CLTI Correlation (2A)

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Please send corrections (or suggestions) to youngwlim@hotmail.com.

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Correlation

How signals move
relative to each other

Positively correlated — the same direction

Average of product > product of averages

Negatively correlated — the opposite direction

Average of product < product of averages

Uncorrelated

CrossCorrelation for Power Signals

Energy Signal

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t+\tau)\, dt$$
$$= \int_{-\infty}^{+\infty} x(t-\tau)\, y^*(t)\, dt$$

Power Signal

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)\, y^*(t+\tau)\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_T x(t-\tau)\, y^*(t)\, dt$$

Energy Signal real $x(t),\ y(t)$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt$$
$$= \int_{-\infty}^{+\infty} x(t-\tau)\, y(t)\, dt$$

Power Signal real $x(t),\ y(t)$

$$R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_T x(t)\, y(t+\tau)\, dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_T x(t-\tau)\, y(t)\, dt$$

Periodic Power Signal

$$R_{xy}(\tau) = \frac{1}{T} \int_T x(t)\, y(t+\tau)\, dt$$

Correlation and Convolution

real $x(t), \; y(t)$

Correlation $$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt = \int_{-\infty}^{+\infty} x(t-\tau)\, y(t)\, dt$$

Convolution $$x(t) * y(t) = \int_{-\infty}^{+\infty} x(t-\tau)\, y(\tau)\, d\tau$$

$$R_{xy}(\tau) = x(-\tau) * y(\tau)$$

$$x(-t) \Longleftrightarrow X^*(f)$$

$$R_{xy}(\tau) \Longleftrightarrow X^*(f)\, Y(f)$$

Correlation for Periodic Power Signals

$$R_{xy}(\tau) = \frac{1}{T}\int_T x(t)\,y(t+\tau)\,dt$$

Periodic Power Signal

$$R_{xy}(\tau) = \frac{1}{T}[x(-\tau) \circledast y(\tau)]$$

$$R_{xy}(\tau) \overset{\text{CTFS}}{\Longleftrightarrow} X^*[k]Y[k]$$

Circular Convolution

$$x(t) * y(t) \overset{\text{CTFS}}{\Longleftrightarrow} T\,X[k]Y[k]$$

$$x[n] * y[n] \overset{\text{CTFS}}{\Longleftrightarrow} N_0 Y[k]X[k]$$

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\,y(t+\tau)\,dt$$

$$R_{xy}(\tau) = \frac{1}{T}\int_T x(t)\,y(t+\tau)\,dt$$

Correlation for Power & Energy Signals

One signal – a power signal
The other – an energy signal

Use the Energy Signal Version

$$R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y(t+\tau)\, dt$$

Autocorrelation

Energy Signal

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(t)x(t+\tau)\,dt$$

total signal energy

$$R_{xx}(0) = \int_{-\infty}^{+\infty} x^2(t)\,dt$$

$$R_{xx}(0) \ge R_{xx}(\tau)$$

max at zero shift

$$R_{xx}(-\tau) = \int_{-\infty}^{+\infty} x(t)x(t-\tau)\,dt$$

$$R_{xx}(+\tau) = \int_{-\infty}^{+\infty} x(s+\tau)x(s)\,ds$$

$$s = t-\tau$$

$$ds = dt$$

$$R_{yy}(\tau) = \int_{-\infty}^{+\infty} x(t-t_0)x(t-t_0+\tau)\,dt$$

$$R_{xx}(\tau) = \int_{-\infty}^{+\infty} x(s)x(s+\tau)\,ds$$

$$y(t) = x(t-t_0)$$

Power Signal

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_T x(t)x(t+\tau)\,dt$$

average signal power

$$R_{xx}(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-\infty}^{+\infty} x^2(t)\,dt$$

$$R_{xx}(0) \ge R_{xx}(\tau)$$

$$R_{xx}(-\tau) = \lim_{T\to\infty} \frac{1}{T}\int_T x(t)x(t-\tau)\,dt$$

$$R_{xx}(+\tau) = \lim_{T\to\infty} \frac{1}{T}\int_T x(s+\tau)x(s)\,ds$$

$$R_{yy}(\tau) = \lim_{T\to\infty} \int^{T} x(t-t_0)x(t-t_0+\tau)\,dt$$

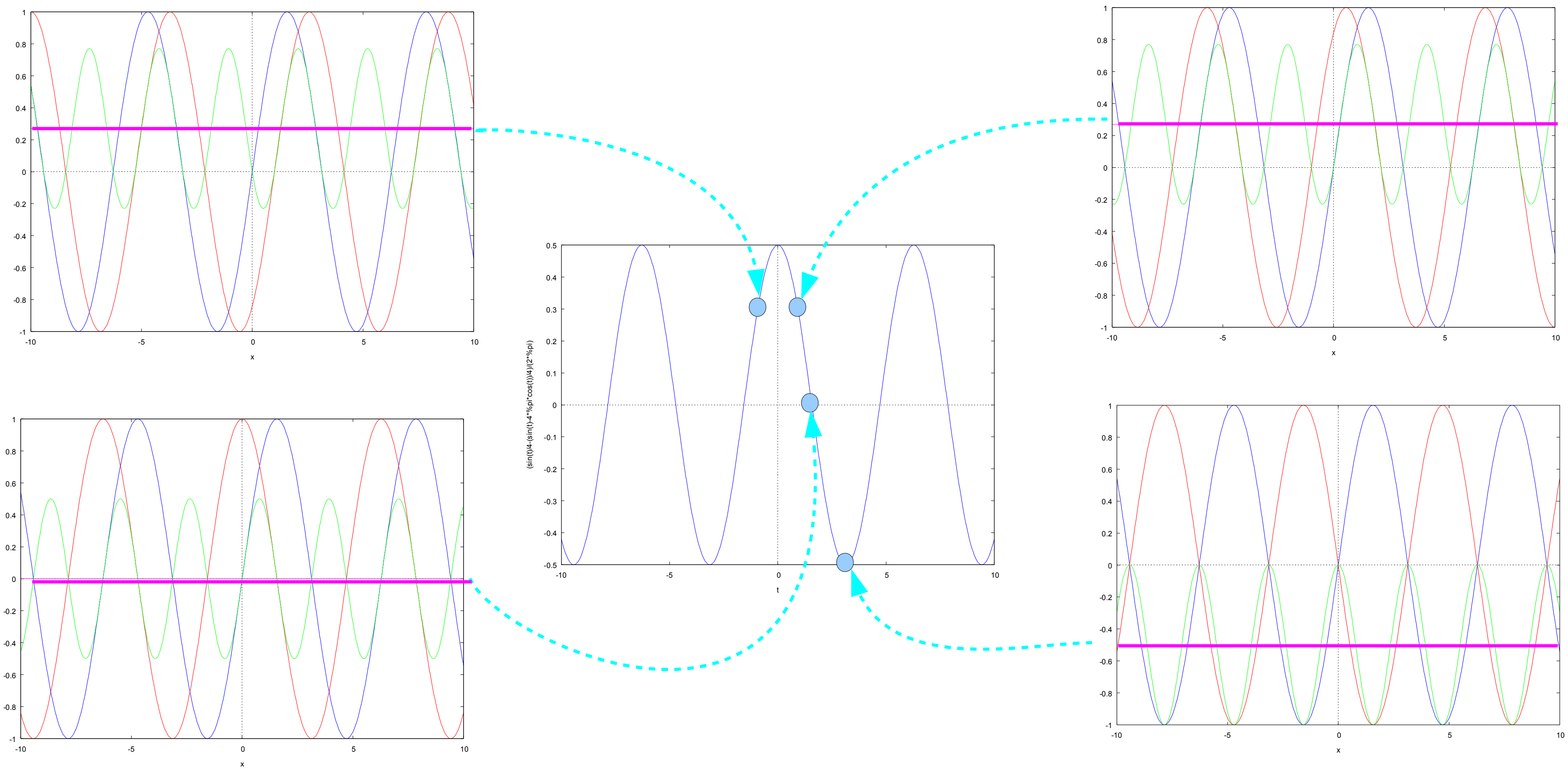
$$R_{xx}(\tau) = \lim_{T\to\infty} \int^{T} x(s)x(s+\tau)\,ds$$

AutoCorrelation for Power Signals

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{2\pi} \sin(t)\sin(t+\tau)\,dt$$

Positively correlated

Positively correlated



Uncorrelated

Negatively correlated

Autocorrelation of Sinusoids

$$x(t) = A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2) = x_1(t) + x_2(t)$$

$$x(t)x(t+\tau) = \{A_1\cos(\omega_1 t + \theta_1) + A_2\cos(\omega_2 t + \theta_2)\}\{A_1\cos(\omega_1(t+\tau) + \theta_1) + A_2\cos(\omega_2(t+\tau) + \theta_2)\}$$

$$= A_1\cos(\omega_1 t + \theta_1)A_1\cos(\omega_1(t+\tau) + \theta_1) + A_2\cos(\omega_2 t + \theta_2)A_2\cos(\omega_2(t+\tau) + \theta_2)$$

$$+ \underline{A_1\cos(\omega_1 t + \theta_1)A_2\cos(\omega_2(t+\tau) + \theta_2)} + \underline{A_2\cos(\omega_2 t + \theta_2)A_1\cos(\omega_1(t+\tau) + \theta_1)}$$

$$\int_T A_1\cos(\omega_1 t + \theta_1)A_2\cos(\omega_2(t+\tau) + \theta_2)\,dt = 0$$

$$\int_T A_2\cos(\omega_2 t + \theta_2)A_1\cos(\omega_1(t+\tau) + \theta_1)\,dt = 0$$

$$R_x(\tau) = R_{x1}(\tau) + R_{x2}(\tau) \qquad x_k(t) = A_k\cos(2\pi f_k t + \theta_k)$$

Autocorrelation of Random Signals

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_k t + \theta_k)$$

$$R_x(\tau) = \sum_{k=1}^{N} R_k(\tau)$$

autocorrelation of $a_k \cos(\omega_k t + \theta_k)$

independent of choice of θ_k

random phase shift θ_k
the same amplitudes a
the same frequencies ω

$x_k(t)$ different look
$R_k(\tau)$ similar look

the amplitudes a
the frequencies ω

can be observed
in the autocorrelation $R_k(\tau)$

similar look but not exactly the same

describes a signal generally, but not exactly
– suitable for a random signal

Autocorrelation Examples

AWGN signal

changes rapidly with time

current value has no correlation with past or future values

even at very short time period

random fluctuation except large peak at $\tau = 0$

ASK signal : sinusoid multiplied with rectangular pulse

regardless of sin or cos, the autocorrelation is always even function

cos wave multiplied by a rhombus pulse

CrossCorrelation

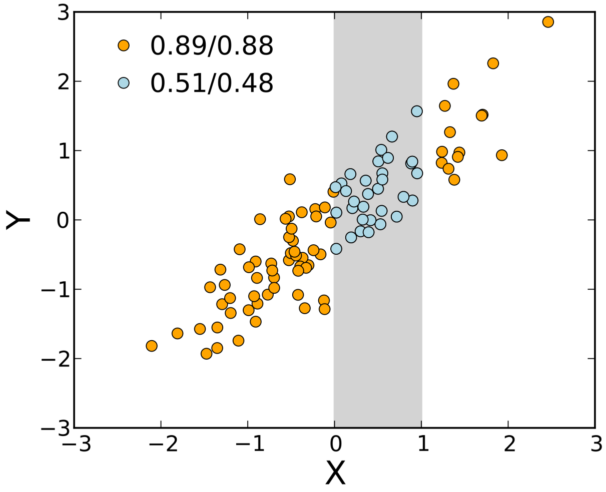
$$R_{xy}(\tau) = R_{xy}(-\tau)$$

The largest peak occurs at a shift which is exactly the amount of shift
Between x(t) and y(t)

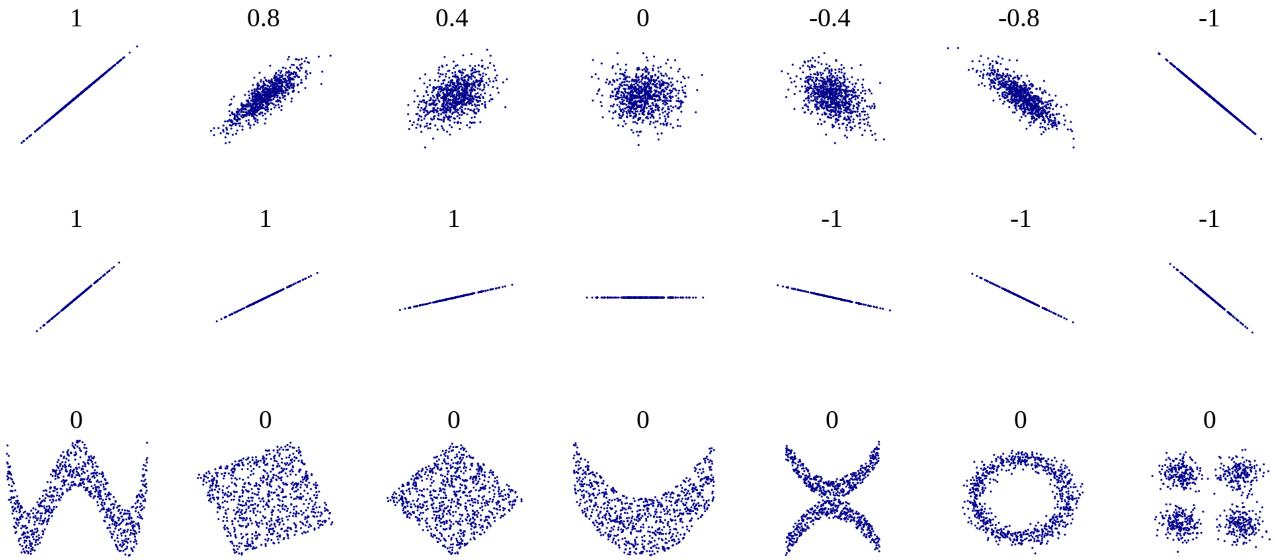
The signal power of the sum depends strongly on whether two signals are correlated

Positively correlated vs. uncorrelated

Pearson's product-moment coefficient



$$\rho_{XY} = \frac{\boldsymbol{E}[(X - m_x)(Y - m_Y)]}{\sigma_X \sigma_Y}$$



CrossCorrelation Example (1)

$$x_1(t) = \sin(\omega t)$$
$$x_2(t) = \sin(\omega t + \tfrac{\pi}{2}) = \cos(\omega t)$$
$$x_3(t) = \sin(\omega t + \tfrac{\pi}{4})$$
$$x_4(t) = \sin(\omega t + \pi)$$

$$R_{12}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\cos(\omega t)\, dt = 0$$
$$R_{13}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi/4)\, dt = 0.354$$
$$R_{14}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi)\, dt = -0.5$$

$$f(t) = x_1(t) + x_2(t) = \sin(\omega t) + \sin(\omega t + \frac{\pi}{2})$$
$$= \sin(\tfrac{(2\omega t + \pi/2)}{2})\cos(-\pi/4)$$
$$= \sin(\omega t + \tfrac{\pi}{4})\cos(\tfrac{-\pi}{4}) = 0.707 \sin(\omega t + \tfrac{\pi}{4})$$

sum of uncorrelated signals

$$g(t) = x_1(t) + x_3(t) = \sin(\omega t) + \sin(\omega t + \frac{\pi}{4})$$
$$= \sin(\tfrac{(2\omega t + \pi/4)}{2})\cos(-\pi/8)$$
$$= \sin(\omega t + \tfrac{\pi}{8})\cos(\tfrac{-\pi}{8}) = 0.924 \sin(\omega t + \tfrac{\pi}{4})$$

sum of positively correlated signals

The signal power of the sum depends strongly on whether two signals are correlated

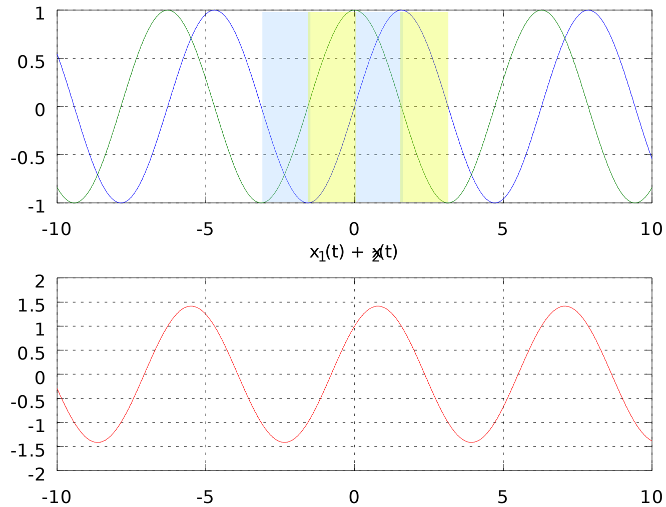
positively correlated vs. uncorrelated

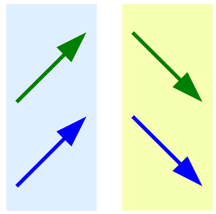
$$h(t) = x_1(t) + x_4(t) = \sin(\omega t) + \sin(\omega t + \pi)$$
$$= \sin(\omega t) - \sin(\omega t)$$
$$= 0$$

sum of negatively correlated signals

CrossCorrelation Example (2)

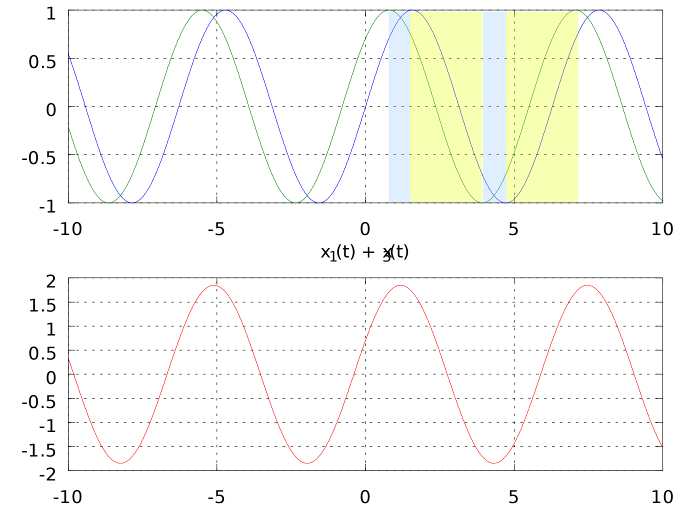
$$R_{12}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\cos(\omega t)\, dt = 0$$



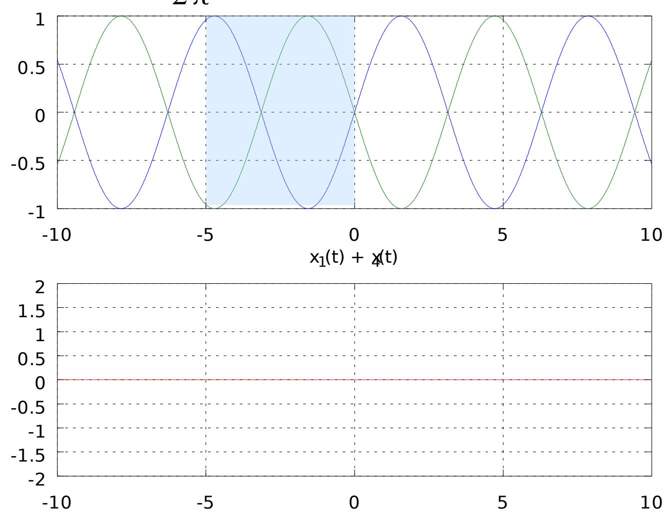


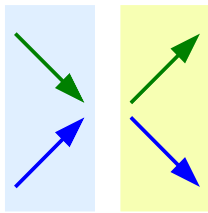
both signals increase or decrease

$$R_{13}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi/4)\, dt = 0.354$$



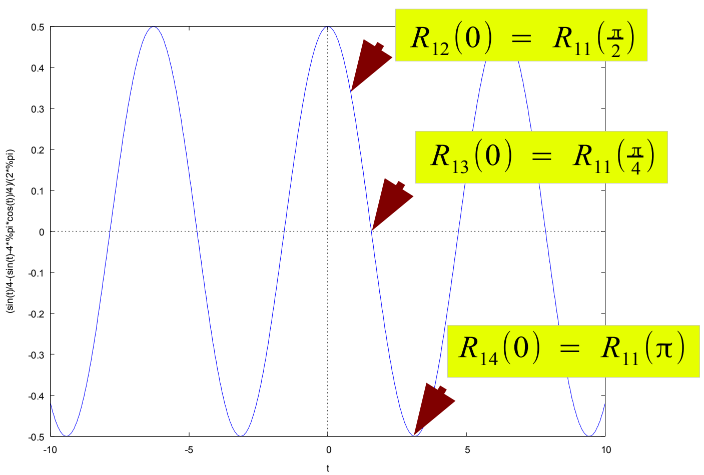
$$R_{14}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi)\, dt = -0.5$$





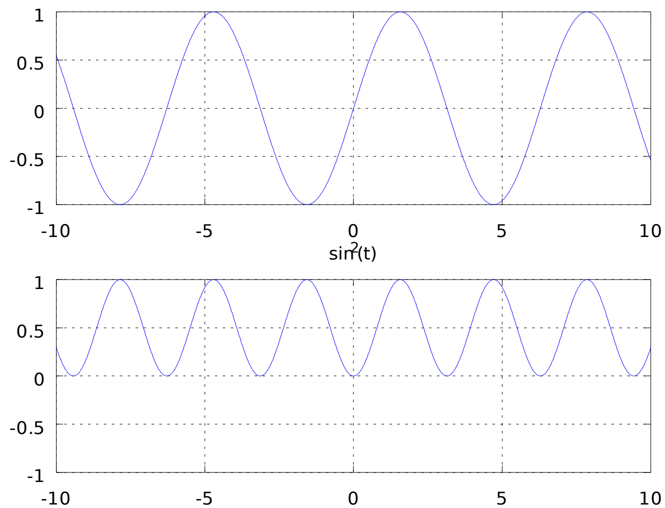
one increases, the other decreases

autocorrealation $R_{11}(\tau)$

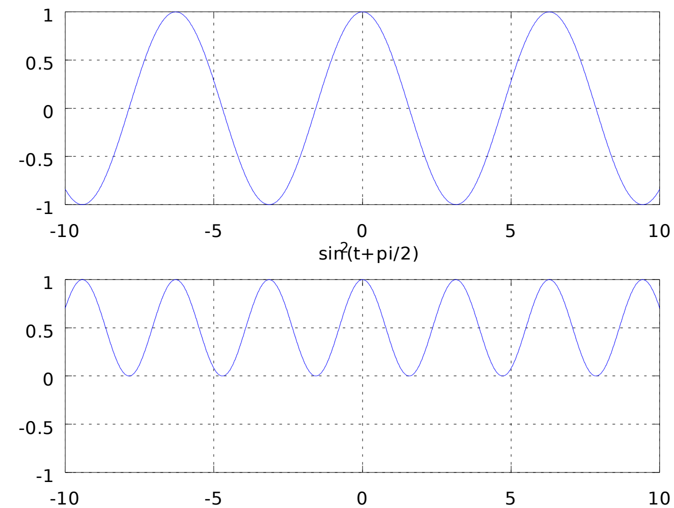


CrossCorrelation Example (3)

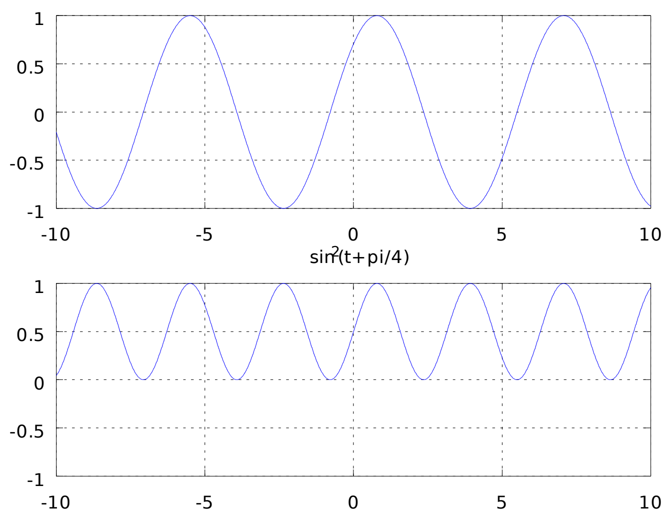
$$\sigma_1^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t)\, dt = 0.5 \qquad m_1 = 0$$



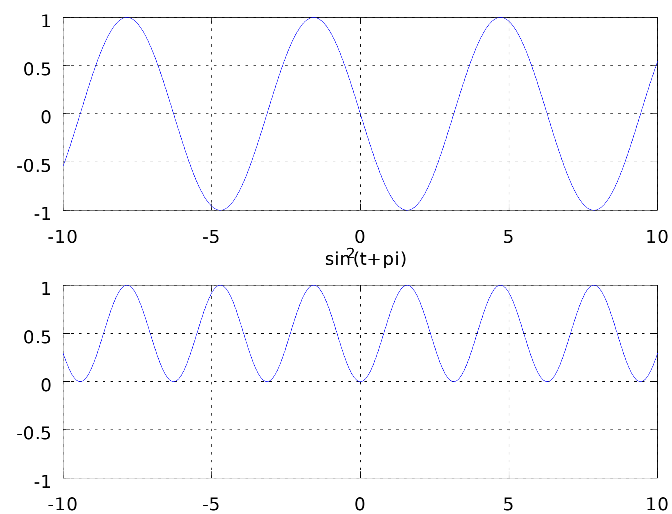
$$\sigma_2^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \frac{\pi}{2})\, dt = 0.5 \qquad m_1 = 0$$



$$\sigma_3^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \frac{\pi}{4})\, dt = 0.5 \qquad m_3 = 0$$



$$\sigma_4^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \pi)\, dt = 0.5 \qquad m_4 = 0$$



CrossCorrelation Example (4)

$$\sigma_1^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t)\, dt = 0.5 \quad m_1 = 0$$

$$\sigma_2^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \frac{\pi}{2})\, dt = 0.5 \quad m_1 = 0$$

$$\sigma_3^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \frac{\pi}{4})\, dt = 0.5 \quad m_3 = 0$$

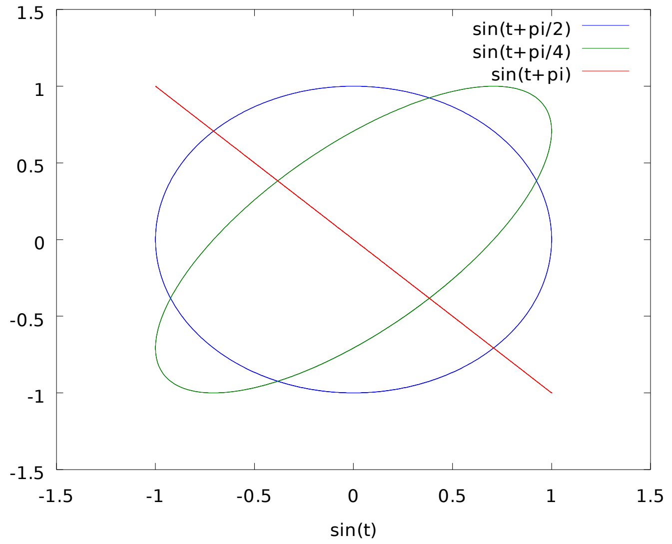
$$\sigma_4^2 = \frac{1}{2\pi}\int_{2\pi} \sin^2(\omega t + \pi)\, dt = 0.5 \quad m_4 = 0$$

$$R_{12}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\cos(\omega t)\, dt = 0$$

$$R_{13}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi/4)\, dt = 0.354$$

$$R_{14}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi)\, dt = -0.5$$

$$\rho_{XY} = \frac{\boldsymbol{E}[(X - m_x)(Y - m_Y)]}{\sigma_X \sigma_Y}$$



$$\rho_{12} = \frac{\boldsymbol{E}[x_1(t)x_2(t)]}{\sigma_1 \sigma_2} = \frac{R_{12}(0)}{0.5} = 0$$

$$\rho_{13} = \frac{\boldsymbol{E}[x_1(t)x_3(t)]}{\sigma_1 \sigma_3} = \frac{R_{13}(0)}{0.5} = 0.177$$

$$\rho_{14} = \frac{\boldsymbol{E}[x_1(t)x_4(t)]}{\sigma_1 \sigma_4} = \frac{R_{14}(0)}{0.5} = -1$$

CrossCorrelation Example (5)

$$x_1(t) = \sin(\omega t)$$

$$x_2(t) = \sin(\omega t + \tfrac{\pi}{2}) \quad \Leftrightarrow \quad x_1(t + \tfrac{\pi}{2\omega}) = \sin(\omega(t + \tfrac{\pi}{2\omega}))$$

$$x_3(t) = \sin(\omega t + \tfrac{\pi}{4}) \quad \Leftrightarrow \quad x_1(t + \tfrac{\pi}{4\omega}) = \sin(\omega(t + \tfrac{\pi}{4\omega}))$$

$$x_4(t) = \sin(\omega t + \pi) \quad \Leftrightarrow \quad x_1(t + \tfrac{\pi}{\omega}) = \sin(\omega(t + \tfrac{\pi}{\omega}))$$

$$R_{12}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\cos(\omega t)\, dt = 0 \quad \Leftrightarrow \quad R_{11}(\tfrac{\pi}{2\omega})$$

$$R_{13}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi/4)\, dt = 0.354 \quad \Leftrightarrow \quad R_{11}(\tfrac{\pi}{4\omega})$$

$$R_{14}(0) = \frac{1}{2\pi}\int_{2\pi} \sin(\omega t)\sin(\omega t + \pi)\, dt = -0.5 \quad \Leftrightarrow \quad R_{11}(\tfrac{\pi}{\omega})$$

CrossCorrelation

AutoCorrelation

Random Signal

No exact description of the signal

But we can estimate
- Autocorrelation
- Energy spectral densities (**ESD**)
- Power spectral densities **PSD**)

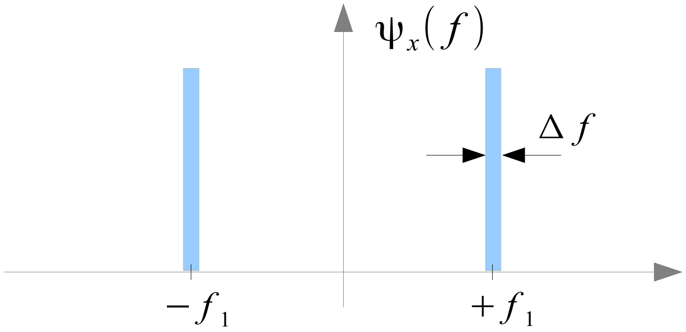
Total Energy

$$E_x = \int_{-\infty}^{+\infty} \Psi_x(f)\, d f$$

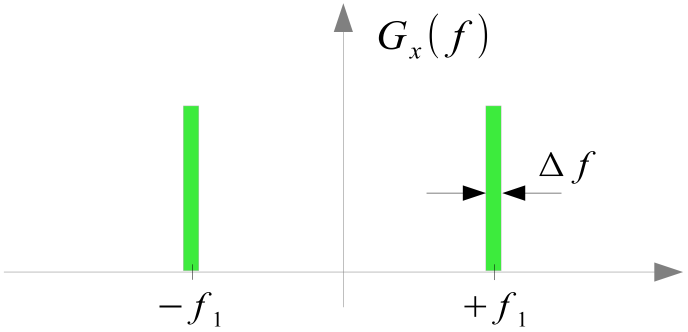
Average Power

$$P_x = \int_{-\infty}^{+\infty} G_x(f)\, d f$$

energy spectral densities



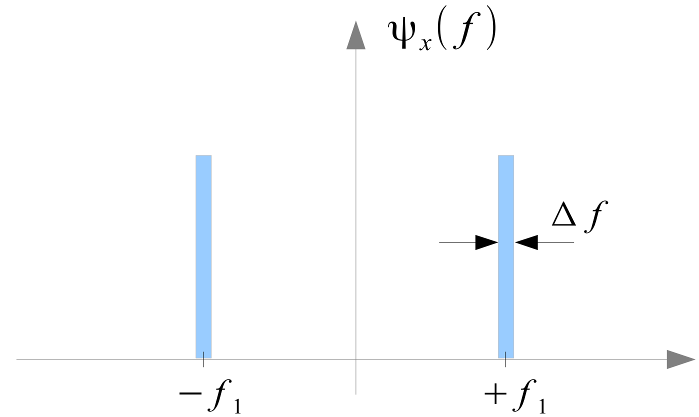
power spectral densities



Parseval's Theorem

$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

Energy Spectral Density $|X(f)|^2 = \Psi_x(f)$



The distribution of signal energy versus frequency

Real x(t) ➡ even, non-negative, real $\Psi_x(f)$

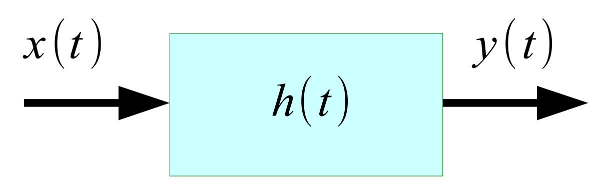
$$E_x = 2\int_{0}^{+\infty} \Psi_x(f) \, df$$

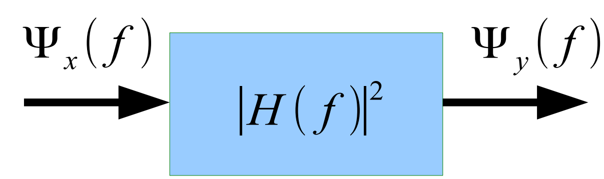
$$E_y = 2\int_{0}^{+\infty} \Psi_y(f) \, df = 2\int_{0}^{+\infty} |Y(f)|^2 \, df$$

$$= 2\int_{0}^{+\infty} |H(f)\, X(f)|^2 \, df = 2\int_{0}^{+\infty} |H(f)|\, |X(f)|^2 \, df$$

$$= 2\int_{0}^{+\infty} |H(f)|^2 \, \Psi_x(f) \, df$$

$$\Psi_y(f) = H(f) H^*(f) \Psi_x(f)$$



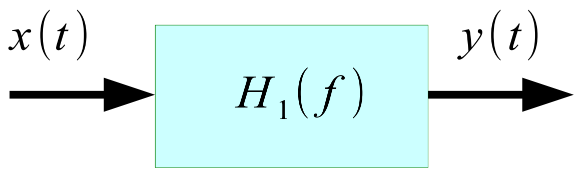


Conceptual ESD Estimation

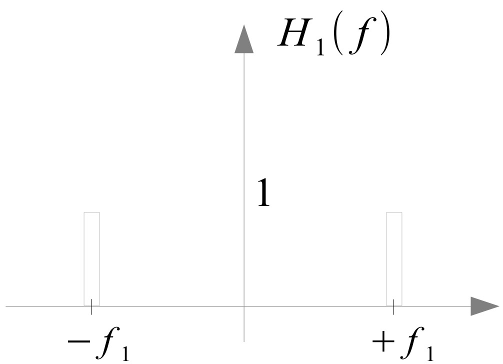
Parseval's Theorem

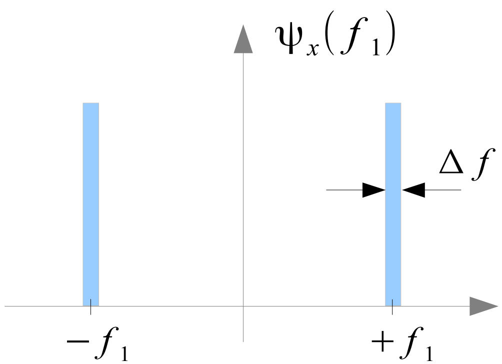
$$E_x = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X(f)|^2 \, df$$

Energy Spectral Density $\quad |X(f)|^2 = \Psi_x(f)$



$$E_y = \int_{-\infty}^{+\infty} |y(t)|^2 \, dt \qquad E_y / \Delta f \approx \Psi_x(f_1)$$





ESD and Autocorrelation

$R_x(t) \Longleftrightarrow \Psi_x(f) \quad \left(= |X(f)|^2 \right)$

$R_x(t) \Longleftrightarrow X^*(f)\,X(f)$

$$R_x(t) = x(-t) * x(t) = \int_{-\infty}^{+\infty} x(-\tau)\,x(t-\tau)\,d\tau \quad \Rightarrow \quad R_x(t) = \int_{-\infty}^{+\infty} x(\tau)\,x(\tau+t)\,d\tau$$

Power Spectral Density (PSD)

Many signals are considered as a **power signal**
The **steady state** signals activated for a long time ago will expected to continue

$$x_T(t) = rect\left(\frac{t}{T}\right)x(t) = \begin{cases} x(t) & |t| < \frac{T}{2} \\ 0 & \end{cases}$$

The **truncated version** of x(t)

$$X_T(f) = \int_{-\infty}^{+\infty} x_T(\tau)e^{-2\pi f t}\, dt = \int_{-T/2}^{+T/2} x_T(\tau)e^{-2\pi f t}\, dt$$

$$\Psi_{x_T}(f) = |X_T(f)|^2$$

The **ESD** of a **truncated version** of x(t)

$$G_{X_T}(f) = \frac{\Psi_{X_T}}{T} = \frac{1}{T}|X_T(f)|^2$$

The **PSD** of a **truncated version** of x(t)

$$G_x(f) = \lim_{T\to\infty} G_{X_T}(f) = \lim_{T\to\infty} \frac{1}{T}|X_T(f)|^2$$

The estimated **PSD**

$$2\int_{f_L}^{f_H} G(f)\,df$$

The power of a finite signal power signal in a bandwidth f_L to f_H

Conceptual PSD Estimation

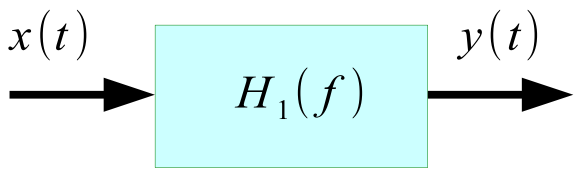
Parseval's Theorem

$$E_{x_T} = \int_{-\infty}^{+\infty} |x_T(t)|^2 \, dt = \int_{-\infty}^{+\infty} |X_T(f)|^2 \, df$$

Energy Spectral Density

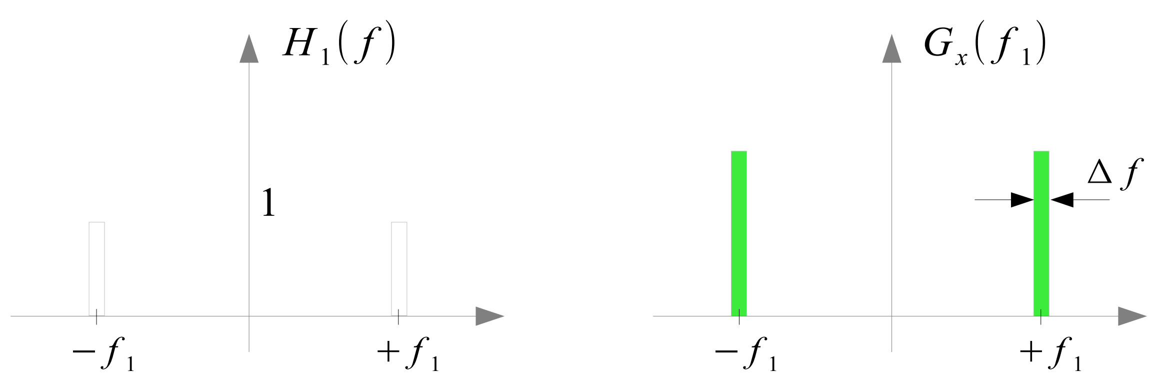
$$|X_T(f)|^2 = \Psi_{x_T}(f)$$

$$G_{X_T}(f) = \frac{\Psi_{X_T}}{T} = \frac{1}{T}|X_T(f)|^2$$



$$P_y = \frac{1}{T}\int_{-\infty}^{+\infty} |y(t)|^2 \, dt$$

$$P_y / \Delta f \approx G_x(f_1)$$



ESD and Band-pass Filtering

$$E_y = 2\int_0^{+\infty} \Psi_y(f)\, df = 2\int_0^{+\infty} |Y(f)|^2\, df = 2\int_0^{+\infty} |H(f)X(f)|^2\, df$$

$$E_y = 2\int_0^{+\infty} |H(f)|^2 \Psi_x(f)\, df = 2\int_{f_L}^{f_H} \Psi_x(f) df$$

$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f) H^*(f) \Psi_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

PSD and Band-pass Filtering

$$G_y(f) = |H(f)|^2 G_x(f) = H(f)H^*(f)G_x(f)$$

A description of the signal energy versus frequency
How the signal energy is distributed in frequency

References

[1] http://en.wikipedia.org/
[2] M.J. Roberts, Signals and Systems,