

Signals and Spectra (1A)

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Energy and Power

Instantaneous Power

$$p(t) = x^2(t) \quad \text{real signal}$$

Energy dissipated during
(-T/2, +T/2)

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t) dt$$

Affects the performance
of a communication system

Average power dissipated during
(-T/2, +T/2)

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

The rate at which energy is dissipated
Determines the voltage

Energy and Power Signals (1)

Energy dissipated during

$(-T/2, +T/2)$

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t) dt$$

Average power dissipated during

$(-T/2, +T/2)$

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Energy Signal

Nonzero but finite energy

$0 < E_x < +\infty$ for all time

$$\begin{aligned} E_x &= \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} x^2(t) dt < +\infty \end{aligned}$$

Power Signal

Nonzero but finite power

$0 < P_x < +\infty$ for all time

$$\begin{aligned} P_x &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt \\ &< +\infty \end{aligned}$$

Energy and Power Signals (2)

Energy Signal

Nonzero but finite energy

$$0 < E_x < +\infty \text{ for all time}$$

$$\begin{aligned} E_x &= \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} x^2(t) dt < +\infty \end{aligned}$$

Power Signal

Nonzero but finite power

$$0 < P_x < +\infty \text{ for all time}$$

$$\begin{aligned} P_x &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt \\ &< +\infty \end{aligned}$$

$$\begin{aligned} P_x &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \lim_{T \rightarrow +\infty} \frac{B}{T} \rightarrow 0 \end{aligned}$$

$$\begin{aligned} E_x &= \lim_{T \rightarrow +\infty} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \lim_{T \rightarrow +\infty} B \cdot T \rightarrow +\infty \end{aligned}$$

Non-periodic signals
Deterministic signals

Periodic signals
Random signals

Energy and Power Spectral Densities (1)

Total Energy, Non-periodic

$$E_x^T = \int_{-\infty}^{+\infty} x^2(t) dt$$

Average power, Periodic

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Parseval's Theorem, Non-periodic

$$= \int_{-\infty}^{+\infty} |X(f)|^2 df$$

$$= \int_{-\infty}^{+\infty} \Psi(f) df$$

$$= 2 \int_0^{+\infty} \Psi(f) df$$

Parseval's Theorem, Periodic

$$= \sum_{n=-\infty}^{+\infty} |c_n|^2$$

$$= \int_{-\infty}^{+\infty} G_x(f) df$$

$$= 2 \int_0^{+\infty} G_x(f) df$$

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - n f_0)$$

Energy and Power Spectral Densities (2)

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Total Energy, Non-periodic

$$\begin{aligned} E_x^T &= \int_{-\infty}^{+\infty} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} \Psi(f) df \end{aligned}$$

Parseval's Theorem, Non-periodic

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - n f_0)$$

Average power, Periodic

$$\begin{aligned} P_x^T &= \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt \\ &= \int_{-\infty}^{+\infty} G_x(f) df \end{aligned}$$

Parseval's Theorem, Periodic

Non-periodic power signal
(having infinite energy) ?

Energy and Power Spectral Densities (3)

Power Spectral Density

$$G_x(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X_T(f)|^2$$

Non-periodic power signal
(having infinite energy) ?

→ No Fourier Series

truncate $(-\frac{T}{2} \leq t \leq +\frac{T}{2})$

$$x(t) \rightarrow x_T(t)$$

→ Fourier Transform $X_T(f)$

$$P_x^T = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$
$$= \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{|X(f)|^2}{T} df$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - n f_0)$$

Average power, Periodic

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$
$$= \int_{-\infty}^{+\infty} G_x(f) df$$

Parseval's Theorem, Periodic

Autocorrelation of Energy and Power Signals

Autocorrelation of an Energy Signal

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau) dt$$

$(-\infty \leq \tau \leq +\infty)$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow \Psi(f)$$

$$R_x(0) = \int_{-\infty}^{+\infty} x^2(t) dt$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau) dt$$

$(-\infty \leq \tau \leq +\infty)$

Autocorrelation of a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau) dt$$

$(-\infty \leq \tau \leq +\infty)$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt$$

Ensemble Average

Random Variable

$$\begin{aligned} m_x &= \mathbf{E}\{X\} \\ &= \int_{-\infty}^{+\infty} \mathbf{x} p_X(\mathbf{x}) d\mathbf{x} \end{aligned}$$

$$\begin{aligned} \mathbf{E}\{X^2\} &= \sigma_x^2 + m_x^2 \\ &= \int_{-\infty}^{+\infty} \mathbf{x}^2 p_X(\mathbf{x}) d\mathbf{x} \end{aligned}$$

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} \mathbf{x} p_{X_k}(\mathbf{x}) d\mathbf{x} \end{aligned}$$

for a given time t_k

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathbf{x}_1 \mathbf{x}_2 p_{X_1, X_2}(\mathbf{x}_1, \mathbf{x}_2) d\mathbf{x}_1 d\mathbf{x}_2 \end{aligned}$$

WSS (Wide Sense Stationary)

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} \mathbf{x} p_{X_k}(\mathbf{x}) d\mathbf{x} \end{aligned}$$

for a given time t_k

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 p_{X_1, X_2}(x_1, x_2) dx_1 dx_2 \end{aligned}$$



WSS Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= m_x \end{aligned}$$



$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= R_x(t_1 - t_2) \end{aligned}$$

Ergodicity and Time Averaging

Random Process

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= \int_{-\infty}^{+\infty} \mathbf{x} p_{X_k}(\mathbf{x}) d\mathbf{x} \end{aligned}$$

for a given time

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 p_{X_1, X_2}(x_1, x_2) dx_1 dx_2 \end{aligned}$$

WSS Process by ensemble average

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} \\ &= m_x \end{aligned}$$

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} \\ &= R_x(t_1 - t_2) = R_x(\tau) \end{aligned}$$

Ergodic Process by time average

$$\begin{aligned} m_x(t_k) &= \mathbf{E}\{X(t_k)\} = \\ m_x &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t) dt \end{aligned}$$

$$\begin{aligned} R_x(t_1, t_2) &= \mathbf{E}\{X(t_1) X(t_2)\} = \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t) X(t+\tau) dt \end{aligned}$$

Autocorrelation of Power Signals

Autocorrelation of a Random Signal

$$R_x(\tau) = \mathbf{E}\{X(t) X(t + \tau)\}$$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \mathbf{E}\{X^2(t)\}$$

Autocorrelation of a Power Signal

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t) x(t + \tau) dt$$

$(-\infty \leq \tau \leq +\infty)$

Autocorrelation of a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x(t) x(t + \tau) dt$$

$(-\infty \leq \tau \leq +\infty)$

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x^2(t) dt$$

Autocorrelation of Random Signals

Autocorrelation of a Random Signal

$$R_x(\tau) = \mathbf{E}\{X(t) X(t + \tau)\}$$

$$= \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t) X(t + \tau) dt$$

if *ergodic* in the autocorrelation function

$$R_x(\tau) = R_x(-\tau)$$

$$R_x(\tau) \leq R_x(0)$$

$$R_x(\tau) \Leftrightarrow G_x(f)$$

$$R_x(0) = \mathbf{E}\{X^2(t)\}$$

Power Spectral Density of a Random Signal

$$G_x(f) = \lim_{T \rightarrow +\infty} \frac{1}{T} |X_T(f)|^2$$

$$G_x(f) = G_x(-f)$$

$$G_x(f) \geq 0$$

$$G_x(f) \Leftrightarrow R_x(\tau)$$

$$P_x(0) = \int_{-\infty}^{+\infty} G_x(f) df$$

Ergodic Random Process

$m_X = \mathbf{E}\{X(t)\}$ DC level

m_X^2 normalized power in the dc component

$\mathbf{E}\{X^2(t)\}$ total average normalized power (mean square value)

$\sqrt{\mathbf{E}\{X^2(t)\}}$ rms value of voltage or current

σ_X^2 average normalized power in the ac component
 σ_X

$m_X = m_X^2 = 0 \Rightarrow \sigma_X^2 = \mathbf{E}\{X^2\}$ var = total average normalized power
= mean square value (rms²)

σ_X rms value of the ac component

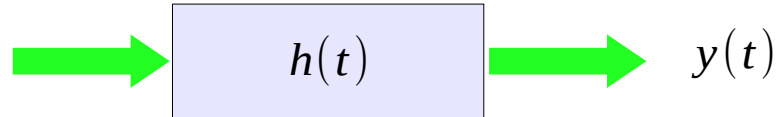
$m_X = 0$ rms value of the signal

Single Tone Input

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

$$x(t) = \underline{A e^{j\Phi} e^{j\omega t}}$$

$$\begin{cases} \text{amplitude} = A \\ \text{phase} = \Phi \\ \text{frequency} = \omega \end{cases}$$



$$y(t) = \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega(t-\tau)} d\tau$$

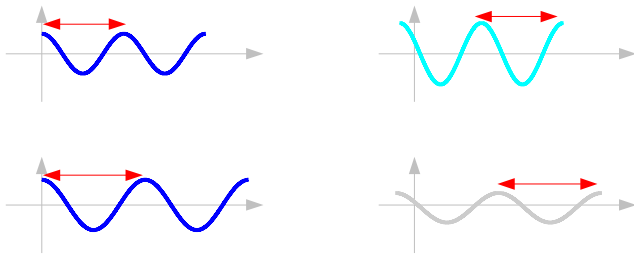
$$= \int_{-\infty}^{+\infty} h(\tau) A e^{j\Phi} e^{j\omega t} e^{-j\omega\tau} d\tau$$

$$= \underline{A e^{j\Phi} e^{j\omega t}} \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

$$= \underline{A e^{j\Phi} e^{j\omega t}} \boxed{H(j\omega)} \quad \text{complex number given } \omega \text{ and } t$$

$$= \underline{A e^{j\Phi} e^{j\omega t}} \boxed{\rho e^{j\theta}}$$

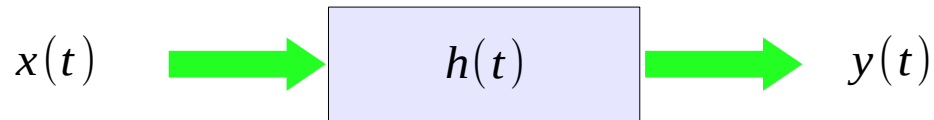
Fourier Transform



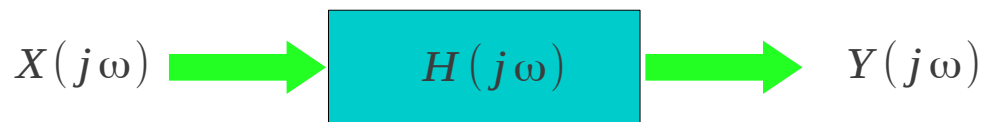
changes the amplitude and the phase of the single tone input but *not the frequency*

Impulse Response & Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$



Fourier Transform

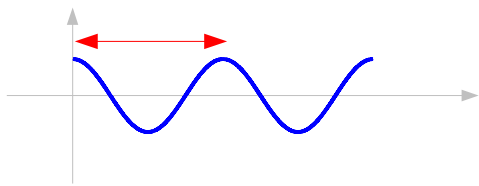
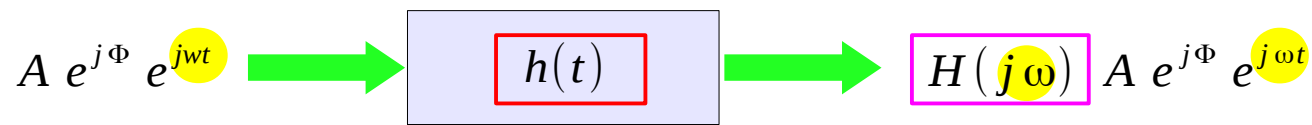
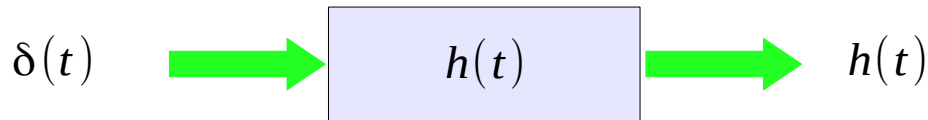
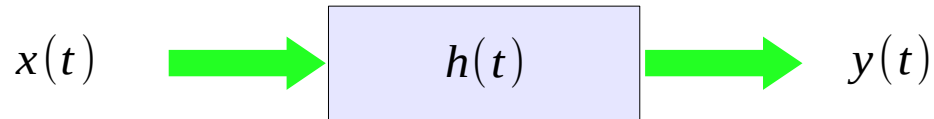


$$Y(j\omega) = H(j\omega)X(j\omega)$$

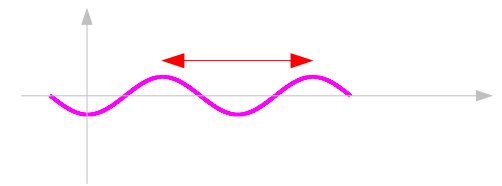
$$Y(j\omega) = \int_{-\infty}^{+\infty} y(\tau) e^{-j\omega\tau} d\tau \quad H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$
$$X(j\omega) = \int_{-\infty}^{+\infty} x(\tau) e^{-j\omega\tau} d\tau$$

Linear System

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$

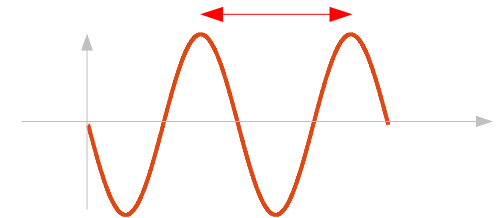


single frequency component : ω



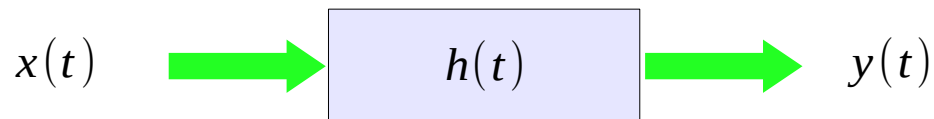
Frequency Response

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$



Transfer Function & Frequency Response

$$y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau) x(t - \tau) d\tau$$



Transfer Function

$$H(s) = H(\sigma + j\omega)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$$

Laplace Transform

initial state
differential equation

$$\sigma = 0$$



Frequency Response

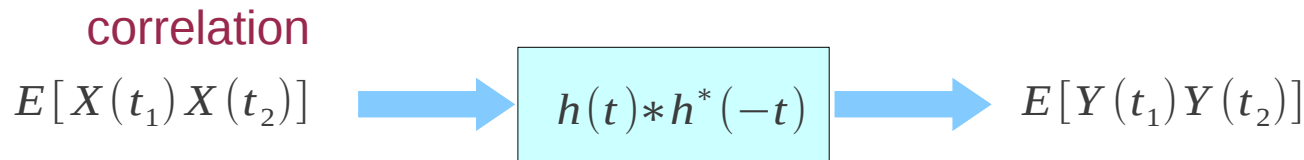
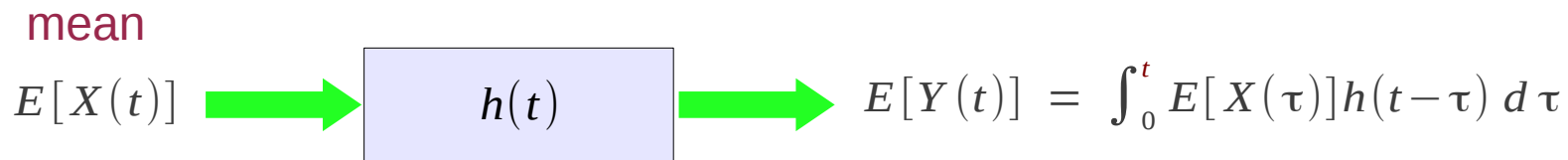
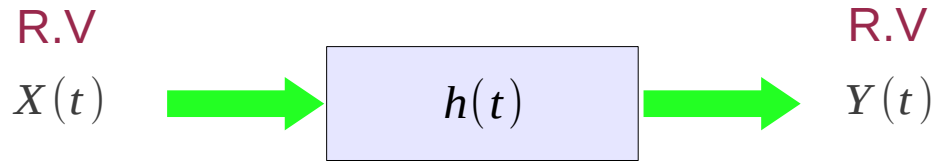
$$H(\omega) = H(j\omega)$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(\tau) e^{-j\omega\tau} d\tau$$

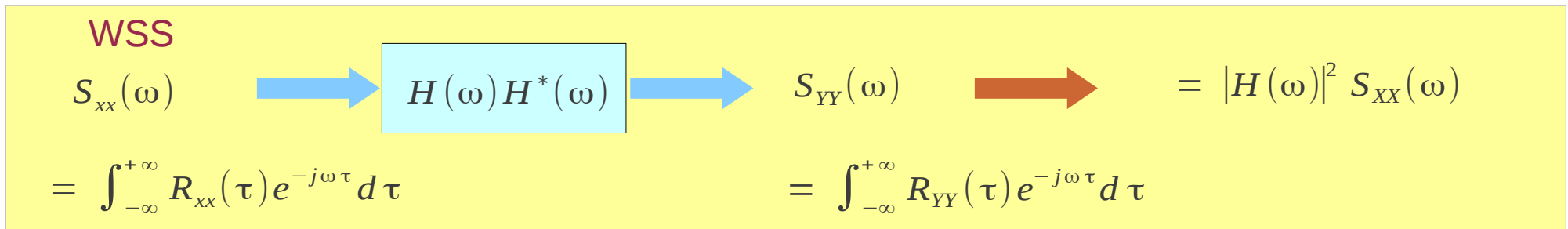
Fourier Transform

steady state
frequency response

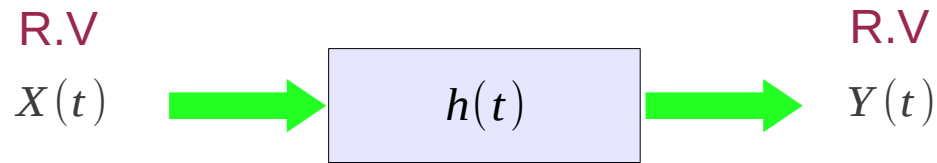
Linear System & Random Variables (1)



Fourier Transform



Linear System & Random Variables (2)



$$\rho(t) = h(t) * h^*(-t)$$

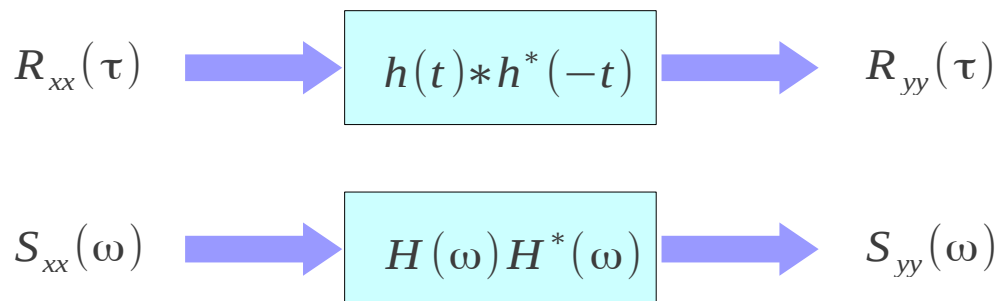
$$|H(\omega)|^2 = H(\omega) H^*(\omega)$$

$$R_{xy}(\tau) = R_{xx}(\tau) * h^*(-\tau) \quad R_{yy}(\tau) = R_{xy}(\tau) * h(\tau)$$

$$S_{xy}(\omega) = S_{xx}(\omega) H^*(\omega) \quad S_{yy}(\omega) = S_{xy}(\omega) H(\omega)$$

$$R_{yy}(\tau) = R_{xx}(\tau) * \rho(\tau)$$

$$S_{yy}(\omega) = S_{xx}(\omega) |H(\omega)|^2$$



Summary (1)

Non-periodic signals

Energy Signal

$$E_x^T = \int_{-T/2}^{+T/2} x^2(t) dt$$

Energy Spectral Density

$$\Psi(f) = |X(f)|^2$$

Total Energy

$$\int_{-\infty}^{+\infty} \Psi(f) df$$

Periodic signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Power Spectral Density

$$G_x(f) = \sum_{n=-\infty}^{+\infty} |c_n|^2 \delta(f - nf_0)$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f) df$$

Random signals

Power Signal

$$P_x^T = \frac{1}{T} \int_{-T/2}^{+T/2} x^2(t) dt$$

Power Spectral Density

$$G_x(f) = \lim_{T \rightarrow \infty} \frac{1}{T} |X_T(f)|^2$$

Average Power

$$\int_{-\infty}^{+\infty} G_x(f) df$$

Summary (2)

Energy Signal Autocorrelation

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau) dt$$

Power Signal Autocorrelation

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} x(t)x(t+\tau) dt$$

Random Signal Autocorrelation

$$R_x(\tau) = \mathbf{E}\{X(t)X(t+\tau)\}$$

Non-periodic signals

$$R_x(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau) dt$$

Periodic signals

for a Periodic Signal

$$R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{+T_0/2} x(t)x(t+\tau) dt$$

Random signals

for a Ergodic Signal

$$R_x(\tau) = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_{-T/2}^{+T/2} X(t)X(t+\tau) dt$$

References

- [1] <http://en.wikipedia.org/>
- [2] <http://planetmath.org/>
- [3] B. Sklar, "Digital Communications: Fundamentals and Applications"