1+2^{-2\cdot 0}}} \cdot \frac{1}{\sqrt{1+2^{-2\cdot 1}}} \cdots \frac{1}{\sqrt{1+2^{-2\cdot n}}} \quad \begin{pmatrix} +1 & \mp 2^{-0} \\ \pm 2^{-0} & +1 \end{pmatrix}\begin{pmatrix} +1 & \mp 2^{-1} \\ \pm 2^{-1} & +1 \end{pmatrix} \cdots \begin{pmatrix} +1 & \mp 2^{-n} \\ \pm 2^{-n} & +1 \end{pmatrix}$$

$$\Rightarrow \quad 1/K = 0.607$$

$$\Rightarrow \quad \begin{pmatrix} +\cos(\sum\alpha_i) & -\sin(\sum\alpha_i) \\ +\sin(\sum\alpha_i) & +\cos(\sum\alpha_i) \end{pmatrix}$$

$$\begin{pmatrix} +\cos(\sum\alpha_i) & -\sin(\sum\alpha_i) \\ +\sin(\sum\alpha_i) & +\cos(\sum\alpha_i) \end{pmatrix} = K \cdot \begin{pmatrix} +\cos\alpha & -\sin\alpha \\ +\sin\alpha & +\cos\alpha \end{pmatrix}$$

$$K = \prod\left\{\sqrt{1+\tan^2\alpha_i}\right\} = 1.647$$

CORDIC Iteration Equations (5)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

$$x'_{i+1} = (x'_i - y'_i \tan\alpha_i) = (x'_i \cos\alpha_i - y'_i \sin\alpha_i)\sqrt{1 + \tan^2\alpha_i}$$

$$y'_{i+1} = (x'_i \tan\alpha_i + y'_i) = (x'_i \sin\alpha_i + y'_i \cos\alpha_i)\sqrt{1 + \tan^2\alpha_i}$$

$$x'_n = \{x_0 \cos(\Sigma\alpha_i) - y_0 \sin(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$y'_n = \{x_0 \sin(\Sigma\alpha_i) + y_0 \cos(\Sigma\alpha_i)\} \cdot \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

$$\alpha_n = \alpha - \Sigma\alpha_i$$

Choose α_i such that $\tan\alpha_i = \sigma_i 2^{-i}$ $\quad \sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$

$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$

$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos\alpha - y_0 \sin\alpha)$$

$$y'_n = K(x_0 \sin\alpha + y_0 \cos\alpha)$$

$$\alpha_n = \alpha - \Sigma\tan^{-1}(\sigma_i 2^{-i})$$

$$K = \Pi\{\sqrt{1 + \tan^2\alpha_i}\}$$

CORDIC Iteration Equations (4)

Pseudo-rotation

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_i \\ y'_i \end{pmatrix} \rightarrow \begin{pmatrix} x'_{i+1} \\ y'_{i+1} \end{pmatrix} \rightarrow \cdots \rightarrow \begin{pmatrix} x'_n \\ y'_n \end{pmatrix}$$

Choose α_i such that $\tan\alpha_i = \sigma_i 2^{-i}$ $\quad \sigma_i \in \{+1, -1\}$

$$x'_{i+1} = (x'_i - y'_i \sigma_i 2^{-i})$$
$$y'_{i+1} = (x'_i \sigma_i 2^{-i} + y'_i)$$
$$\alpha_{i+1} = \alpha_i - \tan^{-1}(\sigma_i 2^{-i})$$

$$x'_n = K(x_0 \cos\alpha - y_0 \sin\alpha)$$
$$y'_n = K(x_0 \sin\alpha + y_0 \cos\alpha)$$
$$\alpha_n = \alpha - \sum \tan^{-1}(\sigma_i 2^{-i})$$

pre-compute $K = \prod \{\sqrt{1 + \tan^2\alpha_i}\} = 1.647$

set $x_0 = 1 / K = 0.607$

$y_0 = 0$

then $x'_n = K(1/K \cos\alpha - 0 \cdot \sin\alpha) = \cos\alpha$

$y'_n = K(1/K \sin\alpha + 0 \cdot \cos\alpha) = \sin\alpha$

References

[1] http://en.wikipedia.org/
[2] CORDIC FAQ, www.dspguru.com
[3] R. Andraka, A survey of CORDIC algorithms for FPGA based computers
[4] J. S. Walther, A Unified Algorithm for Elementary Functions
[5] J. P. Deschamps, G. A. Bioul, G.D. Sutter, Synthesis of Arithmetic Circuits