

# Spectra (1A)

---

-

Copyright (c) 2011 Young W. Lim.

Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.2 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

Please send corrections (or suggestions) to [youngwlim@hotmail.com](mailto:youngwlim@hotmail.com).

This document was produced by using OpenOffice and Octave.

# Single-Sided Spectrum

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k \omega_0 t) + b_k \sin(k \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = +1, +2, \dots$$

$$x(t) = g_0 + \sum_{k=1}^{\infty} g_k \cos(k \omega_0 t + \phi_k)$$

$$g_0 = a_0$$

$$g_k = \sqrt{a_k^2 + b_k^2}$$

$$\phi_k = \tan^{-1} \left( -\frac{b_k}{a_k} \right)$$

$$k = +1, +2, \dots$$

$$\cos(\alpha + \beta) = \underline{\cos(\alpha) \cos(\beta)} - \underline{\sin(\alpha) \sin(\beta)}$$

$$g_k \cos(k \omega_0 t + \phi_k) = \underline{g_k \cos(\phi_k) \cos(k \omega_0 t)} - \underline{g_k \sin(\phi_k) \sin(k \omega_0 t)}$$

$$\underline{a_k \cos(k \omega_0 t)} + \underline{b_k \sin(k \omega_0 t)}$$

$$a_k = g_k \cos(\phi_k)$$

$$-b_k = g_k \sin(\phi_k)$$

$$a_k^2 + b_k^2 = g_k^2$$

$$-\frac{b_k}{a_k} = \tan(\phi_k)$$

# Two-Sided Spectrum

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}\sqrt{a_k^2 + b_k^2} & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} \tan^{-1}(-b_k/a_k) & (k > 0) \\ \tan^{-1}(+b_k/a_k) & (k < 0) \end{cases}$$

Power Spectrum *Two-Sided*

$$\underline{|C_k|^2 + |C_{-k}|^2} = \frac{1}{2}g_k^2 = \frac{1}{2}(a_k^2 + b_k^2)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} C_k e^{+jk\omega_0 t}$$

$$C_k = \frac{1}{T} \int_0^T x(t) e^{-jk\omega_0 t} dt$$

$$k = \dots, -2, -1, 0, +1, +2, \dots$$

$$C_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}g_k e^{+j\phi_k} & (k > 0) \\ \frac{1}{2}g_k e^{-j\phi_k} & (k < 0) \end{cases}$$

$$|C_k| = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}|g_k| & (k \neq 0) \end{cases}$$

$$\text{Arg}(C_k) = \begin{cases} +\phi_k & (k > 0) \\ -\phi_k & (k < 0) \end{cases}$$

Periodogram *One-Sided*

$$2 \cdot |C_k| = \underline{g_k} = \underline{\sqrt{a_k^2 + b_k^2}}$$

# Power Spectrum (1)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n \omega_0 t) + b_n \sin(n \omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_k = \frac{2}{T} \int_0^T x(t) \cos(k \omega_0 t) dt$$

$$b_k = \frac{2}{T} \int_0^T x(t) \sin(k \omega_0 t) dt$$

$$k = 1, 2, \dots$$

Average Power

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$P = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \frac{1}{2}(a_k^2 + b_k^2) & (k > 0) \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+jk \omega_0 t}$$

$$c_k = \frac{1}{T} \int_0^T x(t) e^{-jk \omega_0 t} dt$$

$$k = -2, -1, 0, +1, +2, \dots$$

$$c_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

Average Power

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt$$

$$P = |c_0|^2 + \sum_{k=1}^{\infty} (|c_k|^2 + |c_{-k}|^2)$$

Power Spectrum

$$P_k = \begin{cases} |c_0|^2 & (k = 0) \\ (|c_k|^2 + |c_{-k}|^2) & (k > 0) \end{cases}$$

# Power Spectrum (2)

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+jk\omega_0 t}$$

Average Power  $P = \frac{1}{T} \int_0^T |x(t)|^2 dt$

Average Power  $P = \frac{1}{T} \int_0^T |x(t)|^2 dt$

$$P = a_0^2 + \frac{1}{2} \sum_{k=1}^{+\infty} (a_k^2 + b_k^2)$$

$$P = |c_0|^2 + \sum_{k=1}^{+\infty} (|c_k|^2 + |c_{-k}|^2)$$

$$\begin{aligned} &|c_k|^2 + |c_{-k}|^2 \\ &= \frac{1}{2} (a_k^2 + b_k^2) \end{aligned}$$



$$\begin{aligned} |c_k|^2 &= \frac{1}{4} (a_k^2 + b_k^2) \\ |c_{-k}|^2 &= \frac{1}{4} (a_k^2 + b_k^2) \end{aligned}$$



$$c_k = \begin{cases} a_0 & (k = 0) \\ \frac{1}{2}(a_k - jb_k) & (k > 0) \\ \frac{1}{2}(a_k + jb_k) & (k < 0) \end{cases}$$

Power Spectrum

Power Spectrum

$$P_k = \begin{cases} a_0^2 & (k = 0) \\ \frac{1}{2} (a_k^2 + b_k^2) & (k > 0) \end{cases}$$



$$P_k = \begin{cases} |c_0|^2 & (k = 0) \\ (|c_k|^2 + |c_{-k}|^2) & (k > 0) \end{cases}$$

# Periodogram

$$x(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t))$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos(n\omega_0 t) dt$$

$$b_n = \frac{2}{T} \int_0^T x(t) \sin(n\omega_0 t) dt$$

$$n = 1, 2, \dots$$

Positive Frequency Only

$$f = \frac{n}{T} \quad n = 1, 2, \dots$$

Single-sided Spectrum

$$\sqrt{a_n^2 + b_n^2} \quad \text{sometimes } (a_n^2 + b_n^2)$$

An estimate of the spectral density of a signal

$$\frac{T}{2} a_n = \int_{t_1}^{t_1+T} x(t) \cos(kt) dt$$

$$\frac{T}{2} b_n = \int_{t_1}^{t_1+T} x(t) \sin(kt) dt$$

T: integer multiple  $\frac{2\pi}{k}$

$$T = \frac{2\pi}{k} \cdot n$$

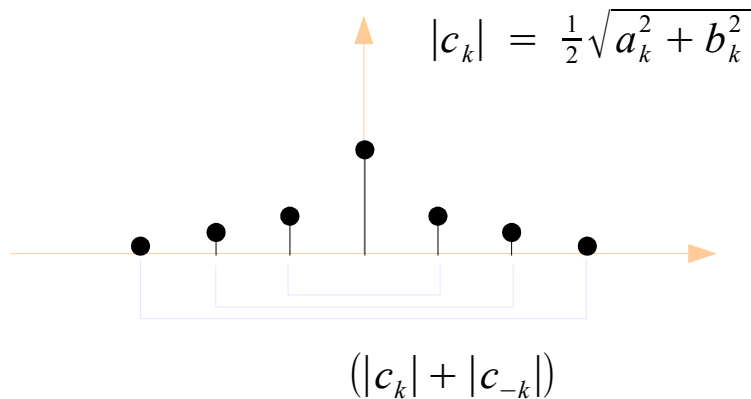
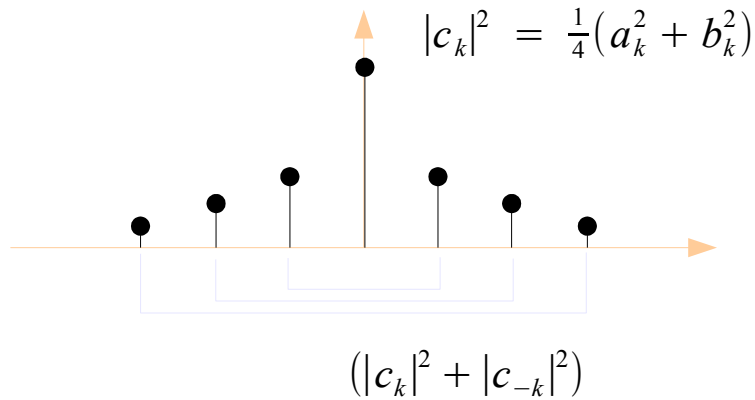
$$k = \frac{2\pi}{T} \cdot n = n\omega_0$$

abscissa  $\frac{2\pi}{k}$

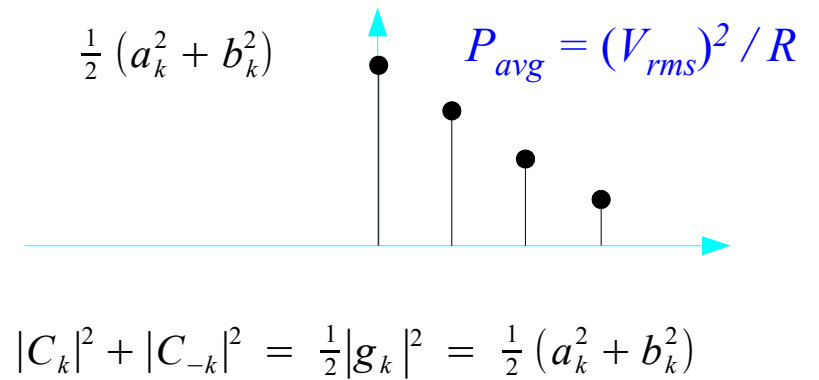
ordinates  $r = \sqrt{a_n^2 + b_n^2}$

# Periodogram and Power Spectrum

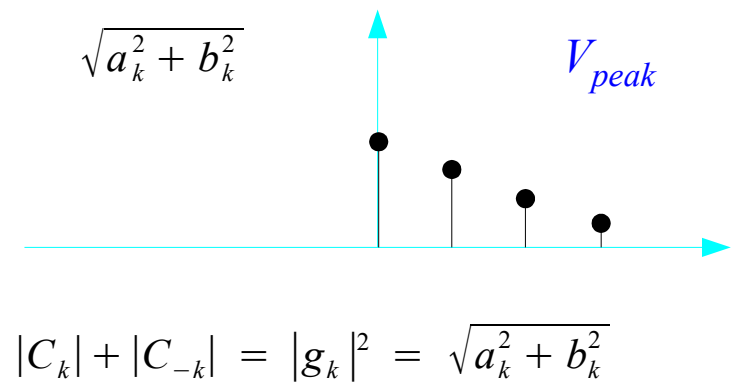
## Power Spectrum Single-Sided



## Power Spectrum Two-Sided



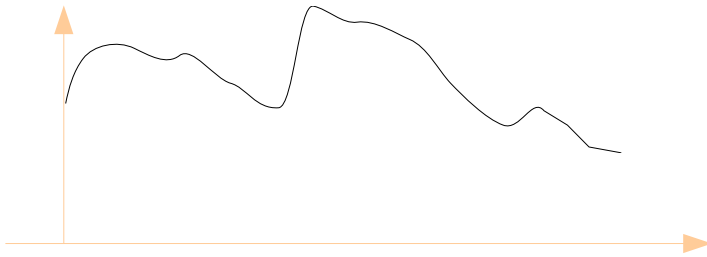
## Periodogram





# Root Mean Square

## Continuous Time



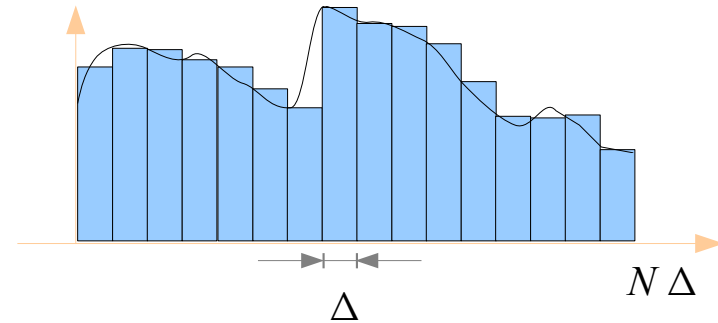
mean squared amplitude

$$\frac{1}{T} \int_0^T g^2(t) dt$$

root mean squared

$$g_{rms} = \sqrt{\frac{1}{T} \int_0^T g^2(t) dt}$$

## Discrete Time



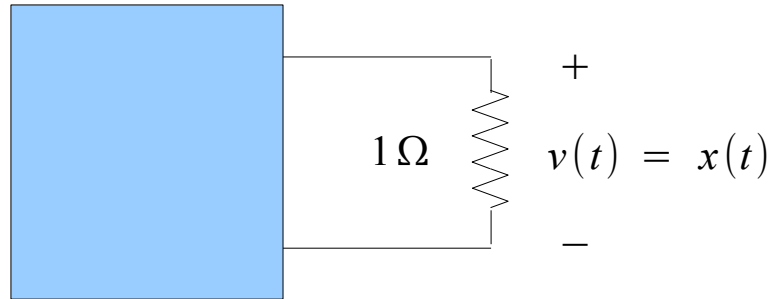
mean squared amplitude

$$\frac{1}{N\Delta} \sum_{k=0}^{N-1} |g[k]|^2 \Delta = \frac{1}{N} \sum_{k=0}^{N-1} |g[k]|^2$$

root mean squared

$$g_{rms} = \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} |g[k]|^2}$$

# Continuous Periodic Signal



instantaneous power

$$x^2(t)$$

average power

$$\frac{1}{T} \int_0^T x^2(t) dt$$

$T$  : period

$v(t) = x(t)$  Continuous Periodic



CTFS (Fourier Series)

Parseval's Theorem

$$x(t) = \sum_{n=-\infty}^{+\infty} C_n e^{+jn\omega_0 t}$$

$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt$$

$$n = -2, -1, 0, +1, +2, \dots$$

$$\frac{1}{T} \int_0^T x^2(t) dt = \sum_{n=-\infty}^{+\infty} |C_n|^2$$

average power

sum of  
power spectrum  $C_n$

# CTFS and CTFT

## Continuous Time Fourier Series

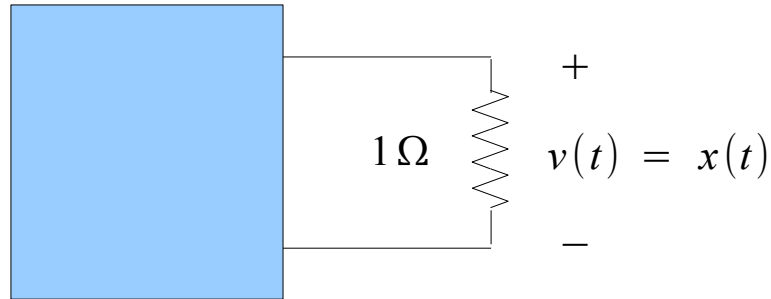
$$C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt \quad \longleftrightarrow \quad x(t) = \sum_{n=0}^{\infty} C_n e^{+jn\omega_0 t}$$

## Continuous Time Fourier Transform

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{+j\omega t} d\omega$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt \quad \longleftrightarrow \quad x(t) = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi f t} df$$

# Continuous Aperiodic Signal



instantaneous power

$$x^2(t)$$

total energy

$$\int_{-\infty}^{+\infty} x^2(t) dt$$

$v(t) = x(t)$  Continuous Aperiodic



CTFT (Fourier Integral)

Parseval's Theorem

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi f t} df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi f t} dt$$

$$\int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

total energy

integral of  
energy spectral  
density  $|X(f)|^2$

# Average Power of Random Signals

## A truncated sample function

$$\begin{aligned}x_T(t) &= x(t) & -\frac{T}{2} < t < +\frac{T}{2} \\ &= 0 & \textit{otherwise}\end{aligned}$$

## Fourier Transform

$$X_T(\omega) = \int_{-\infty}^{+\infty} x_T(t) e^{-j\omega t} dt \quad \longleftrightarrow \quad x_T(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X_T(\omega) e^{+j\omega t} d\omega$$

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(t) e^{-j2\pi f t} dt \quad \longleftrightarrow \quad x_T(t) = \int_{-\infty}^{+\infty} X_T(f) e^{+j2\pi f t} df$$

## Parseval's Theorem

$$\int_{-\infty}^{+\infty} |x_T(t)|^2 dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 df \quad \text{total energy}$$

## Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} dt \quad \text{total energy} / T$$

# Power Spectral Density of Random Signals

## Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \Rightarrow \int_{-\infty}^{+\infty} \hat{S}_{xx}(f) df$$

## Raw Power Spectral Density

$$\frac{|X_T(f)|^2}{T} = \hat{S}_{xx}(f)$$

## Power Spectral Density

$$\lim_{T \rightarrow \infty} \frac{E[|X_T(f)|^2]}{T} = S_{xx}(f)$$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt = \lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \quad \rightarrow \quad \text{not converge}$$

$$E \left[ \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt \right] = E \left[ \lim_{T \rightarrow \infty} \int_{-\infty}^{+\infty} \frac{|X_T(f)|^2}{T} df \right] \quad \text{Random Signal}$$

$$\text{Var}(x(t)) = \sigma_x^2 = \int_{-\infty}^{+\infty} \lim_{T \rightarrow \infty} \frac{E[|X_T(f)|^2]}{T} df \Rightarrow \int_{-\infty}^{+\infty} S_{xx}(f) df$$

# Parseval's Theorem – CTFS

## Average Power

$$E[x^2(t)] = \frac{1}{T} \int_{-\infty}^{+\infty} x^2(t) dt$$

## CTFS

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{+jk\omega t} \quad x^2(t) = x(t) \cdot x^*(t) = \left( \sum_{k=-\infty}^{+\infty} c_k e^{+jk\omega t} \right) \left( \sum_{l=-\infty}^{+\infty} c_l^* e^{-jl\omega t} \right)$$

$$x^2(t) = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} c_k c_l^* e^{+j(k-l)\omega t}$$

$$\frac{1}{T} \int_{-\infty}^{+\infty} x^2(t) dt = \sum_{k=-\infty}^{+\infty} \sum_{l=-\infty}^{+\infty} c_k c_l^* \left( \frac{1}{T} \int_{-\infty}^{+\infty} e^{+j(k-l)\omega t} dt \right)$$

## Average Power

$$\frac{1}{T} \int_{-\infty}^{+\infty} x^2(t) dt = \sum_{k=-\infty}^{+\infty} |c_k|^2$$

## Power Spectrum

$$|c_k|^2$$

*power associated with individual frequency components*

# Parseval's Theorem – CTFT

## Total Energy

$$\int_{-\infty}^{+\infty} x^2(t) dt$$

## CTFT

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{+j2\pi ft} df \quad x^2(t) = x(t) \cdot x^*(t) = \left( \int_{-\infty}^{+\infty} X(f) e^{+j2\pi ft} df \right) \left( \int_{-\infty}^{+\infty} X^*(\nu) e^{-j2\pi \nu t} d\nu \right)$$

$$x^2(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |X(f)|^2 e^{+j2\pi(f-\nu)t} df d\nu$$

$$\int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |X(f)|^2 \int_{-\infty}^{+\infty} e^{+j2\pi(f-\nu)t} dt df d\nu$$

## Total Energy

$$\int_{-\infty}^{+\infty} x^2(t) dt = \int_{-\infty}^{+\infty} |X(f)|^2 df$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 d\omega$$

## Energy Spectral Density

$$|X(f)|^2$$

*energy associated with individual frequency components*



# Parseval's Theorem – DTFT

Total Energy

$$\int_{-\infty}^{+\infty} x^2(t) dt$$

Total energy over all discrete time  $n$

Total energy in fundamental period of frequency

CTFT

$$x[n] = \int_{-0.5}^{+0.5} X(e^{j\hat{f}}) e^{+j2\pi\hat{f}n} d\hat{f}$$

$$x^2[n] = x[n] \cdot x^*[n] = \left( \int_{-0.5}^{+0.5} X(e^{j\hat{f}}) e^{+j2\pi\hat{f}n} d\hat{f} \right) \left( \int_{-0.5}^{+0.5} X^*(e^{j\hat{v}}) e^{+j2\pi\hat{v}n} d\hat{v} \right)$$

$$x^2[n] = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 e^{+j2\pi(\hat{f}-\hat{v})n} d\hat{f} d\hat{v}$$

$$\sum_{n=-\infty}^{+\infty} x^2[n] dt = \int_{-0.5}^{+0.5} \int_{-0.5}^{+0.5} |X(e^{j\hat{f}})|^2 \sum_{n=-\infty}^{+\infty} e^{+j2\pi(\hat{f}-\hat{v})n} d\hat{f} d\hat{v}$$

Total Energy

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \int_{-0.5}^{+0.5} |X(\hat{f})|^2 d\hat{f}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} |X(\hat{\omega})|^2 d\hat{\omega}$$

Energy Spectral Density

$$|X(\hat{f})|^2$$

energy associated with individual frequency components

# Parseval's Theorem – DFT

## Average Power

$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] \approx \frac{1}{NT} \sum_0^{NT} x^2(t) dt$$

## CTFT

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{+j\left(\frac{2\pi}{N}\right)kn} \quad x^2[n] = x[n] \cdot x^*[n] = \frac{1}{N^2} \left( \sum_{k=0}^{N-1} X[k] e^{+j\left(\frac{2\pi}{N}\right)kn} \right) \left( \sum_{l=0}^{N-1} X^*[l] e^{-j\left(\frac{2\pi}{N}\right)ln} \right)$$

$$x^2[n] = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X[k] X^*[l] e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

$$\frac{1}{N} \sum_{n=0}^{N-1} x^2[n] dt = \frac{1}{N^3} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} X[k] X^*[l] \sum_{n=0}^{N-1} e^{+j\left(\frac{2\pi}{N}\right)(k-l)n}$$

## Average Power

$$\frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N^2} \sum_{k=0}^{N-1} |X[k]|^2$$

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X[k]|^2$$

## Periodogram

$$\frac{1}{N} |X[k]|^2$$

# Power and Power Density Spectra

## Average Power

$$E[x^2(t)] = \frac{1}{T} \int_{-\infty}^{+\infty} x_T^2(t) dt$$

## Average Power of N sample of x(t)

$$\frac{1}{N} \sum_{n=0}^{N-1} x_n^2 \approx \frac{1}{N \Delta t} \int_0^{N \Delta t} x^2(t) dt = \frac{1}{N^2} \sum_{m=0}^{N-1} |X_m|^2$$

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} |X_m|^2 \quad m = 0, 1, \dots, N-1$$

## Periodogram:

$$P_{xx}(m) = \frac{1}{N} |X_m|^2$$

$$m = 0, 1, \dots, N-1$$

## Average Power

$$\sum_{n=0}^{N-1} x_n^2 = \frac{1}{N} \sum_{m=0}^{N-1} P_{xx}(m)$$

## Total Energy

$$T \sum_{n=0}^{N-1} x_n^2 = T \sum_{m=0}^{N-1} P_{xx}(m)$$

$$\approx \int_0^{N \Delta t} x^2(t) dt$$

Periodic Signals

Aperiodic Signals

Random Signals

Frequency Spacing

$$\Delta f = \frac{1}{N\Delta t}$$

$$\Delta f = \frac{1}{N\Delta t}$$

$$\sum S \Delta f = \frac{1}{N\Delta t} \sum S \quad \frac{1}{N\Delta t} \sum x^2 \Delta t$$

Two Sided

$$\frac{1}{N} X(k)$$

$$\frac{\Delta t}{N} X(k)$$

$$S(k) = \frac{\Delta t}{N} |X(k)|^2 \quad P = \sum_{k=0}^{N-1} S(k) \Delta f$$

One Sided

$$k=0, \frac{N}{2}$$

$$\frac{1}{N} X(k)$$

$$\frac{\Delta t}{N} X(k)$$

$$S_1(k) = 2S(k) \quad P = \sum_{k=0}^{N/2} S_1(k) \Delta f$$

$$k=1, \dots, \frac{N}{2}-1$$

$$\frac{2}{N} X(k)$$

$$\frac{2\Delta t}{N} X(k)$$

$$S_1(k) = S(k)$$

Frequency Scale

$$k \Delta f$$

$$k \Delta f$$

$$k \Delta f$$

## References

- [1] <http://en.wikipedia.org/>
- [2] J.H. McClellan, et al., Signal Processing First, Pearson Prentice Hall, 2003
- [3] K.H. Shin, J.K. Hammond, Fundamentals of Signal Processing for Sound and Vibration Engineers, Wiley, 2008
- [4] Brian D. Storey, "Computing Fourier Series and Power Spectrum with MATLAB"
- [5] Univ. of Rhode Island, ELE436, "FFT Tutorial"
- [6] Guy Beale, "Signal & Systems Fourier Series Example #2"
- [7] National Instrument, "The Fundamentals of FFT-Based Signal Analysis and Measurement in LabVIEW and LabWindows/CVI"
- [8] Cambridge University Press, "Numerical Recipes"
- [9] E. Overman, "A MATLAB Tutorial"