Surface Integrals (6A)

- Surface Integral
- Stokes' Theorem

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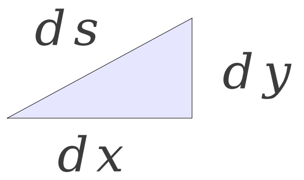
Arc Length In the Plane

$$y = f(x)$$

$$s = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$



Surface Area In the Space

$$z = f(x, y)$$

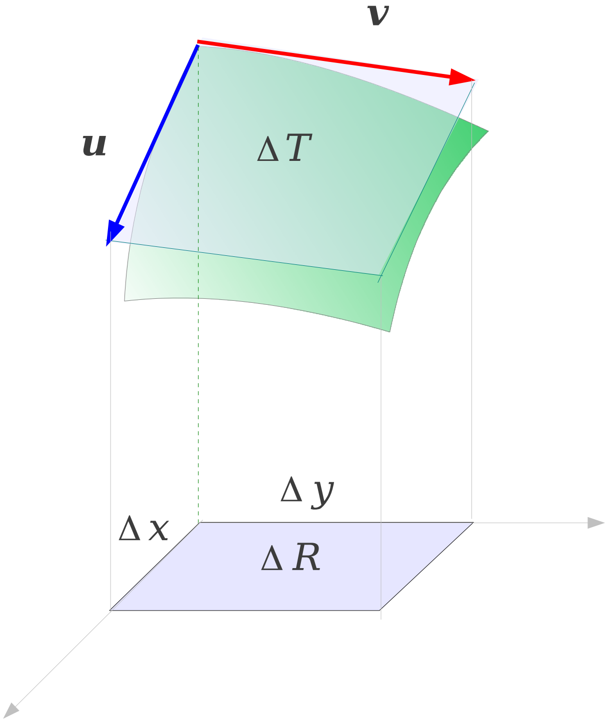
Area of the surface over R

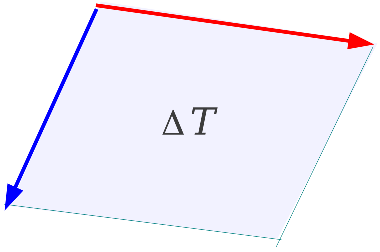
$$A(S) = \iint_R \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

$$= \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA$$

Differential of surface area

$$dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$



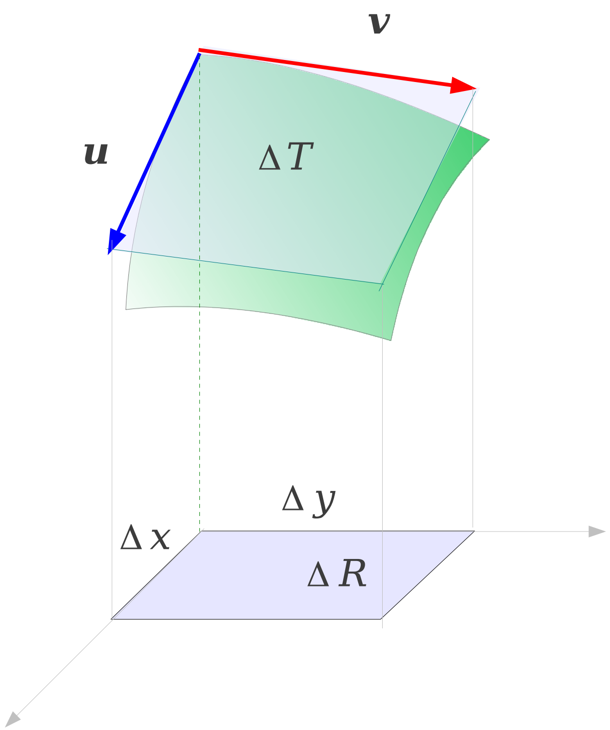


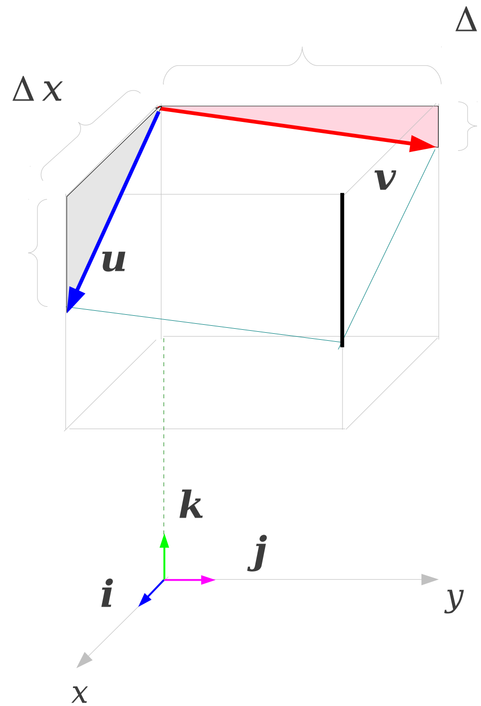
Differential of Surface Area (1)

Differential of surface area

$$dS = \sqrt{1 + \left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2}\, dA$$

Slope along x direction	Slope along y direction
$\frac{\partial f}{\partial x} = -0.4$	$\frac{\partial f}{\partial y} = -0.2$





$\frac{\partial f}{\partial x}\Delta x$

$\frac{\partial f}{\partial y}\Delta y$

$$\boldsymbol{u} = \Delta x\, \boldsymbol{i} + \frac{\partial f}{\partial x}\Delta x\, \boldsymbol{k}$$

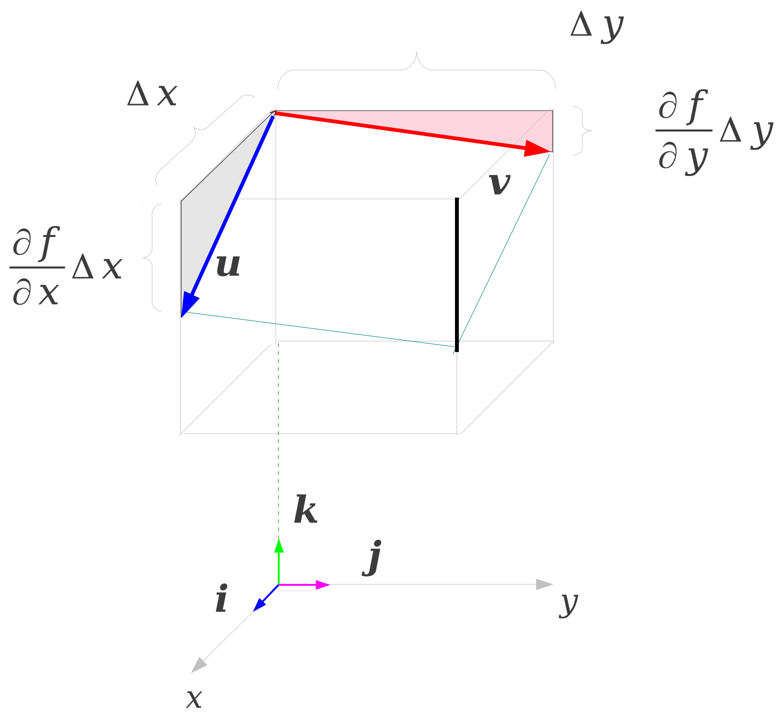
$$\boldsymbol{v} = \Delta y\, \boldsymbol{j} + \frac{\partial f}{\partial y}\Delta y\, \boldsymbol{k}$$

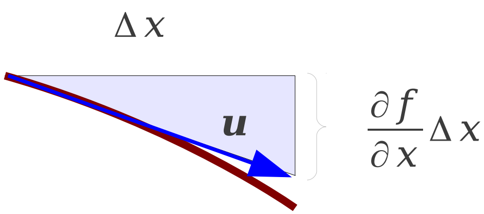
Differential of Surface Area (2)

Differential of surface area

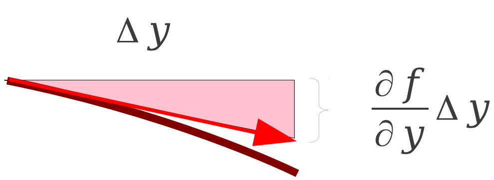
$$dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

Slope along x direction	Slope along y direction
$\dfrac{\partial f}{\partial x} = -0.4$	$\dfrac{\partial f}{\partial y} = -0.2$





$$\boldsymbol{u} = \Delta x\,\boldsymbol{i} + \frac{\partial f}{\partial x}\Delta x\,\boldsymbol{k}$$



$$\boldsymbol{v} = \Delta y\,\boldsymbol{j} + \frac{\partial f}{\partial y}\Delta y\,\boldsymbol{k}$$

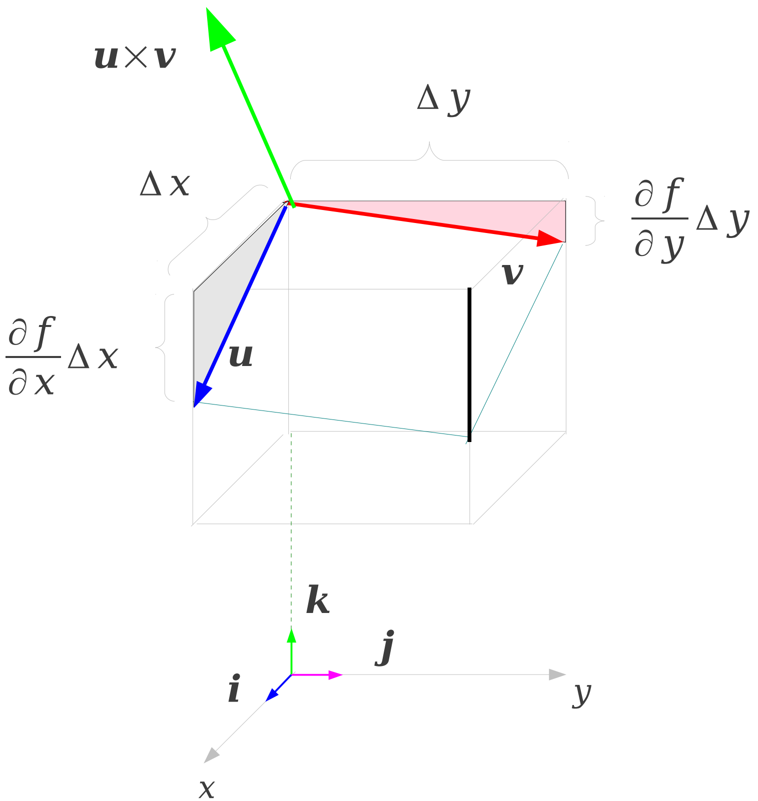
Differential of Surface Area (3)

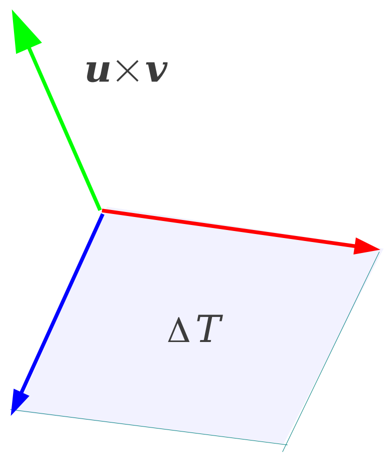
Differential of surface area

$$dS = \sqrt{1 + \left[f_x(x,y)\right]^2 + \left[f_y(x,y)\right]^2}\, dA$$

$$\boldsymbol{u} = \Delta x\, \boldsymbol{i} + \frac{\partial f}{\partial x}\Delta x\, \boldsymbol{k}$$

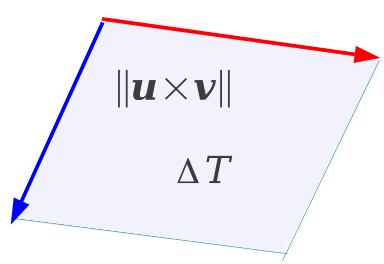
$$\boldsymbol{v} = \Delta y\, \boldsymbol{j} + \frac{\partial f}{\partial y}\Delta y\, \boldsymbol{k}$$





$$\boldsymbol{u}\times\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \Delta x & 0 & \frac{\partial f}{\partial x}\Delta x \\ 0 & \Delta y & \frac{\partial f}{\partial y}\Delta y \end{vmatrix}$$

$$= \left[-\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}\right]\Delta x \Delta y$$



$$\|\boldsymbol{u}\times\boldsymbol{v}\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\;\Delta x \Delta y$$

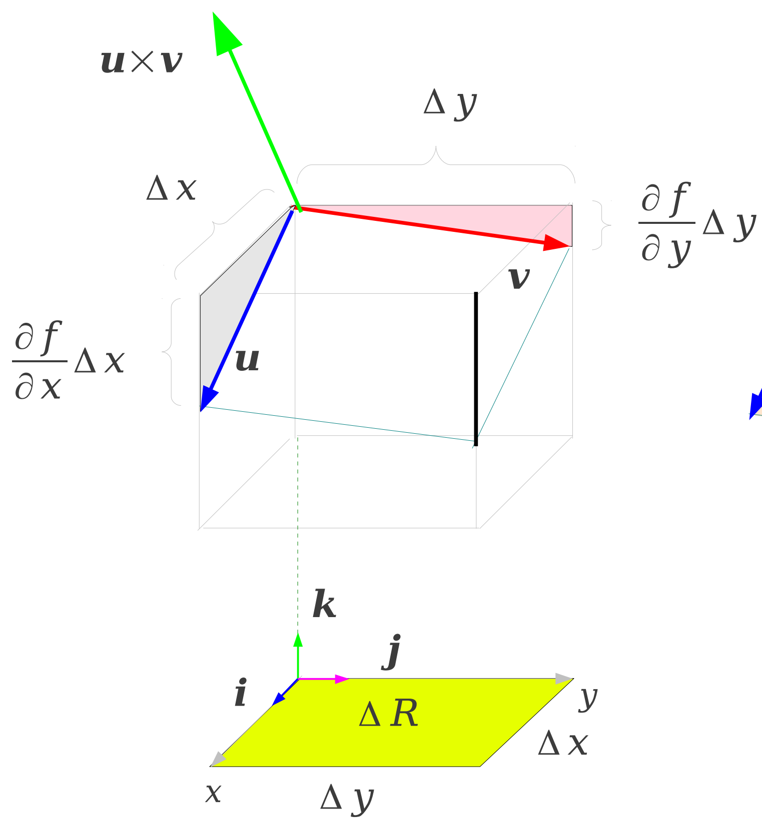
Differential of Surface Area (4)

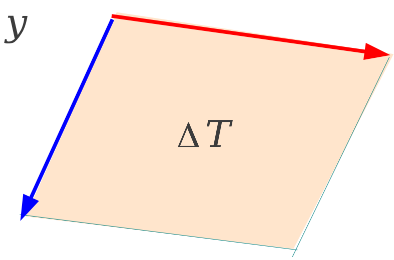
Differential of surface area

$$dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

$$\mathbf{u} = \Delta x\, \mathbf{i} + \frac{\partial f}{\partial x} \Delta x\, \mathbf{k}$$

$$\mathbf{v} = \Delta y\, \mathbf{j} + \frac{\partial f}{\partial y} \Delta y\, \mathbf{k}$$





$$\mathbf{u} \times \mathbf{v} = \left[-\frac{\partial f}{\partial x} \mathbf{i} - \frac{\partial f}{\partial y} \mathbf{j} + \mathbf{k} \right] \Delta x \Delta y$$

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\ \Delta x \Delta y$$

S **R**

$$\Delta T = \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ \Delta x \Delta y$$

$$\Delta S = \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ \Delta A$$

Line Integral with an Explicit Curve Function

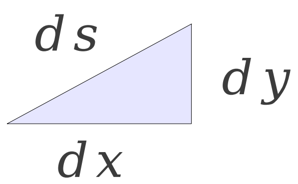
$y = f(x)$ ⇒ $\dfrac{dy}{dx} = f'(x)$ ⇒ $dy = f'(x)\, dx$

$a \le x \le b$

Curve C

$$ds = \sqrt{[dx]^2 + [dy]^2}$$

$$ds = \sqrt{1 + [f'(x)]^2}\, dx$$



$$\int_C G(x, y)\, dx = \int_a^b G(x, f(x))\, dx$$

$$\int_C G(x, y)\, dy = \int_a^b G(x, f(x))\, f'(x)\, dx$$

$$\int_C G(x, y)\, ds = \int_a^b G(x, f(x)) \sqrt{1 + [f'(x)]^2}\, dx$$

Surface Integral with an Explicit Surface Function

$z = f(x, y)$

$a \le x \le b$

$c \le y \le d$

Region R

$$\frac{df}{dx} = f_x(x, y)$$

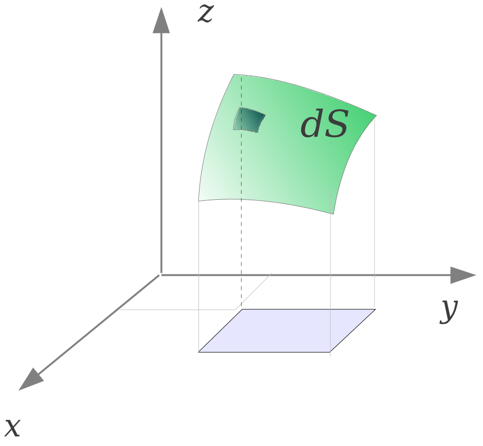
$$\frac{df}{dy} = f_y(x, y)$$

$$dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

$$\iint_S G(x, y, z)\, dS = \iint_R G(x, y, f(x,y)) \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

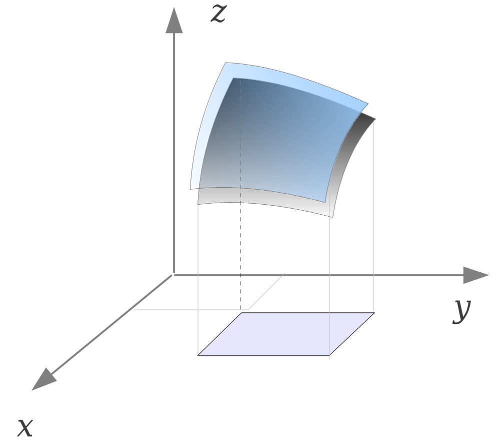
Surface Integral Over XY (1)

domain R^3



$G(x, y, f(x,y))$

range R^4

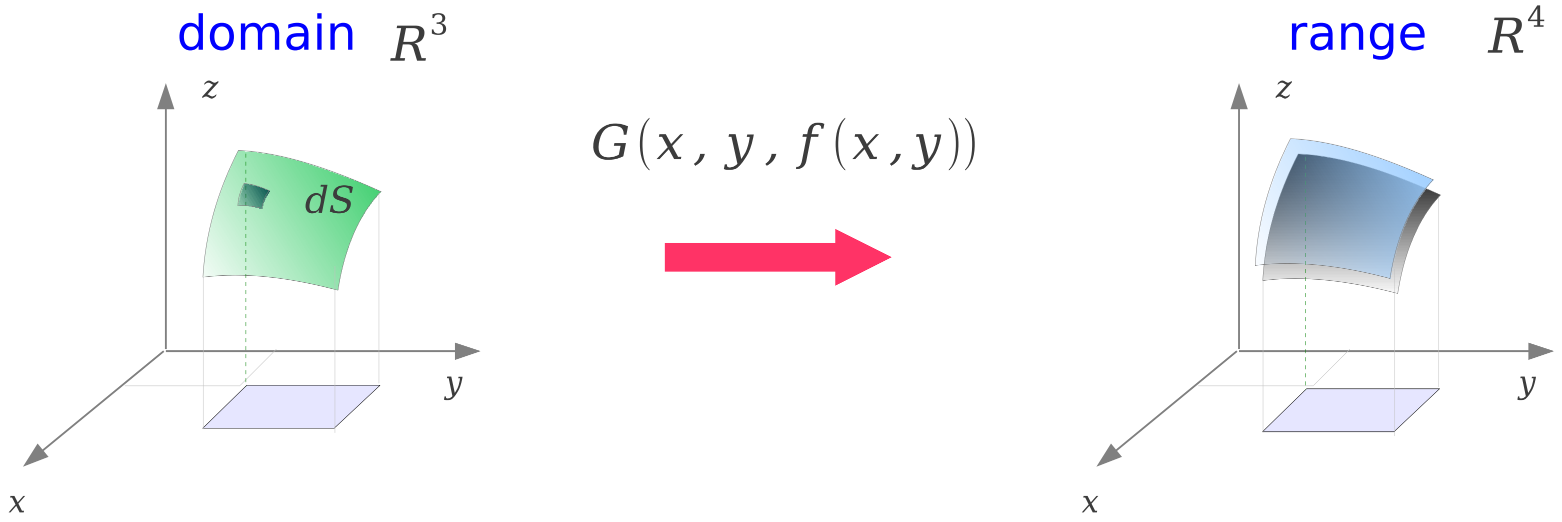


$z = f(x, y)$ $\cap$ $G(x, y, z)$ → $G(x, y, f(x,y))$

$a \le x \le b$

$c \le y \le d$

Surface Integral Over XY (2)

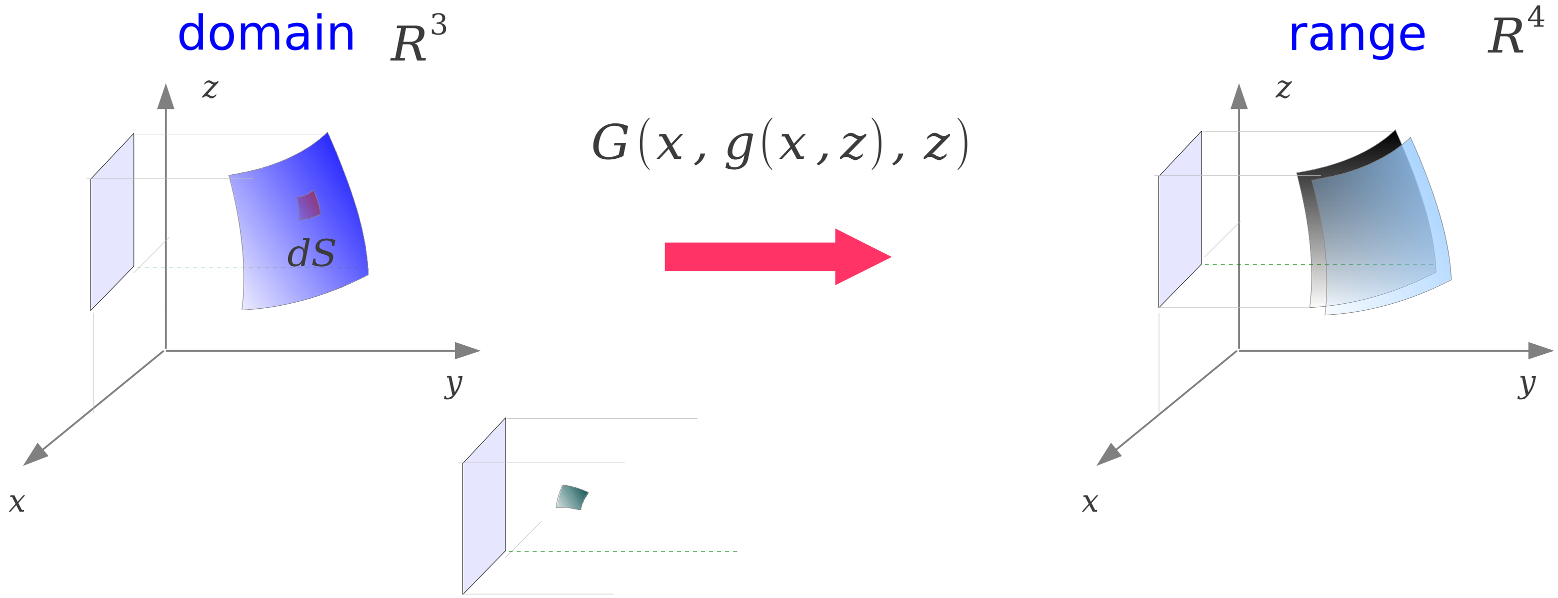


$$z = f(x, y)$$

$$\iint_S G(x, y, z)\, dS$$

$$= \iint_R G(x, y, f(x,y)) \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

Surface Integral Over XZ (1)

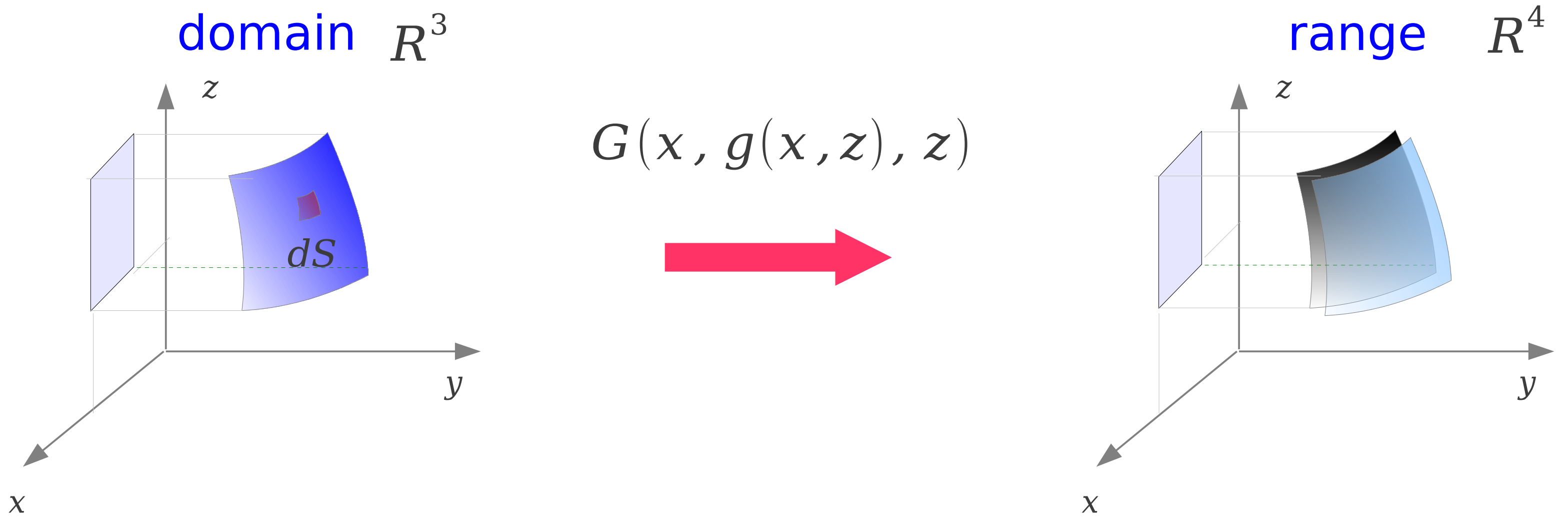


$y = g(x, z)$ $\cap$ $G(x, y, z)$ $\rightarrow$ $G(x, g(x,z), z)$

$a \leq x \leq b$

$c \leq y \leq d$

Surface Integral Over XZ (2)



$$y = g(x, z)$$

$$\iint_S G(x, y, z)\, dS$$

$$= \iint_R G(x, g(x,z), z) \sqrt{1 + [g_x(x,z)]^2 + [g_z(x,z)]^2}\, dA$$

Surface Integral with an Explicit Surface Function

$z = f(x, y)$ ⇒ $\frac{df}{dx} = f_x(x, y)$ ⇒ $dS = \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$

Region R $\quad \frac{df}{dy} = f_y(x, y)$

$$\iint_S G(x, y, z)\, dS = \iint_R G(x, y, f(x,y)) \sqrt{1 + [f_x(x,y)]^2 + [f_y(x,y)]^2}\, dA$$

$y = g(x, z)$ ⇒ $\frac{dg}{dx} = g_x(x, z)$ ⇒ $dS = \sqrt{1 + [g_x(x,z)]^2 + [g_z(x,z)]^2}\, dA$

Region R $\quad \frac{dg}{dz} = g_y(x, z)$

$$\iint_S G(x, y, z)\, dS = \iint_R G(x, g(x,z), z) \sqrt{1 + [g_x(x,z)]^2 + [g_z(x,z)]^2}\, dA$$

$x = h(y, z)$ ⇒ $\frac{dh}{dy} = f_y(y, z)$ ⇒ $dS = \sqrt{1 + [h_y(y,z)]^2 + [h_z(y,z)]^2}\, dA$

Region R $\quad \frac{dh}{dz} = f_z(y, z)$

$$\iint_S G(x, y, z)\, dS = \iint_R G(h(y,z), y, z) \sqrt{1 + [h_y(y,z)]^2 + [h_z(y,z)]^2}\, dA$$

Mass of a Surface

density $\rho(x, y, z)$

mass $m = \iint_S \rho(x, y, z)\, dS$

Orientation of a Surface

Oriented Surface — there exists a continuous unit normal vector function **n** defined at each point (x,y,z) on the surface

Orientation — the vector field **n**(x, y, z) or –**n**(x, y, z)

Upward orientation — z = f(x,y) : positive **k** component

Downward orientation — z = f(x,y) : negative **k** component

Surface $g(x,y,z) = 0$

Unit Normal Vector $\mathbf{n} = \frac{1}{\|\nabla g\|}\nabla g$ $\qquad \nabla g = \frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}$

Surface $z = f(x,y)$

$$g(x,y,z) = z - f(x,y) = 0$$

$$g(x,y,z) = f(x,y) - z = 0$$

Level Curve and Surface

Functions of two variables	$z = F(x, y)$	R^3	$G(x, y, z) = 0$
Level Curve	$c_i = F(x, y)$	R^2	$F(x, y) = 0$ when $z = 0$
	$0 = \nabla F(x, y) \cdot \boldsymbol{r}'(t)$		
Functions of three variables	$w = G(x, y, z)$	R^4	$H(x, y, z, w) = 0$
Level Surface	$c_i = G(x, y, z)$	R^3	$G(x, y, z) = 0$ when $w = 0$
	$0 = \nabla G(x, y, z) \cdot \boldsymbol{r}'(t)$		

Gradient & Normal Vector

Functions of three variables $w = F(x, y, z)$ $\mathbf{R}^4$

Level Surface $c_0 = F(x, y, z)$ $\mathbf{R}^3$

$$x = f(t)$$
$$y = g(t)$$
$$z = h(t)$$

$$\frac{d c_0}{d t} = \frac{d F}{d t}(x, y, z)$$

$$0 = \frac{\partial F}{\partial x}\frac{d x}{d t} + \frac{\partial F}{\partial y}\frac{d y}{d t} + \frac{\partial F}{\partial z}\frac{d z}{d t}$$

$$0 = \left(\frac{\partial F}{\partial x}\mathbf{i} + \frac{\partial F}{\partial y}\mathbf{j} + \frac{\partial F}{\partial z}\mathbf{k}\right) \cdot \left(\frac{d x}{d t}\mathbf{i} + \frac{d y}{d t}\mathbf{j} + \frac{d z}{d t}\mathbf{k}\right)$$

$$0 = \nabla F(x, y, z) \cdot \mathbf{r}'(t)$$

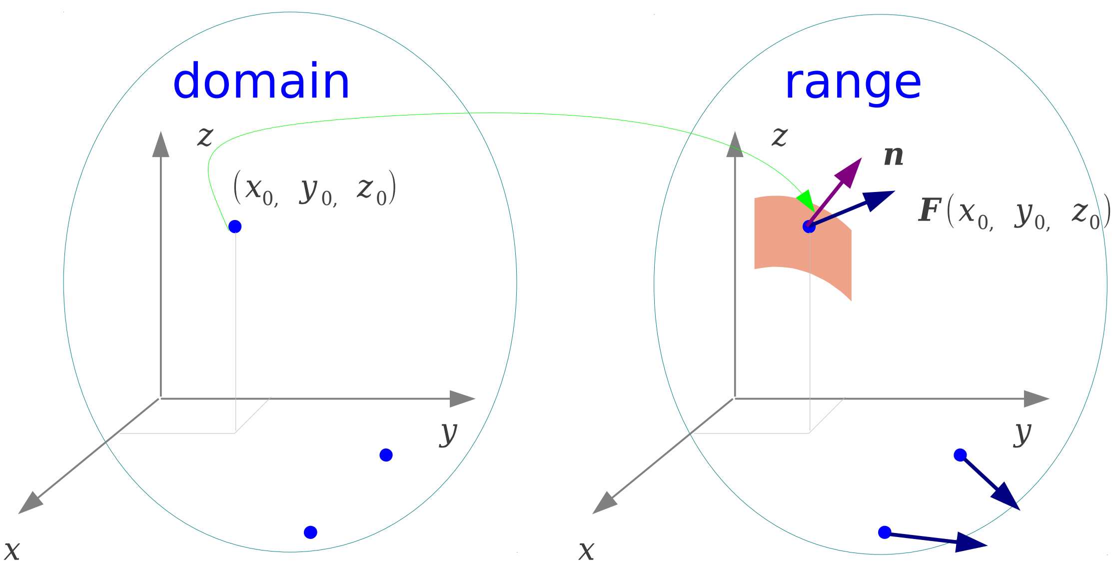
tangent vector

∇F **normal** to the level surface at $P(x_0, y_0, z_0)$

Surface Integral over a 3-D Vector Field (1)

A given point in a 3-d space ➡ A vector

$(x_0,\ y_0,\ z_0)$ $\langle\, P(x_0, y_0, z_0),\ Q(x_0, y_0, z_0),\ R(x_0, y_0, z_0)\,\rangle$



$(x_0,\ y_0,\ z_0)$

3 functions

$(x_0,\ y_0,\ z_0) \longrightarrow P(x_0, y_0, Z_0)$

$(x_0,\ y_0,\ z_0) \longrightarrow Q(x_0, y_0, z_0)$

$(x_0,\ y_0,\ z_0) \longrightarrow R(x_0, y_0, z_0)$

only points that are on the <u>surface</u> $\longrightarrow$ $\boldsymbol{F}(x_0,\ y_0,\ z_0) = P(x_0, y_0, z_0)\boldsymbol{i} + Q(x_0, y_0, z_0)\boldsymbol{j} + R(x_0, y_0, z_0)\boldsymbol{k}$

consider only the component of **F** along **n** $\longrightarrow$ $\boldsymbol{F} \cdot \boldsymbol{n}$

Surface Integral over a 3-D Vector Field (2)

$$\boldsymbol{F}(x, y, z) = P(x, y, z)\,\boldsymbol{i} + Q(x, y, z)\,\boldsymbol{j} + R(x, y, z)\,\boldsymbol{k}$$

Line Integral over a 3-d Vector Field

$$\boldsymbol{r}(t) = f(t)\,\boldsymbol{i} + g(t)\,\boldsymbol{j} + h(t)\,\boldsymbol{k}$$

$$\int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C P(x, y, z)\,dx + Q(x, y, z)\,dy + R(x, y, z)\,dy$$

Surface Integral over a 3-d Vector Field

$$\boldsymbol{n} = \frac{\nabla g}{\|\nabla g\|}$$

$$\iint_S (\boldsymbol{F} \cdot \boldsymbol{n})\,dS = \text{flux}$$

total volume of a fluid passing through S per unit time

$\boldsymbol{F}$ velocity field of a fluid

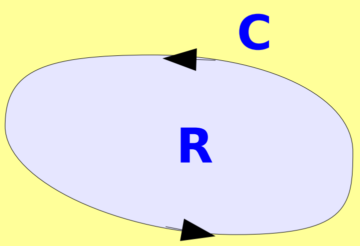
Vector Form of Green's Theorem

A force field $\boldsymbol{F}(x, y) = P(x, y)\boldsymbol{i} + Q(x, y)\boldsymbol{j}$

A smooth curve $C : x = f(t),\ \ y = g(t),\ \ a \le t \le b$

Work done by **F** along C $W = \int_c \boldsymbol{F} \cdot d\boldsymbol{r} = \int_c \boldsymbol{F} \cdot \boldsymbol{T}\, ds$

$$= \int_C P(x, y)\, dx + Q(x, y)\, dy$$



$$\oint_C P\, dx + Q\, dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

curl **F** $= \nabla \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \boldsymbol{k}$

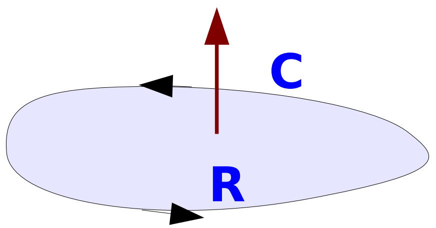
$$(\nabla \times \boldsymbol{F}) \cdot \boldsymbol{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

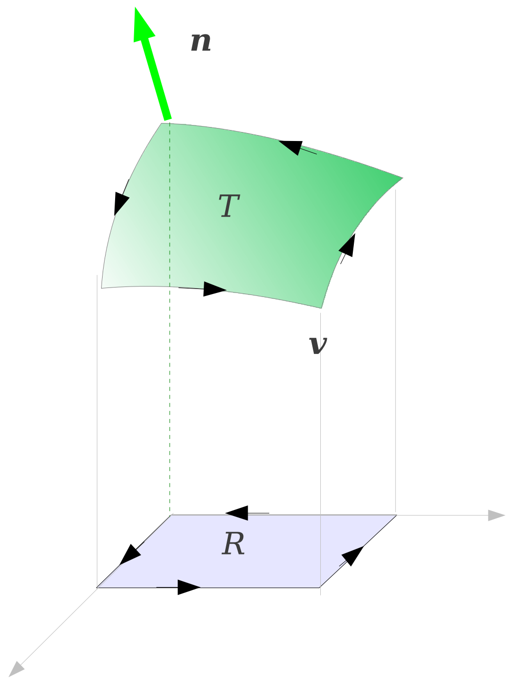
$$\oint_C P\, dx + Q\, dy = \iint_R (\nabla \times \boldsymbol{F}) \cdot k\, dA$$

Stokes' Theorem (1)

A force field $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$

Work done by **F** along C $W = \int_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds$





2-space $= \iint_R (\nabla\times\boldsymbol{F})\cdot\boldsymbol{k}\,dA$

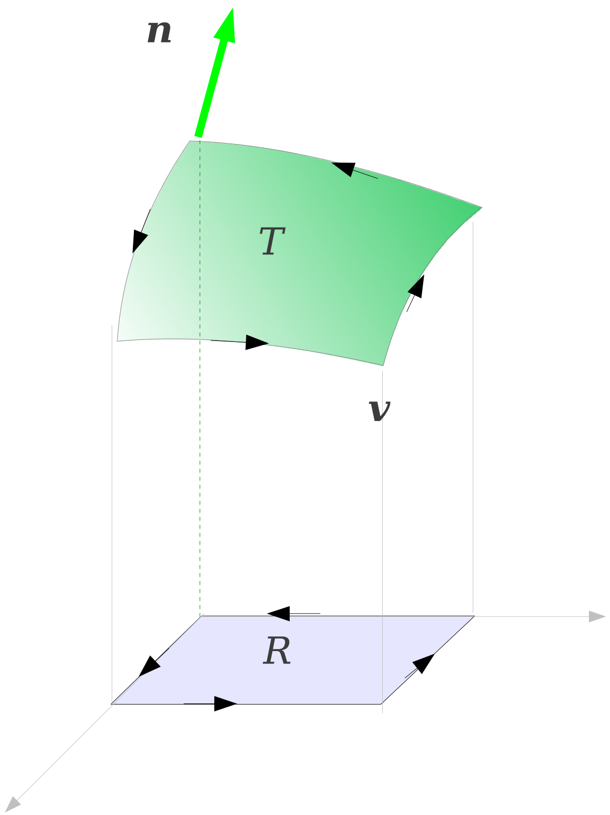
3-space $= \iint_R (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dA$

Stokes' Theorem (2)

A force field $\boldsymbol{F}(x,y) = P(x,y)\boldsymbol{i} + Q(x,y)\boldsymbol{j}$

Work done by **F** along C $W = \int_c \boldsymbol{F}\cdot d\boldsymbol{r} = \int_c \boldsymbol{F}\cdot\boldsymbol{T}\,ds$

3-space $= \iint_R (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dA$



curl **F** $= \nabla\times\boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\boldsymbol{k}$$

$$\mathbf{n} = \frac{\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k}}{\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}$$

surface
$g(x,y,z) = z - f(x,y)$

Gradient of a 2 Variable Function

Function of two variables $f(x, y)$

$$\nabla f(x,y) = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}$$

⇨ vector

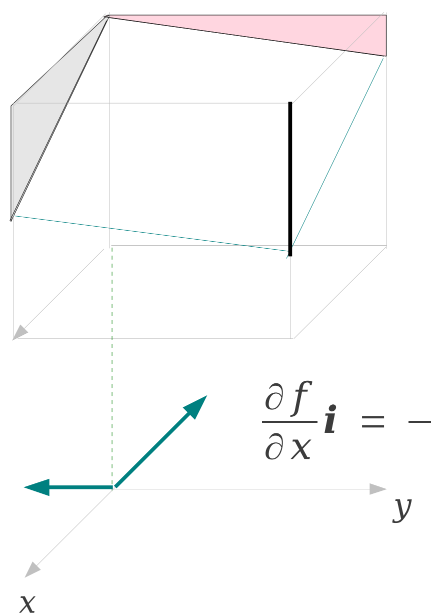
Rate of change of **f** in the **x** direction
Rate of change of **f** in the **y** direction

Slope in the x direction

$$\frac{\partial f}{\partial x} = -2$$

Slope in the y direction

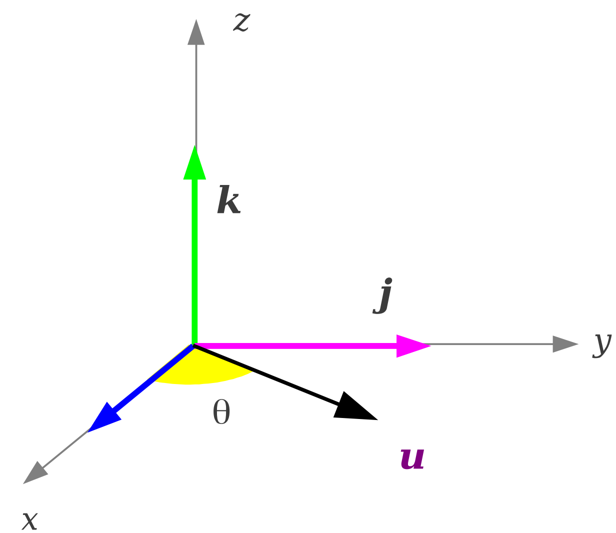
$$\frac{\partial f}{\partial y} = -1$$



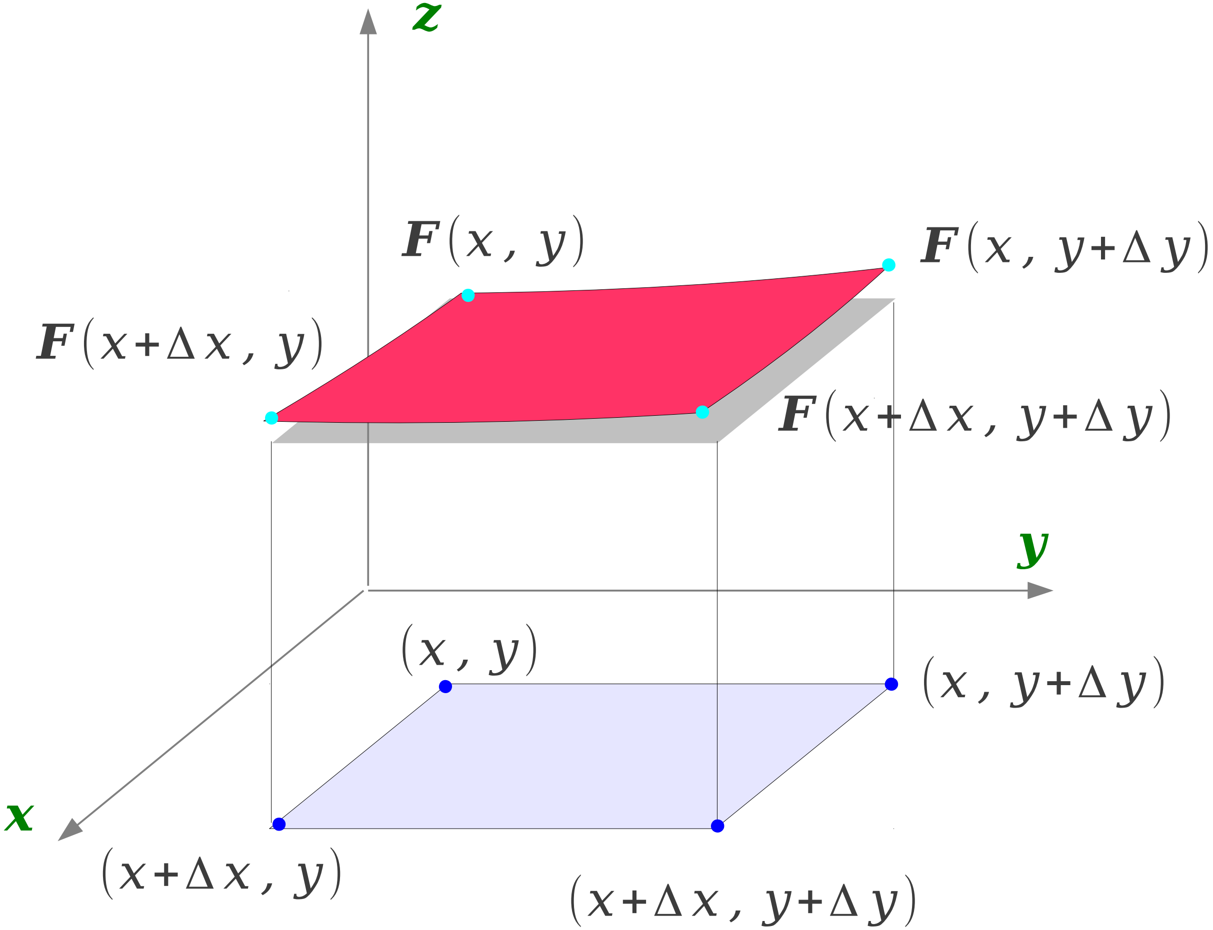
$$\frac{\partial f}{\partial x}\mathbf{i} = -2\mathbf{i}$$

$$\frac{\partial f}{\partial y}\mathbf{j} = -1\,\mathbf{j}$$

$\mathbf{i}$



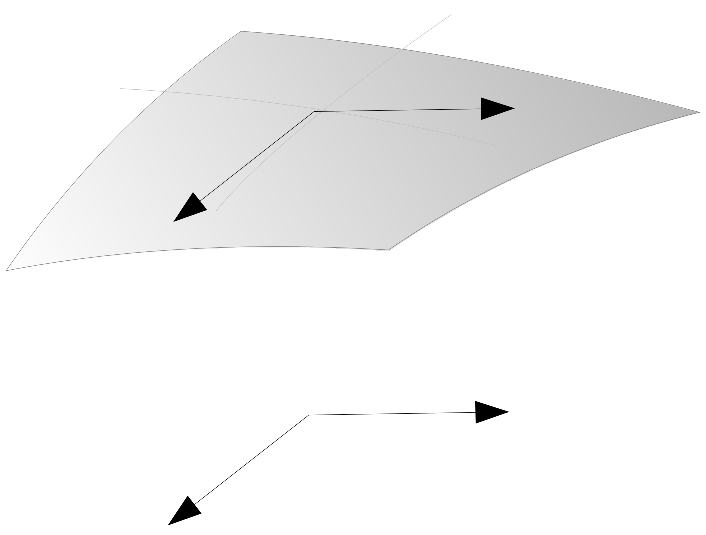
2-D Divergence

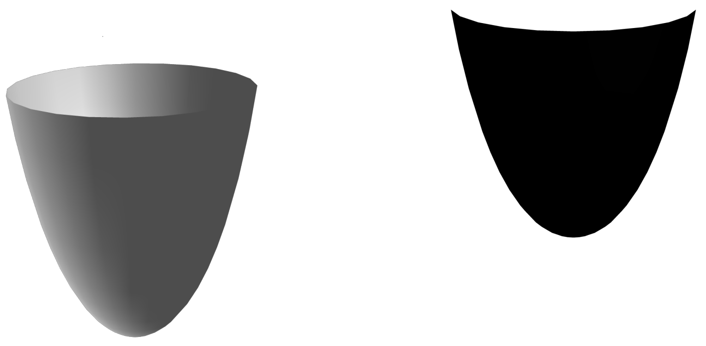


Chain Rule

Function of two variable $y = f(u, v)$

$u = g(x, y) \qquad v = h(x, y)$





References

[1] http://en.wikipedia.org/
[2] http://planetmath.org/
[3] M.L. Boas, “Mathematical Methods in the Physical Sciences”
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